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Abstract

Automatic structures are a subject which has gained a lot of attention in the "logic in computer science" community during the last fifteen years. Roughly speaking, a structure is automatic if its domain, relations and functions can be recognized by finite automata on strings or trees. In particular, such structures are finitely presentable. The investigation of automatic structures is largely motivated by the fact that their first-order theories are uniformly decidable. The corresponding decision procedure takes an automatic presentation of some structure and a first-order sentence as input and checks whether the structure satisfies the sentence by means of constructions and algorithms for finite automata.

In this thesis, we study the model-theoretic complexity of automatic linear orders from two perspectives: in terms of the finite-condensation rank and by means of the Ramsey degree. Intuitively, the finite-condensation rank of a linear order is an ordinal which indicates how far the linear order is away from being dense. Our corresponding main results establish optimal upper bounds on the finite-condensation ranks of automatic linear orders with respect to several notions of automaticity. In this regard, we focus particularly on subclasses of automatic structures which are obtained by restricting language-theoretic properties

of the underlying domains. We further show that the separating line between string-automatic and tree-automatic scattered linear orders can also be drawn in terms of the finite-condensation rank. As an application of this result, we further provide a partial solution to the isomorphism problem for tree-automatic ordinals.

The Ramsey degree of an ordinal measures its model-theoretic complexity by means of the partition relations studied in combinatorial set theory. We investigate this concept in a purely set-theoretic setting as well as in the context of automatic structures. Concerning the set-theoretic case, we show that all ordinals below ω^ω possess a finite Ramsey degree and provide a range of ordinals beyond ω^ω whose Ramsey degrees are infinite. The results in the automatic setting are very similar, except that we prove that all automatic ordinals beyond ω^ω have an infinite Ramsey degree. Last but not least, we conclude this thesis by providing a tree-automatic version of Ramsey's theorem.

Zusammenfassung

Automatische Strukturen sind auf dem Forschungsgebiet „Logik in der Informatik“ seit etwa 15 Jahren ein viel beachtetes Thema. Eine Struktur ist, vereinfacht gesagt, genau dann automatisch, wenn ihre Trägermenge, ihre Relationen und ihre Funktionen allesamt durch endliche Automaten auf Wörtern oder Bäumen erkennbar sind. Insbesondere sind derartige Strukturen endlich darstellbar. Die Hauptmotivation zur Untersuchung automatischer Strukturen liegt in der uniformen Entscheidbarkeit ihrer prädikatenlogischen Theorien erster Stufe. Die zugrundeliegende Entscheidungsprozedur bekommt eine automatische Darstellung einer Struktur und einen prädikatenlogischen Satz als Eingabe und überprüft mithilfe von Konstruktionen und Algorithmen für endliche Automaten, ob der Satz in der Struktur gültig ist.

In dieser Dissertation untersuchen wir die modelltheoretische Komplexität automatischer linearer Ordnungen bezüglich der zwei Komplexitätsmaße Kondensationsrang und Ramsey-Grad. Der Kondensationsrang einer linearen Ordnung misst ihre Abweichung von der Eigenschaft der Dichtheit durch eine Ordinalzahl. Unsere Hauptergebnisse in diesem Zusammenhang leiten für verschiedene Begriffe von Automatizität optimale obere Schranken für die Kondensationsränge automatischer linearer Ordnungen her. Dabei liegt der Fokus vor allem auf Teilklassen

automatischer Strukturen, die die zugrundeliegenden Trägermengen anhand sprachtheoretischer Eigenschaften einschränken. Des Weiteren zeigen wir, dass die Trennlinie zwischen wort- und baumautomatischen verteilten linearen Ordnungen auch vermittels des Kondensationsranges gezogen werden kann. Eine Anwendung dieses Ergebnisses ermöglicht uns eine teilweise Lösung des Isomorphieproblems für baumautomatische Ordinalzahlen.

Der Ramsey-Grad einer Ordinalzahl misst ihre modelltheoretische Komplexität mithilfe von Partitionsrelationen aus der kombinatorischen Mengenlehre. Wir untersuchen dieses Konzept sowohl aus rein mengentheoretischer Sicht als auch im Kontext automatischer Strukturen. Im mengentheoretischen Fall zeigen wir, dass alle Ordinalzahlen unterhalb von ω^ω einen endlichen Ramsey-Grad besitzen und geben einen Bereich von Ordinalzahlen oberhalb von ω^ω an, deren Ramsey-Grade unendlich sind. Die Ergebnisse im automatischen Fall sind sehr ähnlich, mit Ausnahme der Tatsache, dass die Ramsey-Grade aller Ordinalzahlen oberhalb von ω^ω unendlich sind. Zu guter Letzt schließen wir diese Dissertation mit dem Beweis einer baumautomatischen Version des Satzes von Ramsey ab.

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Martin Huschenbett
November 2015

Contents

1 Introduction

At first glance, computers seem to be one of the greatest tools for mathematical problem-solving ever invented. Yet a second glance unveils a huge mismatch: Many mathematical problems involve questions about *infinite* objects in some way, whereas the two most important resources of computers—memory space and computation time—are inherently *finite*. This mismatch manifests, for instance, in *Gödel's first incompleteness theorem* [Göd31] and the negative answer to the *Entscheidungsproblem* given by Church [Chu36a] and Turing [Tur37]. Remarkably enough, all these results predate the invention of the computer in the 1940s. Accordingly, they are not based on the computational power of any real device but rather on "the intuitive notion of effective calculability" [Chu36b] or an abstraction of "a man in the process of computing" [Tur37].

The incompleteness theorem basically states that any logical theory which is generated by a computable set of axioms and includes certain basic facts about elementary arithmetic is incomplete, that is to say it contains statements which can neither be proved nor disproved from the axioms. An immediate consequence is that the first-order theory of arithmetic $(\mathbb{N}; +, \times)$ itself cannot be decided by a computer, cf. [Chu36b]. The Entscheidungsproblem was posed by Hilbert [AH28] and asks for an algorithm which takes a statement and a finite list of axioms, both formalized in first-order logic, as input and decides whether the statement follows from the axioms or not. According to Church and Turing, such an algorithm does not exist.

Despite those limitations, mathematicians and theoretical computer scientists succeeded to find decision procedures for many logical theories. A very prominent result is due to Presburger [Pre30], who demonstrated that the first-order theory of $(\mathbb{N}; +)$, nowadays known as *Presburger's arithmetic*, can be decided using the method of *quantifier elimination*. Another noteworthy application of this technique is Tarski's proof that the first-order theories of $(\mathbb{R}; +, \times, \leqslant)$ and Euclidean geometry are decidable [Tar51].[1]

In the beginning of the 1960s, the recently established field of *automata theory* gave a boost to the development of decision procedures for logical theories. Using results and methods from this new field, Büchi [Büc60], Elgot [Elg61] and Trakhtenbrot [Tra62] independently showed the weak monadic theory[2] of $(\mathbb{N}; \leqslant)$ to be decidable. Later on, this approach was extended to the (non-weak) monadic theories of $(\mathbb{N}; \leqslant)$ and all other countable well-orders by Büchi [Büc62, Büc65] and to the weak monadic theory of the full binary tree by Doner [Don65] and, independently, by Thatcher and Wright [TW68]. Eventually, this development culminated in *Rabin's tree theorem* [Rab69], which states that the monadic theory of the full binary tree is decidable.

In addition to his aforementioned result, Büchi [Büc60] provided an alternative proof of Presburger's decidability result,

[1] Tarski claims that his method was "found in 1930 but previously unpublished" [Tar51, 2].

[2] (Weak) monadic logic, also called (weak) monadic second-order logic, extends first-order logic by variables which range over (finite) subsets of the domain.

which is based on a syntactic reduction to the weak monadic theory of $(\mathbb{N}; \leqslant)$ by means of a logical interpretation. Actually, the syntactic reduction and the subsequent automata-theoretic decision procedure can easily be merged into one "purely" automata-theoretic algorithm without loosing conceptual clarity. Abstracting from how the structure $(\mathbb{N}; +)$ is implicitly presented to the merged algorithm, Hodgson [Hod82, Hod83] introduced the concept of *automatic structures* as a systematic approach towards deciding first-order theories.

Roughly speaking, a structure is *automatic* if its domain is a regular language of strings and its relations are recognizable by synchronous finite multi-tape automata. Such automata take tuples of strings as input, each entry initially written on its own read-only tape, and processes the tapes from left to right with all heads moving at the same speed. For instance, the implicit presentation of $(\mathbb{N}; +)$ mentioned above works as follows: Each number is encoded by its binary representation (least significant bit first) and a finite automaton with three tapes implements the usual ripple-carry addition in order to recognize the relation "$x + y = z$". Just like intended by Hodgson's definition, the first-order theory of any automatic structure can be decided by the automata-theoretic algorithm. In fact, this decision procedure is uniform in the automatic structure, that is to say it still works if the finite automata presenting the structure are not fixed but given as part of the input.

More than a decade later, Khoussainov and Nerode [KN95] independently rediscovered the concept of automatic structures. Unlike Hodgson, their motivation originated in *computable model theory*, cf. [EGNR98]. More precisely, they were interested in a formalism for presenting infinite structures which is more feasible than *(polynomial time) computable structures*. Accordingly, they restricted the model of computation, which is allowed in presentations of structures, from Turing machines to finite (multi-tape) automata. Despite this second discovery of automatic structures, they did not become an active field of research until Blumensath and Grädel [BG00] introduced them to the "logic in computer science" community a few years later. Recalling the efforts from the 1960s, the ensuing research also took the generalizations to *ω-string-automatic* and *tree-automatic structures* into account, although the main focus remained on *string-automatic structures*.[3] Most of the progress which has been made since that time is covered by the two surveys [Rub08, BGR11].

In view of its importance for Hodgson's as well as Khoussainov and Nerode's motivation, the problem of characterizing the automatic members of certain classes of structures, such as groups or linear orders, gained much attention. One of the first results in this line of research was obtained by Delhommé [Del04], who showed that the string-automatic and tree-automatic ordinals are precisely those below ω^ω and ω^{ω^ω}, respectively. Moreover, the string-automatic members of several other classes were completely characterized, including finitely generated groups [OT05], Boolean algebras and fields [KNRS07]. In contrast, for string-automatic linear orders and order trees only partial characterizations in terms of upper bounds on some model-theoretic rank are known [KRS05]. Later on, it turned out that ω-string-automatic ordinals and, more generally, scattered linear orders[4] are effectively string-automatic and hence the (partial) characterizations carry over [Kus11].

Characterizing the automatic members of some class is closely related to its *isomorphism problem*: Given two automatic presentations of structures from this class, decide whether the presented structures are isomorphic. As a matter of fact, the characteriza-

[3] In fact, *ω-tree-automatic structures* were also considered but are still lacking remarkable results.

[4] A linear order is *scattered* if it does not embed the linear order of the rationals.

tions of the string-automatic ordinals, Boolean algebras and fields immediately led to decision procedures for the corresponding isomorphism problems [KRS05, KNRS07]. In contrast, the general isomorphism problem for string-automatic structures is highly undecidable. To be exact, it is complete for the first existential level Σ^1_1 of the analytical hierarchy and hence as hard as the isomorphism problem for arbitrary computable structures [KNRS07].[5] This complexity remains the same even for some subclasses, such as semigroups [Nie07], linear orders or order trees [KLL13b]. Obviously, this Σ^1_1-completeness is also inherited by the isomorphism problem for tree-automatic structures. The isomorphism problem for ω-string-automatic structures is even harder and not contained in the analytical hierarchy at all [KLL13a].

Apparently, string-automatic linear orders gained quite some attention, whereas there is only little knowledge of tree-automatic linear orders. In chapter 3, we improve this situation in two ways.[6] First of all, we partially characterize the tree-automatic linear orders in terms of an upper bound on the same model-theoretic rank mentioned above. In addition, we establish similar bounds for two natural hierarchies of subclasses inside the string-automatic and tree-automatic structures. Roughly speaking, these hierarchies are obtained by restricting certain language-theoretic properties of the permitted domains. Our second contribution investigates the relationship between string-automaticity and tree-automaticity in the context of scattered linear orders. More precisely, we give a decidable characterization of those tree-automatic scattered linear orders which are already string-automatic. As a consequence of this result we further obtain that the isomorphism problem for

[5] Intuitively, this result says that the only way to establish isomorphy is to find an isomorphism.

[6] A detailed overview of the current knowledge of automatic linear orders and our results can be found in the introduction to chapter 3 starting on page 55. The same applies to the subjects of chapters 4 and 5.

tree-automatic ordinals below ω^{ω^2} is decidable. Even if this result might seem very limited, it marks actual progress: While the decision procedure for the string-automatic case heavily builds on the fact that first-order logic plays well with ordinals below ω^ω, this nice interplay is no longer available beyond ω^ω, cf. [Büc65].

The correctness of our new decision procedure relies on an argument involving the infinitary version of *Ramsey's theorem*. Unfortunately, this argument cannot be extended beyond ω^{ω^2} since the guarantees given by Ramsey's theorem are in a way too weak for that purpose. The outcome of our efforts to find a more adequate variant of Ramsey's theorem is quite ambivalent: On the one hand, none of the results we obtained actually helped to improve the limitations of our decision procedure. On the other hand, the collection of these results soon evolved into a subject being of interest on its own. Although answering the questions which subsequently popped up led us astray from the isomorphism problem for tree-automatic ordinals and into *combinatorial set theory*, we took this lead. The result of this deviation is the (purely set-theoretic) *polychromatic Ramsey theory for ordinals* presented in chapter 4. Roughly speaking, this theory studies which ordinals α admit a natural number n with the following property: Every complete graph, whose nodes form a well-order of type at least α and whose edges are colored by finitely many colors, contains a subset of order type α whose internal edges use at most n different colors.

One of the main concerns of computable model theory is the effective content of (purely set-theoretic) mathematical results. As a matter of course, this concern has also played a certain role in the investigation of automatic structures. Remarkable results in this context include string-automatic versions of Cantor's theorem [Kus03], Kőnig's lemma [KRS05] and Ramsey's theorem [Rub08]. The latter result, also known as *Rubin's theorem*, states that every string-automatic edge coloring of the countably infinite complete graph by finitely many colors contains an infinite subset (of the set of nodes) being monochromatic and regular at the same time. Given a string-automatic presentation of the coloring and a color, one can even decide whether there is an infinite subset using only this color and, in case of a positive answer, compute a finite automaton recognizing such a subset. In chapter 5, we revisit our polychromatic Ramsey theory for ordinals from this point of view and establish similar automatic versions of its main results. Last but not least, we reuse the techniques developed in the course of these investigations in order to contribute a new tree-automatic version of Ramsey's theorem, which complements Kartzow's result [Kar11]. While he established decidability of the existence of (possibly non-regular) infinite subsets using a certain color only, we focus on the existence of infinite subsets which are monochromatic and regular.

2 Preliminaries

In this chapter, we present the fundamental concepts required for the results in the subsequent chapters. These fundamentals center around automatic structures and linear orders. For the most part, we assume basic familiarity with the presented topic and the primary purpose is to fix notation. We accompany this rather minimalistic approach by providing references to the literature on all these topics. In the last section of this chapter, we prove a first result on automatic structures which is not particularly connected to the investigations taken out in the next chapters.

2.1 Logic

This section presents the fundamental notions of logic needed in the later chapters. For a detailed overview, we refer the reader to the book "Model theory" by Hodges [Hod93].

2.1.1 Relational Structures and First-Order Logic

Throughout this thesis, we only deal with logic over relational structures, i.e., structures which do neither possess constants nor functions. A *(relational) structure* is a tuple

$$\mathscr{A} = (A; R_1^{\mathscr{A}}, \ldots, R_n^{\mathscr{A}})$$

consisting of an arbitrary set A and relations $R_i \subseteq A^{r_i}$ for some $r_i \in \mathbb{N}$. The set A is called *domain* (or *universe*) of $\mathscr{A}$. We agree on the convention that whenever a structure is named by some capital calligraphic letter, its domain is named by the very same letter in Roman. The symbols R_i are called *relation symbols*, the actual relation $R_i^{\mathscr{A}}$ is the *interpretation* of R_i in $\mathscr{A}$ and r_i is the *arity* of both R_i and $R_i^{\mathscr{A}}$. Whenever there is only one structure in scope which uses a certain relation symbol R, we usually omit the superscript $\mathscr{A}$ from its interpretation $R^{\mathscr{A}}$. Accordingly, we usually introduce $\mathscr{A}$ as "the structure $\mathscr{A} = (A; R_1, \ldots, R_n)$".

Two structures $\mathscr{A}$ and $\mathscr{B}$ are *isomorphic* if they use the same relation symbols $R_1, \ldots, R_n$ with the same arities $r_1, \ldots, r_n$, respectively, and there is a bijection $f\colon A \to B$ such that, for each $i \in [1, n]$ and all $\boldsymbol{u} \in A^{r_i}$,

$$\boldsymbol{u} \in R_i^{\mathscr{A}} \iff f(\boldsymbol{u}) \in R_i^{\mathscr{B}}.$$

In this situation, the map f is called an *isomorphism between $\mathscr{A}$ and $\mathscr{B}$*.

We define *first-order logic* as usual, including the equality predicate. We fix an infinite set of *(individual) variables*. It is customary to denote these variables by small letters such as x, y, z or $x_1, x_2, \ldots$. The *atomic formulas* of first-order logic are $R(x_1, \ldots, x_r)$ and $x = y$, where R is a relation symbol and $x_1, \ldots, x_r, x, y$ are variables. These atomic formulas are composed to more complex *(first-order) formulas* by means of the Boolean connectives disjunction $\vee$, conjunction $\wedge$, negation $\neg$, implication $\rightarrow$ and equivalence $\leftrightarrow$ as well as existential and universal quantification, written as $\exists x \ldots$ and $\forall x \ldots$, respectively. In some situations, we further consider the quantifier "there are infinitely many", written as $\exists^\infty x \ldots$.

Let ϕ be a first-order formula and $\mathscr{A}$ a structure. We say that ϕ and $\mathscr{A}$ are *suitable* for one another if ϕ uses only relation symbols which appear in $\mathscr{A}$ with the same arity. For a formula ϕ and individual variables $x_1, \ldots, x_r$, we write $\phi(x_1, \ldots, x_r)$ to put that the free variables of ϕ are among $x_1, \ldots, x_r$. A formula without free variables is called *(first-order) sentence*. Conventionally, we name sentences by capital Greek letters. For a formula $\phi(x_1, \ldots, x_r)$, a structure $\mathscr{A}$ and elements $u_1, \ldots, u_r \in A$, we write

$$\mathscr{A} \models \phi[u_1, \ldots, u_r]$$

to denote the fact that ϕ is suitable for $\mathscr{A}$ *and* $\mathscr{A}$ satisfies the formula ϕ when the free occurrences of x_i are interpreted by u_i. The *first-order theory* of a structure $\mathscr{A}$ is a the set of all first-order sentences Φ with $\mathscr{A} \models \Phi$.

2.1.2 Monadic Second-Order Logic and Interpretations

Monadic second-order logic or, for short, *mso logic* extends first-order logic by a new kind of variables along with quantifiers and atomic formulas for these variables. More precisely, the new variables are called *set variables* and range over subsets of the domain of the structure under consideration. To emphasize the difference between the two kinds of variables, the "old" variables from first-order logic are called *individual variables* as they range over individual elements of the structure. It is customary to name set variables by capital letters like X, Y, Z and $X_1, X_2, \ldots$. In order to make set variables accessible, mso logic contains existential and universal quantifiers for these variables, written as $\exists X \ldots$ and $\forall X \ldots$, respectively. In addition, mso logic adds the new atomic formula $X(x)$ which evaluates to true if the interpretation of the individual variable x is a member of the interpretation of the set variable X. Moreover, we freely use abbreviations such as $X = Y$, $X \subseteq Y$, $X \cup Y = Z$ and $X \cap Y = \emptyset$, which are easily expressible by mso formulas.

In contrast to first-order logic, mso logic can express transitive closure. More precisely, for every mso formula $\phi(x, y)$, there is an mso formula $\phi^*(x, y)$ such that, for any structure $\mathscr{A}$ suitable for ϕ and all $u, v \in A$, we have $\mathscr{A} \models \phi^*[u, v]$ if and only if there are $n \geqslant 0$ and $w_0, w_1, \ldots, w_n \in A$ with $u = w_0$, $v = w_n$ and $\mathscr{A} \models \phi[w_{i-1}, w_i]$ for each $i \in [1, n]$. For instance, the formula

$$\forall X \Big(X(x) \wedge \forall z, z' \big(X(z) \wedge \phi(z, z') \to X(z') \big) \to X(y) \Big) \tag{2.1}$$

is a possible choice for $\phi^*(x, y)$.

In section 2.4.4, our investigations are based on the idea of defining one structure in another. This idea is formalized by the notion of an *interpretation*. Let $\mathscr{A} = (A; R_1, \ldots, R_n)$ and $\mathscr{B}$ be structures and r_i the arity of R_i. A *monadic second-order interpretation* or, for short, *mso interpretation* of $\mathscr{A}$ in $\mathscr{B}$ is a tuple $\mathcal{I} = (\delta; \varphi_{R_1}, \ldots, \varphi_{R_n})$ of mso formulas suitable for $\mathscr{B}$ satisfying the following conditions:

(1) δ has precisely one free individual variable, each φ_{R_i} has precisely r_i free individual variables and no set variables are free in any of these formulas.

(2) There is an injective map $f \colon A \to B$ such that, for all $v \in B$,

$$v \in f(A) \iff \mathscr{B} \models \delta[v]$$

and, for all $i \in [1, n]$ and $\boldsymbol{u} \in A^{r_i}$,

$$\boldsymbol{u} \in R_i \iff \mathscr{B} \models \varphi_{R_i}[f(\boldsymbol{u})]\,.$$

Put another way, condition (2) ensures that the map f is an isomorphism between $\mathscr{A}$ and the structure

$$\mathcal{I}(\mathscr{B}) := (A'; R'_1, \ldots, R'_n)$$

defined by

$$A' := \{ v \in B \mid \mathscr{B} \models \delta[v] \}$$

and

$$R'_i := \{ \boldsymbol{v} \in (A')^{r_i} \mid \mathscr{B} \models \varphi_{R_i}[\boldsymbol{v}] \} .$$

Whenever we want to emphasize the map f, we say that $\mathcal{I}$ is an mso interpretation of $\mathscr{A}$ in $\mathscr{B}$ *via* f.

The main benefit of mso interpretations is that they provide a way to reduce the mso theory of $\mathscr{A}$ to the mso theory of $\mathscr{B}$. More precisely, there is a *syntactic* transformation which assigns to every mso sentence Φ suitable for $\mathscr{A}$ an mso sentence $\Phi^{\mathcal{I}}$ suitable for $\mathscr{B}$ with the property that $\mathscr{A} \models \Phi$ holds true if and only if $\mathscr{B} \models \Phi^{\mathcal{I}}$. Roughly speaking, $\Phi^{\mathcal{I}}$ is obtained from Φ by relativizing all quantifiers to (sets of) elements satisfying the formula δ and replacing each atomic subformula $R_i(x_1, \ldots, x_{r_i})$ with $\varphi_{R_i}(x_1, \ldots, x_{r_i})$.

2.2 Linear Orders

The purpose of this section is to recall the fundamentals on linear orders and ordinals. Moreover, we provide the necessary background on the finite-condensation rank. For the most part, we loosely follow the presentation in the book “Linear orderings” by Rosenstein [Ros82].

2.2.1 Basic Notations

A *linear order* is a relational structure $(A; \leqslant_A)$ where $\leqslant_A$ is a *linear ordering* of A, i.e., a reflexive, transitive, anti-symmetric and total relation on A. The corresponding *strict linear ordering* of A is denoted by $<_A$. As is customary, we identify the domain A with the linear order $(A; \leqslant_A)$ in many situations and simply call A a linear order then. Whenever we do so, we denote the linear ordering of A by $\leqslant_A$ or even just by $\leqslant$ if there is no danger of confusion. The *order type* or sometimes just *type* of a linear order A is the isomorphism type of A, i.e., the class of all structures which are isomorphic to A. In order to slightly simplify notation, we use the phrase "type τ linear order A" for "linear order A of type τ". The order types of the linear orders $(\mathbb{N}; \leqslant)$, $(\mathbb{N}; \geqslant)$, $(\mathbb{Z}; \leqslant)$ and $(\mathbb{Q}; \leqslant)$ are denoted by ω, $\omega^\star$, ζ and η. The order type of a finite linear order with n elements is simply denoted by n as well.[1]

Let A and B be linear orders. An *embedding* of A into B is an injective map $f\colon A \to B$ such that $u \leqslant_A v$ implies $f(u) \leqslant_B f(v)$ for all $u, v \in A$. Equivalently, a map $f\colon A \to B$ is an embedding if $u <_A v$ implies $f(u) <_B f(v)$ for all $u, v \in A$. Notice that there might be embeddings $f\colon A \to B$ and $g\colon B \to A$ although A and B are *not* isomorphic.

Let I be a linear order and A_i a linear order for each $i \in I$. The *I-sum* of the A_i, denoted by $\sum_{i \in I} A_i$, is the linear order A defined by

$$A := \biguplus_{i \in I} A_i$$

and $u \leqslant_A v$ if either there are $i, j \in I$ with $i <_I j$, $u \in A_i$ and $v \in A_j$ or there is $i \in I$ with $u, v \in A_i$ and $u \leqslant_{A_i} v$. If I is finite, say $I = \{1, \ldots, n\}$ ordered naturally, we also write $A_1 + A_2 + \cdots + A_n$ for the I-sum of the A_i. Clearly, replacing the A_i by isomorphic linear orders yields an isomorphic I-sum. Put another way, we can also build sums of order types.

The *product* of two linear orders A and B is the linear order $A \cdot B$ defined by

$$A \cdot B := A \times B$$

[1] In order to limit potential confusion, we use this notation without further notice only in arithmetical expressions of order types, e.g., $\eta + 1$, $\zeta \cdot n$ or ω^n, where $n \in \mathbb{N}$.

and $\langle u_1, v_1\rangle \leqslant_{A\cdot B} \langle u_2, v_2\rangle$ if either $v_1 <_B v_2$ or both $v_1 = v_2$ and $u_1 \leqslant_A u_2$. Notice that $A \cdot B$ is isomorphic to the sum $\sum_{v\in B} A$. Moreover, the order type of $A \cdot B$ is completely determined by the order types of A and B. Accordingly, we extend this product from linear orders to order types as well.

A linear order A is *dense* if, for all $u, v \in A$ with $u <_A v$, there is $w \in A$ with $u <_A w <_A v$. In fact, there are only very few isomorphism types of dense countable linear orders:

Theorem 2.2.1 (Cantor's theorem)**.** *A non-empty countable linear order is dense if and only if its order type is among* 1, η, $1+\eta$, $\eta+1$ *and* $1+\eta+1$. □

The complete opposite of being dense is being *scattered.* Formally, a linear order A is *scattered* if $(\mathbb{Q}; \leqslant)$ cannot be embedded into A. In some sense, dense and scattered linear orders are the basic building blocks of countable linear orders.

Theorem 2.2.2 (Hausdorff's theorem)**.** *Every countable linear order A is a dense sum of scattered linear orders, i.e., there are a dense linear order I and scattered linear orders A_i for each $i \in I$ such that* $A = \sum_{i\in I} A_i$. □

2.2.2 Well-Orders and Ordinals

We assume familiarity with ordinals and their arithmetic. We regard ordinals as order types of *well-orders.* In order to avoid ambiguities we do *not* identify an ordinal α with the set $\{\,\beta \mid \beta < \alpha\,\}$ of all smaller ordinals. The first uncountable ordinal is denoted by ω_1. The *Cantor normal form* of an ordinal α is its unique representation as a finite sum $\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_s}$ with $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_s$. If A is a type α well-order, its *decomposition into Cantor normal form* is the unique decomposition as a sum

$A_1 + A_2 + \cdots + A_s$ such that A_i has order type ω^{γ_i} for each $i \in [1, s]$.

In addition to the standard arithmetic of ordinals, we need the *natural arithmetic*. To this end, let α and β be two ordinals and $\alpha = \omega^{\gamma_1} + \cdots + \omega^{\gamma_s}$ and $\beta = \omega^{\delta_1} + \cdots + \omega^{\delta_t}$ their Cantor normal forms. Moreover, let $\epsilon_1 \geqslant \epsilon_2 \geqslant \cdots \geqslant \epsilon_{s+t}$ be the sequence of ordinals obtained from sorting the sequence $\gamma_1, \ldots, \gamma_s, \delta_1, \ldots, \delta_t$. The *natural sum* of α and β is the ordinal $\alpha \oplus \beta$ defined by

$$\alpha \oplus \beta := \omega^{\epsilon_1} + \omega^{\epsilon_2} + \cdots + \omega^{\epsilon_{s+t}} .$$

The *natural product* of α and β is the ordinal $\alpha \otimes \beta$ defined by

$$\alpha \otimes \beta := \bigoplus_{\substack{1 \leqslant i \leqslant s \\ 1 \leqslant j \leqslant t}} \omega^{\gamma_i \oplus \delta_j} .$$

In contrast to the standard ordinal sum and product, the natural sum and product both are commutative and strictly monotonic in both arguments.

2.2.3 The Finite-Condensation Rank

As indicated in the introduction, the only known partial characterization of the string-automatic linear orders is an upper bound on their *finite-condensation ranks*. Roughly speaking, the finite-condensation rank of a linear order A is an ordinal which measures how far A is away from being dense. Our presentation of the definition of this rank loosely follows chapters 4 and 5 of Rosenstein's book [Ros82], although we make one fundamental change in notation: We prefer to describe the underlying condensation process in terms of equivalence relations and *not* in terms of natural homomorphisms. However, one can easily show that both variants are equivalent. If A is a linear order and $X, Y \subseteq A$

are subsets, we write $X \ll Y$ to denote the fact that $u < v$ for all $u \in X$ and $v \in Y$.

Let A be a linear order. A *condensation relation* on A is an equivalence relation $\sim$ on A whose equivalence classes are convex subsets of A. In this situation, the set of all $\sim$-classes is strictly linearly ordered by $\ll$. We denote the resulting linear order by $A/{\sim}$. Figuratively speaking, this linear order is obtained from A by contracting (or *condensing*) each $\sim$-class into a single point. To put it the other way round,

$$A = \sum_{X \in A/\sim} X . \tag{2.2}$$

An important example of a condensation relation on A is the relation of being *finitely distant (in A)*: $u, v \in A$ are finitely distant in A if there are only finitely many $w \in A$ with $u \leqslant w \leqslant v$ (if $u \leqslant v$) or $v \leqslant w \leqslant u$ (if $v \leqslant u$). This condensation is called the *finite-condensation relation (on A)*. For the purpose of later use, we note that this condensation relation can be defined in A by means of the $\exists^\infty$-quantifier.

We now formalize the process of transfinitely iterating the finite-condensation relation. To this end, we define for each ordinal α a condensation relation $\sim^\alpha$ on A, which is called the α^{th} *iterated finite-condensation relation*:

(1) $\sim^0$ is the identity relation on A.

(2) If α is a successor ordinal, say $\alpha = \beta+1$, then $u \sim^\alpha v$ whenever the $\sim^\beta$-classes of u and v are finitely distant in $A/{\sim^\beta}$.

(3) If α is a limit ordinal, then $u \sim^\alpha v$ whenever there is $\beta < \alpha$ with $u \sim^\beta v$.

Notice that $\sim^1$ is precisely the finite-condensation relation itself. For reasons of cardinality, there is always an ordinal α such that $\sim^\alpha$ and $\sim^\beta$ coincide for each $\beta \geqslant \alpha$. In fact, there is even a countable α with this property whenever A is countable [Ros82, theorem 5.9]. The former fact justifies the following definition:

Definition 2.2.3. Let A be a linear order. The *finite-condensation rank* or FC-*rank* of A, denoted by $\mathrm{FC}(A)$, is the least ordinal α with the property that $\sim^\alpha$ and $\sim^\beta$ coincide for each $\beta \geqslant \alpha$.

The lemma below lists various properties of the FC-rank that we require later on, cf. [Ros82, chapter 5].

Lemma 2.2.4. *Let A be a linear order and $X \subseteq A$ a suborder.*

(1) If A is a scattered linear order or X is a convex subset of A, then $\mathrm{FC}(X) \leqslant \mathrm{FC}(A)$.

(2) If A is a type ω^γ well-order, then $\mathrm{FC}(A) = \gamma$.

(3) If A is a type $\omega^\gamma + 1$ well-order, then $\mathrm{FC}(A) = \gamma + 1$.

(4) If $\alpha = \mathrm{FC}(A)$, then every $\sim^\alpha$-class is scattered and $A/{\sim^\alpha}$ is dense. □

In view of eq. (2.2) on the preceding page, the last statement demonstrates theorem 2.2.2. In addition, A is scattered if and only if $A/{\sim^\alpha}$ is a singleton linear order. In the remainder of this section, we present an alternative characterization of the class of countable scattered linear orders which evolved in the context of theorem 2.2.2. For each *countable* ordinal α, the class $\mathcal{VD}_\alpha$ of linear orders is defined inductively as follows:

(1) $\mathcal{VD}_0$ consists of the empty linear order and all singleton linear orders.

(2) For $\alpha > 0$, $\mathcal{VD}_\alpha$ consists of all ζ-sums of linear orders from the class

$$\mathcal{VD}_{<\alpha} := \bigcup_{\beta<\alpha} \mathcal{VD}_\beta \,.$$

Finally, the class $\mathcal{VD}$ of *very discrete* linear orders is defined as

$$\mathcal{VD} := \bigcup_{\alpha<\omega_1} \mathcal{VD}_\alpha \,.$$

For any linear order $A \in \mathcal{VD}$, the VD-*rank* of A, which is denoted by $\mathrm{VD}(A)$, is the least ordinal α such that $A \in \mathcal{VD}_\alpha$. The aforementioned characterization of scatteredness is as follows:

Theorem 2.2.5 (Hausdorff's theorem (continued))**.** *A countable linear order A is scattered if and only if $A \in \mathcal{VD}$. In case that A is scattered,*

$$\mathrm{FC}(A) = \mathrm{VD}(A) \,. \qquad \square$$

The classes $\mathcal{VD}_\alpha$ have the disadvantage of not being closed under taking finite sums. However, for our purposes this property is crucial. Accordingly, for each countable ordinal α, we further take the class

$$\mathcal{VD}^\star_\alpha := \{\, A_1 + \cdots + A_n \mid n \geqslant 0, A_1, \ldots, A_n \in \mathcal{VD}_\alpha \,\}$$

into account. Obviously,

$$\mathcal{VD} = \bigcup_{\alpha<\omega_1} \mathcal{VD}^\star_\alpha \,.$$

The VD_*-rank of a scattered linear order A, denoted by $\mathrm{VD}_*(A)$, is the least ordinal α such that $A \in \mathcal{VD}^\star_\alpha$. Using almost the same proof as for theorem 2.2.5, one can show that $\mathrm{VD}_*(A)$ is the least ordinal α such that $A/{\sim^\alpha}$ is finite.

2.3 Automata Theory

In this section, we present the necessary background on finite automata on strings [Eil74, KN01] and trees [TW68, CDG+08], their connection to monadic second-order logic [Tho97] and algebraic recognizability [Eil76]. As deterministic models of finite automata are strong enough and considerably more convenient for the elaborations to follow, we refrain from introducing non-deterministic automata. In the end of this section, we further present some basic results on regular languages of polynomial growth. In particular, we provide a new short proof of the characterization of these languages.

2.3.1 Finite Automata on Strings

Let Σ be an alphabet, i.e., a non-empty finite set. From now on, the letter Σ always refers to an alphabet. The set of all *(finite) strings (over Σ)* is denoted by Σ^*, the *empty string* by ε and the *length* of some string $u \in \Sigma^*$ by $|u|$. The set of non-empty strings is Σ^+, i.e., $\Sigma^+ := \Sigma^* \setminus \{\varepsilon\}$. For $u \in \Sigma^*$ and $a \in \Sigma$, the symbol $|u|_a$ counts the number of a-symbols in u. The *concatenation* of two string $u, v \in \Sigma^*$ is written $u \cdot v$ or just uv. Subsets of Σ^* are called *languages (of strings)*. The *concatenation* of two languages $K, L \subseteq \Sigma^*$ is denoted by $K \cdot L$ or just KL, the *(Kleene) iteration* of $L \subseteq \Sigma^*$ by L^*.

A *deterministic finite automaton on strings (over Σ)* or, for short, *string-automaton (over Σ)* is a 4-tuple $\mathcal{M} = (Q, \iota, \delta, F)$ consisting of a *finite* set Q, an element $\iota \in Q$, a map $\delta \colon Q \times \Sigma \to Q$ and a subset $F \subseteq Q$. The elements of Q are called *states*, ι is the *initial state*, δ is the *transition map* and the states in F are *final states*. We extend δ to a map $\hat{\delta} \colon Q \times \Sigma^* \to Q$ by inductively defining, for all $q \in Q$, $a \in \Sigma$ and $u \in \Sigma^*$,

$$\hat{\delta}(q, \varepsilon) := q \qquad \text{and} \qquad \hat{\delta}(q, ua) := \delta(\hat{\delta}(q, u), a)\,. \tag{2.3}$$

Abusing notation, we omit the accent on $\hat{\delta}$ and just write $\delta(q, u)$ for $\hat{\delta}(q, u)$ in what follows. The automaton $\mathcal{M}$ is said to *accept* a string $u \in \Sigma^*$ if $\delta(\iota, u) \in F$. The language *recognized* by $\mathcal{A}$ is the language $\mathscr{L}(\mathcal{M}) \subseteq \Sigma^*$ of all strings accepted by $\mathcal{M}$, i.e.,

$$\mathscr{L}(\mathcal{M}) := \{\, u \in \Sigma^* \mid \delta(\iota, u) \in F \,\} .$$

A language $L \subseteq \Sigma^*$ is called *regular* if it is recognized by some string-automaton. It is well-known that the class of regular languages is *effectively* closed under Boolean operations, concatenation, iteration and (inverse) projections.

2.3.2 Algebraic Automata Theory

An alternative characterization of the class of regular languages is given by means of algebraic recognizability. Our interest in this characterization is primarily due to the concise pumping arguments it brings into scope. A *semigroup* is a set S together with an associative binary operation $\cdot$ on S, called the *semigroup operation*. It is customary to denote the semigroup operation by juxtaposition, i.e., we write st for $s \cdot t$ with $s, t \in S$. Two important examples of a semigroup are formed by the sets Σ^* and Σ^+ both equipped with concatenation as semigroup operation. Notice that either of this semigroups is finitely generated, the former by $\Sigma \cup \{\varepsilon\}$ and the latter by Σ. The *direct product* of semigroups $S_1, \dots, S_n$ is the Cartesian product $S_1 \times \cdots \times S_n$ with component-wise application of the semigroup operations, i.e.,

$$\langle s_1, \dots, s_n \rangle \cdot \langle t_1, \dots, t_n \rangle := \langle s_1 t_1, \dots, s_n t_n \rangle .$$

A *morphism (of semigroups)* is a map $\eta \colon S \to S'$ between two semigroups S and S' which respects the semigroup operations, i.e., $\eta(st) = \eta(s)\,\eta(t)$ for all $s, t \in S$. Let $L \subseteq \Sigma^*$ be a language and S a *finite* semigroup. A morphism $\eta \colon \Sigma^* \to S$ *recognizes* the language L if one of the following two equivalent conditions is satisfied:

(1) There is a subset $F \subseteq S$ such that $L = \eta^{-1}(F)$.

(2) For all $u, v \in \Sigma^*$ with $\eta(u) = \eta(v)$, we have $u \in L$ if and only if $v \in L$.

A language L is called *algebraically recognizable* if it is recognized by some morphism into a finite semigroup. Throughout this thesis, we use the phrase “a morphism $\eta\colon \Sigma^* \to S$ recognizing L” as an abbreviation for “a morphism $\eta\colon \Sigma^* \to S$ *into a finite semigroup* S which recognizes L”. In particular, we implicitly assume S to be finite. The connection between regularity and algebraic recognizability is as follows:

Theorem 2.3.1 (Myhill's theorem)**.** *Let $L \subseteq \Sigma^*$ be a language. The following conditions are* effectively *equivalent:*

(1) L is regular.

(2) L is algebraically recognizable. □

Suppose that $L_1, \ldots, L_n \subseteq \Sigma^*$ are regular languages and each L_i is recognized by the morphism $\eta_i\colon \Sigma^* \to S_i$. Then the morphism $\eta\colon \Sigma^* \to S_1 \times \cdots \times S_n$ defined by

$$\eta(s_1, \ldots, s_n) := \langle \eta_1(s_1), \ldots, \eta_n(s_n) \rangle$$

recognizes all the L_i. To emphasize this, any finite number of regular languages admit *one common morphism* which *simultaneously recognizes all of them.*

Recall that our interest in algebraic recognizability is mainly motivated by concise pumping arguments. These arguments are formalized by means of the notion of *idempotency.* To this end, we fix a semigroup S. An element $s \in S$ is *idempotent* if $s^2 = s$. Henceforth, we additionally assume that S is finite. Then every $s \in S$ admits some $k(s) \geqslant 1$ such that $s^{k(s)}$ is idempotent. In fact, there is even some $k \geqslant 1$ such that s^k is idempotent for all $s \in S$, e.g., the least common multiple of all the $k(s)$. The least k

with this property is called the *exponent* of S. The choice of the term “exponent” reflects that the role idempotent elements play in semigroups is in some sense similar to the role the neutral element plays in groups. Notice that any multiple k of the exponent of S has the property that s^k is idempotent for all $s \in S$ as well. As a matter of fact, one can even show that no k other than these multiples have this property. To get a taste of the concise pumping arguments we have in mind, we provide a simple showcase:

Example 2.3.2. Let L be a language, $\eta\colon \Sigma^* \to S$ a morphism recognizing L and k the exponent of S. Suppose we have $u, v \in \Sigma^*$ and $m, n \geqslant 2k$ with $u^m v^n \in L$. Using the idempotency of $\eta(u)^k$ and $\eta(v)^k$, we obtain

$$\begin{aligned}
\eta(u^{m+k} v^{n-k}) &= \eta(u^{m-k}) \cdot (\eta(u)^k)^2 \cdot \eta(v)^k \cdot \eta(v^{n-2k}) \\
&= \eta(u^{m-k}) \cdot \eta(u)^k \cdot (\eta(v)^k)^2 \cdot \eta(v^{n-2k}) \\
&= \eta(u^m v^n)
\end{aligned}$$

and hence $u^{m+k} v^{n-k} \in L$. The interesting point about this calculation is that we added as many u’s as we removed v’s. $\square$

Although it is possible to achieve similar results by ordinary pumping arguments applied to finite automata, these arguments would not be as concise. The advantage of resorting to algebraic recognizability becomes even more apparent in our actual applications in chapter 5.

2.3.3 Finite Automata on Trees

The *prefix relation* on $\{0, 1\}^*$ is the partial ordering $\preccurlyeq$ defined by $u \preccurlyeq v$ if there is $w \in \{0, 1\}^*$ with $uw = v$. A subset $U \subseteq \{0, 1\}^*$ is an *anti-chain* if its elements are mutually incomparable wrt $\preccurlyeq$. A *tree-domain* is a non-empty finite subset $D \subseteq \{0, 1\}^*$ which is downward closed wrt $\preccurlyeq$, i.e., the premises $u \preccurlyeq v$ and $v \in D$

always imply $u \in D$.[2] The elements of D are called *nodes* and are of two kinds: A node $u \in D$ is a *leaf* if $u\mathtt{0}, u\mathtt{1} \notin D$ and an *inner node* otherwise. The *boundary* of D is the least (wrt inclusion) set $\partial D \subseteq \{\mathtt{0},\mathtt{1}\}^*$ such that $ui \in D \cup \partial D$ for all $u \in D$ and $i \in \{\mathtt{0},\mathtt{1}\}$. More precisely,

$$\partial D := \{\, ui \mid u \in D, i \in \{\mathtt{0},\mathtt{1}\}, ui \notin D \,\}\,.$$

Notice that $D \cup \partial D$ is a tree-domain as well. Its inner nodes are those in D and its leaves the elements of ∂D.

A *(finite labeled) tree (over Σ)* is a map $t\colon D \to \Sigma$ where $\mathrm{dom}(t) := D$ is a tree-domain. The set of all trees over Σ is denoted by T_Σ. Its subsets are called *languages (of trees)*. Let $t \in T_\Sigma$ be a tree. The *height* of t is the number

$$\mathrm{h}(t) := \max\{\, |u| \mid u \in \mathrm{dom}(t) \,\}\,.$$

The *subtree of t rooted at $u \in \mathrm{dom}(t)$* is the tree $t{\restriction_u} \in T_\Sigma$ defined by

$$\mathrm{dom}(t{\restriction_u}) := \{\, v \in \{\mathtt{0},\mathtt{1}\}^* \mid uv \in \mathrm{dom}(t) \,\}$$

and

$$t{\restriction_u}(v) := t(uv)\,.$$

For an anti-chain $\{u_1,\dots,u_n\} \subseteq \mathrm{dom}(t)$ and trees $t_1,\dots,t_n \in T_\Sigma$, we consider the tree $t[u_1/t_1,\dots,u_n/t_n] \in T_\Sigma$ which is obtained from t by simultaneously replacing, for each $i \in [1,n]$, the subtree rooted at u_i with t_i. Formally,

$$\begin{aligned}&\mathrm{dom}(t[u_1/t_1,\dots,u_n/t_n]) := \\ &\qquad \mathrm{dom}(t) \setminus \{u_1,\dots,u_n\}\{\mathtt{0},\mathtt{1}\}^* \cup \bigcup_{1\leqslant i\leqslant n} u_i\,\mathrm{dom}(t_i)\end{aligned}$$

[2] Some authors additionally require that $u\mathtt{0} \in D$ whenever $u\mathtt{1} \in D$ or even that $u\mathtt{0} \in D$ if and only if $u\mathtt{1} \in D$. As a matter of fact, remark 2.4.4 establishes that such requirements would not reduce the expressive power in the context of automatic structures anyway.

and

$$t[u_1/t_1, \dots, u_n/t_n](u) := \begin{cases} t_i(v) & \text{if } u = u_i v \text{ for some } i \text{ and } v, \\ t(u) & \text{otherwise.} \end{cases}$$

A *bottom-up deterministic finite automaton on trees (over Σ)* or, for short, *tree-automaton (over Σ)* is a 4-tuple $\mathcal{T} = (Q, \iota, \delta, F)$ consisting of a *finite* set Q, an element $\iota \in Q$, a map $\delta \colon Q \times \Sigma \times Q \to Q$ and a subset $F \subseteq Q$. Again, the elements of Q are called *states*, ι is the *initial state*, δ is the *transition map* and the states in F are *final states*. Similar to eq. (2.3) on page 20, we define for each $t \in T_\Sigma$ and $u \in \operatorname{dom}(t) \cup \partial \operatorname{dom}(t)$ a state $\hat{\delta}(\iota, t, u) \in Q$ by

$$\hat{\delta}(\iota, t, u) := \begin{cases} \iota & \text{if } u \in \partial \operatorname{dom}(t), \\ \delta(\hat{\delta}(\iota, t, u\mathtt{0}), t(u), \hat{\delta}(\iota, t, u\mathtt{1})) & \text{if } u \in \operatorname{dom}(t). \end{cases}$$

Notice that

$$\hat{\delta}(\iota, t, u) = \hat{\delta}(\iota, t{\restriction_u}, \varepsilon)$$

whenever $u \in \operatorname{dom}(t)$. Abusing notation in the same way as before, we write $\delta(\iota, t, u)$ for $\hat{\delta}(\iota, t, u)$ in what follows. In addition, we omit the parameter u from $\delta(\iota, t, u)$ whenever $u = \varepsilon$. Intuitively, $\delta(\iota, t)$ is the state the automaton $\mathcal{T}$ reaches at the root when processing t.[3] The language *recognized* by $\mathcal{T}$ is the set

$$\mathscr{L}(\mathcal{T}) := \{\, t \in T_\Sigma \mid \delta(\iota, t) \in F \,\}$$

of all trees *accepted* by $\mathcal{T}$. A language $L \subseteq \Sigma^*$ is called *regular* if it is recognized by some tree-automaton. The class of regular languages of trees is also *effectively* closed under Boolean operations and (inverse) projections.

[3] Although we always use the initial state ι as the first parameter of $\delta(\iota, t)$, we do not omit this parameter for the sake of a notation which treats finite automata on strings and on trees uniformly.

2.3.4 Monadic Second-Order Definability

In order to describe languages of strings or trees by means of mso formulas, we need to represent every string and every tree by a relational structure. The representation of a string $u = a_1 a_2 \dots a_n \in \Sigma^*$ is the structure

$$\mathscr{M}_u := \big(\{1, \dots, n\}; \leqslant_u, (P_a^u)_{a \in \Sigma}\big)$$

where $\leqslant_u$ is the natural ordering of $\{1, \dots, n\}$ and

$$P_a^u := \{\, i \in \{1, \dots, n\} \mid a_i = a \,\}\,.$$

In the following, we identify u with its representation $\mathscr{M}_u$. In particular, we say that an mso sentence Φ is suitable for u if it is suitable for $\mathscr{M}_u$ and write $u \models \Phi$ instead of $\mathscr{M}_u \models \Phi$. A language $L \subseteq \Sigma^*$ is *monadic second-order definable* or, for short, *mso definable* if there is an mso sentence Φ such that

$$L = \{\, u \in \Sigma^* \mid u \models \Phi \,\}\,.$$

In this situation, we say that the sentence Φ *defines* L.

The representation of a tree $t \in T_\Sigma$ is the structure

$$\mathscr{M}_t := \big(\mathrm{dom}(t); (S_i^t)_{i=0,1}, (P_a^t)_{a \in \Sigma}\big)$$

given by

$$S_i^t := \{\, \langle u, v \rangle \in \mathrm{dom}(t)^2 \mid ui = v \,\}$$

and

$$P_a^t := \{\, u \in \mathrm{dom}(t) \mid t(u) = a \,\}\,.$$

Notice that we did not include the prefix relation $\preccurlyeq$ on $\mathrm{dom}(t)$ in $\mathscr{M}_t$. However, this is of no importance since $\preccurlyeq$ is mso definable by means of eq. (2.1) on page 12 as the transitive closure of the formula $S(x, y) := S_0(x, y) \lor S_1(x, y)$. Just like for strings, we identify t with its representation $\mathscr{M}_t$ as well. Accordingly, a language $L \subseteq T_\Sigma$ is *monadic second-order definable* if there is an mso sentence Φ which *defines* L, i.e.,

$$L = \{\, t \in T_\Sigma \mid t \models \Phi \,\}\,.$$

Our interest in mso definability is owed to its close connection to regularity given by the following theorem. The version for strings is sometimes called Büchi–Elgot–Trakhtenbrot theorem [Büc60, Elg61, Tra61] and the version for trees is due to Doner [Don65, Don70] and, independently, Thatcher and Wright [TW68].

Theorem 2.3.3 (cf. [Tho97])**.** *Let L be a language of strings or trees. The following conditions are* effectively *equivalent:*

(1) L is regular.

(2) L is monadic second-order definable. □

2.3.5 Regular Languages of Polynomial Growth

Preparing a natural restriction of the class of automatic structures, this section deals with regular languages of polynomial growth. Basically, we provide a new characterization of this class of languages in terms of *unambiguously rational expressions.* Our proof is very short and subsumes the characterization from [SYZS92].

Definition 2.3.4. Let $L \subseteq \Sigma^*$ be a language. The *growth* of L is the map $g_L \colon \mathbb{N} \to \mathbb{N}$ defined by

$$g_L(n) := |L \cap \Sigma^{\leqslant n}|\,.$$

We say that L has *polynomial growth* or *grows polynomially* if $g_L(n) \in O(n^k)$ for some $k \in \mathbb{N}$. Conversely, we say that L *grows exponentially* if $g_L(n) \in 2^{\Omega(n)}$.

Notice the trivial upper bound $g_L(n) \in 2^{O(n)}$. Thus, every exponentially growing language even satisfies $g_L(n) \in 2^{\Theta(n)}$. The standard example of a polynomially growing language is as follows:

Example 2.3.5. Let $m \geqslant 0$ and, for each $i \in [1, m]$, $k_i \geqslant 0$, $u_{i0}, u_{i1}, \dots, u_{ik_i} \in \Sigma^*$ and $v_{i1}, \dots, v_{ik_i} \in \Sigma^+$. We demonstrate that the language

$$L := \bigcup_{1 \leqslant i \leqslant m} u_{i0} v_{i1}^* u_{i1} \cdots v_{ik_i}^* u_{ik_i} \tag{2.4}$$

grows polynomially by establishing the bound $g_L(n) \in O(n^k)$ for $k := \max\{k_1, \dots, k_m\}$.

To this end, let $L_i := u_{i0} v_{i1}^* u_{i1} \cdots v_{ik_i}^* u_{ik_i}$ for each i. Since $L = \bigcup_{1 \leqslant i \leqslant m} L_i$, we obtain

$$g_L(n) \leqslant \sum_{1 \leqslant i \leqslant m} g_{L_i}(n) \,. \tag{2.5}$$

Now, fix $i \in [1, m]$, $n \in \mathbb{N}$ and consider some $w \in L_i \cap \Sigma^{\leqslant n}$. There are $n_1, \dots, n_{k_i} \in \mathbb{N}$ satisfying $w = u_{i0} v_{i1}^{n_1} u_{i1} \cdots v_{ik_i}^{n_{k_i}} u_{ik_i}$. Notice that

$$n_1, \dots, n_{k_i} \leqslant |u_{i0} v_{i1}^{n_1} u_{i1} \cdots v_{ik_i}^{n_{k_i}} u_{ik_i}| = |w| \leqslant n \,.$$

Thus, we may conclude

$$\begin{aligned} g_{L_i}(n) &\leqslant \left|\{\, \langle n_1, \dots, n_{k_i} \rangle \in \mathbb{N}^{k_i} \mid n_1, \dots, n_{k_i} \leqslant n \,\}\right| \\ &= (n+1)^{k_i} \in O(n^{k_i}) \,. \end{aligned}$$

Finally, we obtain $g_L(n) \in O(n^k)$ according to eq. (2.5) and the choice of k. □

In fact, it is already known that all polynomially growing *regular* languages are of the form in example 2.3.5 [SYZS92]. Moreover, we have the following dichotomy: Every regular language L grows either polynomially or exponentially. In case that L grows polynomially, there is even some $k \geqslant 0$ such that $g_L(n) \in \Theta(n^k)$. Using the notion of *unambiguously rational expressions*, we now give a new proof of these results that is substantially shorter than those in [SYZS92] and slightly strengthens the characterization of polynomially growing regular languages.

A language $L \subseteq \Sigma^*$ is *rational* if it can be constructed from the finite languages using union $\cup$, concatenation $\cdot$ and iteration * only. According to Kleene's theorem, a language is rational if and only if it is regular, i.e., can be recognized by a finite automaton [Kle56]. In fact, this characterization is effective, i.e., one can compute a *rational expression* L from a finite automaton recognizing L and vice versa. It is folklore that applying this construction to a *deterministic* finite automaton yields a rational expression with a special property which is commonly called *unambiguity* and defined as follows: Let $A, B \subseteq \Sigma^*$ be languages.

(1) The union $A \cup B$ is *unambiguous* if A and B are disjoint.

(2) The concatenation $A \cdot B$ is *unambiguous* if every $u \in A \cdot B$ admits precisely one factorization $u = vw$ with $v \in A$ and $w \in B$.

(3) The iteration A^* is *unambiguous* if every $u \in A^*$ admits precisely one factorization $u = v_1 \cdots v_n$ with $n \geqslant 0$ and $v_1, \ldots, v_n \in A \setminus \{\varepsilon\}$.

A language $L \subseteq \Sigma^*$ is *unambiguously rational* if it can be constructed from the finite languages using unambiguous unions, concatenations and iterations only. Using this notion of unambiguity, Kleene's theorem reads as follows:

Theorem 2.3.6 (Kleene's theorem [Kle56])**.** *For every language $L \subseteq \Sigma^*$, the following are* effectively *equivalent:*

(1) L is regular.

(2) L is rational.

(3) L is unambiguously *rational.* □

The next theorem is the announced characterization of the class of polynomially growing regular languages.

Theorem 2.3.7. *Let $L \subseteq \Sigma^*$ be a regular language. Then L grows either polynomially or exponentially. In case L grows polynomially, there are $m \geqslant 0$ and, for each $i \in [1, m]$, $k_i \geqslant 0$, $u_{i0}, u_{i1}, \ldots, u_{ik_i} \in \Sigma^*$ and $v_{i1}, \ldots, v_{ik_i} \in \Sigma^+$ such that*

$$L = \bigcup_{1 \leqslant i \leqslant m} u_{i0} v_{i1}^* u_{i1} \cdots v_{ik_i}^* u_{ik_i} \tag{2.6}$$

and this rational expression is unambiguous. In particular, if L is non-empty, then $g_L(n) \in \Theta(n^k)$ for $k := \max\{k_1, \ldots, k_m\}$.

Proof. The claim for $L = \emptyset$ is trivial. Henceforth, we assume $L \neq \emptyset$. According to theorem 2.3.6, L is unambiguously rational. Using the algebraic properties of $\cup$ and $\cdot$ (associativity, distributivity, neutral/absorbing elements, etc.) and the relationship $\emptyset^* = \varepsilon^* = \{\varepsilon\}$, we can write L as

$$L = \bigcup_{1 \leqslant i \leqslant m} u_{i0} E_{i1}^* u_{i1} \cdots E_{ik_i}^* u_{ik_i} \tag{2.7}$$

with $m \geqslant 1$, $k_i \geqslant 0$, $u_{ij} \in \Sigma^*$ and rational languages $E_{ij} \not\subseteq \{\varepsilon\}$ such that the whole expression is unambiguous.

For each E_{ij}, let $v_{ij} \in E_{ij} \setminus \{\varepsilon\}$ be of minimal length. First, suppose there is E_{ij} with $E_{ij}^* \neq v_{ij}^*$. Then there exists $w \in E_{ij} \setminus v_{ij}^*$. Since E_{ij}^* is an unambiguous iteration, we have $v_{ij} w \neq w v_{ij}$ and hence the subset

$$u_{i0} u_{i1} \cdots u_{i,j-1} \{v_{ij} w, w v_{ij}\}^* u_{ij} \cdots u_{ik_i} \subseteq L$$

grows exponentially. Thus, L grows exponentially itself.

Now, suppose that $E_{ij}^* = v_{ij}^*$ for each E_{ij}. Replacing all the E_{ij}^* with v_{ij}^* in eq. (2.7) establishes eq. (2.6) and the resulting expression is clearly still unambiguous. Due to example 2.3.5, this particularly implies that L grows polynomially and, more precisely, $g_L(n) \in O(n^k)$.

In order to see that $g_L(n) \in \Omega(n^k)$, we fix some i with $k = k_i$. We consider the map $f \colon \mathbb{N}^k \to L$ defined by

$$f(\boldsymbol{x}) := u_{i0} v_{i1}^{x_1} u_{i1} \cdots v_{ik}^{x_k} u_{ik} \,.$$

Since the expression in eq. (2.6) is unambiguous, f is injective. Let

$$p := |u_{i0}| + \cdots + |u_{ik}|$$

and

$$q := \max\{\, |v_{i1}|, \ldots, |v_{ik}| \,\} \,.$$

For all $n \geqslant 0$ and $\boldsymbol{x} \in \mathbb{N}^k$ with $x_1, \ldots, x_k \leqslant \frac{n}{k}$, we have

$$|f(\boldsymbol{x})| \leqslant q \cdot (x_1 + \cdots + x_k) + p \leqslant q \cdot n + p \,.$$

Thus,

$$\begin{aligned} g_L(q \cdot n + p) &\geqslant \left|\{\, \boldsymbol{x} \in \mathbb{N}^k \mid x_1, \ldots, x_k \leqslant \tfrac{n}{k} \,\}\right| \\ &= (\lfloor \tfrac{n}{k} \rfloor + 1)^k \in \Omega(n^k) \,. \end{aligned}$$

Clearly, this implies $g_L(n) \in \Omega(n^k)$. □

Using the previous characterization along with counting arguments very similar to those in example 2.3.5, we obtain:

Corollary 2.3.8. *Let $k \geqslant 1$ and $L \subseteq \Sigma^*$ be a regular language with $g_L(n) \in O(n^k)$. Then*

$$|L \cap \Sigma^{=n}| \in O(n^{k-1}) \,.$$ □

2.4 Automatic Structures

In this section, we provide the required notation concerning automatic structures. We try to avoid repeating definitions for the string-automatic and the tree-automatic case but rather approach them in a more uniform way. For a more detailed overview on string-automatic structures, we refer the reader to the survey [Rub08]. The best reference on tree-automatic structures we are aware of is [BGR11].

2.4.1 String-Automatic and Tree-Automatic Structures

In order to employ finite automata to recognize relations of strings, we need to encode tuples of strings by single strings. To this end, let $\diamond \notin \Sigma$ be a new symbol, called *padding symbol*, and put $\Sigma_\diamond := \Sigma \cup \{\diamond\}$. We encode any tuple $\boldsymbol{w} \in (\Sigma^*)^r$ by its *convolution* $\otimes\boldsymbol{w} \in (\Sigma_\diamond^r)^*$ which is defined as follows: $|\otimes\boldsymbol{w}| = \max\{\,|w_1|, \dots, |w_r|\,\}$ and the i^{th} symbol of $\otimes\boldsymbol{w}$ is the tuple $\langle a_1, \dots, a_r\rangle$ where, for $j \in [1, r]$, a_j is the i^{th} symbol of w_j if $i \leqslant |w_j|$ and $\diamond$ otherwise. Whenever $r = 2$, we also write $\otimes$ as an infix operator, i.e., we write $w_1 \otimes w_2$ for $\otimes\langle w_1, w_2\rangle$. The *convolution* of a whole relation of strings $R \subseteq (\Sigma^*)^r$ is the language

$$\otimes R := \{\, \otimes\boldsymbol{w} \mid \boldsymbol{w} \in R \,\} \subseteq (\Sigma_\diamond^r)^* \,.$$

Regarding a language L as a unary relation and taking its convolution has no effect at all, i.e., $\otimes L = L$. To resolve possible ambiguities, we agree that the convolution operator $\otimes$ has a lower precedence than taking Cartesian powers, i.e., the term $\otimes L^r$ means $\otimes(L^r)$, where L is some language.

Similarly, we encode any tuple of trees $\boldsymbol{t} \in T_\Sigma^r$ by its *convolu-*

tion $\otimes \boldsymbol{t} \in T_{\Sigma_\diamond^r}$ which is defined by

$$\operatorname{dom}(\otimes \boldsymbol{t}) := \bigcup_{1 \leqslant j \leqslant r} \operatorname{dom}(t_j)$$

and

$$(\otimes \boldsymbol{t})(u) := \langle t'_1(u), \dots, t'_r(u) \rangle ,$$

where

$$t'_j(u) := \begin{cases} t_j(u) & \text{if } u \in \operatorname{dom}(t_j), \\ \diamond & \text{otherwise.} \end{cases}$$

Again, the *convolution* of a relation of trees $R \subseteq T_\Sigma^r$ is the language

$$\otimes R := \{\, \otimes \boldsymbol{t} \mid \boldsymbol{t} \in R \,\} \subseteq T_{\Sigma_\diamond^r} .$$

Definition 2.4.1. A relation of strings or trees R is *automatic* if its convolution $\otimes R$ is a regular language. If $\mathcal{M}$ is a finite automaton (on strings or on trees) recognizing $\otimes R$, then we also say that $\mathcal{M}$ *recognizes* R.

Using the notion of an automatic relation, we can provide the very fundamental definition of an automatically presentable structure.

Definition 2.4.2. A structure $\mathscr{A} = (A; R_1, \dots, R_n)$ is *automatically presentable* if there is an injective map $f\colon A \to \Sigma^*$ or $f\colon A \to T_\Sigma$, called *encoding*, satisfying the following two conditions:

(1) The language $f(A)$ is regular.

(2) The relation

$$f(R_i) := \{\, f(\boldsymbol{x}) \mid \boldsymbol{x} \in R_i \,\}$$

is automatic for each $i \in [1, n]$.

In this situation, an *automatic presentation* of $\mathscr{A}$ is a tuple $\mathcal{P} = (\mathcal{M}_0; \mathcal{M}_1, \ldots, \mathcal{M}_n)$ of finite automata such that $\mathcal{M}_0$ recognizes $f(A)$ and $\mathcal{M}_i$ recognizes $f(R_i)$ for each $i \in [1, n]$.

Whenever f maps into Σ^* and we want to emphasize this circumstance, we say that $\mathscr{A}$ is *string-automatically presentable* and call $\mathcal{P}$ a *string-automatic presentation* of $\mathscr{A}$. Similarly, $\mathscr{A}$ is *tree-automatically presentable* and $\mathcal{P}$ a *tree-automatic presentation* if f maps into T_Σ.

The class of string-automatically presentable structures is denoted by SA and the class of tree-automatically presentable structures by TA.

First of all, notice that the injectivity of f in the definition above immediately implies that f is an isomorphism between the structures $\mathscr{A}$ and

$$f(\mathscr{A}) := \big(f(A); f(R_1), \ldots, f(R_n)\big)\,.$$

If we want to show that a certain structure $\mathscr{A}$ is automatically presentable, we mostly do so by specifying the encoding $f(u)$ of each $u \in A$ and verifying that the map f defined this way does actually satisfy the conditions of definition 2.4.2. We note that there is another notion of automatic presentability where every single element of $\mathscr{A}$ might have several encodings and the relation of "encoding the same element" is automatic. However, it is known that every structure which is automatically presentable in this more general sense is also automatically presentable in the sense of definition 2.4.2 [KN95, CL07].

Whenever we investigate properties invariant under isomorphism of automatically presentable structures, we resort to investigating the structure $f(\mathscr{A})$ instead of $\mathscr{A}$ itself. In order to avoid clumsy notation in these situations, structures of the form $f(\mathscr{A})$ have a catchy name:

Definition 2.4.3. A structure $\mathscr{A} = (A; R_1, \dots, R_n)$ is *automatic* if it satisfies the following two conditions:

(1) The domain A is a regular language of strings or of trees.

(2) Each relation R_i is automatic.

A *presentation* of $\mathscr{A}$ then is a tuple $\mathcal{P} = (\mathcal{M}_0; \mathcal{M}_1, \dots, \mathcal{M}_n)$ of finite automata such that $\mathcal{M}_0$ recognizes A and $\mathcal{M}_i$ recognizes R_i for each $i \in [1, n]$.

More precisely, $\mathscr{A}$ is *string-automatic* if A is a language of strings and *tree-automatic* if A is a language of trees.

Put another way, a structure is automatic if it is automatically presentable by encoding each element by itself. Moreover, a structure is automatically presentable if and only if it is isomorphic to an automatic structure. Finally, we note a subtle difference between an automatic presentation and a presentation (without the prefixed "automatic") of an automatic structure: Whereas the former might correspond to an arbitrary encoding, the latter always requires the encoding to be the identity map.

Remark 2.4.4. As already mentioned, some authors put higher requirements on tree-domains, the strongest of them being that $u\mathtt{0} \in D$ if and only if $u\mathtt{1} \in D$. However, every structure which is tree-automatically presentable in the sense of definition 2.4.2 is also tree-automatically presentable in this more restricted sense. To see this, let $\mathscr{A}$ be a tree-automatic structure with $A \subseteq T_\Sigma$ and $\perp \notin \Sigma$ a fresh symbol. Encoding every tree $t \in A$ by the tree $t^\perp \in T_{\Sigma \cup \{\perp\}}$ given by

$$\operatorname{dom}(t^\perp) := \operatorname{dom}(t) \cup \partial \operatorname{dom}(t)$$

and

$$t^\perp(u) := \begin{cases} t(u) & \text{if } u \in \operatorname{dom}(t), \\ \perp & \text{otherwise,} \end{cases}$$

effectively yields a tree-automatic presentation of $\mathscr{A}$ satisfying the strongest requirement on tree-domains. $\square$

It is well-known that every string-automatic structure $\mathscr{A}$ is also tree-automatically presentable, i.e., $\mathsf{SA} \subseteq \mathsf{TA}$. To see this, one fixes an arbitrary symbol $a_0 \in \Sigma$ and encodes each string $w = a_1 \dots a_n \in A$ by the unique tree $t_w \in T_\Sigma$ with $\mathrm{dom}(t_w) = \mathsf{0}^{\leqslant n}$ and $t_w(\mathsf{0}^i) = a_i$. It is a matter of routine to check that this encoding satisfies the conditions of definition 2.4.2. A prominent and very useful example of a string-automatic structure is a linear order which plays a role in several proofs to follow.

Example 2.4.5. Let $\leqslant_\Sigma$ be an arbitrary linear ordering of the alphabet Σ. The *length-lexicographic ordering (wrt $\leqslant_\Sigma$)* of Σ^* is the linear ordering $\leqslant_{\mathrm{llex}}$ defined by $u <_{\mathrm{llex}} v$ if either $|u| < |v|$ or both $|u| = |v|$ and there are $x, y, z \in \Sigma^*$ and $a, b \in \Sigma$ with $a <_\Sigma b$, $u = xay$ and $v = xbz$. It is well-known and easy to check that $(\Sigma^*; \leqslant_{\mathrm{llex}})$ is a string-automatic type ω linear order. $\square$

The interest in automatic structures is mostly owed to the following fundamental theorem and its corollary, cf. [Hod83, KN95, Blu99]. In fact, this theorem is fundamental to such an extent that we use it without further reference. Usually, its application is indicated by arguing that some relation R is first-order definable and concluding that R is hence automatic.

Theorem 2.4.6 (fundamental theorem, cf. [Blu99])**.** *Let $\mathscr{A}$ be an automatic structure and $\phi(x_1, \dots, x_r)$ a first-order formula suitable for $\mathscr{A}$. Then the relation*

$$\phi^{\mathscr{A}} := \{\, \boldsymbol{u} \in A^r \mid \mathscr{A} \models \phi[\boldsymbol{u}] \,\}$$

defined *by ϕ is* effectively *automatic. More precisely, given a presentation of $\mathscr{A}$ and the formula ϕ, one can compute a finite automaton recognizing $\phi^{\mathscr{A}}$.* $\square$

Corollary 2.4.7 (cf. [Blu99])**.** *The first-order theory of every automatically presentable structure is* uniformly *decidable. More precisely, given an automatic presentation of some structure $\mathscr{A}$ and a first-order sentence Φ suitable for $\mathscr{A}$, one can decide whether $\mathscr{A} \models \Phi$ holds true or not.* □

Remark 2.4.8. It is well-known that theorem 2.4.6 and corollary 2.4.7 remain valid if first-order logic is extended by the "there are infinitely many" quantifier $\exists^\infty$ [Blu99].

We note that any decision procedure which verifies corollary 2.4.7 is inherently non-elementary. This is caused by the circumstance that there are string-automatic structures possessing a first-order theory of non-elementary complexity, including the full binary tree $(\{\mathtt{0},\mathtt{1}\}^*; S_0, S_1, \preccurlyeq)$ [CH90] and the extension $(\mathbb{N}; +, |_2)$ of Presburger's arithmetic where $x \mid_2 y$ if x is a power of 2 which divides y [Grä90].

We conclude our introduction to automatic structures by providing a result that can be regarded as a pumping lemma for string-automatic structures. As a matter of fact, this lemma turned out to be highly useful for showing that certain structures are *not* string-automatically presentable and we use it in the same way here.

Definition 2.4.9. Let $r \in \mathbb{N}$ and A be a set. A relation $R \subseteq A^{r+1}$ is *finitely valued* at $\boldsymbol{u} \in A^r$ if there only finitely many $v \in A$ such that $\langle \boldsymbol{u}, v \rangle \in R$. The relation R is *locally finite* if it is finitely valued at every $\boldsymbol{u} \in A^r$.

Lemma 2.4.10 ([EM65])**.** *Let $R \subseteq (\Sigma^*)^{r+1}$ be an automatic relation. There exists a constant $C \in \mathbb{N}$ such that, for all $\langle \boldsymbol{u}, v \rangle \in R$ where R is finitely valued at $\boldsymbol{u}$, the length of v is bounded by*

$$|v| \leqslant |{\otimes}\boldsymbol{u}| + C\,. \qquad \square$$

2.4.2 Automatic Structures on Domains of Polynomial Growth

A very natural and well studied subclass of SA is the class $\mathsf{1SA}$ of *unary string-automatically presentable structures* which is obtained by restricting the alphabet Σ to singleton sets only, cf. [Blu99, Rub04]. A more general but lesser studied class is formed by those structures which are string-automatically presentable on a domain of polynomial growth, cf. [Bár07]. Remarkably enough, imposing this restriction on the domain of a string-automatic structure leads to a first-order theory in PSPACE.

Definition 2.4.11. For every $k \in \mathbb{N}$, the class $\mathsf{pSA}[k]$ contains all structures that are isomorphic to a string-automatic structure $\mathscr{A}$ with $g_A(n) \in O(n^k)$. The class pSA contains all structures that are isomorphic to a string-automatic structure $\mathscr{A}$ whose domain A grows polynomially, i.e.,

$$\mathsf{pSA} := \bigcup_{k \geqslant 0} \mathsf{pSA}[k]\,.$$

Notice that the classes $\mathsf{pSA}[k]$ form a hierarchy

$$\mathsf{pSA}[0] \subseteq \mathsf{pSA}[1] \subseteq \mathsf{pSA}[2] \subseteq \cdots \subseteq \mathsf{pSA} \subseteq \mathsf{SA}$$

inside pSA and SA. Obviously, $\mathsf{pSA}[0]$ is the class of finite structures. Furthermore, the class $\mathsf{pSA}[1]$ contains precisely the unary string-automatically presentable structures [Bár07].

A similar hierarchy inside TA was proposed under the name "finite-rank tree-automatic presentations" in [BGR11]. Intuitively, the idea behind this hierarchy is to restrict the branching complexity of the trees involved in a tree-automatic presentation. Formally, this branching complexity can be captured by means of the Cantor–Bendixson rank, cf. [KRS05]. However, it is possible to introduce the *same* restriction in terms of the growth of languages of strings. To this end, we assign to every language $L \subseteq T_\Sigma$ the set

$$T(L) := \bigcup_{t \in L} \mathrm{dom}(t) \subseteq \{\mathtt{0}, \mathtt{1}\}^* .$$

One can easily show that $T(L)$ is regular whenever L is regular. In particular, theorem 2.3.7 applies to $T(L)$ then.

Definition 2.4.12. For every $k \in \mathbb{N}$, the class $\mathsf{pTA}[k]$ contains all structures that are isomorphic to a tree-automatic structure $\mathscr{A}$ with $g_{T(A)}(n) \in O(n^k)$. The class pTA contains all structures that are isomorphic to a tree-automatic structure $\mathscr{A}$ such that $T(A)$ grows polynomially, i.e.,

$$\mathsf{pTA} := \bigcup_{k \geqslant 0} \mathsf{pTA}[k] .$$

Again, we have a hierarchy

$$\mathsf{pTA}[0] \subseteq \mathsf{pTA}[1] \subseteq \mathsf{pTA}[2] \subseteq \cdots \subseteq \mathsf{pTA} \subseteq \mathsf{TA} .$$

As a matter of fact, the class $\mathsf{pTA}[1]$ coincides with SA. The inclusion $\mathsf{SA} \subseteq \mathsf{pTA}[1]$ is sketched right below remark 2.4.4. The converse inclusion can be shown by “compressing” any tree-automatic structure $\mathscr{A}$ with $g_{T(A)}(n) \in O(n^1)$ into an isomorphic string-automatic structure, cf. theorem 2.4.17.

2.4.3 Slim Languages of Trees

The last two sections of this chapter are devoted to the aforementioned compression technique for showing $\mathsf{pTA}[1] \subseteq \mathsf{SA}$. To be exact, we demonstrate a more general technique which works for tree-automatic structures on *slim* domains and is needed in this generality in section 3.5. For this purpose, we first introduce the notion of *slim* languages of trees and show that slimness is a decidable property. Afterwards, we describe how to actually compress a tree-automatic structure on a slim domain into an isomorphic string-automatic structure in the next section.

Definition 2.4.13. The *diameter* $\varnothing(t) \in \mathbb{N}$ of a tree $t \in T_\Sigma$ is the maximal number of nodes on any level, i.e.,

$$\varnothing(t) := \max\{\, |\mathrm{dom}(t) \cap \{\mathtt{0},\mathtt{1}\}^{=\ell}| \mid \ell \in \mathbb{N} \,\} .$$

For every $d \in \mathbb{N}$, the set of all $t \in T_\Sigma$ with $\varnothing(t) \leqslant d$ is denoted by $T_{\Sigma,d}$. A language $L \subseteq T_\Sigma$ of trees is *slim* if there exists $d \in \mathbb{N}$ such that $L \subseteq T_{\Sigma,d}$.

Remark 2.4.14. Let $L \subseteq T_\Sigma$ be a regular language of trees with $g_{T(L)}(n) \in O(n^1)$. According to corollary 2.3.8, we have

$$|T(L) \cap \{\mathtt{0},\mathtt{1}\}^{=n}| \in O(n^0) .$$

Put another way, there is some $d \in \mathbb{N}$ such that

$$|T(L) \cap \{\mathtt{0},\mathtt{1}\}^{=n}| \leqslant d$$

for all $n \in \mathbb{N}$. Since $\mathrm{dom}(t) \subseteq T(L)$ for each $t \in L$, this particularly implies $L \subseteq T_{\Sigma,d}$. Thus, L is slim. □

As a first step, we show that it is decidable whether the language recognized by a given tree-automaton is slim. To this end, we need the notion of *reachable* and *infinitely reachable* states: Let $\mathcal{T} = (Q, \iota, \delta, F)$ be a tree-automaton. A state $q \in Q$ is *reachable* if there is a tree $t \in T_\Sigma$ with $\delta(\iota, t) = q$. If there are infinitely many such t, then q is *infinitely reachable*. Using a simple marking algorithm, one can compute the set of all reachable states of $\mathcal{T}$ as follows: In the beginning mark ι and as long as there are unmarked states $q \in Q$ which admit marked states $r, s \in Q$ and

$a \in \Sigma$ with $\delta(r, a, s) = q$ mark these states q. Since removing unreachable states from $\mathcal{T}$ does not affect its language, we assume all states of $\mathcal{T}$ to be reachable as of now. Tree-automata with this property are called *reduced*.

Using graph algorithms, one can even compute the set of all infinitely reachable states of $\mathcal{T}$. These algorithms inspect the directed graph $G_{\mathcal{T}} = (Q, E_{\mathcal{T}})$ whose edge relation is given by

$$\langle p, q \rangle \in E_{\mathcal{T}} \quad :\Longleftrightarrow \quad \exists r \in Q, a \in \Sigma\colon \\ \delta(p, a, r) = q \vee \delta(r, a, p) = q\,. \tag{2.8}$$

Notice that, for all $t \in T_{\Sigma}$, $u \in \mathrm{dom}(t)$ and $i \in \{\mathtt{0}, \mathtt{1}\}$, there is an edge

$$\langle \delta(\iota, t, ui), \delta(\iota, t, u) \rangle \in E_{\mathcal{T}}\,,$$

which is verified by choosing $j \in \{\mathtt{0}, \mathtt{1}\} \setminus \{i\}$, $r = \delta(\iota, t, uj)$ and $a = t(u)$. It is well-known that the following conditions are equivalent for all $q \in Q$:

(1) q is infinitely reachable.

(2) There is a tree $t \in T_{\Sigma}$ with $\mathrm{h}(t) \geqslant |Q|$ and $\delta(\iota, t) = q$.

(3) $G_{\mathcal{T}}$ contains a cycle from which q is reachable.

In order to decide whether a tree-automaton recognizes a slim language, we employ the characterization given by the next lemma. Therein, an edge $\langle p, q \rangle \in E_{\mathcal{T}}$ is *fat* if one can choose r to be infinitely reachable in eq. (2.8).

Lemma 2.4.15. *Let $\mathcal{T} = (Q, \iota, \delta, F)$ be a reduced tree-automaton. The following conditions are equivalent:*

(1) The language $\mathscr{L}(\mathcal{T})$ recognized by $\mathcal{T}$ is not *slim.*

(2) There is a tree $t \in \mathscr{L}(\mathcal{T})$ with $\varnothing(t) > 2^{|Q|-1}$.

(3) $G_{\mathcal{T}}$ contains a cycle which includes a fat edge and from which some state in F is reachable.

Proof. The implication (1) $\Rightarrow$ (2) is trivial and hence it suffices to establish the implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1).

Implication (2) $\Rightarrow$ (3). We fix a tree $t \in T_\Sigma$. In order to keep notation concise, we put $t[u] := \delta(\iota, t, u)$ for each $u \in \mathrm{dom}(t)$. We further consider the set

$$Q_t := \{\, t[u] \mid u \in \mathrm{dom}(t) \,\}\,.$$

For $u, v \in \mathrm{dom}(t)$ such that $u \preccurlyeq v$, say $v = ui_1 \ldots i_k$ with $k \geqslant 0$ and $i_1, \ldots, i_k \in \{\mathtt{0}, \mathtt{1}\}$, we use $t[v, u]$ to denote the path through $G_{\mathcal{T}}$ from $t[v]$ to $t[u]$ along the states $q_k, q_{k-1}, \ldots, q_1, q_0$ with $q_\ell = t[ui_1 \cdots i_\ell]$. Notice that $t[v, u]$ visits only states in Q_t.

Using induction on $n \geqslant 0$, we show that whenever $\varnothing(t) > 2^{n-1}$ and $|Q_t| \leqslant n$, there are $u, v \in \mathrm{dom}(t)$ such that $u \prec v$ and $t[v, u]$ is a cycle which includes a fat edge. In the end, choosing $n = |Q|$ and $t \in \mathscr{L}(\mathcal{T})$ with $\varnothing(t) > 2^{|Q|-1}$ verifies condition (3) because $t[\varepsilon] \in F$ is reachable from the cycle $t[v, u]$ along the path $t[u, \varepsilon]$.

The base case $n = 0$ of the induction is trivial because the premise $|Q_t| \leqslant 0$ is never met. Henceforth, assume that $n > 0$, $\varnothing(t) > 2^{n-1}$ and $|Q_t| \leqslant n$. Let $\ell \in \mathbb{N}$ be such that the set

$$U := \mathrm{dom}(t) \cap \{\mathtt{0}, \mathtt{1}\}^{=\ell}$$

satisfies $|U| > 2^{n-1}$. Moreover, let $u \in \mathrm{dom}(t)$ be the longest common prefix of all elements in U. Clearly,

$$\ell \geqslant |u| + n \geqslant |u| + |Q_t|\,.$$

Depending on whether there exists $v \in \mathrm{dom}(t)$ with $u \prec v$ and $t[u] = t[v]$, we distinguish two cases.

First, suppose there is such v. We assume without loss of generality that $u\mathtt{0} \preccurlyeq v$. Due to the choice of u, there is some $w \in U$ with $u\mathtt{1} \preccurlyeq w$. The path $t[w, u]$ contains $\ell - |u| \geqslant |Q_t|$

edges and hence a cycle. The state $t[u\mathtt{1}]$ is infinitely reachable since it is located on or after this cycle. Thus, the edge $t[u\mathtt{0}, u]$ is fat and included in the cycle $t[v, u]$.

Now, suppose there is no $v \in \mathrm{dom}(t)$ with $u \prec v$ and $t[u] = t[v]$. We have $2 \leqslant |Q_t| \leqslant n$. Since

$$\varnothing(t{\restriction_u}) \geqslant |U| > 2^{n-1} \, ,$$

there is $i \in \{\mathtt{0}, \mathtt{1}\}$ such that $\varnothing(s) > 2^{n-2}$ for $s = t{\restriction_{ui}}$. We have $Q_s \subseteq Q_t$ and $t[u] \in Q_t \setminus Q_s$. Thus,

$$|Q_s| \leqslant |Q_t| - 1 \leqslant n - 1 \, .$$

According to the induction hypothesis, there are $v, w \in \mathrm{dom}(s)$ such that $v \prec w$ and $s[w, v]$ is a cycle which includes a fat edge. The claim of the induction follows from $uiv \prec uiw$ and $t[uiw, uiv] = s[w, v]$.

Implication (3) ⇒ (1). Using induction on $n \geqslant 0$, we show the following: If $G_{\mathcal{T}}$ contains a path which ends in $q \in Q$ and includes n fat edges, there is a tree $t \in T_\Sigma$ with $\delta(\iota, t) = q$ and $\varnothing(t) > n$. In the end, this proves statement (1) because the cycle in $G_{\mathcal{T}}$ induces paths which end in F and include arbitrarily many fat edges.

The base case $n = 0$ is trivial since $\mathcal{T}$ is reduced and every $t \in T_\Sigma$ satisfies $\varnothing(t) > 0$. Henceforth, assume $n > 0$ and consider a path π which ends in q and includes n fat edges. Let $\langle p, r\rangle$ be the last fat edge in π. Applying the induction hypothesis to everything of π before $\langle p, r\rangle$ yields a tree $s \in T_\Sigma$ with $\delta(\iota, s) = p$ and $\varnothing(s) > n - 1$. Let $\ell \in \mathbb{N}$ be such that

$$|\mathrm{dom}(s) \cap \{\mathtt{0}, \mathtt{1}\}^{=\ell}| > n - 1 \, .$$

Since $\langle p, r\rangle$ is a fat edge, there are an infinitely reachable $p' \in Q$ and $a \in \Sigma$ with $\delta(p, a, p') = r$ or $\delta(p', a, p) = r$. Due to the symmetry of both cases, we assume without loss of generality that $\delta(p, a, p') = r$. As p' is infinitely reachable, there is $s' \in T_\Sigma$ with $\delta(\iota, s') = p'$ and $\mathrm{h}(s') \geqslant \ell$.

We now consider the tree $t' = a(s, s')$, i.e., the unique $t' \in T_\Sigma$ with $t'(\varepsilon) = a$, $t'\restriction_0 = s$ and $t'\restriction_1 = s'$. Due to the choices made above, we have $\delta(\iota, t') = r$ and $\varnothing(t') > n$. Since everything in π after $\langle p, r\rangle$ forms a path from r to q, there are $t \in T_\Sigma$ and $u \in \mathrm{dom}(t)$ with $\delta(\iota, t) = q$ and $t\restriction_u = t'$. Clearly, $\varnothing(t') > n$ implies $\varnothing(t) > n$ as well. This completes the induction. $\square$

Notice that condition (3) of lemma 2.4.15 is decidable. Thus, removing all unreachable states from a tree-automaton and applying lemma 2.4.15 yields the subsequent decidability result.

Theorem 2.4.16. *Given a tree-automaton $\mathcal{T}$, one can decide whether the language recognized by $\mathcal{T}$ is slim or not. In case $\mathscr{L}(\mathcal{T})$ is slim, then $\mathscr{L}(\mathcal{T}) \subseteq T_{\Sigma,2^{n-1}}$ for n the number of reachable states of $\mathcal{T}$.* $\square$

2.4.4 Tree-Automatic Structures on Slim Domains

The sole purpose of this section is to prove the subsequent theorem. To this end, we demonstrate how any tree-automatic structure on a slim domain can be compressed into an isomorphic string-automatic structure.

Theorem 2.4.17. *Given a presentation of a tree-automatic structure $\mathscr{A}$ on a slim domain, one can compute a string-automatic presentation of $\mathscr{A}$.*

First of all, we fix an alphabet Σ and two distinct symbols $\bot, \$ \in \Sigma$. A tree $t \in T_\Sigma$ is called *special* if its root is not a leaf and every node $u \in \mathrm{dom}(t)$ has the following three properties: (1) $t(u) \neq \$$, (2) if u is an inner node, then $t(u) \neq \bot$ and u has precisely two children and (3) if u is a leaf, then $t(u) = \bot$. A tuple $\langle t_1, \ldots, t_n \rangle \in T_\Sigma^n$ is *special* if each t_i is special. We say that a relation on T_Σ is *special* if all its elements are special.

Due to remark 2.4.4, every tree-automatic structure $\mathscr{A}$ with $A \subseteq T_{\Sigma \setminus \{\bot, \$\}}$ is effectively isomorphic to a tree-automatic structure $\mathscr{B}$ such that $B \subseteq T_\Sigma$ is special. Whenever A is slim, say $A \subseteq T_{\Sigma \setminus \{\bot, \$\}, d}$, then $B \subseteq T_{\Sigma, 2d}$, i.e., B is also slim. Accordingly, we only take tree-automatic structures on special domains into account.

For the remainder of this section, we further fix some $d \in \mathbb{N}$. The translation from tree-automaticity to string-automaticity consists of two parts:

(1) We provide an encoding of any *special* tree $t \in T_{\Sigma, d}$ by a string $C(t) \in \Sigma^*$.

(2) We demonstrate that this encoding preserves automaticity.

Before defining this encoding formally, we give an intuitive description. Consider a special tree $t \in T_{\Sigma, d}$ of height h. Its encoding $C(t) = \sigma_0 \sigma_1 \cdots \sigma_h$ consists of $h+1$ blocks $\sigma_0, \sigma_1, \ldots, \sigma_h \in \Sigma^{=d}$ describing the individual levels of t. More precisely, σ_i consists of the labels of the i^{th} level from left to right and is padded up to length d by \$-symbols. For example, the special tree $t_0 \in T_{\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \bot, \$\}}$ in fig. 2.1 on the following page satisfies $\varnothing(t_0) = 6$ and is, provided that $d = 8$, encoded as

$$C(t_0) = \mathtt{a}\$^7 \ \mathtt{bc}\$^6 \ \mathtt{cb}\bot\mathtt{a}\$^4 \ \bot\bot\mathtt{a}\bot\bot\bot\$^2 \ \bot\bot\$^6 .$$

Definition 2.4.18. Let $t \in T_{\Sigma, d}$ be a special tree of height h. The *encoding* of t is the string

$$C(t) := \sigma_0 \sigma_1 \cdots \sigma_h \in \Sigma^*$$

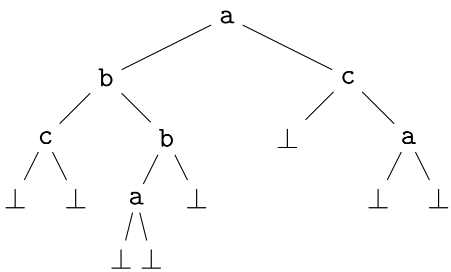


Figure 2.1: An example tree t_0

where, for each $i \in [0, h]$,

$$\sigma_i := t(u_{i1})t(u_{i2}) \cdots t(u_{is_i})\$^{d-s_i} ,$$

provided that

$$\mathrm{dom}(t) \cap \{\mathtt{0}, \mathtt{1}\}^{=i} = \{\, u_{i1} <_{\mathrm{lex}} u_{i2} < \cdots <_{\mathrm{lex}} u_{is_i} \,\} .$$

We lift this encoding to special tuples and special relations in the obvious way.

Given a tree-automaton $\mathcal{T}_A$ which recognizes a special language $A \subseteq T_{\Sigma,d}$, it seems quite reasonable to construct a string-automaton which recognizes $C(A)$ by simulating $\mathcal{T}_A$. However, it turned out to be far more complicated to implement the analogous simulation for binary relations $R \subseteq T^2_{\Sigma,d}$. The reason for this disparity is as follows: Every position of $C(t)$ refers to a unique node of t whereas distinct positions of $C(t_1) \otimes C(t_2)$ might refer to the same node of $t_1 \otimes t_2$. Thus, a string-automaton which simulates a tree-automaton recognizing R would also have to keep track of which positions refer to the same nodes. Unfortunately, it appears to be too intricate to handle this construction properly.

In view of this intricacy, we resort to the connection between recognizability by finite automata and mso definability along with mso interpretations in order to prove that the encoding C preserves automaticity. Our main tool in this proof is the following lemma. Basically, it states that t can be recovered from $C(t)$ by means of an mso interpretation which is independent from t. Recall that the modulo quantifier "there are n many for some $n \in \mathbb{N}$ with $n \equiv r \pmod{d}$", written as $\exists^{r \bmod d}$, can be expressed in mso logic over strings.

Lemma 2.4.19. *There is an mso interpretation $\mathcal{I}_C$ of Σ-trees in Σ-strings such that $t \cong \mathcal{I}_C(C(t))$ for every special tree $t \in T_{\Sigma,d}$.*

Proof. Let $t \in T_{\Sigma,d}$ be a special tree of height h. We write $C(t) = \sigma_0\sigma_1 \cdots \sigma_h$ with $\sigma_i, s_i, u_{i1}, \ldots, u_{is_i}$ for $i \in [0,h]$ as in definition 2.4.18. We construct the interpretation

$$\mathcal{I}_C = (\delta; (\varphi_{S_b})_{b=0,1}, (\varphi_{P_a})_{a\in\Sigma})$$

by describing how to interpret t in $C(t)$. The node u_{ij} shall be represented by the j^{th} position of σ_i, i.e., the one which is labeled by $t(u_{ij})$. Accordingly, we choose

$$\delta(x) := \neg P_{\$}(x)$$

and

$$\varphi_{P_a}(x) := P_a(x)\,.$$

Concerning the construction of $\varphi_{S_b}(x,y)$ for $b = 0,1$, recall that u_{ij} is an inner node of t precisely if $t(u_{ij}) \neq \perp$. Now, consider an inner node u_{ij}. The children of u_{ij} are the nodes $u_{i+1,2k+1}$ and $u_{i+1,2k+2}$, where k is the number of inner nodes among $u_{i1}, \ldots, u_{i,j-1}$. Notice that $0 \leqslant k < d$. Suppose we had positions p, q, r in $C(t)$ such that $C(t) \models \psi[p,q,r]$ for the formula

$$\psi(x,y,z) := \exists^{0 \bmod d} z' \, (z' < z) \wedge z \leqslant x < z + d \leqslant y \leqslant z + 2d - 1\,.$$

The first conjunct ensures that r is the first position of σ_i for some $i \in [0, h]$. The second conjunct in turn ensures that p and q are positions in σ_i and σ_{i+1}, respectively. Using this formula $\psi(x, y, z)$, we finally choose $\varphi_{S_b}(x, y)$ as follows:

$$\varphi_{S_b}(x,y) := \neg P_\perp(x) \wedge \exists z \left(\begin{array}{l} \psi(x,y,z) \wedge \\ \displaystyle\bigvee_{0 \leqslant k < d} \left(\begin{array}{l} \exists^{=k} z' \, (z \leqslant z' < x) \wedge \\ y = z + d + 2k + b \end{array} \right) \end{array} \right)$$

This completes the construction of $\mathcal{I}_C$. $\square$

As a first consequence, $C(t_1) = C(t_2)$ implies

$$t_1 \cong \mathcal{I}_C(C(t_1)) = \mathcal{I}_C(C(t_2)) \cong t_2$$

and hence $t_1 = t_2$. Put another way, the encoding C is injective. The previous lemma along with the next one shows that C preserves regularity.

Lemma 2.4.20. *There is an mso sentence Φ_C which is suitable for Σ-strings and such that any $w \in \Sigma^*$ satisfies Φ_C if and only if there is a special $t \in T_{\Sigma,d}$ with $C(t) = w$.*

Proof. First of all, we characterize those $w \in \Sigma^*$ which shall satisfy Φ_C: There is a special tree $t \in T_{\Sigma,d}$ with $C(t) = w$ if and only if w admits a factorization $w = \sigma_0\sigma_1 \cdots \sigma_h$ with $h \in \mathbb{N}$ which satisfies the following conditions:

(1) $\sigma_0, \sigma_1, \ldots, \sigma_h \in (\Sigma \setminus \{\$\})^+\$^* \cap \Sigma^{=d}$,

(2) $|\sigma_0|_{\Sigma\setminus\{\perp,\$\}} = 1$ and $|\sigma_0|_\perp = 0$,

(3) $|\sigma_i|_{\Sigma\setminus\{\$\}} = 2 \cdot |\sigma_{i-1}|_{\Sigma\setminus\{\perp,\$\}}$ for each $i \in [1, n]$ and

(4) $|\sigma_h|_{\Sigma\setminus\{\perp,\$\}} = 0$.

It is a matter of routine to verify this characterization provided the following ideas behind the four conditions are taken into account: (1) w has the overall shape of an encoding of a special tree of height h, (2) the 0^{th} level contains precisely one node which is no leaf, namely the root, (3) every inner node on the $(i-1)^{\text{st}}$ level induces two nodes on the i^{th} level and (4) there are no inner nodes on the last level.

Using the ideas from the proof of lemma 2.4.19 and the formula

$$\psi(x, y) := \exists^{0 \bmod d} x' \, (x' < x) \wedge x \leqslant y \leqslant x + d - 1 \,,$$

which ensures that x refers to the first position of some σ_i and y to a position in the same σ_i, it is another matter of routine to translate the four conditions above into the desired mso sentence Φ_C. □

A simple consequence of the previous two lemmas is that the encoding C preserves regularity. We note that the inverse of C does *not* preserve regularity.

Proposition 2.4.21. *Let $A \subseteq T_{\Sigma,d}$ be a special language. If A is regular, then $C(A)$ is also regular.*

Proof. Suppose that A is regular. According to theorem 2.3.3, there is an mso sentence Ψ defining A. Due to the choice of $\mathcal{I}_C$ and Φ_C, the mso sentence $\Phi_C \wedge \Psi^{\mathcal{I}_C}$ defines the language $C(A)$. This implies that $C(A)$ is regular by theorem 2.3.3 once more. □

We now prove that the encoding C does not only preserve regularity but also automaticity of n-ary relations. Basically, the main idea behind this proof is the same as above although it is more involved. Consider a special tuple $\boldsymbol{t} \in T^n_{\Sigma,d}$. In general, the structure $\otimes\boldsymbol{t}$ contains more elements than the structure $\otimes C(\boldsymbol{t})$ and is hence *not* directly mso interpretable therein. We solve this problem by means of the unique homomorphism $\mu\colon (\Sigma_\diamond^n)^* \to \Sigma_\diamond^*$ which extends the inclusion $\Sigma_\diamond^n \hookrightarrow \Sigma_\diamond^*$, i.e.,

$$\mu(\boldsymbol{a}_1\boldsymbol{a}_2\ldots\boldsymbol{a}_\ell) := a_{11}a_{12}\ldots a_{1n}\; a_{21}a_{22}\ldots a_{2n}\;\cdots\; a_{\ell 1}a_{\ell 2}\ldots a_{\ell n}\,.$$

Intuitively, μ turns a string of column vectors into a string of individual letters by turning each column vector into a row vector and concatenating all of them. The interpretation of $\otimes\boldsymbol{t}$ in $\mu(\otimes C(\boldsymbol{t}))$ embraces two aspects that we consider separately. To this end, we regard the tuple $\boldsymbol{t}$ as a forest $\mathscr{F}(\boldsymbol{t})$ augmented by the same-level relation L and unary relations Q_i marking the t_i. Formally,

$$F(\boldsymbol{t}) := \bigcup_{1\leqslant i\leqslant n} \{i\}\times \operatorname{dom}(t_i)$$

and

$$\begin{aligned}
\langle\langle i,u\rangle,\langle j,v\rangle\rangle \in S_b^{\mathscr{F}(\boldsymbol{t})} \quad &:\Longleftrightarrow \quad i=j\ \&\ \langle u,v\rangle \in S_b^{t_i}\,,\\
\langle i,u\rangle \in P_a^{\mathscr{F}(\boldsymbol{t})} \quad &:\Longleftrightarrow \quad u\in P_a^{t_i}\,,\\
\langle\langle i,u\rangle,\langle j,v\rangle\rangle \in L^{\mathscr{F}(\boldsymbol{t})} \quad &:\Longleftrightarrow \quad |u|=|v|\,,\\
\langle i,u\rangle \in Q_k^{\mathscr{F}(\boldsymbol{t})} \quad &:\Longleftrightarrow \quad i=k\,,
\end{aligned}$$

where $b\in\{0,1\}$, $a\in\Sigma$ and $k\in[1,n]$. The next two lemmas demonstrate how to interpret $\otimes\boldsymbol{t}$ in $\mathscr{F}(\boldsymbol{t})$ and $\mathscr{F}(\boldsymbol{t})$ in $\mu(\otimes C(\boldsymbol{t}))$. Combining these interpretations yields an interpretation of $\otimes\boldsymbol{t}$ in $\mu(\otimes C(\boldsymbol{t}))$.

Lemma 2.4.22. *There is an mso interpretation $\mathcal{I}_{\mathscr{F}}$ of $\Sigma_\diamond^n$-trees in forests such that $\otimes\boldsymbol{t}\cong\mathcal{I}_{\mathscr{F}}(\mathscr{F}(\boldsymbol{t}))$ for every $\boldsymbol{t}\in T_\Sigma^n$.*

Proof. Let $\boldsymbol{t}\in T_\Sigma^n$. Our first goal is to show that the equivalence relation $\equiv$ on $F(\boldsymbol{t})$ defined by $\langle i,u\rangle\equiv\langle j,v\rangle$ if $u=v$ is mso definable in $\mathscr{F}(\boldsymbol{t})$. For this purpose, we consider the partial ordering $\sqsubseteq$ of $F(\boldsymbol{t})$ given by $\langle i,u\rangle\sqsubseteq\langle j,v\rangle$ if $i=j$ and $u\preccurlyeq v$. It is well-known that there is *one* mso formula which defines the prefix relation $\preccurlyeq$ in *every* Σ-tree. Hence, $\sqsubseteq$ is mso definable in $\mathscr{F}(\boldsymbol{t})$. The following formula $\varepsilon(x, y)$ defines the relation $\equiv$ by encoding an easy characterization of the condition $u = v$:

$$\varepsilon(x, y) := L(x, y) \wedge \forall x', y' \begin{pmatrix} (x' \sqsubseteq x \wedge y' \sqsubseteq y \wedge L(x', y')) \rightarrow \\ (\omega_0(x') \leftrightarrow \omega_0(y')) \end{pmatrix}$$

where

$$\omega_0(z) := \exists z' \, S_0(z', z) \, .$$

Now, we devise the interpretation

$$\mathcal{I}_{\mathscr{F}} = (\delta; (\varphi_{S_b})_{b=0,1}, (\varphi_{P_{\boldsymbol{a}}})_{\boldsymbol{a} \in \Sigma_\diamond^n})$$

by describing how to interpret $\otimes \boldsymbol{t}$ in $\mathscr{F}(\boldsymbol{t})$. Recall that

$$\mathrm{dom}(\otimes \boldsymbol{t}) = \mathrm{dom}(t_1) \cup \cdots \cup \mathrm{dom}(t_n) \, .$$

The node $u \in \mathrm{dom}(\otimes \boldsymbol{t})$ shall be represented by the unique pair $\langle u, i \rangle \in F(\boldsymbol{t})$ where i is minimal with $u \in \mathrm{dom}(t_i)$. Using the formula $\varepsilon(x, y)$ constructed above, we choose

$$\delta(x) := \forall y \Big(\varepsilon(x, y) \rightarrow \bigvee_{1 \leqslant i \leqslant j \leqslant n} Q_i(x) \wedge Q_j(y) \Big)$$

and

$$\varphi_{S_b}(x, y) := \exists z \left(\varepsilon(x, z) \wedge S_b(z, x) \right) .$$

Concerning $\varphi_{P_{\boldsymbol{a}}}(x)$, we define two auxiliary formulas, where $a \in \Sigma$:

$$\psi_{i,a}(x) := \exists y \left(\varepsilon(x, y) \wedge Q_i(y) \wedge P_a(y) \right)$$

and

$$\psi_{i,\diamond}(x) := \neg \exists y \left(\varepsilon(x, y) \wedge Q_i(y) \right) .$$

Finally,

$$\varphi_{P_{\langle a_1, \ldots, a_n \rangle}}(x) := \bigwedge_{1 \leqslant i \leqslant n} \psi_{i,a_i}(x) \, . \qquad \square$$

Lemma 2.4.23. *There is an mso interpretation $\mathcal{I}_\mu$ of forests in $\Sigma_\diamond$-strings such that $\mathscr{F}(\boldsymbol{t}) \cong \mathcal{I}_\mu(\mu(\otimes C(\boldsymbol{t})))$ for every special $\boldsymbol{t} \in T^n_{\Sigma,d}$.*

Proof. Let $\boldsymbol{t} \in T^n_{\Sigma,d}$ be special and $i \in [1,n]$. Notice that taking every n^{th} position of $\mu(\otimes C(\boldsymbol{t}))$ starting with the i^{th} yields the string $C(t_i) \diamond \ldots \diamond$. Thus, one can interpret $\mathscr{F}(\boldsymbol{t})$ in $\mu(\otimes C(\boldsymbol{t}))$ by representing the node $\langle i, u\rangle \in F(\boldsymbol{t})$ by the representative of u in this scattered substring $C(t_i)$. Using the quantifier $\exists^{i \bmod n}$ and the interpretation $\mathcal{I}_C$ from lemma 2.4.19, it is just a matter of routine to obtain all formulas of $\mathcal{I}_\mu$, except for $\varphi_L(x,y)$. In the encoding $C(t_i) = \sigma_0\sigma_1\cdots\sigma_h$, the factor σ_k represents the k^{th} level of t_i. Consequently, if we factorize $\mu(\otimes C(\boldsymbol{t})) = \tau_0\tau_1\cdots\tau_\ell$ such that $\tau_0, \tau_1, \ldots, \tau_\ell \in \Sigma^{=dn}$, then τ_k represents the k^{th} level of $\mathscr{F}(\boldsymbol{t})$. Accordingly, we choose

$$\varphi_L(x,y) := \exists z \left(\exists^{0 \bmod dn} z' \,(z' < z) \wedge z \leqslant x, y \leqslant z + dn - 1\right). \ \square$$

Lemma 2.4.24. *There is an mso sentence Φ_μ which is suitable for $\Sigma_\diamond$-strings and such that any $w \in \Sigma_\diamond^*$ satisfies Φ_μ if and only if there is a special $\boldsymbol{t} \in T^n_{\Sigma,d}$ with $\mu(\otimes(C(\boldsymbol{t}))) = w$.*

Proof. Basically, Φ_μ just needs to verify for each $i \in [1,n]$ that there is a special tree $t_i \in T_{\Sigma,d}$ such that the scattered substring containing every n^{th} position of w starting from the i^{th} is of the form $C(t_i) \diamond \ldots \diamond$. Using the quantifier $\exists^{i \bmod n}$ and the sentence Φ_C from lemma 2.4.20, this is easily accomplished. $\square$

Proposition 2.4.25. *Let $R \subseteq T^n_{\Sigma,d}$ be a special relation. If R is automatic, then $C(R)$ is also automatic.*

Proof. Suppose that R is automatic. According to theorem 2.3.3, there is an mso sentence Ψ defining $\otimes R$. Due to the choice of $\mathcal{I}_{\mathscr{F}}$, $\mathcal{I}_\mu$ and Φ_μ, the mso sentence $\Phi_\mu \wedge (\Psi^{\mathcal{I}_{\mathscr{F}}})^{\mathcal{I}_\mu}$ defines the language $\mu(\otimes C(R))$. This implies that $\mu(\otimes C(R))$ is regular by theorem 2.3.3 once more. Since μ is a homomorphism, this further implies that $\otimes C(R)$ is regular, i.e., $C(R)$ is automatic. $\square$

We are now able to prove theorem 2.4.17 by collecting all the pieces.

Theorem 2.4.17. *Given a presentation of a tree-automatic structure $\mathscr{A}$ on a slim domain, one can compute a string-automatic presentation of $\mathscr{A}$.*

Proof. Let $\mathscr{A} = (A; R_1, \dots, R_n)$ be a tree-automatic structure such that A is slim and $\mathcal{P} = (\mathcal{T}_0; \mathcal{T}_1, \dots, \mathcal{T}_n)$ a presentation of $\mathscr{A}$. Theorem 2.4.16 allows for computing a number $d \in \mathbb{N}$ with $A \subseteq T_{\Sigma,d}$ from $\mathcal{T}_0$. Obviously, the structure

$$C(\mathscr{A}) := \big(C(A); C(R_1), \dots, C(R_n)\big)$$

is isomorphic to $\mathscr{A}$. According to propositions 2.4.21 and 2.4.25, $C(\mathscr{A})$ is also string-automatic. Since all proofs throughout this section are constructive, they actually provide a way to compute a presentation of $C(\mathscr{A})$ from $\mathcal{P}$. $\square$

3 Automatic Linear Orders

A problem that gained a lot of attention in the context of automatic structures is the following: Given a class $\mathcal{C}$ of structures and a formalism $\mathbf{F}$ for presenting structures, characterize all $\mathbf{F}$-presentable members of $\mathcal{C}$ in terms of model-theoretic properties. Instances of this problem where a full characterization was successfully accomplished include unary string-automatic graphs, groups, equivalence relations and linear orders [Blu99, Rub04], string-automatic and tree-automatic well-orders [Del04], string-automatic finitely generated groups [OT05], Boolean algebras and fields [KNRS07]. In some cases, only upper bounds on the model-theoretic complexity of the $\mathbf{F}$-presentable members of $\mathcal{C}$ are known. For example, bounds on the ranks of string-automatic linear orders and order trees were established [KRS05]. In this chapter, we focus on linear orders and the various notions of automaticity introduced in chapter 2.

The first results on automatic linear orders are due to Hodgson [Hod83] as well as Khoussainov and Nerode [KN95]: (i) The order type η is string-automatically presentable, i.e., contained in SA. (ii) The ordinal ω^n belongs to SA for each $n \in \mathbb{N}$. (iii) The class of linear orders in SA is closed under finite sums and products. As an immediate consequence of the latter two facts, every ordinal $\alpha < \omega^\omega$ is a member of SA. In the end of their paper, Khoussainov and Nerode asked for the least ordinal which is *not* contained in SA. Several years later, Delhommé [Del04] came up with the answer: An ordinal α belongs to SA if and only if $\alpha < \omega^\omega$. In addition, he proved ω^{ω^ω} to be the respective bound for TA, the class of tree-automatically presentable structures. In order to obtain these results, Delhommé developed a *decomposition technique* for automatic structures and applied it to the class of well-orders.

Shortly afterwards, Khoussainov, Rubin and Stephan [KRS05] applied this decomposition technique to scattered linear orders and combined it with Hausdorff's theorem: The finite-condensation rank[1] $\mathrm{FC}(A)$ of any linear order A from SA is bounded by $\mathrm{FC}(A) < \omega$. It is well-known that any two ordinals α and γ satisfy $\mathrm{FC}(\alpha) \leqslant \gamma$ precisely if $\alpha \leqslant \omega^\gamma$, cf. lemma 2.2.4 on page 18. Consequently, the upper bound ω on the FC-rank generalizes the upper bound ω^ω on ordinals. In line with this, it has been suspected that $\mathrm{FC}(A) < \omega^\omega$ for any linear order A in TA since then, but a confirmation was missing. In the second half of this chapter, we close this gap by confirming the suspicion:[2]

(1) The FC-rank of any linear order A in TA is bounded by $\mathrm{FC}(A) < \omega^\omega$ (theorem 3.3.19).

Roughly speaking, the proof is another application of the decomposition technique to scattered linear orders in combination with Hausdorff's theorem. In more detail, (the wording of) Delhommé's *decomposition theorem* for tree-automatic structures is slightly

[1] See definition 2.2.3 on page 18 for details on the *finite-condensation rank*.

[2] This result already appeared in [Hus13]. Recently, Jain, Khoussainov, Schlicht and Stephan independently showed $\mathrm{FC}(A) < \omega^\omega$ for any tree-automatic *scattered* linear order A [JKSS14]. Although this implies $\mathrm{FC}(A) \leqslant \omega^\omega$ for every tree-automatic linear order A, there is no obvious way to change their proof to rule out ω^ω as a possible rank.

too weak for this purpose and hence needs some refinement. Since Delhommé did unfortunately not provide a proof of this theorem, we state and prove a *refined decomposition theorem* for tree-automatic structures.[3] However, the main difficulty in confirming the suspicion is to substantiate that scattered linear orders are accessible to this refined decomposition technique at all. Refining the decomposition technique even further allows for *almost* completing the picture of characterizations of automatic well-orders and bounds on the FC-ranks of automatic linear orders:

(2) The FC-rank of any linear order A in $\mathsf{pTA}[k]$ is bounded by $\mathrm{FC}(A) < \omega^k$ (theorem 3.4.1).

(3) An ordinal α is in $\mathsf{pTA}[k]$ if and only if $\alpha < \omega^{\omega^k}$ (corollary 3.4.3).

(4) An ordinal α is in $\mathsf{pSA}[k]$ if and only if $\alpha < \omega^{k+1}$ (theorem 3.2.4).

Regrettably, the FC-rank is too coarse to be bounded on the linear orders in pSA by means of the decomposition technique. In view of this impediment, we take another approach to obtain this bound nevertheless. First, we prove that every linear order in pSA is scattered. Afterwards, we demonstrate how to transform a string-automatic scattered linear order into an automatic well-order on the same domain while preserving the VD_*-rank. The VD_*-rank is a slight variation of the FC-rank on scattered linear orders which deviates by at most 1.[4]

(5) Every linear order A in $\mathsf{pSA}[k]$ is scattered and its VD_*-rank is bounded by $\mathrm{VD}_*(A) \leqslant k$ (theorem 3.2.9).

With all these bounds on ranks of automatic linear orders in mind,

[3]This refined decomposition technique also turned out to be useful in the context of well-founded order trees [HKLL13].

[4]The VD_*-rank is defined right after theorem 2.2.5 on page 19.

one might wonder which of them do actually provide characterizations. In the case of string-automatic linear orders of growth in $O(n^1)$, which are basically just the unary string-automatic linear orders, the answer is affirmative [Blu99, Rub04]. In all other cases, the answer is negative due to a simple reason: There is a scattered linear order with FC-rank 2 whose first-order theory is undecidable. Subsequently, the question arises whether the bounds on the ranks characterize the automatically presentable linear orders among those linear orders whose first-order theories are sufficiently simple to not rule out automaticity. In line with the optimal upper bounds shown by Kuske [Kus09], we call a first-order theory *sufficiently simple for string-automatic decidability* if the Σ_k-theory belongs to $(k-1)$-$\mathsf{EXPSPACE}$ for each $k \geqslant 1$. Again, the answer is negative and for the linear orders in pSA even worse:

(6) There is a computable scattered linear order which is neither contained in SA nor in TA although it has FC-rank 2 and its first-order theory is sufficiently simple for string-automatic decidability (theorem 3.6.3).

(7) There is a scattered linear order of VD_*-rank 2 in SA which is *not* contained in pSA (example 3.6.4).

(8) For each $k \geqslant 2$, there is a scattered linear order of VD_*-rank 2 in $\mathsf{pSA}[k]$ which is *not* contained in $\mathsf{pSA}[k-1]$ (example 3.6.5).

Going one step further, one might ask whether the bound on the FC-rank of linear orders in SA characterizes them among all linear orders in TA. This time, the answer is affirmative for scattered linear orders at least:[5]

(9) A scattered linear order A from TA is contained in SA if and only if $\mathrm{FC}(A) < \omega$ (theorem 3.5.5).

[5] This result already appeared in [Hus12].

In addition, this characterization is effective in the following sense: Given a tree-automatic presentation of a scattered linear order A, one can decide whether A satisfies $\mathrm{FC}(A) < \omega$ and hence belongs to SA. In case of a positive answer, one can even compute a string-automatic presentation of A.

A problem which is closely related to characterizing the automatically presentable linear orders is solving their *isomorphism problem*: Given two automatic presentations of linear orders, decide whether the presented linear orders are isomorphic. In fact, Delhommé's characterization of the ordinals in SA almost immediately led to a decision procedure for the isomorphism problem for string-automatic well-orders [KRS05]. Given string-automatic presentations of two well-orders, this procedure basically computes the Cantor normal forms of their order types and compares these normal forms afterwards. The former of these two steps heavily relies on the fact that first-order logic plays well with ordinals below ω^ω. Since this nice interplay is no longer available beyond ω^ω and no other methods have been found yet, the isomorphism problem for tree-automatic well-orders is still unsolved. Based on the aforementioned decidable characterization of the scattered linear orders A in TA which satisfy $\mathrm{FC}(A) < \omega$, we contribute the following partial solution:

(10) The isomorphism problem for tree-automatic well-orders of order types strictly below ω^{ω^2} is decidable (corollary 3.5.10).

Unfortunately, none of our numerous attempts towards extending the upper bound beyond ω^{ω^2} was crowned with success.

For the sake of completeness, we mention that the isomorphism problem for arbitrary string-automatic linear orders is Σ_1^1-complete and hence highly undecidable [KLL13b]. Obviously, this complexity is inherited by the tree-automatic version. In contrast, isomorphism of unary string-automatic linear orders can be decided in linear time [LM11]. Finally, the isomorphism problem for scattered linear orders is still open in the string-automatic case and undecidable in the tree-automatic case, where the best known lower bound is Π_1^0-hardness [Kus14].

Outline. The current state of research on linear orders in SA is presented in section 3.1. The subsequent section 3.2 is devoted to the positive results on linear orders in pSA. In section 3.3, we present the (refined) decomposition technique and apply it to obtain the aforementioned bounds on linear orders in TA. The analogous results for linear orders in pTA are the subject of section 3.4. The purpose of section 3.5 is twofold: First, we characterize those scattered linear orders in TA which are also contained in SA. Based on this characterization, we further demonstrate our partial solution to the isomorphism problem for tree-automatic well-orders. All results concerning the non-automaticity of various scattered linear orders are finally proved in section 3.6.

3.1 String-Automaticity

Although we sketched the current state of the art concerning the characterization of string-automatic linear orders in the introduction already, we state the two major results for later reference again. Moreover, we present some consequences of these results that are used later as well. In the end, we briefly discuss the isomorphism problem for string-automatic linear orders.

First of all, we provide two useful examples of string-automatic linear orders. The first of them demonstrates that the linear order of the rationals is string-automatically presentable.

Example 3.1.1. We define a linear order $Q = (\{\mathtt{0}, \mathtt{1}\}^*; \leqslant_{\text{in}})$ by $u \leqslant_{\text{in}} v$ if the longest common prefix w of u and v satisfies $w\mathtt{0} \preccurlyeq u\mathtt{0}$ and $w\mathtt{1} \preccurlyeq v\mathtt{1}$, where $\preccurlyeq$ denotes the prefix relation. Intuitively,

$\leqslant_{\text{in}}$ captures the in-order traversal of the full binary tree. It is a matter of routine to check that Q is a string-automatic linear order. Using Cantor's theorem 2.2.1 on page 15, one can prove that Q has order type η. $\square$

The second example shows that SA contains all ordinals $\alpha < \omega^\omega$.

Example 3.1.2. For every $n \in \mathbb{N}$, a string-automatic type ω^n well-order is given by $((\mathtt{1}^*\mathtt{0})^n; \leqslant_{\text{in}})$, where $\leqslant_{\text{in}}$ is the linear ordering from the previous example. A string-automatic well-order of type $\alpha < \omega^n$ is obtained by taking an initial segment thereof. Clearly, this exhausts all ordinals $\alpha < \omega^\omega$.[6] $\square$

The best known partial characterization of the class of string-automatically presentable linear orders is given by the theorem below. Due to the previous example, any $n < \omega$ is in effect the FC-rank of some string-automatic linear order. In particular, the upper bound ω is optimal.

Theorem 3.1.3 ([KRS05])**.** *The* FC*-rank of any linear order A in* SA *is bounded by*

$$\mathrm{FC}(A) < \omega\,. \qquad \square$$

We already mentioned that this theorem is by no means a characterization. The subsequent example provides a reason for this claim.

Example 3.1.4 ([KRS05])**.** Let $M \subseteq \mathbb{N}$ be an undecidable set and consider the order type $\tau_M := \sum_{n \in M} \zeta + n$. On the one hand, the FC-rank of τ_M is 2. On the other hand, τ_M is *not* automatically presentable since M can easily be reduced to the first-order theory of τ_M. $\square$

[6] We refer to the argument provided in the last two sentences as "the last argument from example 3.1.2" in what follows.

In section 3.6, we even give an example of a linear order which is not contained in SA although it has FC-rank 2 and its first-order theory is sufficiently simple for string-automatic decidability. Delhommé's characterization of the string-automatically presentable ordinals is an immediate consequence of example 3.1.2 and theorem 3.1.3.

Corollary 3.1.5 ([Del04]). *An ordinal α is contained in SA if and only if*

$$\alpha < \omega^\omega . \qquad \square$$

Recall that the finite-condensation relation on a string-automatic linear order A is effectively automatic. Consequently, the finite-condensation process on A can be made effective. According to theorem 3.1.3, this process arrives at a dense linear order after finitely many steps and terminates then. Since being dense is a first-order definable property, the termination condition is indeed decidable. These circumstances have two important consequences:

Corollary 3.1.6 ([KRS05]). *Given a presentation of a string-automatic linear order A, one can decide whether A is scattered. In case A is* not *scattered, one can compute a string-automaton recognizing a regular type η subset of A.* $\square$

Corollary 3.1.7 ([KRS05]). *Given a string-automatic presentation of a linear order A, it is decidable whether A is a well-order. In case of a positive answer, one can compute numbers $n_1, \dots, n_s \in \mathbb{N}$ such that $\omega^{n_1} + \cdots + \omega^{n_s}$ is the Cantor normal form of the order type of A.*

Proof sketch. In view of corollary 3.1.6, we may assume that A is scattered. Then, one can easily base a decision procedure on the following equivalence: A scattered linear order A is a well-order if and only if each $\sim$-class contains a least element and

$A/{\sim}$ is a well-order.[7] This procedure always terminates since $0 < \mathrm{FC}(A) < \omega$ implies $\mathrm{FC}(A/{\sim}) < \mathrm{FC}(A)$.

In case that A is a well-order, the Cantor normal form of the order type α of A can be computed similarly: The case $\alpha = 0$ is trivial. If α is a successor ordinal, say $\alpha = \beta + 1$, and $\omega^{n_1} + \cdots + \omega^{n_s}$ the Cantor normal form of β, then $\omega^{n_1} + \cdots + \omega^{n_s} + \omega^0$ is the Cantor normal form of α. If α is a limit ordinal, β the order type of $A/{\sim}$, i.e. the unique ordinal with $\alpha = \omega\,\beta$, and $\omega^{n_1} + \cdots + \omega^{n_s}$ the Cantor normal form of β, then $\omega^{1+n_1} + \cdots + \omega^{1+n_s}$ is the Cantor normal form of α. This procedure always terminates since $\beta < \alpha$ holds in both cases, in the latter case due to $0 < \alpha < \omega^\omega$. $\square$

In chapter 5, we use the following variation of the second part of corollary 3.1.7.

Corollary 3.1.8. *Given a presentation of a string-automatic well-order A, one can compute string-automata recognizing the parts A_i of its decomposition $A_1 + \cdots + A_s$ into Cantor normal form.*

Proof. Basically, we implement the algorithm from the proof of corollary 3.1.7 in terms of string-automata. To this end, let α be the order type of A. If $\alpha = 0$ or α is a successor ordinal, the claim is again trivial. If α is a limit ordinal, B the set of limit points of A and $B = B_1 + \cdots + B_s$ the decomposition of B into Cantor normal form, then $A = A_1 + \cdots + A_s$ with

$$A_i := \{\, u \in A \mid \exists v \in B_i \colon u \sim v \,\}$$

is the decomposition of A into Cantor normal form. Clearly, the set B is effectively regular and one can compute an automaton recognizing A_i from an automaton recognizing B_i. $\square$

[7] The *finite-condensation relation* $\sim$ is defined on page 17.

Last but not least, corollary 3.1.7 immediately implies that the isomorphism problem for string-automatic well-orders is decidable.

Corollary 3.1.9 ([KRS05]). *Given string-automatic presentations of two well-orders A and B, one can decide whether A and B are isomorphic.* □

In contrast, the isomorphism problem for arbitrary string-automatic linear orders is highly undecidable.

Theorem 3.1.10 ([KLL13b]). *Given string-automatic presentations of two linear orders A and B, it is Σ^1_1-complete to decide whether A and B are isomorphic.* □

For the intermediate class of string-automatic *scattered* linear orders, it is still open whether the isomorphism problem is decidable or not. The best known upper bound is a reduction to the first-order theory of $(\mathbb{N}; +, \times)$ [KLL13b].

3.2 String-Automaticity on Polynomial Domains

The objective of this section is twofold: On the one hand, we characterize, for every $k \in \mathbb{N}$, the ordinals in $\mathsf{pSA}[k]$ as those being strictly below ω^{k+1}. On the other hand, we prove that all linear orders in $\mathsf{pSA}[k]$ are scattered and their VD_*-ranks do not exceed k. As we already mentioned in the introduction, the VD_*-rank is too coarse for interacting with Delhommé's decomposition technique for $\mathsf{pSA}[k]$. More precisely, for the decomposition technique to be applicable it would be necessary that the step from $\mathsf{pSA}[k-1]$ to $\mathsf{pSA}[k]$ made infinitely many new VD_*-ranks available. In view of this obstacle, we take the following alternative approach: We use the decomposition technique to characterize the ordinals in $\mathsf{pSA}[k]$

(theorem 3.2.4). Afterwards, we establish that all linear orders in pSA are scattered (corollary 3.2.6). Finally, we demonstrate how to transform any string-automatic scattered linear order into a string-automatic well-order on the same domain and with the same VD_*-rank. By these means, we obtain an optimal upper bound on the VD_*-rank of linear orders in $\mathsf{pSA}[k]$ (theorem 3.2.9).

3.2.1 Well-Orders

This section is devoted to the proof of theorem 3.2.4, which characterizes the ordinals α in $\mathsf{pSA}[k]$ as those satisfying $\alpha < \omega^{k+1}$. The "if"-part is verified by the next example.

Example 3.2.1. For every $m \in \mathbb{N}$, example 3.1.2 provides the string-automatic type $\omega^k m$ well-order $A = (\mathtt{1}^{<m}\mathtt{0}(\mathtt{1}^*\mathtt{0})^k; \leqslant_{\mathrm{in}})$. Since $g_A(n) \in O(n^k)$, the class $\mathsf{pSA}[k]$ contains $\omega^k m$ and hence all ordinals $\alpha < \omega^{k+1}$ by the last argument from example 3.1.2. □

In the remainder of this section, we prove the "only if"-part by applying Delhommé's decomposition technique. To avoid notational overhead, we do not formulate this technique as a standalone result first but rather employ it *ad hoc*. The basic fact on well-orders underlying the proof is a result by Caruth on the natural sum of ordinals.

Theorem 3.2.2 ([Car42])**.** *Let A be a well-order and consider a partition $\{B_1, \ldots, B_n\}$ of A. If α and β_i denote the order types of A and B_i, respectively, then*

$$\alpha \leqslant \beta_1 \oplus \cdots \oplus \beta_n \,. \qquad \square$$

The main ingredient of extending the decomposition technique from SA to pSA is the following technical lemma:

Lemma 3.2.3. *Let $k \in \mathbb{N}$ and $A \subseteq \Sigma^*$ be a regular language with $g_A(n) \in O(n^k)$. There exists a constant $c \in \mathbb{N}$ such that any anti-chain (wrt the prefix relation $\preccurlyeq$) $U \subseteq \Sigma^*$ contains at most c elements $u \in U$ with $g_{u^{-1}A}(n) \in \Theta(n^k)$.*

Proof. Suppose that $\Sigma = \{\sigma_1, \ldots, \sigma_r\}$. If $r = 1$, the claim is trivial since any anti-chain then contains at most one element. Henceforth, we assume $r \geqslant 2$. Let $\mathcal{M} = (Q, \iota, \delta, F)$ be a string-automaton recognizing A and put $m := |Q|$. We prove $c := r^m - 1$ to be a possible choice.

Aiming for a contradiction, suppose there is an anti-chain $U \subseteq \Sigma^*$ such that $|U| \geqslant r^m$ and $g_{u^{-1}A}(n) \in \Theta(n^k)$ for all $u \in U$. We derive a contradiction by constructing a subset $B \subseteq A$ with $g_B(n) \in \Theta(n^{k+1})$. To this end, let T be the set of all $v \in \Sigma^*$ which are the longest common prefix of some non-empty subset of U. The structure $(T; \preccurlyeq)$ forms a finite tree whose set of leaves is U. Our first goal is to show that every inner node $v \in T \setminus U$ branches at most r-ary.

Aiming for another contradiction, suppose that v has at least $r+1$ mutually distinct immediate successors $w_0, \ldots, w_r$. For each $i \in [0, r]$, there is $\sigma_i \in \Sigma$ with $v\sigma_i \preccurlyeq w_i$. By the pigeon hole principle, we have $\sigma_i = \sigma_j$ for some $0 \leqslant i < j \leqslant r$. Let $w' \in T$ be the longest common prefix of w_i and w_j. In particular, $v\sigma_i \preccurlyeq w'$ and hence $w' \neq v$. Consequently, w_i and w_j cannot both be immediate successors of v in the tree T. This proves that inner nodes of T branch at most r-ary.

In view of this bound and since T has $|U| \geqslant r^m$ many leaves, the height of T must be at least m. Consequently, T contains a path $v_0 \prec v_1 \prec \cdots \prec v_m \prec \cdots$. According to the pigeon hole principle, there are $i, j \in [0, m]$ with $i < j$ and $\delta(\iota, v_i) = \delta(\iota, v_j)$. Let $\sigma_1 \in \Sigma$ and $w_1 \in \Sigma^*$ be such that $v_i\sigma_1 w_1 = v_j$. Moreover, let $U' \subseteq U$ be a subset whose longest common prefix happens to be v_i. Since $v_i\sigma_1$ is not the longest common prefix of U', there are $u_0 \in U$ and $\sigma_2 \in \Sigma \setminus \{\sigma_1\}$ with $v_i\sigma_2 \preccurlyeq u_0$, say $v_i\sigma_2 w_2 = u_0$. In the remainder of this proof, we show that the set

$$B := v_1(\sigma_1 w_1)^* \sigma_2 w_2 (u_0^{-1}A) \tag{3.1}$$

is a subset of A with $g_B(n) \in \Theta(n^{k+1})$, which clearly contradicts $g_A(n) \in O(n^k)$.

First, consider some arbitrary $y \in B$, say $y = v_i(\sigma_1 w_1)^\ell \sigma_2 w_2 x$ with $\ell \in \mathbb{N}$ and $x \in u_0^{-1}A$. Using $\delta(\iota, v_i) = \delta(\iota, v_i\sigma_1 w_1)$, we obtain

$$\delta(\iota, y) = \delta(\iota, v_i(\sigma_1 w_1)^\ell \sigma_2 w_2 x) = \delta(\iota, v_i\sigma_2 w x) = \delta(\iota, u_0 x) \in F\,.$$

Thus, $y \in A$ and, more generally, $B \subseteq A$.

Since $g_{u_0^{-1}A}(n) \in \Theta(n^k)$, theorem 2.3.7 on page 30 provides us with an unambiguous rational expression for $u_0^{-1}A$ of the shape

$$u_0^{-1}A = \bigcup_{1 \leqslant i \leqslant m} p_{i0} q_{i1}^* p_{i1} \cdots q_{ik_i}^* p_{ik_i}$$

with $\max\{k_1, \ldots, k_m\} = k$. According to the choice of B in eq. (3.1), we have the following rational expression for B:

$$B = \bigcup_{1 \leqslant i \leqslant m} v_1(\sigma_1 w_1)^* \sigma_2 w_2 p_{i0} q_{i1}^* p_{i1} \cdots q_{ik_i}^* p_{ik_i}\,.$$

Since $\sigma_1 \neq \sigma_2$, this expression is unambiguous as well. Consequently, another application of theorem 2.3.7 yields

$$g_B(n) \in \Theta(n^{k+1})\,. \qquad \square$$

The theorem below is the main result of this section. The first part of the corresponding *ad hoc* application of the decomposition technique bears notable similarities to the proof of [KRS05, proposition 4.6].

Theorem 3.2.4. *Let $k \in \mathbb{N}$. An ordinal α is in* $\mathsf{pSA}[k]$ *if and only if*
$$\alpha < \omega^{k+1} .$$

Proof. The "if"-part has already been verified in example 3.2.1. We prove the "only if"-part by induction on k. The case $k = 0$ is trivial, since $g_A(n) \in O(n^0)$ just says that A is finite. Henceforth, we assume $k > 0$.

Aiming for a contradiction, suppose there is a string-automatic well-order A of type $\alpha \geqslant \omega^{k+1}$ with $g_A(n) \in O(n^k)$. Let $\mathcal{M} = (Q, \iota, \delta, F)$ be a string-automaton recognizing $<_A$. We consider the set

$$M := \{\, \langle u, v \rangle \in \Sigma^* \times A \mid |u| = |v| \,\} .$$

For all $\langle u, v \rangle \in M$, we define a pair of states

$$\boldsymbol{q}_{u,v} := \langle \delta(\iota, u \otimes u), \delta(\iota, u \otimes v) \rangle$$

and a subset

$$A_{u,v} := \{\, w \in A \mid u \preccurlyeq w \text{ and } w <_A v \,\} \subseteq A .$$

The suborder $A_{u,v}$ is also automatic since it is first-order definable in A augmented by the regular language $u\Sigma^* \cap A$. Let $\alpha_{u,v}$ denote the order type of $A_{u,v}$. We derive a contradiction by proving the following two statements:

(1) For all $\langle u, v \rangle, \langle \tilde{u}, \tilde{v} \rangle \in M$, $\boldsymbol{q}_{u,v} = \boldsymbol{q}_{\tilde{u},\tilde{v}}$ implies $\alpha_{u,v} = \alpha_{\tilde{u},\tilde{v}}$.

(2) For every $m \in \mathbb{N}$, there exists $\langle u, v \rangle \in M$ such that
$$\omega^k m < \alpha_{u,v} < \omega^{k+1} .$$

Notice that statement (1) implies that there are only finitely many ordinals of the form $\alpha_{u,v}$, whereas statement (2) amounts to the contrary.

Regarding statement (1). Consider two pairs $\langle u, v\rangle, \langle \tilde{u}, \tilde{v}\rangle \in M$ with $\boldsymbol{q}_{u,v} = \boldsymbol{q}_{\tilde{u},\tilde{v}}$. We show that the injective map $f\colon A_{u,v} \to A_{\tilde{u},\tilde{v}}$ defined by

$$f(uw) := \tilde{u}w$$

is an isomorphism between $A_{u,v}$ and $A_{\tilde{u},\tilde{v}}$. For every $w \in \Sigma^*$, we have

$$\begin{aligned}\delta(\iota, uw \otimes v) &= \delta(\iota, (u \otimes v)(w \otimes \varepsilon))\\ &= \delta(\iota, (\tilde{u} \otimes \tilde{v})(w \otimes \varepsilon))\\ &= \delta(\iota, \tilde{u}w \otimes \tilde{v})\,,\end{aligned}$$

where the first and third equality use the defining property of M and the second equality uses $\delta(\iota, u\otimes v) = \delta(\iota, \tilde{u}\otimes\tilde{v})$. Consequently, we obtain the following chain of equivalences, which establishes that f is surjective and well-defined wrt its image:

$$\begin{aligned}uw \in A_{u,v} \quad &\Longleftrightarrow \quad \delta(\iota, uw \otimes v) \in F\\ &\Longleftrightarrow \quad \delta(\iota, \tilde{u}w \otimes \tilde{v}) \in F\\ &\Longleftrightarrow \quad \tilde{u}w \in A_{\tilde{u},\tilde{v}}\,.\end{aligned}$$

It remains to show that f is order-preserving. Based on the premise $\delta(\iota, u \otimes u) = \delta(\iota, \tilde{u} \otimes \tilde{u})$, we obtain the following chain of equivalences for all $uw_1, uw_2 \in A_{u,v}$:

$$\begin{aligned}uw_1 <_A uw_2 \quad &\Longleftrightarrow \quad \delta(\iota, uw_1 \otimes uw_2) \in F\\ &\Longleftrightarrow \quad \delta(\iota, \tilde{u}w_1 \otimes \tilde{u}w_2) \in F\\ &\Longleftrightarrow \quad f(uw_1) <_A f(uw_2)\,.\end{aligned}$$

This proves statement (1).

Regarding statement (2). We fix some $m \in \mathbb{N}$. Let $c \in \mathbb{N}$ be the constant which exists by lemma 3.2.3. Since $\alpha \geqslant \omega^{k+1}$, there exists some $v \in A$ such that the initial segment

$$I_v := \{\, w \in A \mid w <_A v \,\}$$

of A has order type $\omega^k(mc+1)$. We fix this v and put $\ell := |v|$. Observe that I_v can be partitioned as

$$I_v = (I_v \cap \Sigma^{<\ell}) \uplus \biguplus_{u \in \Sigma^{=\ell}} A_{u,v} \,.$$

Let κ_v be the size of the finite set $I_n \cap \Sigma^{<\ell}$. Due to theorem 3.2.2, we have

$$\omega^k(mc+1) \leqslant \kappa_v \oplus \bigoplus_{u \in \Sigma^{=\ell}} \alpha_{u,v} \,. \tag{3.2}$$

We consider the set

$$U := \left\{ u \in \Sigma^{=\ell} \;\middle|\; g_{A_{u,v}}(n) \in \Theta(n^k) \right\}.$$

For $u \in \Sigma^{=\ell} \setminus U$, we have

$$g_{A_{u,v}}(n) \in O(n^k) \setminus \Theta(n^k)$$

and hence

$$g_{A_{u,v}}(n) \in O(n^{k-1}) \,.$$

This implies $\alpha_{u,v} < \omega^k$ by the induction hypothesis. In contrast, for $u \in U$, we have

$$A_{u,v} \subseteq u(u^{-1}A) \subseteq A$$

and hence both

$$g_{u^{-1}A}(n) \leqslant g_A(n + |u|) \in O(n^k)$$

as well as

$$g_{u^{-1}A}(n) \geqslant g_{A_{u,v}}(n + |u|) \in \Omega(n^k) \,.$$

Thus,

$$g_{u^{-1}A}(n) \in \Theta(n^k) \,.$$

Due to the choice of c, we obtain $|U| \leqslant c$. If we had $\alpha_{u,v} \leqslant \omega^k m$ for each $u \in U$, we would obtain

$$\kappa_v \oplus \bigoplus_{u \in \Sigma^\ell} \alpha_{u,v} = \underbrace{\kappa_v \oplus \bigoplus_{u \in \Sigma^\ell \setminus U} \alpha_{u,v}}_{<\omega^k} \oplus \underbrace{\bigoplus_{u \in U} \alpha_{u,v}}_{\leqslant \omega^k mc} < \omega^k(mc+1)\,.$$

Since this would contradict eq. (3.2), we conclude that there is some $\tilde{u} \in U$ such that $\alpha_{\tilde{u},v} > \omega^k m$. At the same time, $A_{\tilde{u},v} \subseteq I_v$ implies

$$\alpha_{\tilde{u},v} \leqslant \omega^k(mc+1) < \omega^{k+1}\,.$$

This proves statement (2) and hence the whole theorem. □

3.2.2 Dense and Non-scattered Linear Orders

In this section, we prove that there is neither a dense nor any non-scattered *infinite* linear order in pSA. Put another way, every linear order in pSA is scattered. The proof of the first result uses growth arguments based on lemma 2.4.10 on page 37, which are standard in the investigation of automatic structures by now.

Theorem 3.2.5. *The linear order* $(\mathbb{Q}; \leqslant)$ *does not belong to* pSA.

Proof. Let $(A; \leqslant_A)$ be a string-automatic type η linear order. We prove the claim by demonstrating that its domain A grows exponentially. We consider the relation

$$R := \left\{ \langle u, v, w \rangle \in A^3 \;\middle|\; u <_A v \text{ and } w = \min\nolimits_{\mathrm{llex}}(u,v)_A \right\},$$

where $(u,v)_A$ denotes the open interval between u and v. Clearly, R is automatic and locally finite. Thus, lemma 2.4.10 provides us with a constant $C \in \mathbb{N}$ such that

$$|w| \leqslant \max\{\, |u|, |v| \,\} + C$$

for all $\langle u, v, w \rangle \in R$. Moreover, let $D \in \mathbb{N}$ be such that $|u| \leqslant D$ for at least two distinct $u \in A$. In the remainder of this proof, we derive a contradiction by inductively constructing subsets $G_0, G_1, G_2, \ldots \subseteq A$ such that, for each $n \geqslant 0$, $|G_n| = 2^n + 1$ but $|u| \leqslant C \cdot n + D$ for all $u \in G_n$. Obviously, this is only possible if A grows exponentially.

Due to the choice of D, there is a subset $G_0 \subseteq A$ with the desired properties. Henceforth, assume $n \geqslant 1$. Let $G_{n-1} \subseteq A$ be the subset which exists by the induction hypothesis and $u_0 <_A u_1 <_A \cdots <_A u_{2^{n-1}}$ the ascending enumeration of its elements. For each $i \in [1, 2^{n-1}]$, let $v_i \in A$ be such that $\langle u_{i-1}, u_i, v_i \rangle \in R$. Since $|u_{i-1}|, |u_i| \leqslant C \cdot (n-1) + D$ and due to the choice of C, we obtain $|v_i| \leqslant C \cdot n + D$. Consequently, the set

$$G_n := G_{n-1} \cup \{ \, v_i \mid i \in [1, 2^{n-1}] \, \}$$

proves the claim of the inductive step. □

According to corollary 3.1.6, every string-automatic non-scattered linear order contains an automatic suborder of type η. Along with theorem 3.2.5, we conclude:

Corollary 3.2.6. *Every linear order in* pSA *is scattered.* □

As a consequence, we obtain an interesting subtle difference between arbitrary string-automatic structures and string-automatic linear orders. For each $k \geqslant 3$, one can find a structure in $\mathsf{pSA}[k]$ which is *not* string-automatically presentable on any domain growing in $\Theta(n^k)$.[8] In contrast, this situation cannot arise in the context of linear orders.

[8] This result is part of unpublished joint work with Bakhadyr Khoussainov and Jiamou Liu.

Corollary 3.2.7. *Let $k \geqslant 1$. Every infinite linear order in* $\mathsf{pSA}[k]$ *is isomorphic to a string-automatic linear order A with $g_A(n) \in \Theta(n^k)$.*

Proof. Let $(B; \leqslant)$ be a string-automatic linear order satisfying $g_B(n) \in O(n^k)$. Moreover, let $\sim$ be the finite-condensation relation on B. Since B is infinite and scattered, there is some $u \in B$ whose $\sim$-class $[u]$ is infinite. More precisely, $[u]$ has order type ω, $\omega^\star$ or ζ. We only demonstrate the case of order type ω, the other two cases are similar.

Let $\mathtt{0}$ and $\mathtt{1}$ be fresh symbols not appearing in B. Moreover, let A be the string-automatic linear order which is obtained from B by replacing the convex subset $[u]$ with the length-lexicographic ordering of the set $(\mathtt{0}^*\mathtt{1})^k$. Obviously, A and B are isomorphic. The domain

$$A := B \setminus [u] \cup (\mathtt{0}^*\mathtt{1})^k$$

satisfies $g_A(n) \in \Theta(n^k)$ by theorem 2.3.7 on page 30. $\square$

3.2.3 Scattered Linear Orders

We complete the investigation of linear orders in pSA by providing an upper bound on their VD_*-ranks in theorem 3.2.9. Our main tool is the lemma below, which reduces the problem to the characterization of ordinals in theorem 3.2.4. We note that this result bears perfunctory similarities with [KRS05, theorem 7.7] where bounds on the Cantor–Bendixson ranks of string-automatic trees were obtained by means of the Kleene–Brouwer ordering and theorem 3.1.3.

Lemma 3.2.8. *Let $(A; \leqslant)$ be a string-automatic scattered linear order. There exists an automatic well-ordering $\trianglelefteq$ of A such that*

$$\mathrm{VD}_*(A; \leqslant) = \mathrm{VD}_*(A; \trianglelefteq)\,.$$

Proof. Due to theorem 3.1.3 the VD_*-rank of $(A;\leqslant)$ is finite, say $n := \mathrm{VD}_*(A;\leqslant)$. If $n = 0$, then A is finite and the claim is trivial. Henceforth, we assume $n > 0$. For each $k \in \mathbb{N}$, let $\sim^k$ and $[u]_k$ denote the k^{th} iterated finite-condensation relation on $(A;\leqslant)$ and the $\sim^k$-class of $u \in A$, respectively.

Before delving into the details, we provide a brief sketch of the basic idea. Intuitively, we consider a tree whose nodes are all $\sim^k$-classes of A for all $k \in \mathbb{N}$. They are ordered by inclusion. Since $\mathrm{VD}(A;\leqslant) \leqslant n+1$, there is only one $\sim^{n+1}$-class, namely A, which is the root. The leaves are the $\sim^0$-classes, i.e., the singleton sets $\{u\}$ for each $u \in A$. If the children of any node are ordered by $\ll$, the induced ordering of the leaves is isomorphic to $(A;\leqslant)$ via mapping $\{u\}$ to u. Now, suppose that each node, which is a subset of A, is labeled with its length-lexicographically least element. Further suppose that the children of any node are ordered length-lexicographically with respect to their labels. If there are infinitely many children, they are now ordered like ω. In effect, the induced linear ordering of the leaves is a well-ordering. In the remainder of the proof, we formalize this description and show that the resulting linear order is indeed a well-order, which in addition has the same VD_*-rank as $(A;\leqslant)$.

We consider the type ω^{n+1} well-order $(A^{n+1};\sqsubseteq)$ where the relation $\langle u_n,\dots,u_0\rangle \sqsubset \langle v_n,\dots,v_0\rangle$ holds true precisely when the greatest i with $u_i \neq v_i$ satisfies $u_i <_{\mathrm{llex}} v_i$. We further consider the map $f\colon A \to A^{n+1}$ given by

$$f(w) := \langle \min_{\mathrm{llex}}[w]_n,\dots,\min_{\mathrm{llex}}[w]_0\rangle\,.$$

This map is injective since $[w]_0 = \{w\}$. Finally, we define a well-ordering $\trianglelefteqslant$ of A by $u \trianglelefteqslant v$ if $f(u) \sqsubseteq f(v)$. Obviously, $\trianglelefteqslant$ is first-order definable in $(A;\leqslant)$ augmented by the automatic relations $\leqslant_{\mathrm{llex}}, \sim^0,\dots,\sim^n$ and hence automatic itself. Thus, it only remains to show $\mathrm{VD}_*(A;\leqslant) = \mathrm{VD}_*(A;\trianglelefteqslant)$. In terms of the order type α of $(A; \trianglelefteq)$, this amounts to proving

$$\omega^n \leqslant \alpha < \omega^{n+1} .$$

For each $k \in \mathbb{N}$, we obtain a suborder B_k of $(A; \leqslant)$ which is isomorphic to $A/{\sim^k}$ by choosing the length-lexicographic least element of each $\sim^k$-class as its representative, i.e.,

$$B_k := \{ \min_{\text{llex}}[w]_k \mid w \in A \} .$$

Notice that $f(A) \subseteq B_n \times A^n$. Since $\mathrm{VD}_*(A; \leqslant) = n$, the set B_n is finite, say $m := |B_n|$. Hence, $\alpha \leqslant \omega^n\, m < \omega^{n+1}$. This proves the upper bound on α.

Concerning the lower bound, we first recall that

$$(A; \leqslant) = \sum_{X \in A/\sim^n} (X; \leqslant) = \sum_{w \in (B_n; \leqslant)} ([w]_n; \leqslant) .$$

Since $\mathrm{VD}_*(A; \leqslant) = n$ and B_n is finite, there is some $w \in B_n$ with $\mathrm{VD}_*([w]_n; \leqslant) = n$. In the remainder of this proof, we show that the order type of $([w]_n; \trianglelefteq)$ is at least ω^n. In the end, this establishes the lower bound on α. More generally, we show for all $k \in \mathbb{N}$ and $u \in B_k$ the following implication: If $\mathrm{VD}_*([u]_k; \leqslant) = k$, then the order type of $([u]_k; \trianglelefteq)$ is at least ω^k.

We proceed by induction on k. The cases $k = 0$ and $k = 1$ are trivial. Henceforth, assume $k \geqslant 2$ and consider some $u \in B_k$ with $\mathrm{VD}_*([u]_k; \leqslant) = k$. The equation

$$([u]_k; \leqslant) = \sum_{v \in ([u]_k \cap B_{k-1}; \leqslant)} ([v]_{k-1}; \leqslant)$$

captures the condensation of all $\sim^{k-1}$-classes contained in $[u]_k$ into the $\sim^k$-class $[u]_k$. Recall that $\mathrm{VD}_*([v]_{k-1}; \leqslant) \leqslant k - 1$ for each $v \in [u]_k \cap B_{k-1}$. In fact, there are infinitely many v with $\mathrm{VD}_*([v]_{k-1}; \leqslant) = k - 1$ since $\mathrm{VD}_*([u]_k; \leqslant) = k$. Due to the induction hypothesis, the order type of $([v]_{k-1};\trianglelefteq)$ is at least ω^{k-1} for these infinitely many v. Hence, it suffices to prove

$$([u]_k;\trianglelefteq) = \sum_{v\in([u]_k\cap B_{k-1};\trianglelefteq)} ([v]_{k-1};\trianglelefteq)$$

in order to show that the order type of $([u]_k;\trianglelefteq)$ is at least ω^k.

To this end, consider $v,\tilde{v} \in [u]_k \cap B_{k-1}$, $w \in [v]_{k-1}$ and $\tilde{w} \in [\tilde{v}]_{k-1}$ with $v \triangleleft \tilde{v}$. Our goal is to show $w \triangleleft \tilde{w}$. Since $w,\tilde{w} \in [u]_k$, we have $[w]_k = [\tilde{w}]_k$ and hence

$$\min\nolimits_{\text{llex}}[w]_\ell = \min\nolimits_{\text{llex}}[\tilde{w}]_\ell$$

for each $\ell \geqslant k$. Moreover, $v,\tilde{v} \in B_{k-1}$ and $v \neq \tilde{v}$ imply

$$\min\nolimits_{\text{llex}}[w]_{k-1} = v <_{\text{llex}} \tilde{v} = \min\nolimits_{\text{llex}}[\tilde{w}]_{k-1}\,.$$

This proves $w \triangleleft \tilde{w}$. □

The theorem below is the desired analogue of theorem 3.1.3 for the class pSA.

Theorem 3.2.9. *Let $k \in \mathbb{N}$. Every linear order A in $\mathsf{pSA}[k]$ is scattered and satisfies*

$$\mathrm{VD}_*(A) \leqslant k\,.$$

Proof. Let $(A;\leqslant)$ be a string-automatic linear order satisfying $g_A(n) \in O(n^k)$. According to corollary 3.2.6, $(A;\leqslant)$ is scattered. Due to lemma 3.2.8, there exists a well-ordering $\trianglelefteq$ of A such that $\mathrm{VD}_*(A;\leqslant) = \mathrm{VD}_*(A;\trianglelefteq)$. By theorem 3.2.4, the order type α of $(A;\trianglelefteq)$ is bounded by $\alpha < \omega^{k+1}$ and hence $\mathrm{VD}_*(A;\trianglelefteq) \leqslant k$. □

In view of example 3.1.4, this bound on the VD_*-rank does *not* characterize the linear orders in $\mathsf{pSA}[k]$ whenever $k \geqslant 2$. In contrast, theorem 3.2.9 is a characterization if $k \leqslant 1$. This is trivial for $k = 0$ since $\mathsf{pSA}[0]$ and $\mathcal{VD}_0^\star$ are the classes of all finite structures and all finite linear orders, respectively. For $k = 1$, this follows from the circumstance that the unary string-automatically presentable linear orders are precisely those in $\mathcal{VD}_1^\star$ [Rub04, theorem D.1.19].

3.3 Tree-Automaticity

After studying the linear orders from pSA in the previous section, we now turn to those contained in TA. We provide an upper bound on their FC-ranks in theorem 3.3.19. Subsequently, corollary 3.3.21 provides Delhommé's characterization of the ordinals in TA. First of all, we give an example of a tree-automatic linear order.

Example 3.3.1. Let $\leqslant_\Sigma$ be an arbitrary linear ordering of the alphabet Σ. Moreover, let $\leqslant_{\text{in}}$ be the linear ordering of $\{\mathtt{0},\mathtt{1}\}^*$ from example 3.1.1. We define a linear ordering $\trianglelefteq$ of T_Σ by $t_1 \triangleleft t_2$ if the least (wrt $\leqslant_{\text{in}}$) $u \in \text{dom}(t_1) \cup \text{dom}(t_2)$ where t_1 and t_2 differ either satisfies $u \notin \text{dom}(t_1)$ or both $u \in \text{dom}(t_1) \cap \text{dom}(t_2)$ and $t_1(u) <_\Sigma t_2(u)$. It is matter of routine to check that $\trianglelefteq$ is an automatic linear ordering of T_Σ. In addition, one can show that $(T_\Sigma; \trianglelefteq)$ is *not* scattered. $\square$

3.3.1 The Decomposition Technique

In this section, we motivate and prove our refined version of Delhommé's decomposition theorem. As we apply the decomposition technique only to linear orders here, we refrain from presenting its general version but rather focus on its specialization to linear orders. First of all, recall that one of the main ingredients of characterizing the ordinals in pSA and SA is the application of theorem 3.2.2, which is restated below.

Theorem 3.2.2 ([Car42]). *Let A be a well-order and consider a partition $\{B_1, \dots, B_n\}$ of A. If α and β_i denote the order types of A and B_i, respectively, then*

$$\alpha \leqslant \beta_1 \oplus \cdots \oplus \beta_n . \qquad \square$$

The generalization of Delhommé's upper bound on the ordinals in SA to an upper bound on the FC-ranks of linear orders in SA is based on two additional ingredients: Hausdorff's theorem 2.2.2 on page 15 and theorem 3.3.2 below, which in some sense extends theorem 3.2.2 to scattered linear orders.

Theorem 3.3.2 ([KRS05]). *Let A be a scattered linear order and consider a partition $\{B_1, \dots, B_n\}$ of A. Then*

$$\mathrm{VD}_*(A) \leqslant \max\{\, \mathrm{VD}_*(B_1), \dots, \mathrm{VD}_*(B_n) \,\} . \tag{3.3}$$

For a moment, we reverse the point of view on partitions of linear orders. Let A and $B_1, \dots, B_n$ be linear orders and B the partial order that is obtained by taking the disjoint union of the B_i. Then A admits a partition $\{A_1, \dots, A_n\}$ with $A_i \cong B_i$ for each i if and only if A is (isomorphic to) a linear extension of B. In this light, theorem 3.3.2 can be read as follows: Any scattered linear extension A of B satisfies eq. (3.3).[9]

In the context of decomposing tree-automatic linear orders, we are not only confronted with linear extension of disjoint unions but also with linear extensions of direct products. If partitions are regarded as the converse of disjoint unions, the according converse of direct products is given by the definition below.[10]

[9] As a matter of fact, any linear extension of B is scattered.

[10] Earlier publications dealing with the decomposition technique used the term "sum-decomposition" instead of "partition", cf. [Hus13, HKLL13]. However, as we need the notion of a partition anyway and in order to prevent confusion, we refrain from using the term "sum-decomposition" in this meaning here.

Definition 3.3.3. Let A be a linear order. A *box-decomposition* of A is a tuple $(f; B_1, \dots, B_n)$ consisting of finitely many linear orders $B_1, \dots, B_n$ and a bijection $f \colon B_1 \times \cdots \times B_n \to A$ such that $u_1 \leqslant_{B_1} v_1, \dots, u_n \leqslant_{B_n} v_n$ implies

$$f(u_1, \dots, u_n) \leqslant_A f(v_1, \dots, v_n) .$$

Let $\mathcal{S}$ be a set of linear orders. We say that A is *box-decomposable in* $\mathcal{S}$ if there exists a box-decomposition $(f; B_1, \dots, B_n)$ of A with $B_1, \dots, B_n \in \mathcal{S}$.

Notice that box-decompositions are closed under permutations in the following sense: If $(f; B_1, \dots, B_n)$ is a box-decomposition of A and $i_1, \dots, i_n$ a permutation of $1, \dots, n$, then $(f'; B_{i_1}, \dots, B_{i_n})$ with

$$f'(u_{i_1}, \dots, u_{i_n}) := f(u_1, \dots, u_n)$$

is a box-decomposition of A as well. The fundamental result on box-decompositions of well-orders used by Delhommé to prove his upper bound on the ordinals contained in TA is as follows:

Theorem 3.3.4 ([Car42])**.** *Let A be a well-order and consider a box-decomposition $(f; B_1, \dots, B_n)$ of A. If α and β_i denote the order types of A and B_i, respectively, then*

$$\alpha \leqslant \beta_1 \otimes \cdots \otimes \beta_n . \qquad \square$$

The expected extension to scattered linear orders would read as follows: Let A be a well-order and consider a box-decomposition $(f; B_1, \dots, B_n)$ of A. Then

$$\mathrm{VD}_*(A) \leqslant \mathrm{VD}_*(B_1) \oplus \cdots \oplus \mathrm{VD}_*(B_n) . \tag{3.4}$$

However, this assertion is *not* valid as the next example shows:

Example 3.3.5. Let $\gamma > 0$ be a countable ordinal and

$$A = \sum_{k \in \zeta} A_k$$

a sum of scattered linear orders A_k with $\mathrm{VD}_*(A_k) = \gamma$. Clearly, $\mathrm{VD}_*(A) = \gamma + 1$. Moreover, let $f \colon \mathbb{N}^2 \to A$ be a bijection such that, for each $k \in \mathbb{Z}$,

$$f^{-1}(A_k) = \{ \langle x, y \rangle \mid x - y = k \} .$$

Since each of these sets $f^{-1}(A_k)$ forms an anti-chain in the partial order which is the direct product of $(\mathbb{N}; \leqslant)$ and $(\mathbb{N}; \geqslant)$, the tuple $\big(f; (\mathbb{N}; \leqslant), (\mathbb{N}; \geqslant)\big)$ is a box-decomposition of A. However, there is no meaningful bound on $\mathrm{VD}_*(A)$ in terms of $\mathrm{VD}_*(\mathbb{N}; \leqslant)$ and $\mathrm{VD}_*(\mathbb{N}; \geqslant)$ because γ can be chosen arbitrarily. $\square$

Fortunately, it turns out that such “wild” behavior cannot happen in the context of tree-automatic linear orders. Our formalization of “non-wild” behavior is based on the following abstraction from automatic presentations of linear orders.

Definition 3.3.6. Let A be a linear order. A *finite device* for A is a map $\mu \colon A^2 \to Q$ into a finite set Q which admits a subset $F \subseteq Q$ such that, for all $u, v \in A$, $u \leqslant v$ if and only if $\mu(u, v) \in F$.

This notion abstracts from automatic presentations in the following sense: Let A be an automatic linear order and $\mathcal{T} = (Q, \iota, \delta, F)$ an automaton recognizing $\leqslant_A$. Then the map $\mu \colon A^2 \to Q$ defined by

$$\mu(s, t) := \delta(\iota, s \otimes t)$$

is a finite device for A. Using this notion, “non-wild” or, as we call them, “tame” box-decompositions are formalized as follows:

Definition 3.3.7. A box-decomposition $(f; B_1, \dots, B_n)$ of A is called *tame* if there are finite devices $\mu_1, \dots, \mu_n$ for $B_1, \dots, B_n$, respectively, such that the following defines a finite device μ for A:

$$\mu(f(u_1, \dots, u_n), f(v_1, \dots, v_n)) := \langle \mu_1(u_1, v_1), \dots, \mu_n(u_n, v_n) \rangle .$$

Let $\mathcal{S}$ be a set of linear orders. We say that A is *tamely box-decomposable in* $\mathcal{S}$ if there exists a tame box-decomposition $(f; B_1, \dots, B_n)$ of A with $B_1, \dots, B_n \in \mathcal{S}$.

As the words “wild” and “tame” shall suggest, the assertion in eq. (3.4) on page 79 becomes valid if the respective box-decomposition is presumed to be tame. We prove this claim in theorem 3.3.17 in the next section. In the remainder of this section, we demonstrate our refined decomposition theorem 3.3.8 for tree-automatic linear orders. Apart from its specialization to the later use case, the essential difference to Delhommé’s (unproven) version is the addition of the word “tamely”. As a matter of fact, the notion of tameness can be extended to graphs or even arbitrary structures and yields analogous results then, cf. [Hus13, HKLL13].

Theorem 3.3.8 (decomposition theorem, cf. [Del04])**.** *Let A be a tree-automatic linear order. There exists a* finite *set $\mathcal{S}$ of tree-automatic linear orders such that every closed interval in A admits a partition into suborders which are* tamely *box-decomposable in $\mathcal{S}$.*

Before delving into the details of the proof, we sketch how the decomposition of any closed interval $I = [\ell, r]_A \subseteq A$ is carried out, which is also depicted in fig. 3.1 on the following page. Roughly speaking, two trees belong to the same class of the partition of I if they (1) coincide on the domain of $\ell \otimes r$ and (2) lead to the same states in the automaton recognizing $\leqslant_A$ along the boundary of $\ell \otimes r$, i.e., in the nodes $u_1, \dots, u_m$ shown in fig. 3.1. Each class C of this partition is then tamely box-decomposed into

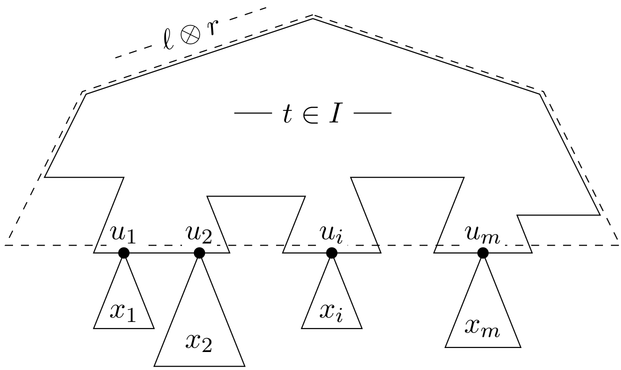


Figure 3.1: Basic idea behind the decomposition of the closed interval $I = [\ell, r]_A \subseteq A$

$(f; X_1, \ldots, X_m)$, where the domain of X_i is the set of all subtrees rooted at u_i within any $t \in C$ and the linear ordering of X_i is chosen in a very natural way. Moreover,

$$f(x_1, \ldots, x_m) := t_0[u_1/x_1, \ldots, u_m/x_m]$$

for some arbitrary $t_0 \in C$. After carrying out this decomposition formally, we conclude the proof by showing that only finitely many distinct order types occur among all the involved X_i.

As we use the same decomposition in sections 3.4 and 3.5 again, we have split the proof of theorem 3.3.8 into several lemmas. For the remainder of this section, we fix a tree-automatic linear order A as well as tree-automata $\mathcal{T} = (Q, \iota, \delta, F)$ and $\mathcal{T}' = (Q', \iota', \delta', F')$ recognizing $\leqslant_A$ and the relation

$$\left\{ \langle t, \ell, r \rangle \in A^3 \;\middle|\; t \in [\ell, r]_A \right\},$$

respectively. Furthermore, we fix a closed interval $I = [\ell, r]_A \subseteq A$ and put

$$D := \mathrm{dom}(\ell \otimes r) = \mathrm{dom}(\ell) \cup \mathrm{dom}(r)\,.$$

Recall that the *boundary* of D is the set

$$\partial D := \{\, ui \mid u \in D, i \in \{\mathtt{0}, \mathtt{1}\}, ui \notin D \,\}\,.$$

The equivalence relation $\equiv_I$ defined below formalizes the partition of I we described above.

Definition 3.3.9. The *I-type* of a tree $t \in T_\Sigma$ is the tree $\vartheta \in T_{\Sigma \uplus (Q \times Q')}$ defined by

$$\mathrm{dom}(\vartheta) := \mathrm{dom}(t) \cap (D \cup \partial D)$$

and

$$\vartheta(u) := \begin{cases} t(u) & \text{if } u \in D, \\ \langle \delta(\iota, t \otimes t, u), \delta'(\iota', \otimes\langle t, \ell, r\rangle, u)\rangle & \text{if } u \in \partial D. \end{cases}$$

Two trees $t_1, t_2 \in T_\Sigma$ are *I-equivalent*, denoted by $t_1 \equiv_I t_2$, if their I-types coincide.

Let ϑ be an I-type, $u_1, \dots, u_m$ an enumeration of $\mathrm{dom}(\vartheta) \cap \partial D$ and $\vartheta(u_i) = \langle q_i, q_i'\rangle$ for each i. Since $\vartheta \notin T_\Sigma$ in general, the convolution $\vartheta \otimes \vartheta$ is not a valid input for $\mathcal{T}$. However, $\vartheta \otimes \vartheta$ provides enough information to be treated as such an input. More precisely, we define

$$\delta(\iota, \vartheta \otimes \vartheta) := \delta_{u_1/q_1, \dots, u_m/q_m}(\iota, \vartheta{\restriction_D} \otimes \vartheta{\restriction_D})\,.$$

Similarly, we treat $\otimes\langle \vartheta, \ell, r\rangle$ as an input for $\mathcal{T}'$ by defining

$$\delta'(\iota, \otimes\langle \vartheta, \ell, r\rangle) := \delta_{u_1/q_1', \dots, u_m/q_m'}(\iota', \otimes\langle \vartheta{\restriction_D}, \ell, r\rangle)\,.$$

In fact, these two conventions along with definition 3.3.9 were just chosen such that

$$\delta(\iota, \vartheta \otimes \vartheta) = \delta(\iota, t \otimes t)$$

and

$$\delta'(\iota', \otimes\langle\vartheta, \ell, r\rangle) = \delta'(\iota', \otimes\langle t, \ell, r\rangle)$$

for every tree $t \in T_\Sigma$ with I-type ϑ. The latter equality particularly implies that ϑ completely determines whether $t \in I$. Put another way, I is a union of $\equiv_I$-classes. Since every I-type ϑ satisfies $\mathrm{dom}(\vartheta) \subseteq D \cup \partial D$, there are only finitely many I-types or, equivalently, $\equiv_I$-classes. The next lemma summarizes these insights.

Lemma 3.3.10. *The closed interval I is a finite union of $\equiv_I$-classes.* $\square$

In the following, $\equiv_I$-classes C with $C \subseteq I$ and their I-types play an important role.

Definition 3.3.11. An $\equiv_I$-class C is *proper* if $C \subseteq I$. An I-type is *proper* if it corresponds to a proper $\equiv_I$-class.

Our next step is to construct a tame box-decomposition of each proper $\equiv_I$-class. The components of this decomposition are given by the next lemma. For $x \in T_\Sigma$, the tree $x^\diamond \in T_{\Sigma^3_\diamond}$ is defined by

$$\mathrm{dom}(x^\diamond) := \mathrm{dom}(x)$$

and

$$x^\diamond(u) := \langle x(u), \diamond, \diamond\rangle\,.$$

Intuitively, $x^\diamond$ is obtained by convolving x with two copies of the "empty tree".

Lemma 3.3.12. *Let ϑ be a proper I-type and $u \in \mathrm{dom}(\vartheta) \cap \partial D$. The structure $(X_{\vartheta u}; \trianglelefteq_{\vartheta u})$ defined by*

$$X_{\vartheta u} := \left\{\, x \in T_\Sigma \;\middle|\; \langle\delta(\iota, x \otimes x), \delta'(\iota', x^\diamond)\rangle = \vartheta(u) \,\right\}$$

and

$$x \trianglelefteq_{\vartheta u} y \quad :\Longleftrightarrow \quad \delta_{u/\delta(\iota, x\otimes y)}(\iota, \vartheta \otimes \vartheta) \in F\,.$$

is a tree-automatic linear order.

Proof. It is a matter of routine to check that $(X_{\vartheta u}; \trianglelefteq_{\vartheta u})$ is indeed tree-automatic. It remains to verify that $\trianglelefteq_{\vartheta u}$ is a linear ordering of $X_{\vartheta u}$. For this purpose, let C be the $\equiv_I$-class belonging to ϑ and fix some arbitrary $t \in C$. We show that mapping $x \in X_{\vartheta u}$ to $t[u/x]$ defines an embedding of $(X_{\vartheta u}; \trianglelefteq_{\vartheta u})$ into $(C; \leqslant_A)$. Due to the choice of $X_{\vartheta u}$, $t[u/x]$ has I-type ϑ as well, i.e., $t[u/x] \in C$. Finally, for all $x, y \in X_{\vartheta u}$, we have

$$\begin{aligned} x \trianglelefteq_{\vartheta u} y \quad &\Longleftrightarrow \quad \delta_{u/\delta(\iota, x\otimes y)}(\iota, \vartheta \otimes \vartheta) \in F \\ &\Longleftrightarrow \quad \delta(\iota, t[u/x] \otimes t[u/y]) \in F \\ &\Longleftrightarrow \quad t[u/x] \leqslant_A t[u/y]\,. \end{aligned}$$ □

The actual tame box-decomposition itself is given by the next lemma.

Lemma 3.3.13. *Let C be a proper $\equiv_I$-class, ϑ its I-type and $u_1, \dots, u_m$ an enumeration of* $\mathrm{dom}(\vartheta) \cap \partial D$. *Furthermore, let $f\colon X_{\vartheta u_1} \times \cdots \times X_{\vartheta u_m} \to C$ be defined by*

$$f(x_1, \dots, x_m) := \vartheta[u_1/x_1, \dots, u_m/x_m]\,.$$

Then $(f; X_{\vartheta u_1}, \dots, X_{\vartheta u_m})$ is a tame box-decomposition of C.

Proof. Obviously, f is injective. Putting together all the related definitions, we easily obtain that a tree is contained in the image of f if and only if its I-type is ϑ. In other words, f is a bijection.

Our next step is to show that f satisfies the condition of definition 3.3.3. To this end, let $X := X_{\vartheta u_1} \times \cdots \times X_{\vartheta u_m}$ and consider $\boldsymbol{x}, \boldsymbol{y} \in X$ with $x_i \trianglelefteq_{\vartheta u_i} y_i$ for each $i \in [1, m]$. We have to show $f(\boldsymbol{x}) \leqslant_A f(\boldsymbol{y})$. For $i \in [0, m]$, we put

$$\boldsymbol{z}_i := \langle y_1, \ldots, y_i, x_{i+1}, \ldots, x_m \rangle \in X\,.$$

In these terms, we have to show $f(\boldsymbol{z}_0) \leqslant_A f(\boldsymbol{z}_m)$. We do so by proving

$$f(\boldsymbol{z}_0) \leqslant_A f(\boldsymbol{z}_1) \leqslant_A \cdots \leqslant_A f(\boldsymbol{z}_m)\,.$$

For this purpose, we fix some $i \in [1, m]$ and observe that

$$\begin{aligned} &f(\boldsymbol{z}_{i-1}) \otimes f(\boldsymbol{z}_i) \\ &\qquad = (\vartheta \otimes \vartheta)[(u_j/y_j \otimes y_j)_{j<i}, u_i/x_i \otimes y_i, (u_j/x_j \otimes x_j)_{j>i}]\,. \end{aligned}$$

For each j, the definition of $X_{\vartheta u_j}$ says that both $\delta(\iota, x_j \otimes x_j)$ and $\delta(\iota, y_j \otimes y_j)$ coincide with the first component of $\vartheta(u_j)$. Thus,

$$\delta(\iota, f(\boldsymbol{z}_{i-1}) \otimes f(\boldsymbol{z}_i)) = \delta_{u_i/\delta(\iota, x_i \otimes y_i)}(\iota, \vartheta \otimes \vartheta) \in F\,,$$

where the membership in F is due to $x_i \trianglelefteq_{\vartheta u_i} y_i$. Consequently, $f(\boldsymbol{z}_{i-1}) \leqslant_A f(\boldsymbol{z}_i)$. So far, we have shown that $(f; X_{\vartheta u_1}, \ldots, X_{\vartheta u_m})$ is a box-decomposition of C.

It remains to show that this box-decomposition is tame. Due to the definition of $\trianglelefteq_{\vartheta u_i}$, the map $\mu_i \colon X_{\vartheta u_i}^2 \to Q$ given by $\mu_i(x_i, y_i) := \delta(\iota, x_i \otimes y_i)$ is a finite device for $X_{\vartheta u_i}$. Thus, it suffices to show that the map $\mu \colon C^2 \to Q^m$ defined by

$$\mu(f(x_1, \ldots, x_m), f(y_1, \ldots, y_m)) := \langle \mu_1(x_1, y_1), \ldots, \mu_m(x_m, y_m) \rangle$$

is a finite device for C.

To this end, we define a map $g \colon Q^m \to Q$ by

$$g(q_1, \ldots, q_m) := \delta_{u_1/q_1, \ldots, u_m/q_m}(\iota, \vartheta \otimes \vartheta)\,.$$

For all $s = f(x_1, \ldots, x_m), t = f(y_1, \ldots, y_m) \in C$, we have

$$\begin{aligned}\delta(\iota, s \otimes t) &= g(\delta(\iota, x_1 \otimes y_1), \ldots, \delta(\iota, x_m \otimes y_m)) \\ &= g(\mu_1(x_1, y_1), \ldots, \mu_m(x_m, y_m)) \\ &= g(\mu(s, t)) \,.\end{aligned}$$

Since $\mathcal{T}$ recognizes $\leqslant_A$, we finally obtain the following chain of equivalences:

$$\begin{aligned} s \leqslant_A t \quad &\iff \quad \delta(\iota, s \otimes t) \in F \\ &\iff \quad \mu(s, t) \in g^{-1}(F) \,. \end{aligned}$$

This proves that μ is a finite device for $\leqslant_A$. $\square$

Now, we are in a position to prove the decomposition theorem.

Theorem 3.3.8 (decomposition theorem, cf. [Del04])**.** *Let A be a tree-automatic linear order. There exists a* finite *set $\mathcal{S}$ of tree-automatic linear orders such that every closed interval in A admits a partition into suborders which are* tamely *box-decomposable in $\mathcal{S}$.*

Proof. In view of lemmas 3.3.10 and 3.3.13, it only remains to show that collecting the $(X_{\vartheta u}; \trianglelefteq_{\vartheta u})$ over all closed intervals $I = [\ell, r]_A \subseteq A$, each I-type ϑ and every $u \in \operatorname{dom}(\vartheta) \cap \partial \operatorname{dom}(\ell \otimes r)$ yields only finitely many distinct linear orders. However, this is almost trivial since the set $X_{\vartheta u}$ is determined by the pair $\vartheta(u) \in Q \times Q'$ and the linear ordering by the set

$$\{\, q \in Q \mid \delta_{u/q}(\iota, \vartheta \otimes \vartheta) \in F \,\} \,.$$

Clearly, there are only $|Q| \cdot |Q'| \cdot 2^{|Q|}$ many choices for these parameters. $\square$

Finally, an easy inspection of the preceding proofs reveals that theorem 3.3.8 is effective in the following regards: (1) Given a tree-automatic presentation of the linear order A, one can compute tree-automatic presentations of the elements of $\mathcal{S}$. (2) Given a closed interval in $I \subseteq A$, one can compute an automatic partition of I and for each part a box-decomposition into members of $\mathcal{S}$.

3.3.2 Tame Box-Decompositions of Scattered Linear Orders

The sole purpose of this section is to prove theorem 3.3.17, which asserts that eq. (3.4) on page 79 is valid for tame box-decompositions. Basically, the proof proceeds by induction on the size n of the box-decomposition. The main part of this section deals with the case $n = 2$ in proposition 3.3.16. We prepare the proof by two technical lemmas.

Let A be a linear order and $\mu\colon A^2 \to Q$ a finite device for A. A subset $X \subseteq A$ is called *homogeneous (wrt μ)* if there are $q_<, q_=, q_> \in Q$ such that, for all $a, b \in X$ and $\theta \in \{<, =, >\}$, $a \, \theta \, b$ if and only if $\mu(a, b) = q_\theta$.

Lemma 3.3.14. *Let A be a linear order and $\mu\colon A^2 \to Q$ a finite device for A.*

(1) If A has no greatest element, then there is a homogeneous cofinal type ω subset $X \subseteq A$.

(2) If A has no least element, then there is a homogeneous coinitial type $\omega^\star$ subset $X \subseteq A$.

Proof. Suppose that A has no greatest element. Hence, there is a cofinal type ω subset $Z \subseteq A$. According to the infinitary pigeon hole principle, there are $q_= \in Q$ and an infinite subset $Y \subseteq Z$ such that $\mu(a, a) = q_=$ for all $a \in Y$. Due to the infinitary version of Ramsey's theorem (cf. theorem 4.1.3 on page 126), there are $q_<, q_> \in Q$ and an infinite subset $X \subseteq Y$ such that

$\mu(a,b) = q_<$ and $\mu(b,a) = q_>$ for all $a, b \in X$ with $a < b$. This proves statement (1). Statement (2) is shown analogously. $\square$

Lemma 3.3.15. *Let α be an ordinal, $A \in \mathcal{VD}_\alpha$ a scattered linear order and $X \subseteq A$.*

(1) If A is an ω-sum of linear orders from $\mathcal{VD}_{<\alpha}$ and X is bounded from above, then $X \in \mathcal{VD}^\star_{<\alpha}$.

(2) If A is an $\omega^\star$-sum of linear orders from $\mathcal{VD}_{<\alpha}$ and X is bounded from below, then $X \in \mathcal{VD}^\star_{<\alpha}$.

Proof. We only prove statement (1), statement (2) is shown analogously. Suppose the premises are satisfied. We write A as an ω-sum $A = \sum_{i\in\omega} A_i$ with $A_i \in \mathcal{VD}_{<\alpha}$ for each i. Let $a \in A$ be an upper bound on X and $k \in \mathbb{N}$ with $a \in A_k$. Then $X \subseteq A_0 + \cdots + A_k \in \mathcal{VD}^\star_{<\alpha}$. $\square$

The next proposition proves the case $n = 2$ of theorem 3.3.17. It is only for technical reasons, that we refrained from stating its claim as $\mathrm{VD}_*(A) \leqslant \mathrm{VD}_*(B) \oplus \mathrm{VD}_*(C)$.

Proposition 3.3.16. *Let A be a scattered linear order, $(f; B, C)$ a tame box-decomposition of A and β, γ ordinals. If $B \in \mathcal{VD}^\star_\beta$ and $C \in \mathcal{VD}^\star_\gamma$, then $A \in \mathcal{VD}^\star_{\beta\oplus\gamma}$.*

Proof. To keep notation simple, we assume without loss of generality that $A = B \times C$ and f is the identity map. Before we delve into the details of an induction on β and γ, we perform a slight simplification. Since $B \in \mathcal{VD}^\star_\beta$, we can write $B = B_1 + \cdots + B_m$ with $B_1, \ldots, B_m \in \mathcal{VD}_\beta$. Analogously, $C = C_1 + \cdots + C_n$ with $C_1, \ldots, C_n \in \mathcal{VD}_\gamma$. Since every ζ-sum can be written as a sum of an ω^*-sum and an ω-sum, we can additionally assume that none of the B_i and C_j is constructed as a ζ-sum. Notice that the set

$$\{\, B_i \times C_j \mid i \in [1,m], j \in [1,n] \,\}$$

forms a partition of A. In view of theorem 3.2.2, it hence suffices to show $(B_i \times C_j; \leqslant_A) \in \mathcal{VD}^\star_{\beta\oplus\gamma}$ for all i and j. Since $(f{\restriction}_{B_i \times C_j}; B_i, C_j)$ is a tame box-decomposition of $(B_i \times C_j; \leqslant_A)$, this amounts to proving the claim of the theorem under the stronger assumptions that $B \in \mathcal{VD}_\beta$, $C \in \mathcal{VD}_\gamma$ and neither B nor C are constructed as a ζ-sum. We demonstrate this modified claim by induction on β and γ.

Base case: $\beta = 0$ or $\gamma = 0$. If $\beta = 0$, then B is either empty or a singleton. In both cases, the claim is trivial. The case $\gamma = 0$ is symmetric.

Inductive step: $\beta > 0$ and $\gamma > 0$. If B is a finite sum of linear orders from $\mathcal{VD}_{<\beta}$, then $B \in \mathcal{VD}^\star_{<\beta}$ and hence $A \in \mathcal{VD}^\star_{<\beta\oplus\gamma}$ by the induction hypothesis. The case of a finite sum C is symmetric. It remains to show the claim under the assumption that both B and C are ω-sums or ω^*-sums of non-empty linear orders from $\mathcal{VD}_{<\beta}$ and $\mathcal{VD}_{<\gamma}$, respectively. In line with this, we distinguish four cases. In each case, let μ_B and μ_C be finite devices for B and C, respectively, witnessing the tameness of the box-decomposition. In addition, let μ_A be the induced finite device for A, i.e.,

$$\mu_A(\langle b_1, c_1\rangle, \langle b_2, c_2\rangle) := \langle \mu_B(b_1, b_2), \mu_C(c_1, c_2)\rangle \,.$$

Case 1: B is an ω-sum and C is an $\omega^\star$-sum. According to lemma 3.3.14, there are a homogeneous (wrt μ_B) cofinal subset $\{\, b_0 < b_1 < b_2 < \cdots \} \subseteq B$ and a homogeneous (wrt μ_C) coinitial subset $\{\, c_0 > c_1 > c_2 > \cdots \} \subseteq C$. Depending on how $\langle b_0, c_0\rangle$ compares to $\langle b_1, c_1\rangle$ in A, we distinguish two cases.

Case 1.1: $\langle b_0, c_0\rangle <_A \langle b_1, c_1\rangle$. Figure 3.2 on page 92 depicts the idea behind the treatment of this case. The horizontal axis describes B and increases from left to right, whereas the vertical axis outlines C and grows from bottom to top. Within the grid, arrows point from smaller to greater elements.

Formally, let $b_{-1} := -\infty$ and put

$$X_i := (b_{i-1}, b_i]_B \times (-\infty, c_0)_C$$

for each $i \in \mathbb{N}$. Moreover, let

$$Y_1 := B \times [c_0, \infty)_C\,, \quad Y_2 := \bigcup_{i \in \mathbb{N}} X_{2i} \quad \text{and} \quad Y_3 := \bigcup_{i \in \mathbb{N}} X_{2i+1}\,.$$

Since the sequence of the b_i is unbounded, we have

$$B \times C = Y_1 \uplus Y_2 \uplus Y_3\,.$$

Our goal is to show $Y_1, Y_2, Y_3 \in \mathcal{VD}^{\star}_{\beta \oplus \gamma}$. Due to theorem 3.3.2, this implies $A \in \mathcal{VD}^{\star}_{\beta \oplus \gamma}$, as desired.

According to lemma 3.3.15, we have $(b_{i-1}, b_i]_B \in \mathcal{VD}^{\star}_{<\beta}$, for each i, as well as $[c_0, \infty)_C \in \mathcal{VD}^{\star}_{<\gamma}$. By the induction hypothesis, we obtain $X_i, Y_1 \in \mathcal{VD}^{\star}_{<\beta \oplus \gamma}$ for each i. Recall that, by definition, $X_i \ll X_j$ holds true precisely if $a <_A a'$ for all $a \in X_i$ and $a' \in X_j$. Our next step is to show

$$X_0 \ll X_2 \ll X_4 \ll \cdots \quad \text{and} \quad X_1 \ll X_3 \ll X_5 \ll \cdots\,. \tag{3.5}$$

To this end, consider $i \in \mathbb{N}$, $\langle b, c\rangle \in X_i$ and $\langle b', c'\rangle \in X_{i+2}$. Since the sequence of the c_j is strictly decreasing and unbounded, there is $j_0 \in \mathbb{N}$ such that $c_{j_0} \leqslant c'$. The choice of the b_i and c_j implies

$$\mu_A(\langle b_0, c_0\rangle, \langle b_1, c_1\rangle) = \mu_A(\langle b_i, c_0\rangle, \langle b_{i+1}, c_{j_0}\rangle)$$

and hence $\langle b_i, c_0\rangle <_A \langle b_{i+1}, c_{j_0}\rangle$. Since $(f; B, C)$ is a box-decomposition of A, we further conclude

$$\langle b, c\rangle <_A \langle b_i, c_0\rangle <_A \langle b_{i+1}, c_{j_0}\rangle <_A \langle b', c'\rangle\,.$$

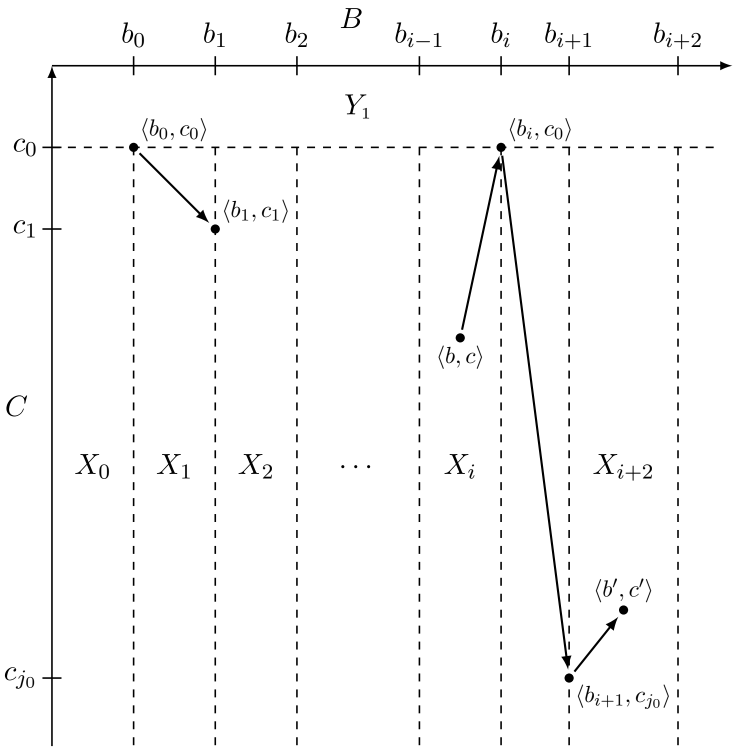


Figure 3.2: Proof sketch for case 1.1

This proves eq. (3.5). As a direct consequence, we obtain

$$Y_2 = \sum_{i\in\omega} X_{2i} \qquad \text{and} \qquad Y_3 = \sum_{i\in\omega} X_{2i+1}\,.$$

Hence, Y_2 and Y_3 are ω-sums of linear orders in $\mathcal{VD}^\star_{<\beta\oplus\gamma}$, i.e., $Y_2, Y_3 \in \mathcal{VD}_{\beta\oplus\gamma}$. Altogether, we have shown $Y_1, Y_2, Y_3 \in \mathcal{VD}^\star_{\beta\oplus\gamma}$ so far. According to theorem 3.3.2, this implies $A \in \mathcal{VD}^\star_{\beta\oplus\gamma}$ and completes case 1.1.

Case 1.2: $\langle b_0, c_0\rangle >_A \langle b_1, c_1\rangle$. This case is symmetric to case 1.1 and depicted in fig. 3.3 on the following page.

Case 2: B and C both are ω-sums. This time, lemma 3.3.14 guarantees the existence of cofinal subsets $\{\, b_0 < b_1 < b_2 < \cdots \} \subseteq B$ and $\{\, c_0 < c_1 < c_2 < \cdots \} \subseteq C$ which are homogeneous wrt μ_B and μ_C, respectively. Depending on how $\langle b_0, c_1\rangle$ compares to $\langle b_1, c_0\rangle$ in A, we distinguish two cases.

Case 2.1: $\langle b_0, c_1\rangle <_A \langle b_1, c_0\rangle$. This case is treated similar to case 1.1 and depicted in fig. 3.4 on page 95.

Case 2.2: $\langle b_0, c_1\rangle >_A \langle b_1, c_0\rangle$. This case is symmetric to case 2.1.

Case 3: B is an $\omega^\star$-sum and C is an ω-sum. This case is symmetric to case 1.

Case 4: B and C both are $\omega^\star$-sums. This case is symmetric to case 2. □

Finally, we are in a position to perform the induction proving theorem 3.3.17.

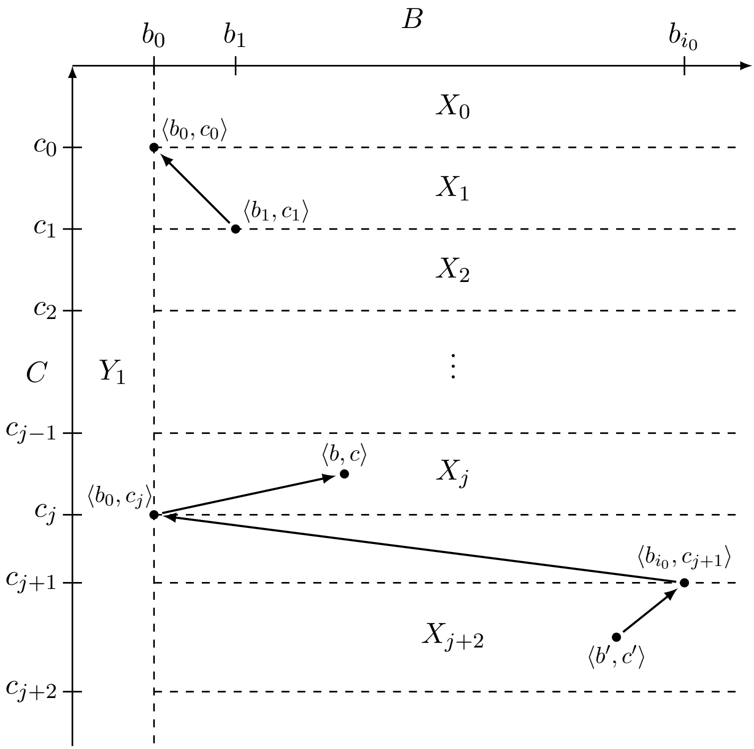


Figure 3.3: Proof sketch for case 1.2

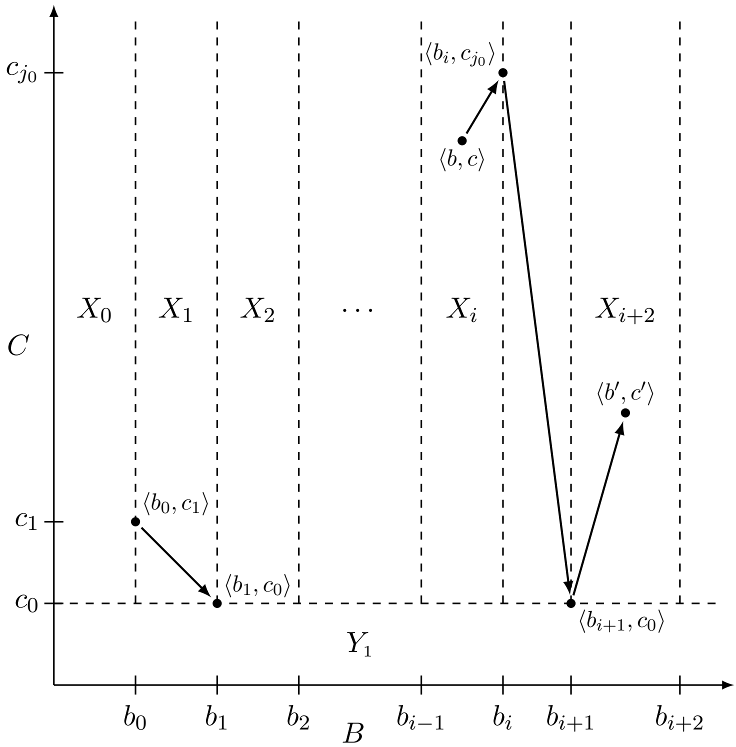


Figure 3.4: Proof sketch for case 2.1

Theorem 3.3.17. *Let A be a scattered linear order and consider a tame box-decomposition $(f; B_1, \ldots, B_n)$ of A. Then*

$$\mathrm{VD}_*(A) \leqslant \mathrm{VD}_*(B_1) \oplus \cdots \oplus \mathrm{VD}_*(B_n) .$$

Proof. We proceed by induction on n.

Base case: $n = 1$. Since $A \cong B_1$ (via f), the claim is trivial.

Inductive step: $n > 1$. Without loss of generality, we assume that $A = B_1 \times \cdots \times B_n$ and f is the identity map. Let $\mu_1, \ldots, \mu_n$ be finite devices for $B_1, \ldots, B_n$, respectively, such that the following defines a finite device μ for A:

$$\mu(\langle x_1, \ldots, x_n\rangle, \langle y_1, \ldots, y_n\rangle) := \langle \mu_1(x_1, y_1), \ldots, \mu_n(x_n, y_n)\rangle .$$

Let $A' := B_1 \times \cdots \times B_{n-1}$. We define an equivalence relation $\sim$ on B_n by $x_n \sim y_n$ if $\mu_n(x_n, x_n) = \mu_n(y_n, y_n)$. Since $\sim$ has finite index and due to theorem 3.3.2, it suffices to show the following upper bound for each $\sim$-class $Z \subseteq B_n$:

$$\mathrm{VD}_*(A' \times Z) \leqslant \mathrm{VD}_*(B_1) \oplus \cdots \oplus \mathrm{VD}_*(B_n) . \tag{3.6}$$

For this purpose, we fix a representative $z \in Z$. Observe that $(g; B_1, \ldots, B_{n-1})$ with $g(\boldsymbol{x}) := \langle \boldsymbol{x}, z\rangle$ is a tame box-decomposition of $A' \times \{z\}$. We complete the proof by showing that $(h; A' \times \{z\}, Z)$ is a tame box-decomposition of $A' \times Z$, where the bijection $h\colon (A' \times \{z\}) \times Z \to A' \times Z$ is defined by

$$h(\langle \boldsymbol{x}, z\rangle, x_n) := \langle \boldsymbol{x}, x_n\rangle .$$

Proposition 3.3.16 and the induction hypothesis then yield

$$\begin{aligned} \mathrm{VD}_*(A' \times Z) &\leqslant \mathrm{VD}_*(A' \times \{z\}) \oplus \mathrm{VD}_*(Z) \\ &\leqslant \mathrm{VD}_*(B_1) \oplus \cdots \oplus \mathrm{VD}_*(B_{n-1}) \oplus \mathrm{VD}_*(B_n) . \end{aligned}$$

Consider some $\boldsymbol{x}, \boldsymbol{y} \in A'$ and $x_n, y_n \in Z$ with $\langle \boldsymbol{x}, z\rangle \leqslant_A \langle \boldsymbol{y}, z\rangle$ and $x_n \leqslant_{B_n} y_n$. Since $\mu_n(z,z) = \mu_n(x_n, x_n)$, we obtain

$$\mu(\langle \boldsymbol{x}, z\rangle, \langle \boldsymbol{y}, z\rangle) = \mu(\langle \boldsymbol{x}, x_n\rangle, \langle \boldsymbol{y}, x_n\rangle)$$

and hence

$$h(\langle \boldsymbol{x}, z\rangle, x_n) = \langle \boldsymbol{x}, x_n\rangle \leqslant_A \langle \boldsymbol{y}, x_n\rangle \leqslant_A \langle \boldsymbol{y}, y_n\rangle = h(\langle \boldsymbol{y}, z\rangle, y_n)\,.$$

This demonstrates that $(h; A' \times \{z\}, Z)$ is indeed a box-decomposition of $A' \times Z$. Using (the restrictions of) the finite devices μ and μ_n for $A' \times \{z\}$ and Z, respectively, it is a matter of routine to verify that this box-decomposition is tame. □

3.3.3 Bounding the Finite-Condensation Rank

In this section, we finally prove the upper bound on the FC-ranks of linear orders in TA. As a corollary, we obtain Delhommé's characterization of the ordinals in TA. The lemma below is a slight variation of [KRS05, proposition 4.5].

Lemma 3.3.18. *Let A be a linear order. There is a scattered closed interval $I \subseteq A$ with $\mathrm{VD}_*(I) = \alpha$ for each $\alpha < \mathrm{FC}(A)$.*

Proof. Consider some $\alpha < \mathrm{FC}(A)$. The proof of [KRS05, proposition 4.5] shows that there is a scattered closed interval $I \subseteq A$ with $\mathrm{VD}(A) = \alpha + 1$. Since I has a least and a greatest element, it is neither an ω-sum nor an ω^*-sum nor a ζ-sum of non-empty linear orders from $\mathcal{VD}_{<\alpha+1} = \mathcal{VD}_\alpha$. Thus, I is a finite sum of linear orders from $\mathcal{VD}_\alpha$, i.e., $I \in \mathcal{VD}_\alpha^\star$. Since $\mathrm{VD}(I) = \alpha + 1$, this implies $\mathrm{VD}_*(I) = \alpha$. □

The main result of this section is as follows:

Theorem 3.3.19. *The FC-rank of any linear order A in TA is bounded by*

$$\mathrm{FC}(A) < \omega^\omega\,.$$

Proof. Aiming for a contradiction, assume there is a tree-automatic linear order A such that $\mathrm{FC}(A) \geqslant \omega^\omega$. According to theorem 3.3.8, there is a finite set $\mathcal{S}$ of linear orders such that every closed interval in A admits a partition into suborders which are tamely box-decomposable in $\mathcal{S}$. We derive a contradiction by showing that $\mathcal{S}$ contains a scattered linear order of VD_*-rank ω^k for each $k \in \mathbb{N}$.

To this end, fix some $k \in \mathbb{N}$. By lemma 3.3.18, there exists a scattered closed interval $I \subseteq A$ such that $\mathrm{VD}_*(I) = \omega^k$. Due to the choice of $\mathcal{S}$, there is a partition Δ of I such that each linear order in Δ is box-decomposable into linear orders from $\mathcal{S}$. By theorem 3.3.2, there is $B \in \Delta$ with $\mathrm{VD}_*(B) = \omega^k$.

Finally, let $(f; C_1, \dots, C_n)$ be a tame box-decomposition of B with $C_1, \dots, C_n \in \mathcal{S}$. Recall that $\mathrm{VD}_*(C_i) \leqslant \omega^k$. If we had $\mathrm{VD}_*(C_i) < \omega^k$ for each i, we would obtain

$$\mathrm{VD}_*(C_1) \oplus \cdots \oplus \mathrm{VD}_*(C_n) < \omega^k \,.$$

However, this would contradict theorem 3.3.17. Put another way, there is some j such that $\mathrm{VD}_*(C_j) = \omega^k$, i.e., $\mathcal{S}$ contains a scattered linear order of VD_*-rank ω^k. $\square$

The next example is folklore and shows that TA contains all ordinals $\alpha < \omega^{\omega^\omega}$. In particular, this proves the bound in theorem 3.3.19 to be optimal.

Example 3.3.20 (cf. [BGR11, example 1.3.6])**.** Let $\gamma \leqslant \omega^n$ be an ordinal and A the string-automatic type γ well-order with $A \subseteq (\mathtt{1}^*\mathtt{0})^n$ from example 3.1.2. The standard example $(\mathbb{N}^{(A)}; \trianglelefteq)$ of a type ω^γ well-order is defined by

$$\mathbb{N}^{(A)} := \left\{ f \colon A \to \mathbb{N} \;\middle|\; f(u) = 0 \text{ for all but finitely many } u \in A \right\}$$

and $f \vartriangleleft g$ if the greatest $u \in A$ with $f(u) \neq g(u)$ satisfies $f(u) < g(u)$. Encoding each $f \in \mathbb{N}^{(A)}$ by the unique tree $t_f \in T_{\{\mathtt{a}\}}$

whose domain $\mathrm{dom}(t_f)$ is the prefix-closure of the set

$$\bigcup_{\substack{u \in A \\ f(u) \neq 0}} u\mathtt{1}^{f(u)}$$

yields a tree-automatic type ω^γ linear order. Consequently, TA contains ω^γ and hence all ordinals $\alpha < \omega^{\omega^\omega}$ by the last argument from example 3.1.2. $\square$

The following characterization of the ordinals in TA is immediately implied by theorem 3.3.19 along with example 3.3.20.

Corollary 3.3.21 ([Del04])**.** *An ordinal α is in TA if and only if*

$$\alpha < \omega^{\omega^\omega} . \qquad \square$$

3.4 Tree-Automaticity on Polynomial Domains

We complete our investigation on ranks of automatic linear orders by studying pTA. The main result in this regard is theorem 3.4.1 below, whose proof combines ideas from the investigations of pSA and TA, namely theorems 3.2.4 and 3.3.19.

Theorem 3.4.1. *Let $k \geqslant 1$. The* FC*-rank of any linear order A in $\mathsf{pTA}[k]$ is bounded by*

$$\mathrm{FC}(A) < \omega^k .$$

Proof. We proceed by induction on k. We add an artificial base case $k = 0$ and use the induction hypothesis only in the following restricted form: The VD_*-rank of any scattered linear order A in $\mathsf{pTA}[k]$ is bounded by $\mathrm{VD}_*(A) < \omega^k$. For $k \geqslant 1$, this assertion easily follows from

$$\mathrm{VD}_*(A) \leqslant \mathrm{FC}(A) < \omega^k .$$

Base case: $k = 0$. Since any structure in $\mathsf{pTA}[0]$ is finite, every scattered linear order A in $\mathsf{pTA}[0]$ trivially satisfies $\mathrm{VD}_*(A) < \omega^0$.

Inductive step: $k \geqslant 1$. Aiming for a contradiction, assume there is a tree-automatic linear order A with $\mathrm{FC}(A) \geqslant \omega^k$ and $g_{T(A)}(n) \in O(n^k)$. According to theorem 3.3.8, there is a finite set $\mathcal{S}$ of linear orders such that every closed interval in A admits a partition into suborders which are tamely box-decomposable in $\mathcal{S}$. We derive a contradiction to the finiteness of $\mathcal{S}$ by showing that $\mathcal{S}$ contains for each $\ell \in \mathbb{N}$ a scattered linear order B with

$$\omega^{k-1}\ell < \mathrm{VD}_*(B) < \omega^k .$$

To this end, we fix some $\ell \in \mathbb{N}$. By lemma 3.2.3, there is a constant $c \in \mathbb{N}$ such that any anti-chain $U \subseteq \{\mathtt{0}, \mathtt{1}\}^*$ contains at most c elements $u \in U$ with

$$g_{u^{-1}T(A)}(n) \in \Theta(n^k) .$$

Due to lemma 3.3.18, there exists a scattered closed interval $I = [\ell, r]_A \subseteq A$ such that

$$\mathrm{VD}_*(I) = \omega^{k-1}(\ell c + 1) .$$

According to lemma 3.3.10, I is a finite union of $\equiv_I$-classes. Thus, there is an $\equiv_I$-class $C \subseteq I$ with

$$\mathrm{VD}_*(C) = \omega^{k-1}(\ell c + 1)$$

by theorem 3.3.2. Let ϑ be the I-type of C, $u_1, \ldots, u_m$ an enumeration of $\mathrm{dom}(\vartheta) \cap \partial\,\mathrm{dom}(\ell \otimes r)$ and $(f; X_{\vartheta u_1}, \ldots, X_{\vartheta u_m})$ the tame box-decomposition of C from lemma 3.3.13. Recall that each $X_{\vartheta u_i}$ is a scattered linear order with

$$\mathrm{VD}_*(X_{\vartheta u_i}) \leqslant \omega^{k-1}(\ell c + 1) .$$

For all $x_1 \in X_{\vartheta u_1}, \ldots, x_m \in X_{\vartheta u_m}$, we have

$$\vartheta[u_1/x_1, \ldots, u_m/x_m] \in C \subseteq A\,.$$

In particular,

$$T(X_{\vartheta u_i}) \subseteq u_i^{-1}T(A)$$

for each i. We consider the set

$$H := \left\{\, i \in [1,m] \;\middle|\; g_{u_i^{-1}T(A)}(n) \in \Theta(n^k) \,\right\}.$$

Due to the choice of c, we have $|H| \leqslant c$. For all $i \in [1,m] \setminus H$, the restricted induction hypothesis applies to $X_{\vartheta u_i}$, i.e.,

$$\mathrm{VD}_*(X_{\vartheta u_i}) < \omega^{k-1}\,.$$

If we also had $\mathrm{VD}_*(X_{\vartheta u_i}) \leqslant \omega^{k-1}\ell$ for all $i \in H$, we would obtain

$$\begin{aligned}\bigoplus_{i\in[1,m]} \mathrm{VD}_*(X_{\vartheta u_i}) &= \underbrace{\bigoplus_{i\in[1,m]\setminus H} \mathrm{VD}_*(X_{\vartheta u_i})}_{<\omega^{k-1}} \oplus \underbrace{\bigoplus_{i\in H} \mathrm{VD}_*(X_{\vartheta u_i})}_{\leqslant\omega^{k-1}\ell c} \\ &< \omega^{k-1}(\ell c+1)\,.\end{aligned}$$

However, this would contradict theorem 3.3.17. Hence, there is some $j \in [1,m]$ such that

$$\omega^{k-1}\ell < \mathrm{VD}_*(X_{\vartheta u_j}) \leqslant \omega^{k-1}(\ell c+1) < \omega^k\,.$$

This proves the claim. □

The following example demonstrates that each ordinal $\alpha < \omega^{\omega^k}$ is contained in $\mathsf{pTA}[k]$.

Example 3.4.2. Let $k \geqslant 1$, $m \in \mathbb{N}$ and

$$A = (\mathtt{1}^{<m}\mathtt{0}(\mathtt{1}^*\mathtt{0})^{k-1}; \leqslant_{\text{in}})$$

be the string-automatic type $\omega^{k-1}m$ well-order from example 3.2.1. Applying the construction from example 3.3.20 yields a tree-automatic type $\omega^{\omega^{k-1}m}$ well-order $(B; \leqslant_B)$. The set $T(B)$ is the prefix-closure of $\mathtt{1}^{<m}\mathtt{0}(\mathtt{1}^*\mathtt{0})^{k-1}\mathtt{1}^*$, i.e.,

$$T(B) = \bigcup_{0 \leqslant i \leqslant k} \mathtt{1}^{<m}(\mathtt{0}\mathtt{1}^*)^i \,.$$

Thus, $g_{T(B)}(n) \in O(n^k)$. Consequently, $\mathsf{pTA}[k]$ contains $\omega^{\omega^{k-1}m}$ and hence all ordinals $\alpha < \omega^{\omega^k}$ by the last argument from example 3.1.2. □

Just like before, theorem 3.4.1 in combination with example 3.4.2 yields a characterization of the ordinals in $\mathsf{pTA}[k]$

Corollary 3.4.3. *Let $k \in \mathbb{N}$. An ordinal α is in $\mathsf{pTA}[k]$ if and only if*

$$\alpha < \omega^{\omega^k} \,.$$ □

Notice that corollaries 3.3.21 and 3.4.3 imply that every ordinal in TA is already contained in pTA. In fact, one can show that the domain of any tree-automatic well-order—or more generally, scattered linear order—is of polynomial growth [JKSS14].

3.5 String-Automaticity versus Tree-Automaticity

This section is devoted to theorems 3.5.5 and 3.5.9. The former characterizes those scattered linear orders in TA which are also contained in SA and the latter provides an algorithm to compute the Cantor normal form of any tree-automatic well-order of type $\alpha < \omega^{\omega^2}$. Recall that every tree-automatic structure on a slim domain belongs to SA by theorem 2.4.17 on page 44. Along with theorem 3.1.3, we obtain that the FC-rank of a tree-automatic linear order A is bounded by $\mathrm{FC}(A) < \omega$ whenever A is slim. The aforementioned two results both rely on the converse of this implication, which is demonstrated in proposition 3.5.4. Although it would be possible to prove this without using the decomposition technique, we partially resort to this technique since we have already introduced it anyway. As our first step, we establish a very restricted converse of theorem 3.3.17.

Lemma 3.5.1. *Let A be a scattered linear order and consider a tame box-decomposition $(f; A_1, \dots, A_n)$ of A. If all the A_i are infinite, then*

$$\mathrm{VD}_*(A) \geqslant n\,.$$

Proof. Without loss of generality, we assume $A = A_1 \times \cdots \times A_n$ and that f is the identity map. We denote the orderings of A and A_i by $\leqslant$ and $\leqslant_i$, respectively. Let $\sqsubseteq$ be the partial order on A defined by $\boldsymbol{x} \sqsubseteq \boldsymbol{y}$ if $x_i \leqslant_i y_i$ for each i. Due to the definition of box-decompositions, $\leqslant$ is a linear extension of $\sqsubseteq$. Let $\mu_1, \dots, \mu_n$ be finite devices for $A_1, \dots, A_n$, respectively, witnessing that the box-decomposition is tame. Let μ be the induced finite device for A, i.e.,

$$\mu(\boldsymbol{x}, \boldsymbol{y}) := \langle \mu_1(x_1, y_1), \dots, \mu_n(x_n, y_n) \rangle\,.$$

Obviously, each A_i contains a suborder of type ω or $\omega^\star$. Applying lemma 3.3.14 to this suborder, yields a homogeneous (wrt μ) suborder $X_i \subseteq A_i$ of the same type. Our goal is to show that the suborder

$$X := X_1 \times \cdots \times X_n \subseteq A$$

satisfies $\mathrm{VD}_*(X) = n$.

For this purpose, fix some $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in X$ with $a_i <_i b_i <_i c_i$ for each $i \in [1, n]$. We define

$$\boldsymbol{e}_i := \langle a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n \rangle\,.$$

Since the permutation of a tame box-decomposition is again a tame box-decomposition, we assume without loss of generality that

$$\boldsymbol{e}_1 < \boldsymbol{e}_2 < \cdots < \boldsymbol{e}_n \,.$$

As a next step, we show that, for all $\boldsymbol{x}, \boldsymbol{y} \in X$, $\boldsymbol{x} < \boldsymbol{y}$ whenever there is some i with $x_i \neq y_i$ and the maximal such i satisfies $x_i <_i y_i$. Suppose that the latter condition is satisfied and let k be maximal with $x_k <_k y_k$. We define $\boldsymbol{p}, \boldsymbol{q} \in X$ by

$$\langle p_i, q_i \rangle := \begin{cases} \langle a_i, b_i \rangle & \text{if } x_i <_i y_i, \\ \langle a_i, a_i \rangle & \text{if } x_i = y_i, \\ \langle b_i, a_i \rangle & \text{if } x_i >_i y_i. \end{cases}$$

Notice that $\mu(\boldsymbol{x}, \boldsymbol{y}) = \mu(\boldsymbol{p}, \boldsymbol{q})$. Hence, it suffices to verify $\boldsymbol{p} < \boldsymbol{q}$. If $k = 1$, this follows from $\boldsymbol{p} \sqsubset \boldsymbol{q}$. Henceforth, assume $k > 0$. For each $i \in [1, k-1]$, we define

$$\boldsymbol{s}_i := \langle a_1, \ldots, a_{i-1}, c_i, b_{i+1}, \ldots, b_{k-1}, a_k, \ldots, a_n \rangle \,.$$

For $i < k-1$, we have $\mu(\boldsymbol{s}_i, \boldsymbol{s}_{i+1}) = \mu(\boldsymbol{e}_i, \boldsymbol{e}_{i+1})$ and hence $\boldsymbol{s}_i < \boldsymbol{s}_{i+1}$. We further have $\mu(\boldsymbol{s}_{k-1}, \boldsymbol{e}_k) = \mu(\boldsymbol{e}_{k-1}, \boldsymbol{e}_k)$ and hence $\boldsymbol{s}_{k-1} < e_k$. Altogether,

$$\boldsymbol{p} \sqsubset \boldsymbol{s}_1 < \boldsymbol{s}_2 < \cdots < \boldsymbol{s}_{k-1} < \boldsymbol{e}_k \sqsubseteq \boldsymbol{q} \,.$$

This proves $\boldsymbol{p} < \boldsymbol{q}$ and hence $\boldsymbol{x} < \boldsymbol{y}$.

Put another way, we have just shown that

$$(X; \leqslant) = (X_1; \leqslant_1) \cdot (X_2; \leqslant_2) \cdots (X_n; \leqslant_n) \,.$$

Since each $(X_i; \leqslant_i)$ has order type ω or $\omega^\star$, we obtain $\mathrm{VD}_*(X) = n$ and hence $\mathrm{VD}_*(A) \geqslant n$. □

It strikes us as if incorporating ideas from chapter 4 into the previous proof would yield the subsequently conjectured generalization of lemma 3.5.1. However, going down that road does not appear to be of any particular help for the investigation of automatically presentable linear orders.

Conjecture 3.5.2. *Let A be a scattered linear order and consider a tame box-decomposition $(f; A_1, \dots, A_n)$ of A. Moreover, let $k_1, \dots, k_n \in \mathbb{N}$ be such that each A_i satisfies $\mathrm{VD}_*(A_i) \geqslant k_i$. Then*

$$\mathrm{VD}_*(A) \geqslant k_1 + \cdots + k_n \,.$$

In contrast to this conjecture, the interplay between tame box-decompositions and lower bounds on the VD_*-rank is definitely quite poor for infinite VD_*-ranks. The following example illustrates this poorness for certain particularly relevant tree-automatically presentable well-orders.

Example 3.5.3. Let $k, n \in \mathbb{N}_+$, A be a type ω^{ω^k} well-order and $A = \sum_{i \in \omega} A_i$ the unique decomposition of A such that each A_i has type $\omega^{\omega^{k-1} i}$. We consider the well-order

$$B := \sum_{i_n \in \omega} \cdots \sum_{i_2 \in \omega} \sum_{i_1 \in \omega} A_{i_1} \cdot A_{i_2} \cdots A_{i_n} \,.$$

It is a matter of routine to verify that the tuple $(f; A, A, \dots, A)$, where f is the identity map on A^n, forms a tame box-decomposition of B. A suitable finite device $\mu\colon A^2 \to Q$ for A is given by $Q := \{<, =, >\}^2$ and $\mu(u, v) = \langle \theta, \theta' \rangle$ precisely if $u \; \theta \; v$ and $i \; \theta' \; j$ for the unique $i, j \in \mathbb{N}$ with $u \in A_i$ and $v \in A_j$. Moreover, B has order type

$$\begin{aligned}\sum_{i_n\in\omega}\cdots\sum_{i_2\in\omega}\sum_{i_1\in\omega}&\omega^{\omega^{k-1}i_1}\cdot\omega^{\omega^{k-1}i_2}\cdots\omega^{\omega^{k-1}i_n}\\ &=\sum_{i_n\in\omega}\cdots\sum_{i_2\in\omega}\omega^{\omega^k}\\ &=\omega^{\omega^k+n-1}\,.\end{aligned}$$

In a way, the finite difference between the VD_*-ranks of A and B, namely $\mathrm{VD}_*(A)=\omega^k$ and $\mathrm{VD}_*(B)=\omega^k+n-1$, is neglectable in the context of infinite VD_*-ranks. □

In the proof of the following proposition, we do not use the decomposition theorem 3.3.8 literally but a variation of the decomposition technique which is adapted to proving lower bounds on FC-ranks.

Proposition 3.5.4. *Let A be a tree-automatic scattered linear order. If A is* not *slim, then*

$$\mathrm{FC}(A)\geqslant\omega\,.$$

Proof. Suppose that A is not slim. Using lemma 3.5.1, we show that, for each $n\in\mathbb{N}$, there is a suborder $C\subseteq A$ with $\mathrm{VD}_*(C)\geqslant n$. This proves $\mathrm{FC}(A)\geqslant\omega$.

To this end, fix some $n\in\mathbb{N}$. Let $\mathcal{T}=(Q,\iota,\delta,F)$ and $\mathcal{T}'=(Q',\iota',\delta',F')$ be tree-automata recognizing A and $\leqslant_A$, respectively. We put $k:=|Q|$. Since A is not slim, there are $t\in A$ and $\ell\in\mathbb{N}$ with

$$|\mathrm{dom}(t)\cap\{\mathtt{0},\mathtt{1}\}^{=\ell}|\geqslant 2^k\,n\,.$$

Let

$$U:=\left\{\,u\in\mathrm{dom}(t)\cap\{\mathtt{0},\mathtt{1}\}^{=\ell-k}\;\middle|\;\exists v\in\{\mathtt{0},\mathtt{1}\}^{=k}\colon uv\in\mathrm{dom}(t)\,\right\}.$$

Due to the choice of t and ℓ, we have $|U| \geqslant n$, say $u_1, \ldots, u_n \in U$ are mutually distinct elements of U. For each $i \in [1, n]$, let $q_i := \delta(\iota, t, u)$. The tree $t{\restriction}_{u_i}$ has height at least k and satisfies $\delta(\iota, t{\restriction}_{u_i}) = q_i$. Using a simple pumping argument, we conclude that there are infinitely many $x_i \in T_\Sigma$ with $\delta(\iota, x_i) = q_i$. According to the infinitary pigeon hole principle, there is a state $q'_i \in Q'$ such that the set

$$X_i := \left\{ x_i \in T_\Sigma \;\middle|\; \delta(\iota, x_i) = q_i \text{ and } \delta'(\iota', x_i \otimes x_i) = q'_i \right\}$$

is infinite. In particular, we have $t[u_i/x_i] \in A$ for every $x_i \in X_i$. We assume without loss of generality that we have chosen t initially such that $t{\restriction}_{u_i} \in X_i$. We define a linear ordering $\trianglelefteq_i$ of X_i by $x_i \trianglelefteq_i y_i$ if $t[u_i/x_i] \leqslant_A t[u_i/y_i]$.

We further consider the injective map $f\colon X_1 \times \cdots \times X_n \to C$ given by

$$f(x_1, \ldots, x_n) := t[u_1/x_1, \ldots, u_n/x_n]\,,$$

where C is chosen such that f is also surjective. Due to the choice of the X_i, we have $C \subseteq A$. Using the same arguments as in the proof of lemma 3.3.13, we may conclude that $(f; X_1, \ldots, X_n)$ is a tame box-decomposition of C. Thus, lemma 3.5.1 implies $\mathrm{VD}_*(C) \geqslant n$. □

Combining theorem 2.4.17 on page 44, theorem 3.1.3 and proposition 3.5.4 immediately yields the first main result of this section:

Theorem 3.5.5. *Any scattered linear order A from* TA *is contained in* SA *if and only if*

$$\mathrm{FC}(A) < \omega\,. \qquad \square$$

Due to theorem 2.4.16, it is decidable whether a given tree-automaton recognizes a slim language or not. Moreover, every tree-automatic structure on a slim domain is effectively string-automatically presentable by theorem 2.4.17. Accordingly, we obtain the following two corollaries:

Corollary 3.5.6 ([Hus12]). *Given a tree-automatic presentation of a scattered linear order A, it is decidable whether A is contained in* SA. *In case of a positive answer, one can compute a string-automatic presentation of A.* □

Corollary 3.5.7. *Given a tree-automatic presentation of a structure $\mathscr{A}$ which admits a first-order definable scattered linear ordering $\leqslant$ of the domain A satisfying* $\mathrm{FC}(A; \leqslant) < \omega$, *one can compute a string-automatic presentation of $\mathscr{A}$.* □

Applied to well-orders, the former corollary says that it is decidable whether a given tree-automatically presentable ordinal α satisfies $\alpha < \omega^\omega$. In the remainder of this section, we use this decidability result to demonstrate how the Cantor normal form of any ordinal $\alpha < \omega^{\omega^2}$ can be computed from a tree-automatic presentation of α.

First of all, recall that the Cantor normal form of (the order type of) any string-automatic well-order A can be computed by carrying out the finite-condensation process, cf. corollaries 3.1.7 and 3.1.8. Basically, the effectiveness of this process relies on two facts: (1) the finite-condensation relation $\sim$ is effectively automatic in every automatic well-order and (2) the process stops after $\mathrm{FC}(A)$ many steps, which are only finitely many according to theorem 3.1.3. Unfortunately, condition (2) does no longer hold for ordinals $\alpha \geqslant \omega^\omega$. However, the next lemma establishes that the ω^{th} iterated finite-condensation relation $\sim^\omega$ is effectively automatic as well. This allows for pushing the upper bound to ω^{ω^2} in theorem 3.5.9.

Lemma 3.5.8. *Given a presentation of a tree-automatic well-order A, one can compute a tree-automaton recognizing the ω^{th} iterated condensation relation $\sim^\omega$ on A.*

Proof. Let A be a tree-automatic well-order. Moreover, let $\mathcal{T} = (Q, \iota, \delta, F)$ and $\mathcal{T}' = (Q', \iota', \delta', F')$ be tree-automata recognizing $\leqslant_A$ and the relation

$$\left\{ \langle t, \ell, r\rangle \in A^3 \;\middle|\; t \in [\ell, r]_A \right\},$$

respectively. Clearly, it suffices to construct a tree-automaton recognizing the relation

$$R := \left\{ \langle \ell, r\rangle \in A^2 \;\middle|\; \ell <_A r \text{ and } \ell \not\sim^{\omega} r \right\}.$$

A tree-automaton recognizing $\sim^{\omega}$ is then easily constructed from the one for R. Our first goal is to characterize the pairs in R in terms of the I-types from definition 3.3.9.

To this end, consider $\ell, r \in A$ with $\ell <_A r$. We put $I := [\ell, r]_A$ and denote the order type of I by β. It is well known that $\ell \not\sim^{\omega} r$ is equivalent to $\beta \geqslant \omega^{\omega}$. Accordingly, a tree-automaton for R would have to check whether I is *not* slim. Although it is possible to construct such an automaton *ad hoc*, we apply the decomposition technique from section 3.3.1 once more to simplify the illustration.

Recall how we first partitioned I into finitely many $\equiv_I$-classes in lemma 3.3.10 and then box-decomposed each proper $\equiv_I$-class into linear orders $X_{\vartheta u}$ in lemma 3.3.13. Due to theorems 3.2.2 and 3.3.4, we have $\beta \geqslant \omega^{\omega}$ if and only if there is some $X_{\vartheta u}$ whose order type is at least ω^{ω}. According to (the proof of) theorem 3.5.5 and the choice of $X_{\vartheta u}$ in lemma 3.3.12, the order type of $X_{\vartheta u}$ is at least ω^{ω} precisely if $\vartheta(u)$ is contained in the subset $N \subseteq Q \times Q'$ given by

$$\begin{aligned} &\langle q, q'\rangle \in N \quad :\Longleftrightarrow \\ &\qquad \left\{ x \in T_{\Sigma} \;\middle|\; \delta(\iota, x \otimes x) = q \text{ and } \delta'(\iota', x^{\diamond}) = q' \right\} \text{ is } \textit{not} \text{ slim}. \end{aligned}$$

Notice that N is computable from $\mathcal{T}$ and $\mathcal{T}'$ due to lemma 2.4.15 on page 41.

Altogether, we have $\ell \not\sim^{\omega} r$ if and only if there are a proper I-type ϑ and some $u \in \operatorname{dom}(\vartheta) \cap \partial \operatorname{dom}(\ell \otimes r)$ with $\vartheta(u) \in N$. If we say that such ϑ *witnesses* $\ell \not\sim^{\omega} r$, we obtain

$$R = \left\{ \langle \ell, r \rangle \in A^2 \;\middle|\; \begin{array}{l} \text{there is some } t \in [\ell, r]_A \text{ whose} \\ [\ell, r]_A\text{-type witnesses } \ell \not\sim^{\omega} r \end{array} \right\}.$$

Finally, it is a matter of routine to translate this characterization into a tree-automaton recognizing R. $\square$

Recall the every ordinal $\gamma < \omega^2$ can be written as $\gamma = \omega\, m + n$ for some $m, n \in \mathbb{N}$. Accordingly, the Cantor normal form of any ordinal $\alpha < \omega^{\omega^2}$ can be represented by a list of pairs of numbers.

Theorem 3.5.9. *Given a tree-automatic presentation of some ordinal $\alpha < \omega^{\omega^2}$, one can compute numbers $m_1, n_1, \ldots, m_s, n_s \in \mathbb{N}$ such that*

$$\omega^{\omega\, m_1 + n_1} + \cdots + \omega^{\omega\, m_s + n_s}$$

is the Cantor normal form of α.

Proof. Let A be a tree-automatic well-order of type $\alpha < \omega^{\omega^2}$. We describe a procedure which computes the Cantor normal form of α by induction on the least $k \in \mathbb{N}$ with $\alpha < \omega^{\omega\, k}$. In fact, we do not compute the precise value of k but only need its existence for the procedure to terminate. If $k = 0$ or, equivalently, $\alpha = 0$, the claim is trivial. Henceforth, assume $k \geqslant 1$. There are unique ordinals $\alpha' < \omega^{\omega\,(k-1)}$ and $\beta < \omega^{\omega}$ with

$$\alpha = \omega^{\omega}\, \alpha' + \beta\,.$$

We are interested in computing automatic presentations of these ordinals α' and β.

If $A/{\sim^{\omega}}$ contains a greatest element X and the order type of this $\sim^{\omega}$-class X is strictly below ω^{ω}, then $\alpha'+1$ and β are the order types of $A/{\sim^\omega}$ and X, respectively. In all other cases, α' is the order type of $A/{\sim^\omega}$ and $\beta = 0$. Since $\sim^\omega$ is effectively automatic by lemma 3.5.8, a tree-automatic presentation of $A/{\sim^\omega}$ is obtained from the given presentation of A by choosing the least element from each $\sim^\omega$-class. Using theorem 3.5.5 and corollary 3.5.6, we further obtain a tree-automatic presentation of α' and a string-automatic presentation of β.

Due to the induction hypothesis, we can compute numbers $m_1, n_1, \ldots, m_s, n_s \in \mathbb{N}$ such that

$$\omega^{\omega\, m_1 + n_1} + \cdots + \omega^{\omega\, m_s + n_s}$$

is the Cantor normal form of α'. According to corollary 3.1.7, we can also compute numbers $\ell_1, \ldots, \ell_r \in \mathbb{N}$ such that $\omega^{\ell_1} + \cdots + \omega^{\ell_r}$ is the Cantor normal form of β. Altogether, we obtain that

$$\omega^{\omega\,(m_1+1)+n_1} + \cdots + \omega^{\omega\,(m_s+1)+n_s} + \omega^{\ell_1} + \cdots + \omega^{\ell_r}$$

is the Cantor normal form of α. $\square$

Since the Cantor normal form of every ordinal is unique, one can decide whether two given tree-automatic well-orders of types below ω^{ω^2} are isomorphic by computing and comparing their Cantor normal forms.

Corollary 3.5.10. *Given tree-automatic presentations of two well-orders A and B of order types strictly below ω^{ω^2}, one can decide whether A and B are isomorphic.* $\square$

Unfortunately, the isomorphism problem for tree-automatically presentable well-orders of types beyond ω^{ω^2} resisted numerous attempts towards a solution. The same applies to the closely related problem of deciding whether a given tree-automatically presentable ordinal α satisfies $\alpha < \omega^{\omega^2}$ at all. It appears to us that the main challenge in solving both problems is to establish further useful lower bounds like in lemma 3.5.1 and proposition 3.5.4.

The isomorphism problem for arbitrary tree-automatic linear orders is Σ_1^1-complete as well: The lower bound is inherited from theorem 3.1.10 and the upper bound holds for the isomorphism problem of computable structures in general. In contrast to the string-automatic case, the isomorphism problem for tree-automatic *scattered* linear orders is known to be undecidable.

Theorem 3.5.11 ([Kus14])**.** *Given tree-automatic presentations of two scattered linear orders A and B, it is Π_1^0-hard to decide whether A and B are isomorphic.* □

3.6 Non-automaticity

We complete our investigation of automatic linear orders by providing some examples of linear orders which are not automatically presentable for reasons other than the known bounds on FC-ranks or the complexity of first-order theories. All of these linear orders are of the following type.

Definition 3.6.1. Let $f\colon \mathbb{N} \to \mathbb{N}_+$ be a map. The order type $\tau_f \in \mathcal{VD}_2$ is defined as

$$\tau_f := \sum_{n\in\mathbb{N}} \zeta + f(n)\,.$$

The subsequent lemma provides necessary conditions on f for τ_f to be contained in SA or in $\mathsf{pSA}[k]$. Afterwards, we use these conditions to show that several linear orders are not contained in SA or $\mathsf{pSA}[k]$.

Lemma 3.6.2. *Let $f\colon \mathbb{N} \to \mathbb{N}_+$ be a map and $k \geqslant 2$.*

(1) If τ_f is contained in SA, then $f(n) \in 2^{O(n)}$.

(2) If τ_f is contained in $\mathsf{pSA}[k]$*, then $f(n) \in O(n^{k-1})$.*

Proof. The proofs of both assertions are the same except for the very last argument. Let $(A; \leqslant)$ be a type τ_f string-automatic linear order. Moreover, let $\sim$ be the finite-condensation relation on A and denote the $\sim$-class of $u \in A$ by $[u]$. Recall that $\sim$ is automatic. We consider the subset

$$B := \{\, \min_{\mathrm{llex}}[u] \mid u \in A, [u] \text{ is finite} \,\} ,$$

which has order type ω. Let $u_0 < u_1 < u_2 < \cdots$ be the ascending enumeration of B. Notice that each $[u_n]$ contains exactly $f(n)$ elements.

Since the successor relation of $(B; \leqslant)$ is locally finite and first-order definable in $(A; \leqslant)$ augmented by $\leqslant_{\mathrm{llex}}$ and $\sim$, lemma 2.4.10 on page 37 provides us with a constant $C \in \mathbb{N}$ such that

$$|u_{n+1}| \leqslant |u_n| + C$$

for each $n \in \mathbb{N}$. Using a simple induction on n, we obtain

$$|u_n| \leqslant C \cdot n + |u_0| \in O(n) .$$

Since $\sim$ is finitely valued at each u_n, applying lemma 2.4.10 again yields another constant $D \in \mathbb{N}$ such that $|v| \leqslant |u_n| + D$ for any $v \in [u_n]$. According to the choice of u_n, we also have $|v| \geqslant |u_n|$ for all $v \in [u_n]$. Thus,

$$f(n) = \big|[u_n]\big| \leqslant \sum_{i=0}^{D} \big|A \cap \Sigma^{=|u_n|+i}\big| .$$

In general, we have $|A \cap \Sigma^{=n}| \in 2^{O(n)}$ and hence $f(n) \in 2^{O(n)}$. This proves (1). If we additionally assume $g_A(n) \in O(n^k)$, corollary 2.3.8 on page 31 implies $|A \cap \Sigma^{=n}| \in O(n^{k-1})$ and hence $f(n) \in O(n^{k-1})$. This shows (2). □

The next theorem is the main result of this section. In view of theorem 3.5.5, it does not matter if we consider string-automatic or tree-automatic presentations. Recall that a first-order theory is *sufficiently simple for string-automatic decidability* if the Σ_k-theory belongs to $(k-1)$-EXPSPACE for each $k \geqslant 1$. This notion is in line with the optimal upper bounds on the complexity of the Σ_k-theories of string-automatic structures [Kus09].

Theorem 3.6.3. *There is a scattered linear order A which is* not *automatically presentable although* $\mathrm{FC}(A) = 2$ *and the first-order theory of A is sufficiently simple for string-automatic decidability.*

Proof. We consider the map $f\colon \mathbb{N} \to \mathbb{N}_+$ given by $f(n) := 2^{2^n}$ and show that any linear order of type τ_f has the desired property. The claim $\mathrm{FC}(\tau_f) = 2$ is obvious and the non-automaticity follows from lemma 3.6.2. Thus, we only have to investigate the complexity of the Σ_k-theories.

To this end, let Φ be a Σ_k-sentence suitable for linear orders and m its quantifier depth. From the investigation of Ehrenfeucht–Fraïssé games, it is well known that first-order sentences of quantifier depth m cannot distinguish between finite linear orders containing at least 2^m elements [Ros82, corollary 6.9]. In line with this, we consider the map $h\colon \mathbb{N} \to \mathbb{N}_+$ given by

$$h(n) := \begin{cases} 2^{2^n} & \text{if } 2^n \leqslant m, \\ 2^m & \text{otherwise.} \end{cases}$$

Since Ehrenfeucht–Fraïssé games also play well with sums of linear orders, we obtain that $\tau_f \models \Phi$ if and only if $\tau_h \models \Phi$ [Ros82, lemma 6.5 (2)]. In the remainder of this proof, we demonstrate how to compute a string-automatic presentation of τ_h in time polynomial in m. In the end, one can decide $\tau_f \models \Phi$ by computing this presentation and deciding $\tau_h \models \Phi$. According to [Kus09, proposition 3.3], the latter step can be done in space $(k-1)$-fold exponential in the size of Φ (and the presentation of τ_h, whose size is polynomial in the size of Φ anyway).

We consider the linear order $(A; \leqslant)$ whose domain is given by

$$A := \bigcup_{n\in\mathbb{N}} \mathtt{a}^n \left(\mathtt{b}^+ \cup \mathtt{c}^+ \cup \{\mathtt{0},\mathtt{1}\}^{\log_2 h(n)}\right) \tag{3.7}$$

and where $\leqslant$ is the lexicographic ordering of A induced by

$$\mathtt{b} < \diamond < \mathtt{c} < \mathtt{0} < \mathtt{1} < \mathtt{a}\,.$$

Using the two ideas below, it is a matter of routine to check that $(A; \leqslant)$ has order type τ_h:

(1) The subset $\mathtt{a}^n\,(\mathtt{b}^+ \cup \mathtt{c}^+)$ corresponds the n^{th} occurrence of ζ in τ_h and is internally ordered as

$$\cdots < \mathtt{a}^n\mathtt{b}^3 < \mathtt{a}^n\mathtt{b}^2 < \mathtt{a}^n\mathtt{b}^1 < \mathtt{a}^n\mathtt{c}^1 < \mathtt{a}^n\mathtt{c}^2 < \mathtt{a}^n\mathtt{c}^3 < \cdots\,.$$

(2) The subset $\mathtt{a}^n\,\{\mathtt{0},\mathtt{1}\}^{\log_2 h(n)}$ corresponds to the occurrence of $h(n)$ in τ_h and is internally ordered lexicographically.

It remains to provide a presentation of $(A; \leqslant)$. A string-automaton recognizing A with $O(m)$ states is depicted in fig. 3.5 on the next page, where $\ell := \lfloor \log_2 m \rfloor$. Since recognizing the lexicographic ordering of $\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{0},\mathtt{1}\}^*$ requires only constantly many states, there is a string-automaton recognizing $\leqslant$ with $O(m^2)$ states. Obviously, this string-automatic presentation of τ_h is computable in time polynomial in m. □

Recall that corollary 3.2.6 says that all linear orders in pSA are scattered. In contrast, SA contains non-scattered linear orders, e.g., the linear order of the rationals. In view of these results, one might wonder whether non-scatteredness is the only cause separating the class of linear orders in SA from those in pSA. In fact, it is not as the following example shows.

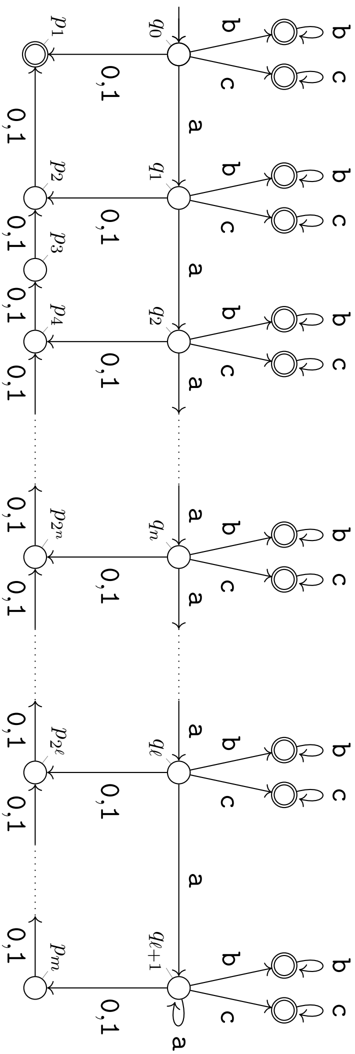


Figure 3.5: A string-automaton recognizing the domain A from eq. (3.7) on the previous page

Example 3.6.4. Consider the map $f(n) := 2^n$. According to lemma 3.6.2, τ_f is not contained in pSA. In contrast, using a very similar idea as in the proof of theorem 3.6.3, we obtain a string-automatic type τ_f linear order on the domain

$$A := \bigcup_{n \geqslant 0} \Big(\mathtt{a}^n (\mathtt{b}^+ \cup \mathtt{c}^+) \cup \{\mathtt{0}, \mathtt{1}\}^n \Big) = \mathtt{a}^* (\mathtt{b}^+ \cup \mathtt{c}^+) \cup \{\mathtt{0}, \mathtt{1}\}^* . \qquad \square$$

According to theorem 3.2.4, the ordinal ω^k separates the class of linear orders in $\mathsf{pSA}[k]$ from those in $\mathsf{pSA}[k-1]$. More generally, every linear order in $\mathsf{pSA}[k]$ of VD_*-rank k provides evidence for this separation. However, there are causes beyond the VD_*-rank for the distinctness of these classes:

Example 3.6.5. Let $k \geqslant 1$ and consider the map $f(n) := \binom{n}{k} + 1$. Obviously, $f(n) \in \Theta(n^k)$. Recall that $\mathrm{VD}_*(\tau_f) = 2$. On the one hand, τ_f is not contained in $\mathsf{pSA}[k]$ by theorem 3.2.9 if $k = 1$ and by lemma 3.6.2 if $k \geqslant 2$. On the other hand, there is a string-automatic linear order of type τ_f on domain

$$A := \mathtt{a}^* \left(\mathtt{b}^+ \cup \mathtt{c}^+\right) \cup \mathtt{0}^* (\mathtt{1}\mathtt{0}^*)^k \cup \mathtt{0}^* .$$

Since

$$g_A(n) = n \cdot (n+1) + \binom{n+1}{k+1} + n + 1 \in O(n^{k+1}) ,$$

this implies that τ_f is contained in $\mathsf{pSA}[k+1]$. Altogether, τ_f separates $\mathsf{pSA}[k+1]$ from $\mathsf{pSA}[k]$. $\square$

3.7 Conclusion

We close our investigation of automatic linear orders by summarizing a large part of the results known so far in two tables. Table 3.1

automaticity	ordinal α	linear order A
1SA	$\alpha < \omega^2$ [Rub04]	$\mathrm{VD}_*(A) \leqslant 1$ [Rub04]
pSA$[k]$	$\alpha < \omega^{k+1}$ (theorem 3.2.4)	$\mathrm{VD}_*(A) \leqslant k$ (theorem 3.2.9)
SA	$\alpha < \omega^\omega$ [Del04]	$\mathrm{FC}(A) < \omega$ [KRS05]
pTA$[k]$	$\alpha < \omega^{\omega^k}$ (corollary 3.4.3)	$\mathrm{FC}(A) < \omega^k$ (theorem 3.4.1)
TA	$\alpha < \omega^{\omega^\omega}$ [Del04]	$\mathrm{FC}(A) < \omega^\omega$ (theorem 3.3.19)

Table 3.1: Upper bounds on the ordinals and the finite-condensation rank of linear orders within certain classes of automatically presentable structures

presents the partial characterizations of the linear orders contained in several classes of automatically presentable structures in terms of upper bounds on their finite-condensation ranks. In the case of ordinals, these bounds are actually complete characterizations. The same holds for arbitrary linear orders in 1SA, the class of unary string-automatically presentable structures. Furthermore, we implicitly point to the fact that all linear orders in 1SA and pSA are scattered by giving a bound on their VD_*-rank instead of their FC-rank.

The current knowledge about the various isomorphism problems for automatic linear orders is shown in table 3.2 on page 120. As already mentioned, the isomorphism problem for arbitrarily large tree-automatic ordinals resisted all our attempts to be solved and is hence still open. However, one of these attempts partially gave rise to the *polychromatic Ramsey theory for ordinals*, whose set-theoretic and automatic variants are presented in the remaining two chapters of this thesis.

automaticity	ordinals	scattered linear orders	arbitrary linear orders
1SA	— decidable in linear time — [LM11]		
SA	decidable [KRS05]	open	Σ^1_1-complete [KLL13b]
TA	decidable below ω^{ω^2} (corollary 3.5.10)	Π^0_1-hard [Kus14]	Σ^1_1-complete [KLL13b]

Table 3.2: Complexities of the isomorphism problems for certain kinds of automatic presentations and classes of linear orders

4 Set-Theoretic Ramsey Theory

A not-so-uncommon situation in mathematics and theoretical computer science is the following: One has a map f on some structure A and is interested in a large substructure $X \subseteq A$ such that the behavior of f on A is easily comprehensible. For instance, we were facing this situation in chapter 3. The structure A was an infinite regular language of trees and the map f the behavior of a tree automaton recognizing a linear ordering of A. We were interested in an infinite subset $X \subseteq A$ such that f takes as few states as possible on X. Our solution was to apply the infinitary version of Ramsey's theorem [Ram30], which reads as follows: Every partition of the edges of an infinite complete graph into finitely many classes admits an infinite induced subgraph all of whose edges belong to the same class. One might wonder whether this statement remains valid if we replace both occurrences of "infinite" by "uncountable". Sierpiński answered this question negatively. More precisely, there is an edge partition of the complete graph on the continuum in two classes such that every uncountable subgraph contains edges of both classes [Sie33]. These two results, particularly the first one, were the starting point for a whole field of research known as *(infinitary) Ramsey theory*, *partition calculus* or *combinatorial set theory*. For a detailed overview of the subject, one might consult the articles [EHR65, EH74] or the monographs [EHMR84, Wil77].

In course of time, not only graphs on unstructured sets were considered but also graphs on linearly ordered sets, notably well-ordered sets. Ramsey's theorem can be rephrased in terms of well-orders as follows: Every edge partition of the complete graph on a type ω linear order into finitely many classes admits a type ω subset whose induced subgraph falls entirely into a single class. Abstracting from this statement, we say that an infinite order type τ has the *Ramsey property* if replacing both occurrences of ω by τ yields a true statement.[1] Of course, ω has the Ramsey property. Another example of an order type with the Ramsey property is $\omega^\star$, the reverse of ω. Using *Sierpiński partitions*, one can show that there are no countable order types with the Ramsey property other than ω and $\omega^\star$. More generally, every order type with the Ramsey property is either a cardinal, regarded as the corresponding initial ordinal, or the reverse of a cardinal [EHMR84]. Sierpiński's aforementioned result however implies that ω_1 and $\omega_1^\star$ do not possess the Ramsey property. In fact, all uncountable cardinals with the Ramsey property are *inaccessible cardinals*, whose existence cannot be proved in Zermelo–Fraenkel set theory with the axiom of choice, cf. [Dra74].

Generally speaking, this is bad news regarding the situation we described initially. However, there is a famous unpublished result by Galvin which provides some hope in the countable case: Every finite edge partition of the complete graph on a type η linear order, alias the rationals, admits a type η subset whose induced subgraph intersects at most two classes. Abstracting from this fact, we say that an order type τ has *Ramsey degree k* if every finite edge partition of the complete graph on τ admits a

[1] As we are not interested in finite order types in this introduction, we implicitly assume all order types under consideration to be infinite.

type τ subset meeting at most k classes and k is minimal with this property. In some sense, the Ramsey degree measures how far an order type is from having the Ramsey property.[2] Of course, every order type with the Ramsey property has Ramsey degree 1 and η has Ramsey degree 2. Since every countable non-scattered linear order contains a type η suborder on the one hand and is embeddable into the rationals on the other hand, all countable non-scattered order types have the same Ramsey degree as η. Using Ramsey's theorem and the infinitary pigeon hole principle, one can easily show that $\omega + 1$ and its reverse $1 + \omega^\star$ also have Ramsey degree 2. Similar but more involved arguments reveal that $\omega + 2$, $\omega \cdot 2$, ω^2 and ζ all have Ramsey degree 4. A result obtained independently by Galvin and Hajnal implies that the Ramsey degree of every ordinal ω^n with $n < \omega$ exists, cf. [Wil77, theorem 7.2.7]. In addition, the proof allows for deriving an upper bound on the Ramsey degree of ω^n in terms of the number of certain lattice paths through the $n \times n$ grid.

This situation naturally raises the question whether the Ramsey degree of every order type does exist. Again, the answer is negative. Several counterexamples can be obtained from more general results: ω^ω from [Tod98, lemma 4], ω_1 from [Tod87] and the initial ordinal of cardinality continuum from [GS73]. Further questions arise immediately, particularly those concerning the countable ordinals we have not mentioned so far. In this chapter, we contribute the following answers, the first two of which already appeared in [HL13]:

(1) The Ramsey degree of every ordinal $\alpha < \omega^\omega$ does exist (theorem 4.5.4).

(2) The precise value of this Ramsey degree can be computed

[2] There are different notions of "Ramsey property" and "Ramsey degree" in finite combinatorics but their relationship is the same as here, cf. [Fou99].

from the Cantor normal form of α (theorem 4.6.7 and corollary 4.6.8).

(3) The Ramsey degree does not exist for any ordinal α with $\omega^\omega \leqslant \alpha < \omega^{\omega^2}$ (theorem 4.7.9).

For the sake of illustration, we were withholding one important aspect of Ramsey's theorem in the presentation so far: Ramsey proved this theorem not only for graphs but also for uniform hypergraphs of any finite arity $r \geqslant 2$. Taking this extra parameter into account, leads to the notions of the *r-ary Ramsey property* and the *r-ary Ramsey degree*. All the negative results mentioned above transfer easily to these extended notions. More precisely, all order types beyond ω and $\omega^\star$ having the r-ary Ramsey property are inaccessible cardinals or their reverses and the r-ary Ramsey degree does still not exist for ω^ω, ω_1 and the initial ordinal of cardinality continuum. On the positive side, Galvin's result on η extends to the r-ary Ramsey degree of η, although the actual values increase as r does [Dev79]. The comment on countable non-scattered order types applies literally.

In view of these circumstances, our contribution in this chapter is not limited to the binary Ramsey degree but provides the answers (1) to (3) above in the more general setting of the r-ary Ramsey degree. More precisely, we prove that, for all ordinals $\alpha < \omega^\omega$ and each $r \geqslant 2$, the r-ary Ramsey degree of α does exist and describe how to compute its exact value by counting certain *box diagrams*. We further demonstrate that none of the r-ary Ramsey degrees of α exists whenever $\omega^\omega \leqslant \alpha < \omega^{\omega^2}$.

Outline. All our results on the r-ary Ramsey degree are obtained in terms of *partition relations*. These relations as well as the formal definition of the Ramsey degree itself are introduced and discussed briefly in section 4.1. The purpose of section 4.2 is to give an overview on the major steps involved in determining Ramsey degrees, namely *polarization*, *canonicalization* and *simplification*. Sections 4.3 to 4.5 detail these three steps. By composing the corresponding results, we obtain optimal upper bounds on Ramsey degrees. Matching lower bounds are established in section 4.6. The resulting exact values of Ramsey degrees are then related to numbers of certain *box diagrams*. In section 4.7, we extend the technique from section 4.6 in order to prove that the Ramsey degrees of ordinals between ω^ω and ω^{ω^2} do not exist. We conclude this chapter by discussing some open problems in section 4.8.

4.1 Basic Definitions and Partition Relations

The objective of this section is to provide a formal definition of the *r-ary Ramsey degree* for ordinals and to discuss its relationship to various *partition relations*. As we are interested in hypergraphs on linearly ordered sets only, we use the set $[A]^r$ defined below as our model of the complete uniform hypergraph of arity r on A.

Definition 4.1.1. Let A be a linear order and $r \in \mathbb{N}$. The set $[A]^r$ is defined as

$$[A]^r := \{\, \langle u_1, u_2, \dots, u_r \rangle \in A^r \mid u_1 < u_2 < \cdots < u_r \,\}.$$

For the sake of technical convenience, we deviate slightly from the usual definition of $[A]^r$, which would be the set of all subsets of A containing precisely r elements. However, there is a very natural bijection between these two sets, namely the one mapping the tuple $\langle u_1, \dots, u_r \rangle \in [A]^r$ to the set $\{u_1, \dots, u_r\}$. Finally, we note that $[A]^r$ is empty whenever $|A| < r$.

In line with the introduction, all partitions in this chapter are assumed to be finite. More precisely, a *(finite) partition* of a set A is a *finite* set Δ of subsets of A, whose elements are called *Δ-classes*, such that each element of A belongs to precisely one Δ-class. If Δ is a partition of A and $B \subseteq A$ a subset, the *restriction* of Δ to B is the partition $\{ D \cap B \mid D \in \Delta \}$ of B.

Definition 4.1.2. Let α be an ordinal and $r \in \mathbb{N}$. The *r-ary Ramsey degree* of α is the least cardinal λ with the following property: For any type α well-order A and every partition Δ of $[A]^r$, there is a type α subset $X \subseteq A$ such that $[X]^r$ intersects at most λ different Δ-classes.

As we consider partitions into finitely many classes only, each r-ary Ramsey degree is either finite or equals the least infinite cardinal $\aleph_0$. We refer to the latter case by simply saying "the r-ary Ramsey degree is infinite".

In order to formulate our intermediate results conveniently, we resort to the notion of *partition relations*. In the subsequent presentation of these relations, we loosely follow [EHMR84]. Throughout this presentation, let α, β be ordinals and $r, \kappa, \lambda \in \mathbb{N}$. Although the case $\kappa = 0$ might seem a bit odd in what follows, we need to take it into account for technical reasons.

The simplest and best-studied partition relation is the *ordinary partition relation*

$$\alpha \longrightarrow (\beta)^r_\kappa \tag{4.1}$$

which denotes the following fact: For any type α well-order A and every partition Δ of $[A]^r$ into κ classes, there is a type β subset $X \subseteq A$ such that $[X]^r$ is contained entirely in a single Δ-class. We refer to this latter property of X as being *homogeneous (wrt Δ)*. In terms of this relation, Ramsey's theorem in its variant for ordinals reads as follows:

Theorem 4.1.3 (Ramsey's theorem [Ram30]). *For all $r, \kappa \in \mathbb{N}$, we have*

$$\omega \longrightarrow (\omega)^r_\kappa \,. \qquad \square$$

More generally, an ordinal α has the *r-ary Ramsey property* mentioned in the introduction if $\alpha \longrightarrow (\alpha)^r_\kappa$ for all $\kappa \in \mathbb{N}$.

The ordinary partition relation in eq. (4.1) is monotonic in various regards: It remains true if one replaces α by a larger ordinal, β by a smaller ordinal or κ by a smaller number. In the following, we refer to this fact as "the monotonicity of the partition relation". Later on, we show that one may also replace r by a smaller number whenever β is infinite (cf. lemma 4.7.1).

The second partition relation we consider is the *square bracket partition relation*

$$\alpha \longrightarrow [\beta]^r_\kappa \tag{4.2}$$

which denotes the following fact: For any type α well-order A and every partition $\varDelta$ of $[A]^r$ into κ classes, there is a type β subset $X \subseteq A$ such that $[X]^r$ does *not* intersect *all* $\varDelta$-classes. We note that eq. (4.2) is monotonic wrt α and β the same way eq. (4.1) is but for κ it is the other way round: The partition relation in eq. (4.2) remains true if we replaces κ by a *larger* number. In the following, we are mainly interested in the negation

$$\alpha \not\longrightarrow [\beta]^r_\kappa$$

which denotes the following fact: There are a type α well-order A and a partition $\varDelta$ of $[A]^r$ into κ classes such that, for each type β subset $X \subseteq A$, the set $[X]^r$ intersects *all* $\varDelta$-classes. Subsets $X \subseteq A$ with this latter property are called *completely inhomogeneous (wrt $\varDelta$)*.

The last partition relation we take into account here is a common generalization of the previous two relations. The *weak square bracket partition relation*

$$\alpha \longrightarrow [\beta]^r_{\kappa,\lambda} \tag{4.3}$$

denotes the following fact: For any type α well-order A and every partition $\varDelta$ of $[A]^r$ into κ classes, there is a type β subset $X \subseteq A$

such that $[X]^r$ intersects at most λ different Δ-classes. Subsets $X \subseteq A$ with this latter property are called *relatively λ-homogeneous (wrt Δ)*. It is obvious that this partition relation respects the same monotonicity properties as the ordinary partition relation. In addition, eq. (4.3) remains true if we replace λ by a larger number. We note that the first two partition relations can be regarded as abbreviations for special cases of the weak square bracket partition relation: $\alpha \longrightarrow (\beta)^r_\kappa$ and $\alpha \longrightarrow [\beta]^r_\kappa$ are equivalent to $\alpha \longrightarrow [\beta]^r_{\kappa,1}$ and $\alpha \longrightarrow [\beta]^r_{\kappa,\kappa-1}$, respectively, whenever $\kappa > 0$.

We conclude this section by discussing the close relationship between the r-ary Ramsey degree and the latter two partition relations. In terms of the weak square bracket partition relation, definition 4.1.2 can be rephrased as follows: The r-ary Ramsey degree of α is either the least $\lambda \in \mathbb{N}$ such that, for all $\kappa \in \mathbb{N}$,

$$\alpha \longrightarrow [\alpha]^r_{\kappa,\lambda}$$

or it is infinite if there is no such λ at all. Due to the monotonicity of the partition relations, the r-ary Ramsey degree of α has another notable characterization: It coincides with the largest $\lambda \in \mathbb{N}$ such that

$$\alpha \not\longrightarrow [\alpha]^r_\lambda \,,$$

provided this maximum exists, and is infinite otherwise. Our strategy to obtain exact values of Ramsey degrees is a mixture of both characterizations and captured by the lemma below, whose proof is trivial:

Lemma 4.1.4. *Let α be an ordinal and $r, \lambda \in \mathbb{N}$. If*

$$\alpha \longrightarrow [\alpha]^r_{\kappa,\lambda} \qquad \textit{and} \qquad \alpha \not\longrightarrow [\alpha]^r_\lambda$$

for all $\kappa \in \mathbb{N}$, then the r-ary Ramsey degree of α is exactly λ. $\square$

4.2 Basic Ideas: A Showcase

Before we delve into the details of showing that certain Ramsey degrees are finite, we sketch how to prove that the binary Ramsey degree of $\omega \cdot 3 = \omega + \omega + \omega$ is exactly 9. The purpose of this sketch is to give an overview of the major steps involved in determining r-ary Ramsey degrees of arbitrary ordinals $\alpha < \omega^\omega$, namely *polarization*, *canonicalization* and *simplification*.

Let A be a type $\omega \cdot 3$ well-order and Δ a partition of $[A]^2$. We consider the decomposition $A = A_1 + A_2 + A_3$ of A into Cantor normal form, i.e., each A_i has order type ω. For every type $\omega \cdot 3$ subset $X \subseteq A$, all the intersections $A_i \cap X$ have order type ω as well. Finding a type $\omega \cdot 3$ subset $X \subseteq A$ such that $[X]^2$ intersects as few Δ-classes as possible therefore amounts to finding type ω subsets $X_i \subseteq A_i$, for $i = 1, 2, 3$, such that $[X_1+X_2+X_3]^2$ intersects as few Δ-classes as possible. The elements of $[X_1 + X_2 + X_3]^2$ then are of six different kinds according to the partition

$$\begin{aligned}[X_1 + X_2 + X_3]^2 = [X_1]^2 \uplus [X_2]^2 \uplus [X_3]^2 \\ \uplus (X_1 \times X_2) \uplus (X_1 \times X_3) \uplus (X_2 \times X_3)\,.\end{aligned}$$

Consequently, our goal is to choose the X_i such that each of the six parts above intersects as few Δ-classes as possible. This choice is accomplished by the following four steps, which are also depicted in fig. 4.1 on the next page:

Step 1: Ramsey's theorem 4.1.3 provides us with type ω subsets $X_i^\circ \subseteq A_i$, for $i = 1, 2, 3$, such that each $[X_i^\circ]^2$ intersects only one Δ-class. We regard these sets X_i° as initial approximations of the final sets X_i.

Step 2: We improve the approximations of X_1 and X_2 by choosing type ω subsets $X_1' \subseteq X_1^\circ$ and $X_2' \subseteq X_2^\circ$ such that $X_1' \times X_2'$ intersects as few Δ-classes as possible, say λ_{12} many.

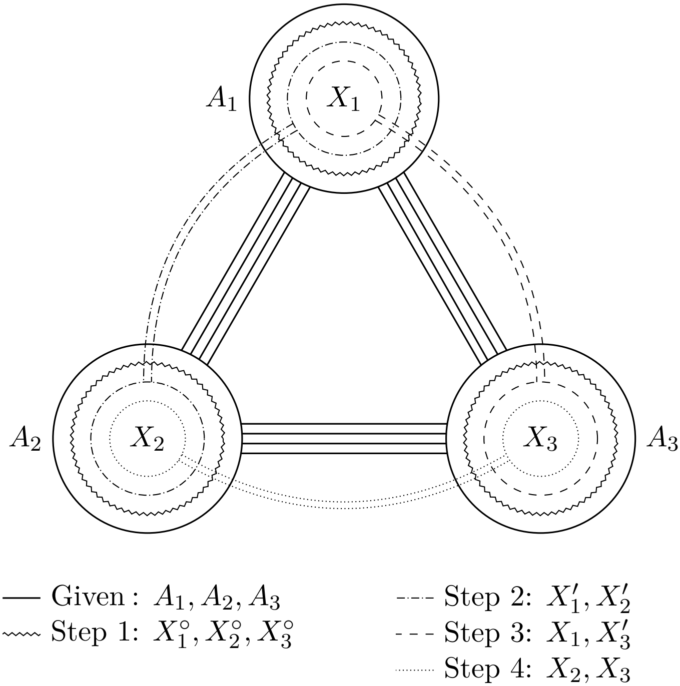


Figure 4.1: The process of polarization

Step 3: We choose the final subset $X_1 \subseteq X_1'$ and a better approximation $X_3' \subseteq X_3^\circ$, both of order type ω, such that $X_1 \times X_3'$ intersects as few Δ-classes as possible, say λ_{13} many.

Step 4: We choose the final type ω subsets $X_2 \subseteq X_2'$ and $X_3 \subseteq X_3'$ such that $X_2 \times X_3$ intersects as few Δ-classes as possible, say λ_{23} many.

Altogether, the set $X_1 + X_2 + X_3 \subseteq A$ has order type $\omega \cdot 3$ and intersects at most $3 + \lambda_{12} + \lambda_{13} + \lambda_{23}$ distinct Δ-classes. In this way, we have reduced the problem of finding a type $\omega \cdot 3$ subset

$X \subseteq A$ which intersects as few $\varDelta$-classes as possible to Ramsey's theorem 4.1.3 and the problem to find the least number $\lambda \in \mathbb{N}$ with the following property: For any type ω well-orders B and C and every partition $\varDelta$ of $B \times C$, there are type ω subsets $Y \subseteq B$ and $Z \subseteq C$ such that $Y \times Z$ intersects at most λ distinct $\varDelta$-classes. A solution of this latter problem can be regarded as a variant of Ramsey's theorem 4.1.3 for complete bipartite graphs on two type ω well-orders.

We call this process of reducing a partition relation on the sum of some ordinals to partition relations (for multipartite graphs) on the summands *polarization*. In section 4.3, we study polarization in the general case.

In order to show that the binary Ramsey degree of $\omega \cdot 3$ is indeed finite, we still need to prove the bipartite version of Ramsey's theorem. Our first goal is to show that every partition $\varDelta$ of $\mathbb{N} \times \mathbb{N}$ admits type ω subsets $Y, Z \subseteq \mathbb{N}$ such that $Y \times Z$ intersects at most 3 distinct $\varDelta$-classes. To this end, we define an equivalence relation $\sim$ on $[\mathbb{N}]^2$ by $\langle x_1, x_2 \rangle \sim \langle y_1, y_2 \rangle$ if, for all $\langle i, j \rangle \in \{1, 2\} \times \{1, 2\}$, $\langle x_i, x_j \rangle$ and $\langle y_i, y_j \rangle$ belong to the same $\varDelta$-class. Since $\sim$ has finite index, Ramsey's theorem 4.1.3 provides us with a type ω subset $H \subseteq \mathbb{N}$ such that $[H]^2$ is completely contained in a single $\sim$-class. In effect, the restriction of $\varDelta$ to $H \times H$ contains only 3 non-empty classes, namely the sets

$$D_\theta = \{ \langle x, y \rangle \in H \times H \mid x \; \theta \; y \}$$

for $\theta \in \{<, =, >\}$.

Subpartitions of this simple form are called *canonical partitions*. In section 4.4, we lift this notion to the general case and extend ideas from [Wil77, section 7.2] in order to show that canonical subpartitions do always exist. This already implies finite upper bounds on certain Ramsey degrees.

Our last goal is to find the exact value of the binary Ramsey degree of $\omega \cdot 3$. For this purpose, we *simplify* the canonical partition $\{D_<, D_=, D_>\}$ of $H \times H$ from above even further. Let $H = Y \uplus Z$ be a partition of H in two infinite subsets Y, Z. Clearly, the set $Y \times Z$ does not intersect the class $D_=$ and hence intersects at most 2 distinct Δ-classes. We extend this simplification of canonical partitions to the general case in section 4.5.

Up to this point, we have established that the type $\omega \cdot 3$ subset $X_1 + X_2 + X_3 \subseteq A$ constructed in the four steps above intersects at most 9 different Δ-classes. Put another way, the Ramsey degree of $\omega \cdot 3$ is at most 9. In order to show that this cannot be improved any further, it suffices to show that neither Ramsey's theorem nor its bipartite version can be improved. Composing the partitions demonstrating these optimalities in a suitable way then, yields a partition of $[A]^2$ into 9 classes which does not admit a relatively 8-homogeneous type $\omega \cdot 3$ subset of A. Obviously, Ramsey's theorem is optimal. To see that the bipartite version is also optimal, we consider the partition $\Delta = \{E_\leqslant, E_>\}$ of $\mathbb{N} \times \mathbb{N}$ given by

$$E_\theta = \{ \langle x, y \rangle \in \mathbb{N} \times \mathbb{N} \mid x \;\theta\; y \} \,.$$

Clearly, $Y \times Z$ intersects both $E_\leqslant$ and $E_>$ for all type ω subsets $Y, Z \subseteq \mathbb{N}$. Finally, notice how much Δ resembles the simplified canonical partition above: The Δ-class of a pair $\langle x, y \rangle \in \mathbb{N} \times \mathbb{N}$ is completely determined by the relative order of x and y in $\mathbb{N}$, without having a separate class for $x = y$. In section 4.6, we demonstrate that simplified canonical partitions are, in a certain sense, the best one can achieve in general.

4.3 Polarization

The fundamental notion needed to formalize the idea of *polarization* is another family of partition relations which do not speak about single ordinals but about tuples of ordinals. Such relations are known as *polarized partition relations* [EHMR84]. Before giving the definition of our variant, we extend some notions from sets to tuples of sets.

Let $s \in \mathbb{N}$. Consider a tuple of sets $\boldsymbol{A} = \langle A_1, \ldots, A_s \rangle$ and some $\boldsymbol{r} = \langle r_1, \ldots, r_s \rangle \in \mathbb{N}^s$. The set $[\boldsymbol{A}]^{\boldsymbol{r}}$ is defined as

$$[\boldsymbol{A}]^{\boldsymbol{r}} := [A_1]^{r_1} \times [A_2]^{r_2} \times \cdots \times [A_s]^{r_s} .$$

A tuple of sets $\boldsymbol{X} = \langle X_1, \ldots, X_s \rangle$ is a *tuple of subsets* of $\boldsymbol{A}$, which we denote by $\boldsymbol{X} \subseteq \boldsymbol{A}$, if each X_k is a subset of A_k. Notice that $[\boldsymbol{X}]^{\boldsymbol{r}}$ is a subset of $[\boldsymbol{A}]^{\boldsymbol{r}}$ whenever $\boldsymbol{X} \subseteq \boldsymbol{A}$. Finally, suppose that $\boldsymbol{A}$ is a tuple of well-orders. The *(order) type* of $\boldsymbol{A}$ is the tuple of ordinals $\langle \alpha_1, \ldots, \alpha_s \rangle$ where each α_k is the order type of A_k.

The notions of homogeneous, completely inhomogeneous and relatively λ-homogeneous subsets (wrt some partition) transfer easily from sets to tuples of sets. For instance, a tuple of subsets $\boldsymbol{X} \subseteq \boldsymbol{A}$ is *relatively λ-homogeneous* wrt some partition Δ of $[\boldsymbol{A}]^{\boldsymbol{r}}$ if $[\boldsymbol{X}]^{\boldsymbol{r}}$ intersects at most λ different Δ-classes.

Definition 4.3.1. Let $s, \kappa, \lambda \in \mathbb{N}$, $\boldsymbol{r} \in \mathbb{N}^s$ and $\alpha_1, \ldots, \alpha_s$, $\beta_1, \ldots, \beta_s$ be ordinals. The *polarized weak square bracket partition relation*

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_s \end{pmatrix} \longrightarrow \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_s \end{bmatrix}^{r_1, \ldots, r_s}_{\kappa, \lambda}$$

denotes the following fact: For any type $\langle \alpha_1, \ldots, \alpha_s \rangle$ tuple of well-orders $\boldsymbol{A}$ and every partition Δ of $[\boldsymbol{A}]^{\boldsymbol{r}}$ into κ classes, there is a relatively λ-homogeneous type $\langle \beta_1, \ldots, \beta_s \rangle$ tuple of subsets of $\boldsymbol{A}$.

Notice that the special case $s = 1$ is precisely the weak square bracket partition relation in eq. (4.3) on page 127. Like the non-polarized relation the polarized variant is also monotonic in various regards: It remains true if one replaces the α_k by larger ordinals, the β_k by smaller ordinals, κ by a smaller number or λ by a larger number. We refer to this fact as "the monotonicity of the polarized partition relation". Recall that one can regard $\alpha \longrightarrow [\beta]^r_\kappa$ as an abbreviation for $\alpha \longrightarrow [\beta]^r_{\kappa,\kappa-1}$. Using the same abbreviation for polarized relations leads to the *polarized square bracket partition relation*. Again, we are mainly interested in its negation

$$\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_s\end{pmatrix} \not\longrightarrow \begin{bmatrix}\beta_1\\ \vdots \\ \beta_s\end{bmatrix}^{r_1,\dots,r_s}_\kappa$$

which denotes the following fact: There are a type $\langle \alpha_1, \dots, \alpha_s \rangle$ tuple of well-orders $\boldsymbol{A}$ and a partition Δ of $[\boldsymbol{A}]^{\boldsymbol{r}}$ into κ classes such that each type $\langle \beta_1, \dots, \beta_s \rangle$ tuple of subsets of $\boldsymbol{A}$ is completely inhomogeneous wrt Δ.

In the remainder of this section, we prove two polarization lemmas, namely the positive polarization lemma 4.3.2 and the negative polarization lemma 4.3.5, which allow for concluding non-polarized partition relations on sums of ordinals from polarized partition relations on the summands. In line with lemma 4.1.4, they deal with the weak and the negated square bracket partition relation. In our applications, the sums are Cantor normal forms. Both lemmas use the set $\mathcal{R}(s,r) \subseteq \mathbb{N}^s$, where $s, r \in \mathbb{N}$, given by

$$\mathcal{R}(s,r) := \left\{\, \tilde{\boldsymbol{r}} \in \mathbb{N}^s \;\middle|\; \tilde{r}_1 + \cdots + \tilde{r}_s = r \,\right\}.$$

For any map $\ell\colon \mathcal{R}(s,r) \to \mathbb{N}$, we define the number $|\ell| \in \mathbb{N}$ as

$$|\ell| := \sum_{\tilde{\boldsymbol{r}} \in \mathcal{R}(s,r)} \ell(\tilde{\boldsymbol{r}}).$$

The positive polarization lemma generalizes the four step process depicted in fig. 4.1 on page 130.

Lemma 4.3.2 (positive polarization lemma). *Let $r, \kappa \in \mathbb{N}$, α be an ordinal, $\alpha = \omega^{\gamma_1} + \cdots + \omega^{\gamma_s}$ its Cantor normal form and $\ell \colon \mathcal{R}(s, r) \to \mathbb{N}$ a map. If*

$$\begin{pmatrix} \omega^{\gamma_1} \\ \vdots \\ \omega^{\gamma_s} \end{pmatrix} \longrightarrow \begin{bmatrix} \omega^{\gamma_1} \\ \vdots \\ \omega^{\gamma_s} \end{bmatrix}^{\tilde{r}_1, \ldots, \tilde{r}_s}_{\kappa, \ell(\tilde{\boldsymbol{r}})}$$

for all $\tilde{\boldsymbol{r}} \in \mathcal{R}(s, r)$, then

$$\alpha \longrightarrow [\alpha]^r_{\kappa, |\ell|} .$$

Proof. Suppose the premise is satisfied. Let $A_1 + \cdots + A_s$ be a type α well-order where each A_k has type ω^{γ_k} and Δ a partition of $[A_1 + \cdots + A_s]^r$ into κ classes. We put $\boldsymbol{A} := \langle A_1, \ldots, A_s \rangle$. Observe that the set

$$\left\{ [\boldsymbol{A}]^{\tilde{\boldsymbol{r}}} \mid \tilde{\boldsymbol{r}} \in \mathcal{R}(s, r) \right\}, \tag{4.4}$$

which was defined as

$$\left\{ [A_1]^{\tilde{r}_1} \times \cdots \times [A_s]^{\tilde{r}_s} \;\middle|\; \tilde{r}_1, \ldots, \tilde{r}_s \in \mathbb{N}, \tilde{r}_1 + \cdots + \tilde{r}_s = r \right\},$$

forms another partition of $[A_1 + \cdots + A_s]^r$. For each $\tilde{\boldsymbol{r}} \in \mathcal{R}(s, r)$, the restriction of Δ to $[\boldsymbol{A}]^{\tilde{\boldsymbol{r}}}$ is a partition of $[\boldsymbol{A}]^{\tilde{\boldsymbol{r}}}$ into κ classes. Thus, one of the presumed polarized partition relations applies. We proceed by applying all these partition relations in a suitable way.

To this end, fix some enumeration $\tilde{\boldsymbol{r}}_1, \ldots, \tilde{\boldsymbol{r}}_m$ of $\mathcal{R}(s, r)$. We construct a chain $\boldsymbol{X}_0 \supseteq \boldsymbol{X}_1 \supseteq \cdots \supseteq \boldsymbol{X}_m$ of type $\langle \alpha_1, \ldots, \alpha_s \rangle$ tuples of subsets of $\boldsymbol{A}$ inductively.

Base case: $t = 0$. We simply choose $\boldsymbol{X}_0 = \boldsymbol{A}$.

Inductive step: $t \in [1, m]$. Assume $\boldsymbol{X}_{t-1}$ to be constructed before. Applying the presumed polarized partition relation for $\tilde{\boldsymbol{r}}_t$ to the restriction Δ_{t-1} of Δ to $[\boldsymbol{X}_{t-1}]^{\tilde{\boldsymbol{r}}_t}$ yields a type $\langle \alpha_1, \ldots, \alpha_s \rangle$ tuple of subsets $\boldsymbol{Y} \subseteq \boldsymbol{X}_{t-1}$ which is relatively $\ell(\tilde{\boldsymbol{r}}_t)$-homogeneous wrt the restriction Δ_{t-1}. Consequently, $[\boldsymbol{Y}]^{\tilde{\boldsymbol{r}}_t}$ intersects at most $\ell(\tilde{\boldsymbol{r}})$ different Δ-classes as well. We complete the inductive step by choosing $\boldsymbol{X}_t := \boldsymbol{Y}$.

We conclude this proof by showing that the type $\alpha_1 + \cdots + \alpha_s$ subset

$$Z := X_{m1} + \cdots + X_{ms}$$

of $A_1 + \cdots + A_s$ is relatively $|\ell|$-homogeneous. Analogously to the set in eq. (4.4), the set

$$\{ [\boldsymbol{X}_m]^{\tilde{\boldsymbol{r}}} \mid \tilde{\boldsymbol{r}} \in \mathcal{R} \}$$

forms a partition of $[Z]^r$. In view of the definition of $|\ell|$, it suffices to show that each $[\boldsymbol{X}_m]^{\tilde{\boldsymbol{r}}}$ intersects at most $\ell(\tilde{\boldsymbol{r}})$ different Δ-classes. To this end, let $t \in [1, m]$ be such that $\tilde{\boldsymbol{r}} = \tilde{\boldsymbol{r}}_t$. We have $\boldsymbol{X}_m \subseteq \boldsymbol{X}_t$ and hence $[\boldsymbol{X}_m]^{\tilde{\boldsymbol{r}}} \subseteq [\boldsymbol{X}_t]^{\tilde{\boldsymbol{r}}_t}$. Thus, $[\boldsymbol{X}_m]^{\tilde{\boldsymbol{r}}}$ intersects indeed at most $\ell(\tilde{\boldsymbol{r}})$ different Δ-classes since $[\boldsymbol{X}_t]^{\tilde{\boldsymbol{r}}_t}$ has this property by choice. □

In order to prove the negative polarization lemma, we need one more step of preparation. In terms of well-orders and suborders, the statement below reads as follows: Let $A = A_1 + \cdots + A_s$ be a well-order and its decomposition into Cantor normal form and $X \subseteq A$. If X has the same order type as A, then $X \cap A_k$ has the same order type as A_k for each k.

Lemma 4.3.3. *Let α be an ordinal, $\alpha = \omega^{\gamma_1} + \cdots + \omega^{\gamma_s}$ its Cantor normal form and $\alpha_1 \leqslant \omega^{\gamma_1}, \ldots, \alpha_s \leqslant \omega^{\gamma_s}$ ordinals. If*

$$\alpha = \alpha_1 + \cdots + \alpha_s\,,$$

then $\alpha_k = \omega^{\gamma_k}$ for each k.

Proof. We show the contraposition of the claimed implication. Suppose there is ℓ such that $\alpha_\ell < \omega^{\gamma_\ell}$. Our first goal is to prove that

$$\alpha_\ell + \omega^{\gamma_{\ell+1}} + \cdots + \omega^{\gamma_s} < \omega^{\gamma_\ell} + \omega^{\gamma_{\ell+1}} + \cdots + \omega^{\gamma_s} \,. \tag{4.5}$$

To this end, let $\omega^{\delta_1} + \cdots + \omega^{\delta_t}$ be the Cantor normal form of α_ℓ. Notice that $\alpha_\ell < \omega^{\gamma_\ell}$ implies either $t = 0$ or $\delta_1 < \gamma_\ell$. There is $m \in [0, t]$ such that

$$\omega^{\delta_1} + \cdots + \omega^{\delta_m} + \omega^{\gamma_{\ell+1}} + \cdots + \omega^{\gamma_s}$$

is the Cantor normal form of the left hand side in eq. (4.5). Thus, eq. (4.5) follows from $\delta_1 < \gamma_\ell$ whenever $m \geqslant 1$ and from $\gamma_\ell \geqslant \gamma_{\ell+1} \geqslant \cdots \geqslant \gamma_s$ otherwise.

Recall that the addition of ordinals is monotonic in both arguments and even strictly monotonic in its second argument. Consequently, eq. (4.5) implies

$$\begin{aligned} \alpha_1 + \cdots + \alpha_s &< \alpha_1 + \cdots + \alpha_\ell + \omega^{\gamma_{\ell+1}} + \cdots + \omega^{\gamma_s} \\ &\leqslant \omega^{\gamma_1} + \cdots + \omega^{\gamma_s} \,. \end{aligned}$$ □

Before we finally turn to the negative polarization lemma, we showcase the main idea behind its proof by demonstrating the following simpler result, which is used in the very end of this chapter.

Lemma 4.3.4. *Let $r, \kappa \in \mathbb{N}$ and α be an ordinal. If the Cantor normal form of α contains a summand ω^γ with*

$$\omega^\gamma \not\longrightarrow [\omega^\gamma]^r_\kappa \,,$$

then

$$\alpha \not\longrightarrow [\alpha]^r_\kappa \,.$$

Proof. The case $\kappa = 0$ is trivial. Henceforth, we assume $\kappa > 0$. Let $A = A_1 + \cdots + A_s$ be a type α well-order and its decomposition into Cantor normal form. Pick $\ell \in [1, s]$ such that A_ℓ has order type ω^γ. Let $\Delta = \{D_1, \ldots, D_\kappa\}$ be a partition of $[A_\ell]^r$ which exemplifies $[\omega^\gamma] \not\rightarrow [\omega^\gamma]^r_\kappa$. Since $[A_\ell]^r \subseteq [A]^r$, there is a partition $\Gamma = \{C_1, \ldots, C_\kappa\}$ of $[A]^r$ such that $D_i \subseteq C_i$ for all $i \in [1, \kappa]$. It suffices to show that every type α subset $X \subseteq A$ is completely inhomogeneous wrt Γ.

For this purpose, we consider a type α subset $X \subseteq A$ and some Γ-class C_i. According to lemma 4.3.3, the set $X \cap A_\ell$ has order type ω^γ and is hence completely inhomogeneous wrt Δ. In particular, $[X \cap A_\ell]^r$ intersects D_i. Since $[X]^r \supseteq [X \cap A_\ell]^r$ and $D_i \subseteq C_i$, this implies that $[X]^r$ intersects C_i. □

In view of lemma 4.1.4, the negative polarization lemma below can be regarded as the contrary of the positive polarization lemma 4.3.2.

Lemma 4.3.5 (negative polarization lemma)**.** *Let $r \in \mathbb{N}$, α be an ordinal, $\alpha = \omega^{\gamma_1} + \cdots + \omega^{\gamma_s}$ its Cantor normal form and $\ell \colon \mathcal{R}(s, r) \to \mathbb{N}$ a map. If*

$$\begin{pmatrix} \omega^{\gamma_1} \\ \vdots \\ \omega^{\gamma_s} \end{pmatrix} \not\rightarrow \begin{bmatrix} \omega^{\gamma_1} \\ \vdots \\ \omega^{\gamma_s} \end{bmatrix}^{\tilde{r}_1, \ldots, \tilde{r}_s}_{\ell(\tilde{\boldsymbol{r}})}$$

for all $\tilde{\boldsymbol{r}} \in \mathcal{R}(s, r)$, then

$$\alpha \not\rightarrow [\alpha]^r_{|\ell|} \, .$$

Proof. Suppose the premise is satisfied. Let $A_1 + \cdots + A_s$ be a type α well-order where each A_k has order type ω^{γ_k}. Our objective is to construct a partition of $[A_1 + \cdots + A_s]^r$ into $|\ell|$

classes which establishes the desired partition relation. We put $\boldsymbol{A} := \langle A_1, \dots, A_s \rangle$ and recall that the set

$$\{ [\boldsymbol{A}]^{\tilde{\boldsymbol{r}}} \mid \tilde{\boldsymbol{r}} \in \mathcal{R}(s,r) \}$$

forms a partition of $[A_1 + \cdots + A_s]^r$.

For each $\tilde{\boldsymbol{r}} \in \mathcal{R}(s,r)$, let $\Delta_{\tilde{\boldsymbol{r}}}$ be a partition of $[\boldsymbol{A}]^{\tilde{\boldsymbol{r}}}$ into $\ell(\tilde{\boldsymbol{r}})$ classes which exemplifies the premise for $\tilde{\boldsymbol{r}}$, i.e., for every type $\langle \omega^{\gamma_1}, \dots, \omega^{\gamma_s} \rangle$ tuple of subsets $\boldsymbol{X} \subseteq \boldsymbol{A}$, the set $[\boldsymbol{X}]^{\tilde{\boldsymbol{r}}}$ intersects all $\Delta_{\tilde{\boldsymbol{r}}}$-classes. We combine all these partitions into one partition Δ of $[A_1 + \cdots + A_s]^r$ into $|\ell|$ classes by putting

$$\Delta := \bigcup_{\tilde{\boldsymbol{r}} \in \mathcal{R}(s,r)} \Delta_{\tilde{\boldsymbol{r}}} \,.$$

In the remainder of this proof, we demonstrate that every type $\omega^{\gamma_1} + \cdots + \omega^{\gamma_s}$ subset $X \subseteq A_1 + \cdots + A_s$ is completely inhomogeneous wrt Δ.

To this end, consider some Δ-class D. There is $\tilde{\boldsymbol{r}} \in \mathcal{R}(s,r)$ with $D \in \Delta_{\tilde{\boldsymbol{r}}}$. For each $k \in [1,s]$, let α_k be the order type of $X \cap A_k$. According to lemma 4.3.3, we have $\alpha_k = \omega^{\gamma_k}$. Due to the choice of $\Delta_{\tilde{\boldsymbol{r}}}$, the set $[X \cap A_1, \dots, X \cap A_s]^{\tilde{\boldsymbol{r}}}$ hence intersects D. Since

$$[X \cap A_1, \dots, X \cap A_s]^{\tilde{\boldsymbol{r}}} \subseteq [X]^r \,,$$

the set $[X]^r$ intersects D as well. $\square$

4.4 Canonicalization

This section is devoted to the investigation of the *canonicalization* step which followed the polarization step in section 4.2. Basically, the canonicalization lemma 4.4.5 extends a result by Hajnal and, independently, Galvin on the existence of *canonical partitions* from sets to tuples of sets. Along with the positive polarization lemma 4.3.2, we can already conclude that the Ramsey degrees of all ordinals $\alpha < \omega^\omega$ are finite. In addition, we obtain upper bounds on their values. However, these bounds turn out to be *not* optimal in the next section.

In our presentation of the existence of canonical partitions, we roughly follow [Wil77, section 7.2]. *As of now, we assume the set* $\mathbb{N}_+^n$ *and its subsets to be ordered lexicographically*, i.e., $\boldsymbol{x} <_{\mathbb{N}_+^n} \boldsymbol{y}$ if there is i with $x_i \neq y_i$ and the least such i satisfies $x_i < y_i$. Although $\mathbb{N}_+^n$ is hence a type ω^n well-order itself, we do *not* use it as our standard model of ω^n. This is mainly due to technical reasons. Instead, we use the suborder

$$\mathcal{W}(n) := \{\, \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{N}_+^n \mid x_1 < x_2 < \cdots < x_n \,\}$$

as our standard model of ω^n. To avoid a level of indices, we treat the elements of $\mathbb{N}_+^n$ and $\mathcal{W}(n)$ as (order-preserving) functions from $[1, n]$ to $\mathbb{N}_+$.[3] Since $\mathcal{W}(n)$ and $\mathbb{N}_+^n$ both are type ω^n well-orders, there is a unique isomorphism between them. We denote the image of $x \in \mathcal{W}(n)$ under this isomorphism by $\triangle x$. It is easy to check that $\triangle x$ is given by

$$\triangle x(\mu) := x(\mu) - x(\mu - 1)\,,$$

where we use $x(0) = 0$. In order to prevent explicitly dealing with some corner cases, we use the convention $x(0) = 0$ for $x \in \mathcal{W}(n)$ in several places without further notice.

In order to work with polarized partition relations in a convenient way, we introduce some notation for tuples of well-orders. Let $s \in \mathbb{N}$, $\boldsymbol{n} = \langle n_1, \ldots, n_s \rangle \in \mathbb{N}^s$ and $\boldsymbol{r} = \langle r_1, \ldots, r_s \rangle \in \mathbb{N}^s$. If not further specified, s, $\boldsymbol{n}$ and $\boldsymbol{r}$ are always of this kind in the remainder of this chapter. We define the tuple of well-orders $\mathcal{W}(\boldsymbol{n})$ by

$$\mathcal{W}(\boldsymbol{n}) := \langle \mathcal{W}(n_1), \ldots, \mathcal{W}(n_s) \rangle\,.$$

[3] This convention does *not* apply to $\mathbb{N}^n$.

In the following, the set

$$[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}} = [\mathcal{W}(n_1)]^{r_1} \times [\mathcal{W}(n_2)]^{r_2} \times \cdots \times [\mathcal{W}(n_s)]^{r_s}$$

plays an important role. Its members are tuples

$$\boldsymbol{x} = \langle x_{11}, \ldots, x_{1r_1}; \ldots; x_{k1}, \ldots, x_{kr_k}; \ldots; x_{s1}, \ldots, x_{sr_s} \rangle \tag{4.6}$$

with $x_{k1}, \ldots, x_{kr_k} \in \mathcal{W}(n_k)$ and $x_{k1} < \cdots < x_{kr_k}$ for all $k \in [1, s]$, where $\mathcal{W}(n_k)$ is ordered lexicographically. Notice how the entries sharing the same first index are grouped by means of semicolons in eq. (4.6). The *entry set* of $\boldsymbol{x}$ is the set

$$\{\, x_{ki}(\mu) \mid k \in [1, s], i \in [1, r_k], \mu \in [1, n_k] \,\} \subseteq \mathbb{N}_+ \, .$$

To avoid repetitively specifying exact ranges for indices, we use the phrase "for all indices k, i, μ" to abbreviate "for all $k \in [1, s]$, $i \in [1, r_k]$ and $\mu \in [1, n_k]$". The meaning of the phrase "for all indices i, k" is analogous.

During the development of the results presented here, we found it helpful to think about elements of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ in terms of *box diagrams* as depicted in fig. 4.2 on the following page, which is explained below.

Example 4.4.1. Let $s = 3$, $\boldsymbol{n} = \langle 3, 2, 4 \rangle$ and $\boldsymbol{r} = \langle 2, 1, 1 \rangle$. Then

$$[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}} = [\mathcal{W}(3)]^2 \times \mathcal{W}(2) \times \mathcal{W}(4) \, .$$

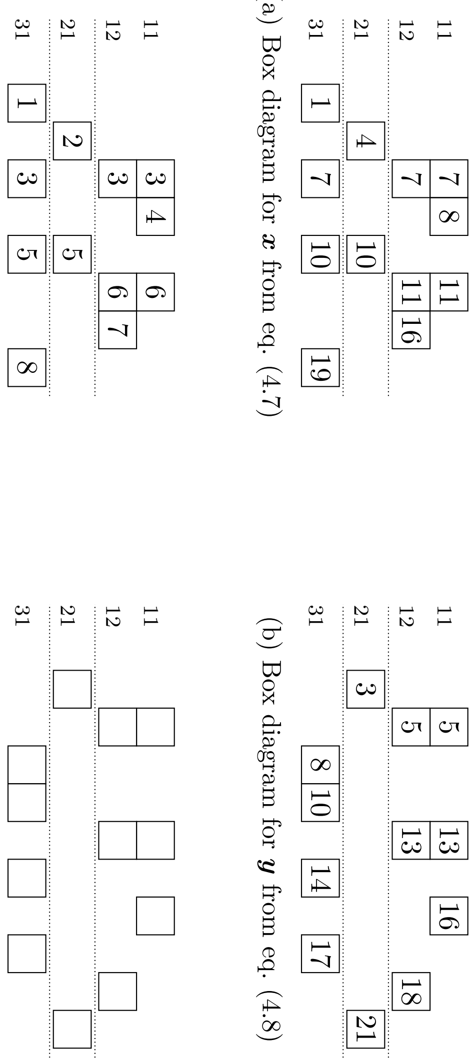
Figure 4.2 depicts the following elements of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ as *box diagrams*:[4]

$$\boldsymbol{x} = \langle \langle 7, 8, 11 \rangle, \langle 7, 11, 16 \rangle; \langle 4, 10 \rangle; \langle 1, 7, 10, 19 \rangle \rangle \tag{4.7}$$

$$\boldsymbol{y} = \langle \langle 5, 13, 16 \rangle, \langle 5, 13, 18 \rangle; \langle 3, 21 \rangle; \langle 8, 10, 14, 17 \rangle \rangle \tag{4.8}$$

$$\boldsymbol{z} = \langle \langle 3, 4, 6 \rangle, \langle 3, 6, 7 \rangle; \langle 2, 5 \rangle; \langle 1, 3, 5, 8 \rangle \rangle \tag{4.9}$$

[4] The semicolons in eqs. (4.7) to (4.9) serve the same purpose as in eq. (4.6).



(a) Box diagram for $\boldsymbol{x}$ from eq. (4.7)

(b) Box diagram for $\boldsymbol{y}$ from eq. (4.8)

(c) Box diagram for $\boldsymbol{z}$ from eq. (4.9)

(d) Shape of the box diagram for $\boldsymbol{y}$

Figure 4.2: Box diagrams for the tuples from eqs. (4.7) to (4.9) on the preceding page

For instance, the box diagram for $\boldsymbol{x}$ in fig. 4.2(a) is obtained as follows: For all indices k, i, the row labeled by ${}_{ki}$ contains precisely n_k boxes with numbers $x_{ki}(1), \ldots, x_{ki}(n_k)$ in them. In each column, all boxes contain the same number and these numbers are strictly increasing from left to right. The dotted lines indicate a change in the first index of ${}_{ki}$, i.e., they serve the same purpose as the semicolons in eq. (4.6) and eqs. (4.7) to (4.9). The box diagrams in figs. 4.2(b) and 4.2(c) are obtained from $\boldsymbol{y}$ and $\boldsymbol{z}$ analogously.

Sometimes, we are not interested in a complete box diagram but only in its *shape*, i.e., the arrangement of its boxes without the numbers. We refer to such diagrams as *box diagram shapes*. For instance, the shape of the box diagram for $\boldsymbol{y}$ is the box diagram shape given in fig. 4.2(d). Finally, notice that the box diagrams of $\boldsymbol{x}$ and $\boldsymbol{z}$ have the same shape. $\square$

Clearly, for all choices of s, $\boldsymbol{n}$ and $\boldsymbol{r}$, any $\boldsymbol{x} \in [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ can be depicted by a box diagram in this way. Recall the condition on $\boldsymbol{x}$ that $x_{ki} < x_{kj}$ whenever $i < j$. This translates in the following condition on the shape of the box diagram for $\boldsymbol{x}$: There is a column in which precisely one of the rows ${}_{ki}$ and ${}_{kj}$ contains a box and in the first such column it is row ${}_{ki}$ which contains the box. Conversely, if you take a diagram of boxes without numbers which satisfies this condition for all indices k, i and k, j and fill the columns with strictly increasing numbers, you obtain the box diagram for some element of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$. Throughout the remainder of this chapter, the relation of having box diagrams with the same shape plays a very important role. We capture this by means of the following equivalence relation on $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$.

Definition 4.4.2. Two tuples $\boldsymbol{x}, \boldsymbol{y} \in [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ are *similar* if all indices k, i, μ and ℓ, j, ν satisfy the following equivalence:

$$x_{ki}(\mu) < x_{\ell j}(\nu) \quad \Longleftrightarrow \quad y_{ki}(\mu) < y_{\ell j}(\nu)\,.$$

Put another way, $\boldsymbol{x}$ and $\boldsymbol{y}$ are similar if and only if mapping $x_{ki}(\mu)$ to $y_{ki}(\mu)$ defines an order-preserving bijection between the entry sets of $\boldsymbol{x}$ and $\boldsymbol{y}$. With this bijection in mind, one easily sees that $\boldsymbol{x}$ and $\boldsymbol{y}$ are similar if and only if their box diagrams have the same shape. Accordingly, box diagram shapes represent similarity classes just like box diagrams represent specific elements of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$. Since every box diagram shape contains precisely $n_1r_1 + \cdots + n_sr_s$ boxes, there are only finitely many similarity classes in $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$.

Example 4.4.1 (continuing)**.** The tuples $\boldsymbol{x}$ and $\boldsymbol{z}$ are similar to each other but not similar to $\boldsymbol{y}$. □

Now, fix some $\boldsymbol{x} \in [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ and let m be the size of its entry set. This size satisfies $m \leqslant n_1r_1 + \ldots + n_sr_s$ and the entry set of any element similar to $\boldsymbol{x}$ has size m as well. Accordingly, we also call m the *entry set size* of the similarity class of $\boldsymbol{x}$. Moreover, for every subset $M \subseteq \mathbb{N}_+$ of size m, there is precisely one element $\boldsymbol{y}$ in the similarity class of $\boldsymbol{x}$ whose entry set is M. In fact, the box diagram for $\boldsymbol{y}$ is obtained from the shape of the box diagram for $\boldsymbol{x}$ by inserting the elements of M in increasing order. This observation justifies the subsequent definition.

Definition 4.4.3. The *least element* of a similarity class in $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ is the unique element therein whose entry set is precisely $\{1, \ldots, m\}$, where m is the entry set size of the class.

Example 4.4.1 (continuing)**.** The least element of the similarity class of $\boldsymbol{x}$ is $\boldsymbol{z}$. □

Finally, consider some $\boldsymbol{x} \in [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ and the least element $\boldsymbol{z}$ of its similarity class. The bijection mapping each $z_{ki}(\mu)$ to $x_{ki}(\mu)$ is in fact an element of $\mathcal{W}(m)$, for m the size of the entry set of $\boldsymbol{x}$. Say $a \in \mathcal{W}(m)$ is this bijection, then

$$x_{ki}(\mu) = a(z_{ki}(\mu)) \tag{4.10}$$

for all indices k, i, μ. This motivates the following notation: Let $N \in \mathbb{N}$, $a \in \mathcal{W}(N)$ and $\boldsymbol{z} \in [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ be such that $z_{ki}(\mu) \leqslant N$ for all indices k, i, μ. We define a tuple $a(\boldsymbol{z}) \in [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ by

$$(a(\boldsymbol{z}))_{ki}(\mu) := a(z_{ki}(\mu))\,.$$

Using this notation, eq. (4.10) can be rephrased as $\boldsymbol{x} = a(\boldsymbol{z})$.

Definition 4.4.4. Let X be a subset of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$. A partition $\varDelta$ of X is *canonical* if it is coarser than similarity, i.e., whenever $\boldsymbol{x}, \boldsymbol{y} \in X$ are similar, they belong to the same $\varDelta$-class.

Put another way, a partition $\varDelta$ of $X \subseteq [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ is canonical if the similarity class of any $\boldsymbol{x} \in X$ already determines its $\varDelta$-class. The existence of canonical subpartitions of any partition of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ is ensured by lemma 4.4.5 below, which extends [Wil77, theorem 7.2.7] beyond the special case $s = 1$ and $r_1 = 2$. For every subset $H \subseteq \mathbb{N}_+$, we define the tuple of sets

$$\mathcal{W}(\boldsymbol{n}) \cap H^{\boldsymbol{n}} := \langle \mathcal{W}(n_1) \cap H^{n_1}, \ldots, \mathcal{W}(n_s) \cap H^{n_s} \rangle\,.$$

Observe that $\mathcal{W}(\boldsymbol{n}) \cap H^{\boldsymbol{n}}$ has the same order type as $\mathcal{W}(\boldsymbol{n})$ if H is infinite.

Lemma 4.4.5 (canonicalization lemma)**.** *Let $\varDelta$ be a partition of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$. There is an infinite subset $H \subseteq \mathbb{N}_+$ such that the restriction of $\varDelta$ to $[\mathcal{W}(\boldsymbol{n}) \cap H^{\boldsymbol{n}}]^{\boldsymbol{r}}$ is canonical.*

Proof. Let $m := n_1 r_1 + \cdots + n_s r_s$ and $I := \{1, \ldots, m\}$. Observe that the set $[\mathcal{W}(\boldsymbol{n}) \cap I^{\boldsymbol{n}}]^{\boldsymbol{r}}$ is finite and all its elements $\boldsymbol{z}$ satisfy $z_{ki}(\mu) \leqslant m$ for all indices k, i, μ. In order to obtain the set H, we first construct a partition of $[\mathbb{N}_+]^m$. Notice that $[\mathbb{N}_+]^m = \mathcal{W}(m)$. We define an equivalence relation $\sim$ on $[\mathbb{N}_+]^m$ by $a \sim b$ if $a(\boldsymbol{z})$ and $b(\boldsymbol{z})$ belong to the same $\varDelta$-class for all $\boldsymbol{z} \in [\mathcal{W}(\boldsymbol{n}) \cap I^{\boldsymbol{n}}]^{\boldsymbol{r}}$. Since $\varDelta$ and $[\mathcal{W}(\boldsymbol{n}) \cap I^{\boldsymbol{n}}]^{\boldsymbol{r}}$ are finite, this equivalence relation induces a finite partition of $\mathcal{W}(m)$. According to theorem 4.1.3, there is an infinite subset $H \subseteq \mathbb{N}_+$ which is homogeneous wrt this partition, i.e., $[H]^m$ is contained in a single $\sim$-class. It remains to show that the restriction of Δ to $[\mathcal{W}(\boldsymbol{n}) \cap H^{\boldsymbol{n}}]^{\boldsymbol{r}}$ is canonical.

Consider $\boldsymbol{x}, \boldsymbol{y} \in [\mathcal{W}(\boldsymbol{n}) \cap H^{\boldsymbol{n}}]^{\boldsymbol{r}}$ which are similar and let $\boldsymbol{z}$ be the least element of their similarity class. Notice that $\boldsymbol{z} \in [\mathcal{W}(\boldsymbol{n}) \cap I^{\boldsymbol{n}}]^{\boldsymbol{r}}$. There are $a, b \in \mathcal{W}(m) \cap H^m$ such that $\boldsymbol{x} = a(\boldsymbol{z})$ and $\boldsymbol{y} = b(\boldsymbol{z})$. Since $\mathcal{W}(m) \cap H^m = [H]^m$, we have $a \sim b$ and hence $\boldsymbol{x}$ and $\boldsymbol{y}$ belong to the same Δ-class. $\square$

An immediate consequence of this lemma is the following polarized partition relation, where $\kappa \in \mathbb{N}$ is arbitrary and $S(\boldsymbol{n}; \boldsymbol{r})$ denotes the number of similarity classes in $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$:

$$\begin{pmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{pmatrix} \longrightarrow \begin{bmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{bmatrix}^{r_1, \ldots, r_s}_{\kappa, S(\boldsymbol{n}; \boldsymbol{r})}$$

Applying the positive polarization lemma 4.3.2 to these partition relations and the Cantor normal form of some ordinal $\alpha < \omega^\omega$ yields that r-ary Ramsey degree of α is finite for each $r \in \mathbb{N}$. However, the corresponding upper bound on this Ramsey degree is not optimal.

4.5 Simplification

In order to obtain optimal bounds, we further *simplify* the subpartitions obtained from the canonicalization lemma 4.4.5.

Definition 4.5.1. A tuple $\boldsymbol{x} \in [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ is called *p-simple*[5] if, for all indices k, i, μ and ℓ, j, ν, the premise $x_{ki}(\mu) = x_{\ell j}(\nu)$ implies $k = \ell$, $\mu = \nu$ and $x_{ki}(\xi) = x_{\ell j}(\xi)$ for each $\xi < \mu$. The number of similarity classes containing a p-simple element is denoted by $P(\boldsymbol{n}; \boldsymbol{r})$.

Example 4.4.1 (continuing)**.** The tuple $\boldsymbol{y}$ is p-simple, but $\boldsymbol{x}$ and $\boldsymbol{z}$ are not.

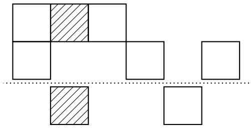
Observe that being p-simple is in fact a property of similarity classes: Whenever a similarity class contains some p-simple element, then all its elements are p-simple. Therefore, $P(\boldsymbol{n}; \boldsymbol{r})$ is just the number of p-simple similarity classes. Obviously, p-simplicity easily translates into a condition on box diagram shapes. More precisely, this translation yields the three forbidden patterns which are depicted in fig. 4.3 on the next page. Accordingly, $P(\boldsymbol{n}; \boldsymbol{r})$ can be computed from $\boldsymbol{n}$ and $\boldsymbol{r}$ by counting the number of box diagram shapes which do not match any of these patterns.

Recall that the canonicalization lemma 4.4.5 states that every partition $\varDelta$ of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ admits an infinite subset $H \subseteq \mathbb{N}_+$ such that the restriction of $\varDelta$ to $[\mathcal{W}(\boldsymbol{n}) \cap H^{\boldsymbol{n}}]^{\boldsymbol{r}}$ is canonical. In partitions of this latter form, non-p-simplicity can be avoided in the the following sense:

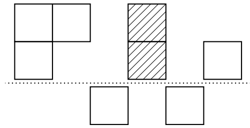
Lemma 4.5.2 (positive simplification lemma)**.** *Let $H \subseteq \mathbb{N}_+$ be an infinite subset. There is a type $\langle \omega^{n_1}, \ldots, \omega^{n_s} \rangle$ tuple of subsets $\boldsymbol{U} \subseteq \mathcal{W}(\boldsymbol{n}) \cap H^{\boldsymbol{n}}$ such that all tuples in $[\boldsymbol{U}]^{\boldsymbol{r}}$ are p-simple.*

Proof. Let $\{G_1, \ldots, G_s\}$ be an arbitrary partition of H consisting entirely of infinite sets. As a first step, we construct the sets U_k.

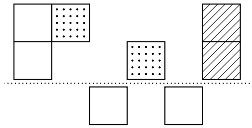
[5]The "p" stands for "prefix": For $s = 1$ and $r_1 = 2$, the shape of the box diagram for any $\boldsymbol{x} \in [\mathcal{W}(n)]^2$ can be regarded as a string over the alphabet $\{$▫, ▫, ⊟$\}$. Then $\boldsymbol{x}$ is p-simple if and only if the ⊟-symbols form a *prefix* of this string.



(a) $k \neq \ell$



(b) $\mu \neq \nu$



(c) $\exists \xi < \mu \colon x_{ki}(\xi) \neq x_{\ell j}(\xi)$

Figure 4.3: The three reasons for non-p-simplicity (the shaded boxes shall contain $x_{ki}(\mu)$ and $x_{\ell j}(\nu)$ whereas the dotted boxes shall contain $x_{ki}(\xi)$ and $x_{\ell j}(\xi)$)

For this purpose, fix some index k and let $p_1, p_2, \dots$ be an arbitrary enumeration of all prime numbers. We put

$$P_k = \left\{ p_1^{a(1)}\, p_2^{a(2)} \cdots p_\mu^{a(\mu)} \;\middle|\; \mu \in [1, n_k], a(1), \dots, a(\mu) \in \mathbb{N}_+ \right\}.$$

Since both P_k and G_k are infinite subsets of $\mathbb{N}_+$, there exists an order-preserving bijection $g_k \colon P_k \to G_k$. We define a map $f_k \colon \mathbb{N}_+^{n_k} \to \mathcal{W}(n_k) \cap G_k^{n_k}$ by

$$(f_k(a))(\mu) := g_k\Big(p_1^{a(1)}\, p_2^{a(2)} \cdots p_\mu^{a(\mu)}\Big).$$

It is a matter of routine to check that f_k is order-preserving as well. Thus, the set

$$U_k := f_k(\mathbb{N}_+^{n_k})$$

has order type ω^{n_k}. We conclude the proof by showing that every tuple $\boldsymbol{x} \in [\boldsymbol{U}]^r$ is p-simple.

Consider indices k, i, μ and ℓ, j, ν with $x_{ki}(\mu) = x_{\ell j}(\nu)$. Since $x_{ki}(\mu) \in G_k$ and $x_{\ell j}(\nu) \in G_\ell$, we conclude $k = \ell$. According to the choice of U_k, there are $a, b \in \mathbb{N}_+^{n_k}$ such that $x_{ki} = f_k(a)$ and $x_{\ell j} = f_k(b)$. Since g_k is a bijection, we obtain

$$p_1^{a(1)} \cdots p_\mu^{a(\mu)} = g_k^{-1}(x_{ki}(\mu)) = g_k^{-1}(x_{\ell j}(\nu)) = p_1^{b(1)} \cdots p_\nu^{b(\nu)} .$$

Due to the unique-prime-factorization theorem, we obtain $\mu = \nu$ and $a(\xi) = b(\xi)$ for each $\xi \leqslant \mu$. Consequently,

$$x_{ki}(\xi) = g_k\left(p_1^{a(1)} \cdots p_\xi^{a(\xi)}\right) = g_k\left(p_1^{b(1)} \cdots p_\xi^{b(\xi)}\right) = x_{\ell j}(\xi) .$$

This verifies the conditions of definition 4.5.1. $\square$

Composing the canonicalization lemma 4.4.5 with the positive simplification lemma 4.5.2 yields a polarized partition relation, which turns out to be optimal in theorem 4.6.5.

Theorem 4.5.3. *For all $s, \kappa \in \mathbb{N}$ and $\boldsymbol{n}, \boldsymbol{r} \in \mathbb{N}^s$, the following holds:*

$$\begin{pmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{pmatrix} \longrightarrow \begin{bmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{bmatrix}^{r_1, \ldots, r_s}_{\kappa, P(\boldsymbol{n}; \boldsymbol{r})}$$

Proof. Let Δ be a partition of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ into κ classes. Due to the canonicalization lemma 4.4.5, there is an infinite subset $H \subseteq \mathbb{N}_+$ such that the restriction of Δ to $[\mathcal{W}(\boldsymbol{n}) \cap H^{\boldsymbol{n}}]^{\boldsymbol{r}}$ is canonical. Applying the simplification lemma 4.5.2 to this restriction yields a type $\langle \omega^{n_1}, \ldots, \omega^{n_s} \rangle$ tuple of subsets $\boldsymbol{U} \subseteq \mathcal{W}(\boldsymbol{n}) \cap H^{\boldsymbol{n}}$ such that all elements of $[\boldsymbol{U}]^{\boldsymbol{r}}$ are p-simple. Since the restriction of Δ to $[\boldsymbol{U}]^{\boldsymbol{r}}$ is still canonical, the tuple $\boldsymbol{U}$ is relatively $P(\boldsymbol{n}; \boldsymbol{r})$-homogeneous wrt Δ. $\square$

Putting the positive polarization lemma 4.3.2 and the polarized partition relations from the previous theorem together, we obtain that the r-ary Ramsey degree of α is finite for each $r \in \mathbb{N}$ and all ordinals $\alpha < \omega^\omega$. If $\alpha = \omega^{n_1} + \cdots + \omega^{n_s}$ is the Cantor normal form of α, then this Ramsey degree is bounded from above by

$$\lambda(\alpha; r) := \sum_{\substack{\tilde{r}_1,\ldots,\tilde{r}_s \in \mathbb{N} \\ \tilde{r}_1+\cdots+\tilde{r}_s = r}} P(n_1, \ldots, n_s; \tilde{r}_1, \ldots, \tilde{r}_s)\,.$$

Theorem 4.5.4. *For all $r, \kappa \in \mathbb{N}$ and every ordinal $\alpha < \omega^\omega$, we have*

$$\alpha \longrightarrow [\alpha]^r_{\kappa,\lambda(\alpha;r)}\,. \qquad \square$$

4.6 Exact Values of Ramsey Degrees

The purpose of this section is to prove that the upper bounds on Ramsey degrees just given in theorem 4.5.4 are optimal. In line with lemma 4.1.4, we hence show in theorem 4.6.6 the negated partition relation

$$\alpha \not\longrightarrow [\alpha]^r_{\lambda(\alpha;r)} \tag{4.11}$$

for all ordinals $\alpha < \omega^\omega$ and $r \in \mathbb{N}$. According to the negative polarization lemma 4.3.5, this amounts to establishing that the polarized partition relations in theorem 4.5.3 are optimal. To this end, we show that the positive simplification lemma 4.5.2 is the best one can achieve in general. In the course of doing so, we use a simple characterization of type ω^n subsets of $\mathcal{W}(n)$ in terms of *free components*, which is taken from [Wil77, section 7.2].

Definition 4.6.1. Let $n \in \mathbb{N}$ and $\mu \in [1, n]$. A subset $U \subseteq \mathcal{W}(n)$ is *free in the μ^{th} component* if for all $a \in U$ and $m \in \mathbb{N}$ there is $b \in U$ with $b(\mu) > m$ and $b(\xi) = a(\xi)$ for each $\xi < \mu$.

Example 4.6.2. Let $n = 3$. The set

$$U := \left\{ a \in \mathcal{W}(3) \;\middle|\; \begin{array}{l} a(1) \text{ is a prime, } a(2) \geqslant 2^{a(1)} \text{ and} \\ a(3) < a(1) + 2 \cdot a(2) \end{array} \right\}$$

is free in the first two components but not free in the last component: Given arbitrary $a \in U$ and $m \in \mathbb{N}$, the elements

$$b_1 = \langle p, 2^p, 2^p + 1 \rangle \in U$$

and

$$b_2 = \langle a(1), 2^{a(1)} + m, 2^{a(1)} + m + 1 \rangle \in U\,,$$

where p is some prime with $p > m$, verify freedom in the 1^{st} and 2^{nd} component, respectively. To see that U is not free in the 3^{rd} component, consider $a = \langle 3, 11, 20 \rangle \in U$ and $m = 25$. Obviously, there is no $b \in U$ which satisfies $b(1) = a(1)$, $b(2) = a(2)$ and $b(3) > m$ at the same time. $\square$

Lemma 4.6.3 ([Wil77])**.** *Let $m \leqslant n$ and $U \subseteq \mathcal{W}(n)$. The order type of U is at least ω^m if and only if there is a non-empty subset of U which is free in m different components.* $\square$

Recall that the positive simplification lemma 4.5.2 states that non-p-simplicity can be avoided in some sense. In the same sense, p-simplicity however cannot be avoided.

Lemma 4.6.4 (negative simplification lemma)**.** *For all type $\langle \omega^{n_1}, \ldots, \omega^{n_k} \rangle$ tuples of subsets $\boldsymbol{U} \subseteq \mathcal{W}(\boldsymbol{n})$, the set $[\boldsymbol{U}]^{\boldsymbol{r}}$ intersects every p-simple similarity class.*

Proof. The basic proof idea is as follows: We consider the box diagram shape representing some p-simple similarity class and fill its columns from left to right with numbers in such a way that we end up with a box diagram for some element of $\boldsymbol{U}$.

To this end, we fix the least element $\boldsymbol{z}$ of an arbitrary p-simple similarity class in $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$. Recall that the entry set of $\boldsymbol{z}$ is of the form $\{1,\ldots,m\}$. According to lemma 4.6.3, there is a tuple of subsets $\boldsymbol{V} \subseteq \boldsymbol{U}$ such that each V_k is non-empty and free in all n_k components. Our goal is to construct $a \in \mathcal{W}(m)$ such that the tuple $a(\boldsymbol{z})$, which is similar to $\boldsymbol{z}$, is contained in $[\boldsymbol{V}]^{\boldsymbol{r}}$ and hence also in $[\boldsymbol{U}]^{\boldsymbol{r}}$. Intuitively, $a(t)$ is just the number we fill into the t^{th} column of the box diagram shape for $\boldsymbol{z}$. Hence, this filling leads to the box diagram for $a(\boldsymbol{z})$.

We construct $a \in \mathcal{W}(m)$ inductively in m steps. In step $t \in [1,m]$, we choose $a(t)$ such that the following invariant is preserved:

($\star$) For all indices k, i, there is some $b \in V_k$ such that the equality $b(\mu) = a(z_{ki}(\mu))$ holds true for all μ with $z_{ki}(\mu) \leqslant t$.

For $t = m$, this condition just says $a(\boldsymbol{z}) \in [\boldsymbol{V}]^{\boldsymbol{r}}$, which proves the claim in the end. For the sake of technical convenience, we add the artificial base case $t = 0$.

Base case. Following our convention, we put $a(0) := 0$. The invariant ($\star$) is then trivially satisfied for $t = 0$ because the sets V_k are non-empty.

Inductive step. Let $t \in [1,m]$ be the number of the current step. Let ℓ, j, ν be indices such that $t = z_{\ell j}(\nu)$. Due to the induction hypothesis, there is some $c \in V_k$ such that $c(\xi) = a(z_{\ell j}(\xi))$ for each $\xi < \nu$. Since V_k is free in the ν^{th} component, there is $d \in V_k$ such that $d(\nu) > a(t-1)$ and $d(\xi) = c(\xi)$ for each $\xi < \nu$. We choose $a(t) := d(\nu)$.

In order to verify that this choice of $a(t)$ preserves the invariant ($\star$), consider indices k, i. We have to find some $b \in V_k$ such that $b(\mu) = a(z_{ki}(\mu))$ for all μ with $z_{ki}(\mu) \leqslant t$. If there is no μ with $z_{ki}(\mu) = t$, the induction hypothesis yields the required b. Henceforth, assume there is μ with $z_{ki}(\mu) = t$. Thus, $z_{ki}(\mu) = z_{\ell j}(\nu)$. Since $\boldsymbol{z}$ is p-simple, we conclude $k = \ell$, $\mu = \nu$ and $z_{ki}(\xi) = z_{\ell j}(\xi)$ for each $\xi \leqslant \mu$. It is a matter of routine to check that $b := d$ is a suitable choice for b. □

The announced negated polarized partition relation is as follows:

Theorem 4.6.5. *For all $s \in \mathbb{N}$ and $\boldsymbol{n}, \boldsymbol{r} \in \mathbb{N}^s$, the following holds:*

$$\begin{pmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{pmatrix} \not\longrightarrow \begin{bmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{bmatrix}^{r_1,\ldots,r_s}_{P(\boldsymbol{n};\boldsymbol{r})}$$

Proof. Let Δ be an arbitrary canonical partition of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ into $P(\boldsymbol{n};\boldsymbol{r})$ classes such that no two p-simple similarity classes fall into the same Δ-class. Consequently, each Δ-class contains precisely one p-simple similarity class. Applying the negative simplification lemma 4.6.4 hence yields that every type $\langle \omega^{n_1}, \ldots, \omega^{n_s} \rangle$ tuple of subsets of $\mathcal{W}(\boldsymbol{n})$ is completely inhomogeneous wrt Δ. □

Applying the negative polarization lemma 4.3.5 to the polarized partition relations just shown yields that theorem 4.5.4 is indeed optimal.

Theorem 4.6.6. *For all ordinals $\alpha < \omega^\omega$ and $r \in \mathbb{N}$, we have*

$$\alpha \not\longrightarrow [\alpha]^r_{\lambda(\alpha;r)} \,. \qquad \square$$

Using lemma 4.1.4, we summarize theorems 4.5.4 and 4.6.6 in terms of the Ramsey degree.

Theorem 4.6.7. *Let $r \in \mathbb{N}$ and $\alpha < \omega^\omega$ be an ordinal. The r-ary Ramsey degree of α is finite and its exact value is given by*

$$\sum_{\substack{\tilde{r}_1,\dots,\tilde{r}_s \in \mathbb{N} \\ \tilde{r}_1+\cdots+\tilde{r}_s = r}} P(n_1,\dots,n_s;\tilde{r}_1,\dots,\tilde{r}_s)\,,$$

provided that $\alpha = \omega^{n_1} + \cdots + \omega^{n_s}$ is the Cantor normal form of α. $\square$

Recall that the numbers $P(n_1,\dots,n_s;\tilde{r}_1,\dots,\tilde{r}_s)$ can be obtained by counting the number of box diagram shapes which belong to p-simple similarity classes. Hence, the r-ary Ramsey degree of $\alpha < \omega^\omega$ is computable from the Cantor normal form of α. We conclude this section by sketching the according calculations for $r = 2$.

Corollary 4.6.8 ([HL13])**.** *Let $\alpha < \omega^\omega$ be an ordinal. The binary Ramsey degree of α is finite and its exact value is given by*

$$\sum_{1\leqslant k\leqslant s}\ \sum_{1\leqslant t\leqslant n_k} \binom{2t-1}{t} + \sum_{1\leqslant k<\ell\leqslant s} \binom{n_k+n_\ell}{n_k},$$

provided that $\alpha = \omega^{n_1} + \cdots + \omega^{n_s}$ is the Cantor normal form of α.

Proof sketch. According to theorem 4.6.7, we only have to determine the values of $P(n_1,\dots,n_s;\tilde{r}_1,\dots,\tilde{r}_s)$ under the assumption $\tilde{r}_1 + \cdots + \tilde{r}_s = 2$. To this end, we regard box diagram shapes representing similarity classes in $[\mathcal{W}(\boldsymbol{n})]^{\tilde{r}}$ as strings over the alphabet $\{{}_{\scriptscriptstyle\square}, {}^{\scriptscriptstyle\square}, \boxminus\}$. We distinguish two cases:

Case 1: There is k with $\tilde{r}_k = 2$. A string w over $\{{}_{\scriptscriptstyle\square}, {}^{\scriptscriptstyle\square}, \boxminus\}$ is a box diagram shape precisely if the following three conditions are satisfied:

(1) $|w|_{{}_{\scriptscriptstyle\square}} + |w|_{\boxminus} = |w|_{{}^{\scriptscriptstyle\square}} + |w|_{\boxminus} = n_k$,

(2) w contains a symbol other than $\boxminus$ and

(3) the first symbol of w different from $\boxminus$ is a $^{\square}$-symbol.

Obviously and as already mentioned in footnote 5 on page 147, w represents a p-simple similarity class precisely if the $\boxminus$-symbols in w form a prefix of w. There are no further restrictions implied on w. Thus, for each $t \in [1, n_k]$, there are precisely $\binom{2t-1}{t}$ strings of this kind whose $\boxminus$-prefix has length $n_k - t$. In total, we obtain

$$P(n_1, \ldots, n_s; \tilde{r}_1, \ldots, \tilde{r}_s) = \sum_{1 \leqslant t \leqslant n_k} \binom{2t-1}{t}.$$

Case 2: There are k and ℓ with $k < \ell$ and $\tilde{r}_k = \tilde{r}_\ell = 1$. This time, a string w is a box diagram shape if and only if $|w|_{^{\square}} + |w|_{\boxminus} = n_k$ and $|w|_{\,_{\square}} + |w|_{\boxminus} = n_\ell$. The represented similarity class is p-simple precisely if w does not contain any $\boxminus$-symbols at all. Accordingly,

$$P(n_1, \ldots, n_s; \tilde{r}_1, \ldots, \tilde{r}_s) = \binom{n_k + n_\ell}{n_k}.$$

Adding all these values yields the claim. $\square$

4.7 Infinite Ramsey Degrees

We complete this chapter by demonstrating that the r-ary Ramsey degree of α is infinite whenever $\omega^\omega \leqslant \alpha < \omega^{\omega^2}$ and $r \geqslant 2$. We accomplish this objective by means of corollary 4.7.8, which basically establishes the negated partition relation

$$\alpha \not\longrightarrow [\alpha]^r_\kappa$$

for all $\kappa \in \mathbb{N}$. As a first step, we show that we can focus on the case $r = 2$.

Lemma 4.7.1. *Let $r, \kappa, \lambda \in \mathbb{N}$ and α, β be* infinite *ordinals. If $r \geqslant 2$ and*

$$\alpha \longrightarrow [\beta]^r_{\kappa,\lambda}\,,$$

then

$$\alpha \longrightarrow [\beta]^2_{\kappa,\lambda}\,.$$

Proof. Suppose the premise is satisfied. We consider a type α well-order A and a partition $\varDelta = \{D_1, \dots, D_\kappa\}$ of $[A]^2$. We define a partition $\varDelta' = \{D'_1, \dots, D'_\kappa\}$ of $[A]^r$ by

$$D'_i := \{\, \langle u_1, \dots, u_r \rangle \in [A]^r \mid \langle u_{r-1}, u_r \rangle \in D_i \,\}\,.$$

Due to the premise, there is a type β subset $X \subseteq A$, which is relatively λ-homogeneous wrt $\varDelta'$. Let $v_1 < \cdots < v_{r-2}$ be the $r-2$ smallest elements of X and put $Y := X \setminus \{v_1, \dots, v_{r-2}\}$. Since β is infinite, Y still has order type β. If $[Y]^2$ intersects some $\varDelta$-class D_i, say $\langle u_1, u_2 \rangle \in D_i \cap [Y]^2$, then $[X]^r$ intersects the $\varDelta'$-class D'_i, namely $\langle v_1, \dots, v_{r-2}, u_1, u_2 \rangle \in D'_i \cap [X]^r$. Consequently, Y is relatively λ-homogeneous wrt $\varDelta$. □

The key ingredient to the negated partition relations we established in the previous section was the negative simplification lemma 4.5.2. In a certain sense, it says that p-simplicity cannot be avoided. Here, we take a similar approach but restrict our attention only to certain p-simple similarity classes called *zigzags*. In the remainder of this section, $m, n \in \mathbb{N}$ are always numbers with $m \leqslant n$. The intuition behind the next definition is depicted in fig. 4.4 on the next page, where we omitted most of the vertical bars for the sake of visual clarity.

Definition 4.7.2. Let $k \in [1, m]$ and $\mu \in \mathcal{W}(m)$ with $\mu(m) \leqslant n$. A pair $\langle x, y \rangle$ in $[\mathcal{W}(n)]^2$ is a *μ-k-zigzag* if it satisfies the following three conditions, which conveniently use $\mu(0) = 0$ and $\mu(m+1) = n+1$:

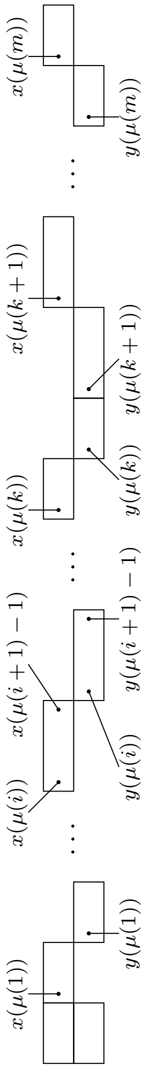


Figure 4.4: The general shape of the box diagram for some μ-k-zigzag

(1) $x(\xi) = y(\xi)$ for all $\xi < \mu(1)$.

(2) $x(\mu(i) - 1) < y(\mu(i))$ and $y(\mu(i) - 1) < x(\mu(i))$ for each i.

(3) $x(\mu(i+1) - 1) < y(\mu(i))$ for all $i \leqslant k$.

(4) $y(\mu(i+1) - 1) < x(\mu(i))$ for all $i > k$.

In view of fig. 4.4, it is almost immediate that the set of μ-k-zigzags forms a p-simple similarity class.[6] Observe that conditions (3) and (4) along with the monotonicity of x, y and μ imply the following two conditions (5) and (6), respectively:

(5) $x(\mu(i)) < y(\mu(i))$ for $i \leqslant k$.

(6) $y(\mu(i)) < x(\mu(i))$ for $i > k$.

The presumed relationship $x < y$ is implicitly also contained in the conditions above: $x < y$ follows from $x(\mu(1)) < y(\mu(1))$ and $x(\xi) = y(\xi)$ for each $\xi < \mu(1)$.

Reasoning by means of box diagrams once more, one can easily see that for every μ-k-zigzag the values of μ and k are unique. Since the proofs to follow rely on this uniqueness, we provide a proof in terms of definition 4.7.2.

Lemma 4.7.3. *Let $m \in \mathbb{N}$. Every pair in $[\mathcal{W}(n)]^2$ is a μ-k-zigzag for at most one choice of $\mu \in \mathcal{W}(m)$ and $k \in [1, m]$.*

Proof. Suppose that $\langle x, y \rangle \in [\mathcal{W}(n)]^2$ is a μ-k-zigzag as well as a ν-ℓ-zigzag for some $\mu, \nu \in \mathcal{W}(m)$ and $k, \ell \in [1, m]$. We show that $\mu = \nu$ and $k = \ell$.

Aiming for a contradiction, assume that $\mu \neq \nu$. Let $i \in [1, m]$ be minimal with $\mu(i) \neq \nu(i)$. Without loss of generality, we assume $\mu(i) > \nu(i)$. If $i = 1$, we have the self-contradictory inequality

$$x(\nu(1)) \overset{(a)}{<} y(\nu(1)) \overset{(b)}{=} x(\nu(1))$$

[6] We refrain from proving this since it does not matter for the correctness of the proofs to follow but is only mentioned for reasons of intuition.

with the following justifications: (a) $\langle x, y\rangle$ is a ν-ℓ-zigzag and $1 \leqslant \ell$; (b) $\langle x, y\rangle$ is a μ-k-zigzag and $\nu(1) < \mu(1)$. If $1 < i \leqslant k+1$, we conclude

$$x(\nu(i)) \overset{(a)}{\leqslant} x(\mu(i)-1) \overset{(b)}{<} y(\mu(i-1)) \overset{(c)}{=} y(\nu(i-1)) \overset{(d)}{\leqslant} y(\nu(i)-1) \overset{(e)}{<} x(\nu(i))$$

using the following arguments: (a) $\nu(i) < \mu(i)$ and x is monotonic; (b) $\langle x, y\rangle$ is a μ-k-zigzag and $i-1 \leqslant k$; (c) $\mu(i-1) = \nu(i-1)$; (d) $\nu(i-1) \leqslant \nu(i)-1$ and y is monotonic; (e) $\langle x, y\rangle$ is a ν-ℓ-zigzag. Clearly, this is also a contradiction. Finally, the case $i > k+1$ is symmetric to the case $1 < i \leqslant k+1$, the only difference is that x and y are interchanged. More precisely, we obtain the contradiction

$$y(\nu(i)) \overset{(a)}{\leqslant} y(\mu(i)-1) \overset{(b)}{<} x(\mu(i-1)) \overset{(c)}{=} x(\nu(i-1)) \overset{(d)}{\leqslant} x(\nu(i)-1) \overset{(e)}{<} y(\nu(i))$$

by the following justifications: (a) $\nu(i) < \mu(i)$ and y is monotonic; (b) $\langle x, y\rangle$ is a μ-k-zigzag and $i-1 > k$; (c) $\mu(i-1) = \nu(i-1)$; (d) $\nu(i-1) \leqslant \nu(i)-1$ and x is monotonic; (e) $\langle x, y\rangle$ is a ν-ℓ-zigzag.

So far, we have shown $\mu = \nu$. Aiming for another contradiction, suppose $k < \ell$. On the one hand, since $\langle x, y\rangle$ is a μ-k-zigzag, $k+1 > k$ and condition (6) imply $y(\mu(k+1)) < x(\mu(k+1))$. On the other hand, since $\langle x, y\rangle$ is also a μ-ℓ-zigzag, $k+1 \leqslant \ell$ and condition (5) imply $x(\mu(k+1)) < y(\mu(k+1))$. Obviously, this is a contradiction. $\square$

The subsequent lemma[7] is an analogue for zigzags of the negative simplification lemma 4.6.4 and states that zigzags cannot be avoided in some sense.

[7] We would have called it "zigzag lemma" if that name were not in use already.

Lemma 4.7.4. *For all type ω^m subsets $U \subseteq \mathcal{W}(n)$ and $k \in [1, m]$, there exists $\mu \in \mathcal{W}(m)$ such that $[U]^2$ contains a μ-k-zigzag.*

Proof. According to lemma 4.6.3, there are a non-empty subset $V \subseteq U$ and a tuple $\mu \in \mathcal{W}(m)$ with $\mu(m) \leqslant n$ such that V is free in the $\mu(i)^{\text{th}}$ component for each $i \in [1, m]$. Like in definition 4.7.2, we conveniently use $\mu(0) = 0$ and $\mu(m+1) = n+1$.

Basically, we now take the same approach as in the proof of lemma 4.6.4: We consider the box diagram shape representing the similarity class of μ-k-zigzags and fill its columns from left to right with numbers in such a way that we end up with a box diagram for some element of $[U]^2$. This time however, we do not fill the boxes column by column but in blocks as indicated in fig. 4.4 on page 157.

To this end, we inductively construct a μ-k-zigzag $\langle x, y\rangle \in [V]^2$ in $m+1$ steps. In step $i \in [0, m]$, we choose $x(\xi)$ and $y(\xi)$ for $\mu(i) \leqslant \xi < \mu(i+1)$ such that the following invariant is preserved:

($\star$) There are $a, b \in V$ such that $a(\xi) = x(\xi)$ and $b(\xi) = y(\xi)$ for all $\xi < \mu(i+1)$.

For $i = m$, this condition simply says $x, y \in V$. In the end, this proves the claim.

Base case: $i = 0$. Since V is not empty, there exists some $a \in V$. We choose $x(\xi) := y(\xi) := a(\xi)$ for each $\xi < \mu(1)$. Clearly, this choice establishes the invariant ($\star$) for $i = 0$ and ensures condition (1) of $\langle x, y\rangle$ being a μ-k-zigzag.

Inductive step: $i > 0$. By the induction hypothesis, there are $a, b \in V$ such that $a(\xi) = x(\xi)$ and $b(\xi) = y(\xi)$ for all $\xi < \mu(i)$.

First, suppose that $i \leqslant k$. Since V is free in the $\mu(i)^{\text{th}}$ component, there are $c, d \in V$ such that $c(\mu(i)) > y(\mu(i)-1)$ and $d(\mu(i)) > c(\mu(i+1)-1)$ as well as $c(\xi) = a(\xi)$ and $d(\xi) = b(\xi)$

for each $\xi < \mu(i)$. We choose $x(\xi) := c(\xi)$ and $y(\xi) := d(\xi)$ for $\mu(i) \leqslant \xi < \mu(i+1)$. It is easy to check that this choice preserves the invariant $(\star)$ for c and d in place of a and b, respectively, and ensures conditions (2), (3) and (4) of $\langle x, y\rangle$ being a μ-k-zigzag.

Finally, the case $i > k$ can be treated with almost the same arguments. The only difference is that the requirements on $c(\mu(i))$ and $d(\mu(i))$ need to be changed to $d(\mu(i)) > x(\mu(i) - 1)$ and $c(\mu(i)) > d(\mu(i+1) - 1)$. □

The omnipresence of zigzags implies the subsequent negated partition relation.

Theorem 4.7.5. *For all $m, n \in \mathbb{N}$ with $m \leqslant n$, we have*

$$\omega^n \not\longrightarrow [\omega^m]^2_m \,.$$

Proof. By lemma 4.7.3, there is a partition $\Delta = \{D_1, \ldots, D_m\}$ of $[\mathcal{W}(n)]^2$ such that D_k contains all μ-k-zigzags for any $\mu \in \mathcal{W}(m)$. Applying lemma 4.7.4 yields that every type ω^m subset $U \subseteq \mathcal{W}(n)$ is completely inhomogeneous. □

Our next step towards corollary 4.7.8 is to compose infinitely many of the partition relations above into one partition relation on ω^ω or, more generally, on ω^γ for $\omega \leqslant \gamma < \omega^2$. We accomplish this by means of the following lemma.

Lemma 4.7.6. *Let $m \in \mathbb{N}$ and β, ν, α_μ be ordinals for $\mu < \nu$. If*

$$\alpha_\mu \not\longrightarrow [\beta + 1]^2_m$$

for each $\mu < \nu$, then

$$\sum_{\mu<\nu} \alpha_\mu \not\longrightarrow [\beta\nu + 1]^2_m \,.$$

Proof. For each $\mu < \nu$, let A_μ be a type α_μ well-order and $\varDelta_\mu = \{D_{\mu 1}, \dots, D_{\mu m}\}$ a partition of $[A_\mu]^2$ exemplifying the presumed partition relation on α_μ, i.e., every type $\beta+1$ subset of A_μ is completely inhomogeneous wrt $\varDelta_\mu$. We consider the well-order

$$A := \sum_{\mu<\nu} A_\mu \,.$$

Notice that the sets $[A_\mu]^2$ are mutually disjoint subsets of $[A]^2$. Thus, there is a partition $\varGamma = \{C_1, \dots, C_m\}$ of $[A]^2$ such that $D_{\mu k} \subseteq C_k$ for all $k \in [1, m]$ and $\mu < \nu$. We conclude the proof by showing that every type $\beta\nu + 1$ subset $X \subseteq A$ is completely inhomogeneous wrt $\varGamma$.

For each $\mu < \nu$, let β_μ be the order type of $X \cap A_n$. Then

$$\sum_{\mu<\nu} \beta_\mu = \beta\nu + 1 \,. \tag{4.12}$$

We cannot have $\beta_\mu \leqslant \beta$ for all μ as this would contradict eq. (4.12). Put another way, there is some $\tilde{\mu} < \nu$ such that $\beta_{\tilde{\mu}} \geqslant \beta + 1$. Consider some arbitrary $k \in [1, m]$. Observe that

$$[X \cap A_{\tilde{\mu}}]^2 \cap D_{\tilde{\mu}k} \subseteq [X]^2 \cap C_k \,.$$

Due to the choice of $\varDelta_{\tilde{\mu}}$, the former set is non-empty and hence the latter set is non-empty as well. Consequently, X is completely inhomogeneous wrt $\varGamma$. □

Applying the lemma above to the partition relation in theorem 4.7.5 yields the following:

Theorem 4.7.7. *For every $m \in \mathbb{N}$ and all ordinals γ with $\omega \leqslant \gamma < \omega^2$, we have*

$$\omega^\gamma \not\rightarrow [\omega^\gamma]^2_m \,.$$

Proof. First, we apply lemma 4.7.6 to $\beta = \omega^m$, $\nu = \omega$ and $\alpha_\mu = \omega^{m+\mu}$ and obtain

$$\omega^\omega \not\to [\omega^{m+1} + 1]^2_m \,.$$

Let δ be such that $\gamma = \omega + \delta$. Applying lemma 4.7.6 to $\beta = \omega^{m+1}$, $\nu = \omega^\delta$ and $\alpha_\mu = \omega^\omega$ yields

$$\omega^\gamma \not\to [\omega^{m+1+\delta} + 1]^2_m \,.$$

Since $\gamma < \omega^2$, we have $m + 1 + \delta < \gamma$ and hence $\omega^{m+1+\delta} + 1 < \omega^\gamma$. Due to the monotonicity of the partition relation, this implies the claim. □

As the Cantor normal form of any ordinal α with $\omega^\omega \leqslant \alpha < \omega^{\omega^2}$ contains a summand ω^γ with $\omega \leqslant \gamma < \omega^2$, theorem 4.7.7 along with lemma 4.3.4 of the negative polarization lemma immediately imply the desired partition relation:

Corollary 4.7.8. *For every $m \in \mathbb{N}$ and all ordinals α with $\omega^\omega \leqslant \alpha < \omega^{\omega^2}$, we have*

$$\alpha \not\to [\alpha]^2_m \,. \qquad \square$$

Putting together lemma 4.7.1 and corollary 4.7.8 and expressing the result in terms of the Ramsey degree, we obtain:

Theorem 4.7.9. *For all $r \geqslant 2$ and ordinals α with $\omega^\omega \leqslant \alpha < \omega^{\omega^2}$, the r-ary Ramsey degree of α is infinite.*

4.8 Open Problems

In view of the results in this chapter, several questions arise immediately. However, with Todorčević's result on ω_1 in mind [Tod87], it seems implausible that there are uncountable order types which possess a (finite or even countable) Ramsey degree. Concerning countable order types, there are basically two open problems:

(1) Are there countable ordinals other than those below ω^ω whose Ramsey degree is finite?

(2) Which countable scattered order types do have a finite Ramsey degree?

With regard to question (1), we particularly wonder whether the technique from the previous section can be extended to ω^{ω^2} and beyond. In the context of question (2), it might be interesting to study the following variation of the Ramsey degree: The *varied r-ary Ramsey degree* of a scattered order type τ is the least cardinal λ which admits another scattered order type τ' of the same VD_*-rank as τ such that $\tau \longrightarrow [\tau']^r_{\kappa,\lambda}$ for all $\kappa \in \mathbb{N}$. The varied r-ary Ramsey degree of an ordinal α with $\omega^n \leqslant \alpha < \omega^{n+1}$ then would coincide with the (non-varied) r-Ramsey degree of ω^n. Consequently, the varied Ramsey degree would be monotonic on ordinals $\alpha < \omega^\omega$; a feature the (non-varied) Ramsey degree regrettably lacks.

5 Automatic Ramsey Theory

As computer scientists, we are not satisfied by the mere *existence* of certain objects but we want to *compute* them. Regarding Ramsey's theorem, for instance, this urge can be expressed as follows: Suppose we are given a *finite presentation* of an infinite graph. How can we compute a *finite presentation* of a homogeneous infinite set of nodes? Is this even possible at all?

As a matter of fact, the answer is manifold and depends heavily on what exactly we do mean by the term "finite presentation". For example, we could mean "presentation by Turing machines". Unfortunately, the answer is negative in this case. More precisely, there is a computable graph which contains no computably enumerable homogeneous infinite set of nodes [Spe71]. Although there might even be no homogeneous infinite subset from $\mathbf{\Sigma}^0_2$, there is always one from $\mathbf{\Pi}^0_2$ [Joc72].

In contrast, the situation is a lot better when "finite presentation" means "string-automatic presentation": Every string-automatic graph admits a regular homogeneous infinite subset and one can actually compute a string-automaton recognizing such a set from a string-automatic presentation of the graph [Rub08]. For automatic presentations using finite automata on other input structures than strings, the situation is more complicated. Although every ω-string-automatic uncountable graph admits a homogeneous uncountable subset, there might be no ω-regular set of this kind [Kus11]. Surprisingly, the first part of this result is no longer valid for ternary hypergraphs [Kus11]. The best known result of this kind for tree-automatic (hyper)graphs is decidability of the existence of an infinite clique, i.e., an infinite complete subgraph [Kar11].

Put in one phrase, the main objective of this chapter is to figure out how much of the theory of Ramsey degrees from the previous chapter can be made effective in the context of automatic structures. To this end, we introduce the *automatic r-ary Ramsey degree* of an ordinal α. Due to the characterizations of automatically presentable ordinals in corollary 3.1.5 on page 62 and corollary 3.3.21 on page 99, this notion is only meaningful for ordinals $\alpha < \omega^{\omega^\omega}$. In addition, corollary 3.5.7 on page 108 implies that tree-automaticity is no more powerful than string-automaticity for presenting well-orders of types below ω^ω and partitions of hypergraphs thereon. Accordingly, we define the automatic Ramsey degree of an ordinal α in terms of *string-automatic partitions* if $\alpha < \omega^\omega$ and in terms of *tree-automatic partitions* if $\omega^\omega \leqslant \alpha < \omega^{\omega^\omega}$.

Our investigations of this automatic Ramsey degree lead to results which strongly resemble those on the (non-automatic) Ramsey degree. Furthermore, all claims on the existence of regular relatively homogeneous sets are *effective*, i.e., one can actually compute automata recognizing such sets. In more detail, the results are the following, the first three of which already appeared for $r = 2$ in [HL13]:

(1) The automatic r-ary Ramsey degree of every ordinal $\alpha < \omega^\omega$ is finite (theorem 5.3.9).

(2) The precise value of this Ramsey degree can be computed from r and the Cantor normal form of α (theorem 5.4.6).

(3) One can compute a string-automaton recognizing a relatively λ-homogeneous type α subset, for λ being this precise value of the Ramsey degree (corollary 5.4.7).

(4) The automatic r-ary Ramsey degree of α is infinite whenever $\omega^\omega \leqslant \alpha < \omega^{\omega^\omega}$ (theorem 5.5.5).

Concerning result (2), it turns out that the automatic r-Ramsey degree of an ordinal $\alpha < \omega^\omega$ always is at least as large as its non-automatic counterpart and in most cases even strictly larger.

Roughly speaking, the notable similarity between the results on non-automatic and automatic Ramsey degrees also carries over to the overall structure of the corresponding proofs. More precisely, the proof of the positive results (1) to (3) also employs the three steps *polarization*, *canonicalization* and *simplification*. Although the automatic versions of the two polarization lemmas are proved almost literally the same way as before, the canonicalization and simplification lemmas both require entirely new proofs. In addition, there is a fourth step, called *standardization*, which resolves one of the most fundamental differences between set-theoretic and automatic Ramsey theory: For any well-orders A and B of the same order type α and every partition Γ of $[A]^r$, there is a unique partition Δ of $[B]^r$ which is isomorphic to Γ. Consequently, we were allowed to freely choose the most suitable type α well-order for demonstrating a partition relation $\alpha \longrightarrow [\beta]^r_{\kappa,\lambda}$. We made extensive use of this freedom by choosing $\mathcal{W}(n)$ as our standard type ω^n well-order in section 4.4. However, the situation is *fundamentally* different in the context of automatic Ramsey theory: There are instances where A, B and Γ are automatic but Δ is *not*. Accordingly, we are no longer free to choose the automatic type α well-order being most easy to handle when investigating an automatic version of the partition relation $\alpha \longrightarrow [\beta]^r_{\kappa,\lambda}$. The sole purpose of the standardization step is to establish that we can still use $\mathcal{W}(n)$ as our standard type ω^n well-order nevertheless.

As a byproduct of these investigations, we obtain a new and quite simple proof of the string-automatic version of Ramsey's theorem, i.e., the fact that every string-automatic uniform hypergraph[1] effectively admits a regular homogeneous infinite subset. Compared to the results in [Rub08], this proof unfortunately has the disadvantage that it does not allow for deciding whether a given string-automatic hypergraph contains an infinite clique. However, its huge advantage is that it *easily* extends to tree-automatic hypergraphs. As a consequence, we obtained the following new results:

(5) Every tree-automatic hypergraph *effectively* admits a regular homogeneous infinite subset (theorem 5.6.8).

(6) It is decidable whether a given tree-automatic hypergraph contains a *regular* infinite clique (theorem 5.6.10).

Notice that the latter result differs from the one in [Kar11] only in the word "regular". Along with an example of a tree-automatic hypergraph containing an infinite clique but no regular infinite clique, this completes the picture on the tree-automatic version of Ramsey's theorem.

Outline. Just like in the non-automatic case, all our results on the automatic Ramsey degree are obtained in terms of (automatic variants of) partition relations. Along with the automatic Ramsey degree itself, these are introduced in section 5.1. The purpose of section 5.2 is to exhibit the aforementioned *standardization* step. The other three steps, namely *polarization*, *canonicalization* and *simplification*, are presented in section 5.3. The implied

[1] As we only deal with uniform hypergraphs, we omit the word "uniform" from now on.

upper bounds on automatic Ramsey degrees are matched by lower bounds in section 5.4. In section 5.5, we demonstrate that the automatic Ramsey degrees of all ordinals between ω^ω and ω^{ω^ω} are infinite. Finally, the announced tree-automatic version of Ramsey's theorem is presented in section 5.6.

5.1 The Automatic Ramsey Degree

Throughout this section, we define string-automatic and tree-automatic variants of several notions. In order to avoid notational overhead, we use the term *automatic linear order* generically to refer to a linear order which is either string-automatic or tree-automatic. Accordingly, symbols involving SA and TA refer to the string-automatic and the tree-automatic version, respectively. Before providing the mentioned definitions, we shortly discuss the notion of *automatic relations* in the context of Ramsey theory. To this end, suppose that A is an automatic linear order and $r \in \mathbb{N}$. Since any relation $D \subseteq [A]^r$ is just a set of tuples from A^r, being *automatic* is a well-defined property of D. Recall that we slightly deviated from the standard when defining $[A]^r$ as

$$[A]^r := \{ \langle u_1, u_2, \ldots, u_r \rangle \in A^r \mid u_1 < u_2 < \cdots < u_r \}$$

and not as the set of all subsets of A having size r. If we had decided in favor of this customary definition, it would seem natural to call a set E of such subsets of A *automatic* whenever the relation

$$\{ \langle u_1, u_2, \ldots, u_r \rangle \mid \{u_1, u_2 \ldots, u_r\} \in E \}$$

is automatic. As a matter of fact, there is no significant difference between these two possible definitions because a relation $D \subseteq [A]^r$ is automatic if and only if its *symmetric closure*

$$\{ \langle u_{i_1}, \ldots, u_{i_r} \rangle \mid \langle u_1, \ldots, u_r \rangle \in D, \{i_1, \ldots, i_r\} = \{1, \ldots, r\} \}$$

is automatic.

Definition 5.1.1. Let A be an automatic linear order and $r \in \mathbb{N}$. A partition Δ of $[A]^r$ is *automatic* if each Δ-class is automatic.

Definition 5.1.2. Let $\alpha, \beta < \omega^{\omega^\omega}$ be ordinals and $r, \kappa, \lambda \in \mathbb{N}$. The *automatic weak square bracket partition relations*

$$\alpha \xrightarrow{\mathsf{SA}} [\beta]^r_{\kappa,\lambda} \qquad \text{and} \qquad \alpha \xrightarrow{\mathsf{TA}} [\beta]^r_{\kappa,\lambda} \tag{5.1}$$

denote the following facts: For any automatic type α well-order A and every automatic partition Δ of $[A]^r$ into κ classes, there is a regular type β subset $X \subseteq A$ which is relatively λ-homogeneous wrt Δ.

First of all, notice that these partition relations possess the same monotonicity properties as the (non-automatic) weak square bracket partition relation. More precisely, the partition relations in eq. (5.1) remain true if we replace α by a larger ordinal (below ω^ω for $\xrightarrow{\mathsf{SA}}$ and below ω^{ω^ω} for $\xrightarrow{\mathsf{TA}}$), β by a smaller ordinal, κ by a smaller number or λ by a larger number. The *automatic ordinary partition relations*

$$\alpha \xrightarrow{\mathsf{SA}} (\beta)^r_{\kappa} \qquad \text{and} \qquad \alpha \xrightarrow{\mathsf{TA}} (\beta)^r_{\kappa}$$

capture the special case $\lambda = 1$ of eq. (5.1). Using this partition relation, the string-automatic version of Ramsey's theorem can be phrased as follows:

Theorem 5.1.3 (Rubin's theorem [Rub08]). *For all $r, \kappa \in \mathbb{N}$, we have*

$$\omega \xrightarrow{\mathsf{SA}} (\omega)^r_{\kappa} .$$

More precisely, given presentations of a string-automatic type ω well-order A and an automatic partition Δ of $[A]^r$ into κ classes, *one can compute a string-automaton recognizing a homogeneous infinite subset of A.*[2]

Similarly to their non-automatic versions, the *automatic square bracket partition relations*

$$\alpha \xrightarrow{\mathsf{SA}} [\beta]^r_\kappa \qquad \text{and} \qquad \alpha \xrightarrow{\mathsf{TA}} [\beta]^r_\kappa$$

refer to the special case $\lambda = \kappa - 1$. Once more, we are primarily interested in the negations

$$\alpha \overset{\mathsf{SA}}{\not\longrightarrow} [\beta]^r_\kappa \qquad \text{and} \qquad \alpha \overset{\mathsf{TA}}{\not\longrightarrow} [\beta]^r_\kappa$$

which denote the following facts: There are an automatic type α well-order A and an automatic partition Δ of $[A]^r$ into κ classes such that each regular type β subset of A is completely inhomogeneous wrt Δ.

Definition 5.1.4. Let $\alpha < \omega^{\omega^\omega}$ be an ordinal and $r \in \mathbb{N}$. The *automatic r-ary Ramsey degree* of α is the least cardinal λ such that, for all $\kappa \in \mathbb{N}$,

$$\alpha \xrightarrow{\mathsf{SA}} [\alpha]^r_{\kappa,\lambda}$$

if $\alpha < \omega^\omega$ and

$$\alpha \xrightarrow{\mathsf{TA}} [\alpha]^r_{\kappa,\lambda}$$

otherwise.

Just like for the non-automatic variant, every automatic Ramsey degree either is finite or equals $\aleph_0$. Analogously to lemma 4.1.4 on page 128, we have the following characterization which again lays down our strategy to obtain precise values of the automatic Ramsey degree.

[2] In fact, Rubin has proved a substantially stronger result, cf. theorem 5.6.1 on page 201 for details.

Lemma 5.1.5. *Let $\alpha < \omega^\omega$ be an ordinal and $r, \lambda \in \mathbb{N}$. If*

$$\alpha \xrightarrow{\mathsf{SA}} [\alpha]^r_{\kappa,\lambda} \qquad \text{and} \qquad \alpha \overset{\mathsf{SA}}{\not\longrightarrow} [\alpha]^r_\lambda$$

for all $\kappa \in \mathbb{N}$, then the automatic r-ary Ramsey degree of α is exactly λ. □

Owing to the fact that our positive result on the automatic Ramsey degrees of ordinals $\alpha < \omega^\omega$ is again based on a polarization step, we also define a string-automatic variant of the most general polarized partition relation.

Definition 5.1.6. Let $s, \kappa, \lambda \in \mathbb{N}$ be numbers, $\boldsymbol{r} \in \mathbb{N}^s$ and $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r < \omega^\omega$ ordinals. The *automatic polarized weak square bracket partition relation*

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_s \end{pmatrix} \xrightarrow{\mathsf{SA}} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_s \end{bmatrix}^{r_1,\dots,r_s}_{\kappa,\lambda}$$

denotes the following fact: For any type $\langle \alpha_1, \dots, \alpha_s \rangle$ tuple of string-automatic well-orders $\boldsymbol{A}$ and every automatic partition $\varDelta$ of $[\boldsymbol{A}]^{\boldsymbol{r}}$ into κ classes, there is a type $\langle \beta_1, \dots, \beta_s \rangle$ tuple of regular subsets of $\boldsymbol{A}$ which is relatively λ-homogeneous wrt $\varDelta$.

5.2 Standardization

The purpose of this section is to demonstrate that we can choose the type ω^n suborder

$$\mathcal{W}(n) := \{\, \boldsymbol{x} \in \mathbb{N}^n_+ \mid x(1) < x(2) < \ldots < x(n) \,\}$$

of $\mathbb{N}^n_+$ as our standard type ω^n well-order again. More precisely, the standardization lemma 5.2.7 establishes that it suffices to consider automatic partitions of the set

$$[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}} := [\mathcal{W}(n_1)]^{r_1} \times \cdots \times [\mathcal{W}(n_s)]^{r_s}$$

when we investigate partition relations of the following kind:

$$\begin{pmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{pmatrix} \xrightarrow{\mathsf{SA}} \begin{bmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{bmatrix}^{r_1,\dots,r_s}_{\kappa,\lambda}$$

First of all, we clarify when a partition of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ is *automatic*. To this end, let $n \in \mathbb{N}$ and $x \in \mathcal{W}(n)$. Recall that $\triangle x(1) = x(1)$ and $\triangle x(\mu) = x(\mu) - x(\mu - 1)$ for $\mu > 1$. The *string representation* of x is the string

$$\sigma = \mathtt{1}^{\triangle x(1)} \mathtt{2}^{\triangle x(2)} \cdots \mathtt{n}^{\triangle x(n)} .$$

Notice that the length of σ is exactly $x(n)$. For each $p \in [1, x(n)]$, let σ_p denote the p^{th} letter of σ. Then we have the following equivalence for all $\mu \in [1, n]$:

$$\sigma_p = \mu \quad \iff \quad x(\mu - 1) < p \leqslant x(\mu) . \tag{5.2}$$

In order to avoid vast quantities of clumsy function applications translating between x and its string representation, we *identify* x with this representation. In line with this, we also identify $\mathcal{W}(n)$ with the set of all string representations of its elements, i.e.,

$$\mathcal{W}(n) = \mathtt{1}^{+} \mathtt{2}^{+} \cdots \mathtt{n}^{+} .$$

Notice that this turns $\mathcal{W}(n)$ into a regular language. Furthermore, this identification allows for speaking about *regular* subsets of and *automatic* relations on $\mathcal{W}(n)$. In particular, it is easy to verify that the only linear ordering of $\mathcal{W}(n)$ we are taking into account is actually automatic, namely the one given by $x < y$ if the least $\mu \in [1, n]$ with $x(\mu) \neq y(\mu)$ satisfies $x(\mu) < y(\mu)$. Put another way, we regard $\mathcal{W}(n)$ as a string-automatic well-order.

Now, we fix some $s \in \mathbb{N}$ and $\boldsymbol{n}, \boldsymbol{r} \in \mathbb{N}^s$. Let $\boldsymbol{x} \in [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$, m be the size of its entry set, $\boldsymbol{z}$ the least element of its similarity class and $a \in \mathcal{W}(m)$ such that $\boldsymbol{x} = a(\boldsymbol{z})$. In line with the identifications above, we denote the convolution of the string representations of all x_{ki} by $\otimes\boldsymbol{x}$. Notice that the length of $\otimes\boldsymbol{x}$ is precisely $a(m)$. For every $p \in [1, a(m)]$, the p^{th} letter $\boldsymbol{\sigma}_p$ of $\otimes\boldsymbol{x}$ is a tuple

$$\boldsymbol{\sigma}_p = \langle \sigma_{p11}, \dots, \sigma_{p1r_1}; \dots; \sigma_{pk1}, \dots, \sigma_{pkr_k}; \dots; \sigma_{ps1}, \dots, \sigma_{psr_s} \rangle$$

with $\sigma_{pki} \in \{1, \dots, n_k, \diamond\}$ for all indices k, i.

Example 5.2.1 (continues example 4.4.1 on page 141)**.** Let $s = 3$, $\boldsymbol{n} = \langle 3, 2, 4 \rangle$ and $\boldsymbol{r} = \langle 2, 1, 1 \rangle$. Recall that

$$[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}} = [\mathcal{W}(3)]^2 \times \mathcal{W}(2) \times \mathcal{W}(4)\,.$$

We consider the elements of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ which are depicted as box diagrams in fig. 4.2 on page 142:

$$\boldsymbol{x} = \langle \langle 7, 8, 11 \rangle, \langle 7, 11, 16 \rangle; \langle 4, 10 \rangle; \langle 1, 7, 10, 19 \rangle \rangle$$

$$\boldsymbol{y} = \langle \langle 5, 13, 16 \rangle, \langle 5, 13, 18 \rangle; \langle 3, 21 \rangle; \langle 8, 10, 14, 17 \rangle \rangle$$

$$\boldsymbol{z} = \langle \langle 3, 4, 6 \rangle, \langle 3, 6, 7 \rangle; \langle 2, 5 \rangle; \langle 1, 3, 5, 8 \rangle \rangle$$

If we write the letters $\boldsymbol{\sigma}_p$ as column vectors in square brackets, the convolutions of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ read as follows:

$$\otimes\boldsymbol{x} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \begin{bmatrix}1\\1\\1\\2\end{bmatrix}^3 \begin{bmatrix}1\\1\\2\\2\end{bmatrix}^3 \begin{bmatrix}2\\2\\2\\3\end{bmatrix} \begin{bmatrix}3\\2\\2\\3\end{bmatrix}^2 \begin{bmatrix}3\\2\\\diamond\\4\end{bmatrix} \begin{bmatrix}\diamond\\3\\\diamond\\4\end{bmatrix}^5 \begin{bmatrix}\diamond\\\diamond\\\diamond\\4\end{bmatrix}^3$$

$$\otimes\boldsymbol{y} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}^3 \begin{bmatrix}1\\1\\2\\1\end{bmatrix}^2 \begin{bmatrix}2\\2\\2\\1\end{bmatrix}^3 \begin{bmatrix}2\\2\\2\\2\end{bmatrix}^2 \begin{bmatrix}2\\2\\2\\3\end{bmatrix}^3 \begin{bmatrix}3\\3\\2\\3\end{bmatrix} \begin{bmatrix}3\\3\\2\\4\end{bmatrix}^2 \begin{bmatrix}\diamond\\3\\2\\4\end{bmatrix} \begin{bmatrix}\diamond\\3\\2\\\diamond\end{bmatrix} \begin{bmatrix}\diamond\\\diamond\\2\\\diamond\end{bmatrix}^3$$

$$\otimes\boldsymbol{z} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \begin{bmatrix}1\\1\\1\\2\end{bmatrix} \begin{bmatrix}1\\1\\2\\2\end{bmatrix} \begin{bmatrix}2\\2\\2\\3\end{bmatrix} \begin{bmatrix}3\\2\\2\\3\end{bmatrix} \begin{bmatrix}3\\2\\\diamond\\4\end{bmatrix} \begin{bmatrix}\diamond\\3\\\diamond\\4\end{bmatrix} \begin{bmatrix}\diamond\\\diamond\\\diamond\\4\end{bmatrix}$$

We note the following connection: If you draw a box around the last occurrence of every letter (except for the $\diamond$-symbol) in each row and write the position of this letter into the box, you end up with the box diagram of the respective element. $\square$

Analogously to eq. (5.2) on page 173, we have the following equivalences for all $p \in [1, a(m)]$ and $\mu \in [1, n_k]$:

$$\sigma_{pki} = \mu \quad \Longleftrightarrow \quad x_{ki}(\mu - 1) < p \leqslant x_{ki}(\mu) \tag{5.3}$$

and

$$\sigma_{pki} = \diamond \quad \Longleftrightarrow \quad x_{ki}(n_k) < p\,. \tag{5.4}$$

Altogether, identifying elements of $\mathcal{W}(n_k)$ with their string representations allows for speaking of *automatic* partitions of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$.

5.2.1 Decomposition of Well-Orders

Our next step towards proving the standardization lemma 5.2.7 is to demonstrate that every string-automatic type ω^n well-order contains a type ω^n suborder which is isomorphic to $\mathcal{W}(n)$ via an isomorphism of a very simple form. In the final proof of the standardization lemma, we show and use the fact that maps of this simple form preserve automaticity in both directions. Formally, "simple form" shall mean the following:

Definition 5.2.2. A map $f\colon \mathcal{W}(n) \to \Sigma^*$ is called *presentable* if there exist strings $u_0, u_1, \ldots, u_n \in \Sigma^*$ and $v_1, \ldots, v_n \in \Sigma^+$ such that, for all $x \in \mathcal{W}(n)$,

$$f(x) = u_0 v_1^{\triangle x(1)} u_1 v_2^{\triangle x(2)} u_2 \cdots v_n^{\triangle x(n)} u_n\,.$$

The tuple $\langle u_0, v_1, u_1, v_2, u_2, \ldots, v_n, u_n\rangle$ then is a *presentation* of f. If there is $p \geqslant 1$ such that $|v_i| = p$ for each i, we say that f is *p-uniformly presentable* and speak of a *p-uniform presentation.*

Phrased in this terminology, our goal is to show that every string-automatic type ω^n well-order A admits a p-uniformly presentable embedding $f\colon \mathcal{W}(n) \to A$ for some $p \geqslant 1$. Before proving this in its whole generality, we showcase one idea behind the proof in the case $n = 1$. This particular idea is also relevant in section 5.6. In order to make the involved calculations easier to follow, we introduce some notation: Suppose that $\eta\colon (\Sigma_\diamond^2)^* \to S$ is a morphism into a finite semigroup. For $u, v \in \Sigma^*$, we define

$$\eta\begin{bmatrix} u \\ v \end{bmatrix} := \eta(u \otimes v)\,.$$

In this notation, we align factors of u and v of the same length. For instance, if $u = u_1 u_2^k u_3$ and $v = v_1 v_2^k v_3 v_4^\ell v_5$ with $|u_i| = |v_i|$ for $i = 1, 2$ and $|u_3| \leqslant |v_3|$, we write

$$\eta\begin{bmatrix} u \\ v \end{bmatrix} = \eta\begin{bmatrix} u_1 & u_2^k & u_3 & \varepsilon & \varepsilon \\ v_1 & v_2^k & v_3 & v_4^\ell & v_5 \end{bmatrix}.$$

In addition, suppose that k, k', ℓ, ℓ' are multiples of the exponent of S.[3] In particular, $s^k = s^{k'}$ and $t^\ell = t^{\ell'}$ for all $s, t \in S$. Choosing $s = \eta(u_2 \otimes v_2)$ and $t = \eta(\varepsilon \otimes v_4)$, we obtain

$$\eta\begin{bmatrix} u_1 & u_2^k & u_3 & \varepsilon & \varepsilon \\ v_1 & v_2^k & v_3 & v_4^\ell & v_5 \end{bmatrix} = \eta\begin{bmatrix} u_1 & u_2^{k'} & u_3 & \varepsilon & \varepsilon \\ v_1 & v_2^{k'} & v_3 & v_4^{\ell'} & v_5 \end{bmatrix}.$$

In the following, we utilize calculations of this kind without any further explanation.

Lemma 5.2.3. *Let A be a string-automatic type ω well-order. There is a presentable embedding $f\colon \mathcal{W}(1) \to A$.*

Proof. First of all, notice that $\mathcal{W}(1) = \mathbb{N}_+$. Let $\eta\colon (\Sigma_\diamond^2)^* \to S$ be a morphism recognizing $\leqslant_A$ and $m \in \mathbb{N}_+$ the exponent of S. A

[3] A definition of the *exponent* of a semigroup can be found on page 23.

simple pumping argument provides us with $u, v, w \in \Sigma^*$ satisfying $|v| \geqslant 1$, $m \cdot |v| \geqslant |w|$ and $uv^k w \in A$ for all $k \in \mathbb{N}_+$.

We show that the map $f\colon \mathbb{N}_+ \to A$ defined by

$$f(k) := u(v^{2m})^k w\,,$$

which is obviously presentable, is an embedding of $\mathbb{N}_+$ into A. Recall that $s^{m'}$ is idempotent for every multiple m' of m and each $s \in S$. In line with the comments above, we can hence perform the following calculations for all $k, \ell \in \mathbb{N}_+$ with $k < \ell$:

$$\begin{aligned}
\eta\begin{bmatrix} f(k) \\ f(\ell) \end{bmatrix} &= \eta\begin{bmatrix} u & v^{2mk} & w & \varepsilon & \varepsilon \\ u & v^{2mk} & v^m & v^{m\,(2(\ell-k)-1)} & w \end{bmatrix} \\
&= \eta\begin{bmatrix} u & v^{2m} & w & \varepsilon & \varepsilon \\ u & v^{2m} & v^m & v^m & w \end{bmatrix} \\
&= \eta\begin{bmatrix} f(1) \\ f(2) \end{bmatrix}
\end{aligned}$$

Since η recognizes $\leqslant_A$, we have $f(k) \leqslant_A f(\ell)$ if and only if $f(1) \leqslant_A f(2)$. Moreover, f is injective because $|v| \geqslant 1$. In particular, $f(1) \neq f(2)$. If we had $f(1) >_A f(2)$, f would be order-reversing, contradicting the fact that A is a well-order. Thus, $f(1) <_A f(2)$ and f is order-preserving. □

Running slightly off the topic, we briefly sketch how to extend the previous proof to a proof of the case $r = 2$ of theorem 5.1.3, i.e., of the partition relation

$$\omega \xrightarrow{\mathsf{SA}} (\omega)^2_\kappa$$

for all $\kappa \in \mathbb{N}$. To this end, consider a partition Δ of $[A]^2$. We may assume that the morphism η does not only recognize $\leqslant_A$ but all Δ-classes as well. Consequently, we have actually shown that all pairs $\langle f(k), f(\ell) \rangle$ with $k < \ell$ belong to the same Δ-class as $\langle f(1), f(2)\rangle$. Put another way, the infinite regular subset $u(v^{2m})^+w \subseteq A$ is homogeneous. In addition, all the constructions involved are effective. In section 5.6, we further extend this idea to the tree-automatic setting and arbitrary $r \in \mathbb{N}$.

Recall that our actual goal is to show that every string-automatic type ω^n well-order A admits a uniformly presentable embedding $f\colon \mathcal{W}(n) \to A$. The basic idea behind the proof for $n \geqslant 2$ is an induction on n which uses that every type ω^n well-order A can be uniquely decomposed into an ω-sum of type ω^{n-1} well-orders. In terms of iterated finite-condensation relations, this decomposition can be obtained as follows: Let $\sim_{n-1}$ be the $(n-1)^{\text{st}}$ iterated finite-condensation relation on A. Then $A/{\sim_{n-1}}$ has order type ω and every $\sim_{n-1}$-class is a type ω^{n-1} suborder of A. In addition, we consider the system of representatives of $\sim_{n-1}$ given by

$$L_{n-1} := \{\, \min[w]_{n-1} \mid w \in A \,\}\,,$$

i.e., we represent each $\sim_{n-1}$-class by its least element.[4] The decomposition of A is now given by

$$A = \sum_{w \in L_{n-1}} [w]_{n-1}\,.$$

This decomposition is automatic in the following sense: The relation $\sim_{n-1}$ and the set L_{n-1} are first-order definable in A and hence automatic whenever A is string-automatic.

We cannot expect the following to work: We take for *every* $\sim_{n-1}$-class $[w]_{n-1}$ a presentable embedding of $\mathcal{W}(n-1)$ into $[w]_{n-1}$ and combine all these embeddings into *one* presentable embedding of $\mathcal{W}(n)$ into A. Accordingly, the following lemma prepares a sensible choice of embeddings that can be combined.

[4] In terms of iterated limit points, L_{n-1} contains precisely the $(n-1)$-limit points of A.

Lemma 5.2.4. *Let $n \geqslant 2$ and A be a string-automatic type ω^n well-order. The relation*

$$R := \left\{ \langle u, \tilde{u} \rangle \in L_{n-1} \times \Sigma^* \;\middle|\; \begin{array}{l} |u| = |\tilde{u}| \text{ and the order type} \\ \text{of } [u]_{n-1} \cap \tilde{u}\Sigma^* \text{ is } \omega^{n-1} \end{array} \right\}$$

is automatic and contains a pair $\langle u, \tilde{u} \rangle$ for each $u \in L_{n-1}$.

Proof. Recall that a well-order B has type ω^{n-1} if and only if $B/{\sim_{n-2}}$ is infinite and $B/{\sim_{n-1}}$ is a singleton. Consequently, the relation R is first-order definable in the automatic structure $(\Sigma^*; A, \leqslant_A, \equiv, \preccurlyeq)$, where $\equiv$ and $\preccurlyeq$ are the same-length and prefix relations, respectively. Thus, R is automatic.

Concerning the second claim, fix some $u \in L_{n-1}$. Recall that $[u]_{n-1}$ has order type ω^{n-1}. We consider the partition

$$[u]_{n-1} = \Sigma^{<|u|} \uplus \biguplus_{\substack{\tilde{u} \in \Sigma^* \\ |u| = |\tilde{u}|}} [u]_{n-1} \cap \tilde{u}\Sigma^* \,.$$

According to theorem 3.2.2 on page 65, this partition contains a class of order type ω^{n-1}. Since the first part is finite, it must be one of the latter parts. Put another way, there is $\tilde{u} \in \Sigma^*$ with $|u| = |\tilde{u}|$ such that $[u]_{n-1} \cap \tilde{u}\Sigma^*$ has order type ω^{n-1}. □

Now, we utilize the sensible choice prepared by lemma 5.2.4 along with the main idea behind the proof of lemma 5.2.3 in order to construct a (possibly non-uniform) presentation of some embedding $f \colon \mathcal{W}(n) \to A$.

Theorem 5.2.5. *Let $n \in \mathbb{N}$ and A be a string-automatic type ω^n well-order. There is a presentable embedding $f \colon \mathcal{W}(n) \to A$. Moreover, given a presentation of A, one can compute a presentation of f.*

Proof. We proceed by induction on n. The claim is trivial for $n = 0$ and has been established for $n = 1$ in lemma 5.2.3. Henceforth, we assume $n \geqslant 2$. According to lemma 5.2.4, the relation

$$R := \left\{ \langle u, \tilde{u} \rangle \in L_{n-1} \times \Sigma^* \;\middle|\; \begin{array}{l} |u| = |\tilde{u}| \text{ and the order type} \\ \text{of } [u]_{n-1} \cap \tilde{u}\Sigma^* \text{ is } \omega^{n-1} \end{array} \right\}$$

is automatic and infinite. An easy pumping argument provides us with strings $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r} \in \left(\Sigma_\diamond^2\right)^*$ such that $|\boldsymbol{q}| \geqslant 1$, $|\boldsymbol{q}| \geqslant |\boldsymbol{r}|$ and $\boldsymbol{p}\boldsymbol{q}^k\boldsymbol{r} \in \otimes R$ for each $k \in \mathbb{N}_+$. Owing to the same-length condition in the definition of R, none of the three strings contains a $\diamond$-symbol. Thus, we can write $\boldsymbol{p} = p \otimes \tilde{p}$, $\boldsymbol{q} = q \otimes \tilde{q}$ and $\boldsymbol{r} = r \otimes \tilde{r}$ for some strings $p, \tilde{p}, q, \tilde{q}, r, \tilde{r} \in \Sigma^*$ with $|p| = |\tilde{p}|$, $|q| = |\tilde{q}|$ and $|r| = |\tilde{r}|$. Notice that $|q| \geqslant 1$, $|q| \geqslant |r|$ and $\langle pq^kr, \tilde{p}\tilde{q}^k\tilde{r} \rangle \in R$ for each $k \in \mathbb{N}$.

Let $\eta\colon \left(\Sigma_\diamond^2\right)^* \to S$ be a morphism simultaneously recognizing $\leqslant_A$ and $\sim_{n-1}$. Furthermore, let $m \in \mathbb{N}_+$ be the exponent of S. We consider the unique language $Z \subseteq \Sigma^*$ with

$$\tilde{p}\tilde{q}^{2m}\tilde{r}\, Z = [pq^{2m}r]_{n-1} \cap \tilde{p}\tilde{q}^{2m}\tilde{r}\, \Sigma^* \,.$$

Due to the choice of R, the subset of A on the right hand side has order type ω^{n-1}. Accordingly, we turn Z into a type ω^{n-1} well-order by defining

$$u \leqslant_Z v \quad :\Longleftrightarrow \quad \tilde{p}\tilde{q}^{2m}\tilde{r}u \leqslant_A \tilde{p}\tilde{q}^{2m}\tilde{r}v \,.$$

Since $\leqslant_A$ and $\sim_{n-1}$ are automatic, the well-order Z is also automatic. Due to the induction hypothesis, there is a presentable embedding $g\colon \mathcal{W}(n-1) \to Z$. For every $x \in \mathcal{W}(n)$, we define $\bar{x} \in \mathcal{W}(n-1)$ by $\triangle\bar{x}(\mu) := \triangle x(\mu + 1)$ for $1 \leqslant \mu \leqslant n-1$. Intuitively, $\triangle\bar{x}$ is obtained from $\triangle x$ by dropping the first element. In the remainder of this proof, we show that the map $f\colon \mathcal{W}(n) \to \Sigma$ defined by

$$f(x) := \tilde{p}\tilde{q}^{2m\,x(1)}\tilde{r}\, g(\bar{x})$$

is a presentable embedding of $\mathcal{W}(n)$ into A.

First, suppose that $\langle u_1, v_2, u_2, \ldots, v_n, u_n\rangle$ is a presentation of g. It is easy to check that $\langle \tilde{p}, \tilde{q}^{2m}, \tilde{r}u_1, v_2, u_2, \ldots, v_n, u_n\rangle$ then is a presentation of f. Our next step is to show that $f(x) \in [pq^{2m\,x(1)}r]_{n-1}$ for all $x \in \mathcal{W}(n)$, which particularly implies $f(x) \in A$.

For this purpose, we consider some $x \in \mathcal{W}(n)$. We have

$$\eta\begin{bmatrix} pq^{2m\,x(1)}r \\ f(x) \end{bmatrix} = \eta\begin{bmatrix} p & q^{2m\,x(1)} & r & \varepsilon \\ \tilde{p} & \tilde{q}^{2m\,x(1)} & \tilde{r} & g(\bar{x}) \end{bmatrix} = \eta\begin{bmatrix} p & q^{2m} & r & \varepsilon \\ \tilde{p} & \tilde{q}^{2m} & \tilde{r} & g(\bar{x}) \end{bmatrix}.$$

Due to the choice of g and Z, we have $g(\bar{x}) \in Z$ and hence $\tilde{p}\tilde{q}^{2m}\tilde{r}\,g(\bar{x}) \in [pq^{2m}r]_{n-1}$. Since η recognizes $\sim_{n-1}$, we may conclude $f(x) \in [pq^{2m\,x(1)}r]_{n-1}$.

Finally, we demonstrate that f is order-preserving. To this end, we consider some $x, y \in \mathcal{W}(n)$ with $x < y$. In order to show $f(x) <_A f(y)$, we distinguish two cases:

Case 1: $x(1) < y(1)$. Using the very same arguments as in lemma 5.2.3, we obtain that the map which sends each $k \in \mathbb{N}_+$ to $pq^{2mk}r \in L_{n-1}$ is order-preserving. In particular,

$$pq^{2m\,x(1)}r <_A pq^{2m\,y(1)}r$$

and hence

$$f(x) \in [pq^{2m\,x(1)}r]_{n-1} \ll [pq^{2m\,y(1)}r]_{n-1} \ni f(y)\,.$$

Case 2: $x(1) \geqslant y(1)$. Since $x < y$, we have $x(1) = y(1)$ and $\bar{x} < \bar{y}$. Consequently, we obtain

$$\eta\begin{bmatrix} f(x) \\ f(y) \end{bmatrix} = \eta\begin{bmatrix} \tilde{p} & \tilde{q}^{2m\,x(1)} & \tilde{r} & g(\bar{x}) \\ \tilde{p} & \tilde{q}^{2m\,y(1)} & \tilde{r} & g(\bar{y}) \end{bmatrix} = \eta\begin{bmatrix} \tilde{p} & \tilde{q}^{2m} & \tilde{r} & g(\bar{x}) \\ \tilde{p} & \tilde{q}^{2m} & \tilde{r} & g(\bar{y}) \end{bmatrix}.$$

Since g is order-preserving and $\bar{x} < \bar{y}$, we have $g(\bar{x}) <_Z g(\bar{x})$. Using the definition of $<_Z$, we conclude

$$\tilde{p}\tilde{q}^{2m}\tilde{r}\,g(\bar{x}) <_A \tilde{p}\tilde{q}^{2m}\tilde{r}\,g(\bar{y})\,.$$

Finally, this implies $f(x) <_A f(y)$ because η recognizes $<_A$. $\square$

5.2.2 Preservation of Automaticity

In order to prove the standardization lemma 5.2.7, we require *uniformly* presentable embeddings for *tuples of* well-orders. Their existence is guaranteed by the next corollary. Therein, a *tuple of maps* $\boldsymbol{f}\colon \mathcal{W}(\boldsymbol{n}) \to \boldsymbol{A}$ is a tuple $\boldsymbol{f} = \langle f_1, \ldots, f_s \rangle$ where each f_k is a map $f_k\colon \mathcal{W}(n_k) \to A_k$.

Corollary 5.2.6. *Let $\boldsymbol{A}$ be a type $\langle \omega^{n_1}, \ldots, \omega^{n_s} \rangle$ tuple of string-automatic well-orders. There are $p \geqslant 1$ and a tuple of p-uniformly presentable embeddings $\boldsymbol{f}\colon \mathcal{W}(\boldsymbol{n}) \to \boldsymbol{A}$.*

Proof. According to theorem 5.2.5, for each $k \in [1, s]$, there is a presentable embedding $f_k\colon \mathcal{W}(n_k) \to A_k$, say

$$\langle u_{k0}, v_{k1}, u_{k1}, \ldots, v_{kn_k}, u_{kn_k} \rangle$$

is a presentation of f_k. Let $p \geqslant 1$ be a common multiple of all the $|v_{k\mu}|$ and put $p_{k\mu} := {}^{p}\!/_{|v_{k\mu}|}$. For each k, we define an embedding $g_k\colon \mathcal{W}(n_k) \to \mathcal{W}(n_k)$ by

$$\triangle(g_k(x))(\mu) = p_{k\mu} \, \triangle x(\mu)\,.$$

Clearly, the map $f_k \circ g_k\colon \mathcal{W}(n_k) \to A_k$ is an embedding too. For all $x \in \mathcal{W}(n_k)$, we have

$$(f_k \circ g_k)(x) = u_{k0} \left(v_{k1}^{p_{k1}} \right)^{\triangle x(1)} u_{k1} \, \cdots \, \left(v_{kn_k}^{p_{kn_k}} \right)^{\triangle x(n_k)} u_{kn_k}\,.$$

Since $|v_{k\mu}^{p_{k\mu}}| = p$ for all k, μ, the tuple $\langle f_1 \circ g_1, \ldots, f_s \circ g_s \rangle$ has the desired properties. $\square$

Finally, we are prepared to prove the standardization lemma. Basically, the proof sandwiches an application of its premise between two translations of automaticity by means of the p-uniformly presentable embeddings from corollary 5.2.6.

Lemma 5.2.7 (standardization lemma)**.** *Let $\kappa, \lambda \in \mathbb{N}$. If every automatic partition of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ into κ classes admits a relatively λ-homogeneous type $\langle \omega^{n_1}, \dots, \omega^{n_s} \rangle$ tuple of regular subsets of $\mathcal{W}(\boldsymbol{n})$, then the following holds:*

$$\begin{pmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_k} \end{pmatrix} \xrightarrow{\mathsf{SA}} \begin{bmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_k} \end{bmatrix}^{r_1, \dots, r_s}_{\kappa, \lambda}$$

Proof. Let $\boldsymbol{A}$ be a type $\langle \omega^{n_1}, \dots, \omega^{n_s} \rangle$ tuple of string-automatic well-orders and $\varDelta$ an automatic partition of $[\boldsymbol{A}]^{\boldsymbol{r}}$ into κ classes. According to corollary 5.2.6, there are $p \geqslant 1$ and a tuple of p-uniformly presentable embeddings $\boldsymbol{f} \colon \mathcal{W}(\boldsymbol{n}) \to \boldsymbol{A}$. For a tuple $\boldsymbol{x} \in [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$, we write $\boldsymbol{f}(\boldsymbol{x})$ for the tuple $\boldsymbol{u} \in [\boldsymbol{A}]^{\boldsymbol{r}}$ given by $u_{ki} = f_k(x_{ki})$ for all indices k, i. As a first step, we show that the partition

$$\varGamma := \{\, \boldsymbol{f}^{-1}(D) \mid D \in \varDelta \,\}$$

of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ is automatic.

To this end, let $\langle u_{k0}, v_{k1}, u_{k1}, \dots, v_{kn_k}, u_{kn_k} \rangle$ be a p-uniform presentation of f_k for each k. We factorize each map f_k into two maps g_k, h_k as follows, where $n = n_k$:

$$\begin{aligned}
f_k \colon \mathcal{W}(n) = \mathtt{1}^+ \cdots \mathtt{n}^+ \quad & \xrightarrow{g_k} (\mathtt{1}^p)^+ \cdots (\mathtt{n}^p)^+ \\
& \xrightarrow{h_k} u_0 v_1^+ u_1 \cdots v_n^+ u_n \subseteq A_k \\
x = \mathtt{1}^{\vartriangle x(1)} \cdots \mathtt{n}^{\vartriangle x(n)} & \xmapsto{g_k} (\mathtt{1}^p)^{\vartriangle x(1)} \cdots (\mathtt{n}^p)^{\vartriangle x(n)} \\
& \xmapsto{h_k} u_0 v_1^{\vartriangle x(1)} u_1 \cdots v_n^{\vartriangle x(n)} u_n \,.
\end{aligned}$$

According to [FS93, corollary 4.2], each map h_k is automatic. In view of this, it is a matter of routine to check that the relation $\boldsymbol{h}^{-1}(D)$ is automatic for each $D \in \varDelta$. Due to the very simple nature of the maps g_k, it is even simpler to verify that the relation $\boldsymbol{g}^{-1}(\boldsymbol{h}^{-1}(D))$ is automatic as well.[5] Since $f_k = h_k \circ g_k$, we actually have

$$\boldsymbol{g}^{-1}(\boldsymbol{h}^{-1}(D)) = \boldsymbol{f}^{-1}(D)\,.$$

Consequently, Γ is automatic.

The premise of this lemma guarantees that there is a type $\langle \omega^{n_1}, \dots, \omega^{n_s} \rangle$ tuple of regular subsets $\boldsymbol{X} \subseteq \mathcal{W}(\boldsymbol{n})$ which is relatively λ-homogeneous wrt Γ. Due to the choice of $\boldsymbol{f}$ and Γ, the tuple $\langle f_1(X_1), \dots, f_s(X_s) \rangle$ is a type $\langle \omega^{n_1}, \dots, \omega^{n_s} \rangle$ tuple of subsets of $\boldsymbol{A}$ which is relatively λ-homogeneous wrt Δ. It only remains to show that each $f_k(X_k)$ is regular. However, since $f_k(X_k) = h_k(g_k(X_k))$, we can use the same arguments as above in reverse order. $\square$

5.3 Polarization, Canonicalization and Simplification

The goal of this section is to show that the automatic Ramsey degrees of all ordinals $\alpha < \omega^\omega$ are finite and to provide their exact values. Similar to sections 4.3 to 4.5, we proceed by proving the positive polarization lemma 5.3.1, the canonicalization lemma 5.3.4 and the positive simplification lemma 5.3.7. As already mentioned, the proof of the first of these three lemmas is almost literally the same as in the non-automatic case whereas the proofs of the other two lemmas are all new.

Concerning the polarization lemma, recall the definition of the set

$$\mathcal{R}(s,r) := \{\, \tilde{\boldsymbol{r}} \in \mathbb{N}^s \mid \tilde{r}_1 + \dots + \tilde{r}_s = r \,\}$$

[5] More precisely, the maps g_k are $\langle 1, p \rangle$-synchronous transductions and such transductions are known to preserve automaticity in both directions, cf. [Bár06, lemma 5 and theorem 2].

and, for any map $\ell\colon \mathcal{R}(s,r) \to \mathbb{N}$, of the number

$$|\ell| := \sum_{\tilde{\boldsymbol{r}} \in \mathcal{R}(s,r)} \ell(\tilde{\boldsymbol{r}})\,.$$

Lemma 5.3.1 (positive polarization lemma). *Let $r, \kappa \in \mathbb{N}$, $\alpha < \omega^\omega$ be an ordinal, $\alpha = \omega^{n_1} + \cdots + \omega^{n_s}$ its Cantor normal form and $\ell\colon \mathcal{R}(s,r) \to \mathbb{N}$ a map. If*

$$\begin{pmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{pmatrix} \xrightarrow{\mathsf{SA}} \begin{bmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{bmatrix}^{\tilde{r}_1,\ldots,\tilde{r}_s}_{\kappa,\ell(\tilde{\boldsymbol{r}})}$$

for all $\tilde{\boldsymbol{r}} \in \mathcal{R}(s,r)$, then

$$\alpha \xrightarrow{\mathsf{SA}} [\alpha]^r_{\kappa,|\ell|}\,.$$

Proof. We employ almost literally the same construction as in the proof of the non-automatic positive polarization lemma 4.3.2 on page 135. The only difference is that we have to ensure that the tuples of sets $\boldsymbol{X}_t$ constructed during the induction contain only regular sets. This follows from corollary 3.1.8 on page 63 for $t = 0$ and from the stronger premises for $t > 0$. □

Our next step is to show the canonicalization lemma 5.3.4. To this end, let $s \in \mathbb{N}$ and $\boldsymbol{n} = \langle n_1, \ldots, n_s\rangle, \boldsymbol{r} = \langle r_1, \ldots, r_s\rangle \in \mathbb{N}^s$. If not further specified, s, $\boldsymbol{n}$ and $\boldsymbol{r}$ are always of this kind in the remainder of this section. Recall definition 4.4.2 on page 143: Two tuples $\boldsymbol{x}, \boldsymbol{y} \in [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ are *similar* if the equivalence

$$x_{ki}(\mu) < x_{\ell j}(\nu) \quad\Longleftrightarrow\quad y_{ki}(\mu) < y_{\ell j}(\nu)$$

is satisfied for all indices k, i, μ and ℓ, j, ν. The *least element* of some similarity class was defined as the unique $\boldsymbol{z}$ therein whose entry set is of the form $\{1,\ldots,m\}$, where m is the size of the entry set. Using eqs. (5.3) and (5.4) on page 175, it is quite easy to show that the letters of $\otimes\boldsymbol{z}$ are mutually distinct.[6] In some sense, that may be regarded as the essence of being the least element of a similarity class. This intuition is backed by the following lemma, which particularly claims the diagram below to commute, provided that the mutual distinctness of the letters in $\otimes\boldsymbol{z}$ is taken for granted:

$$\begin{tikzcd}
\boldsymbol{x} \arrow[r, "\text{least element of}", "\text{similarity class}"'] \arrow[d, "\otimes"'] & \boldsymbol{z} \arrow[d, "\otimes"] \\
\otimes\boldsymbol{x} \arrow[r, "\text{remove duplicate}", "\text{letters}"'] & \otimes\boldsymbol{z}
\end{tikzcd}$$

Lemma 5.3.2. *Let $\boldsymbol{x} \in [\mathcal{W}(\boldsymbol{n})]^r$, m be the size of its entry set, $\boldsymbol{z}$ the least element of its similarity class and $a \in \mathcal{W}(m)$ such that $\boldsymbol{x} = a(\boldsymbol{z})$. If $\otimes\boldsymbol{z} = \boldsymbol{\tau}_1\boldsymbol{\tau}_2\ldots\boldsymbol{\tau}_m$, then*

$$\otimes\boldsymbol{x} = \boldsymbol{\tau}_1^{\triangle a(1)}\boldsymbol{\tau}_2^{\triangle a(2)}\cdots\boldsymbol{\tau}_m^{\triangle a(m)}\,.$$

Proof. Let $\otimes\boldsymbol{x} = \boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_{a(m)}$ be the factorization of $\otimes\boldsymbol{x}$ into its letters. We have to show $\sigma_{pki} = \tau_{qki}$ for all p, q with $1 \leqslant q \leqslant m$ and $a(q-1) < p \leqslant a(q)$ and all indices k, i. First, suppose that $\tau_{qki} = \mu \in [1, n_k]$, i.e., $z_{ki}(\mu-1) < q \leqslant z_{ki}(\mu)$ by eq. (5.3) on page 175. The first part of this inequality is equivalent to $z_{ki}(\mu-1) \leqslant q-1$. The monotonicity of a implies

$$x_{ki}(\mu-1) = a(z_{ki}(\mu-1)) \leqslant a(q-1) < p$$

and

$$p \leqslant a(q) \leqslant a(z_{ki}(\mu)) = x_{ki}(\mu)\,.$$

[6] We refrain from proving this since it does not matter for the correctness of the subsequent proofs but is only mentioned for reasons of intuition.

Applying eq. (5.3) once more, we conclude $\sigma_{pki} = \mu$. Reasoning similarly but using eq. (5.4) instead of eq. (5.3), we obtain that $\tau_{qki} = \diamond$ implies $\sigma_{pki} = \diamond$. □

The following immediate consequence of the previous lemma is needed in the next section only.

Corollary 5.3.3. *Every similarity class in $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ is automatic.*

Proof. Let C be an arbitrary similarity class, $\boldsymbol{z}$ its least element and $\otimes\boldsymbol{z} = \boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_m$. Then lemma 5.3.2 implies

$$\otimes C = \boldsymbol{\tau}_1^+ \cdots \boldsymbol{\tau}_m^+ \,. \qquad \square$$

Recall definition 4.4.4 on page 145: A partition $\varDelta$ of a subset $X \subseteq [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ is called *canonical* if any two $\boldsymbol{x}, \boldsymbol{y} \in X$ which are similar belong to the same $\varDelta$-class. In order to properly phrase the automatic version of the canonicalization lemma, we need to introduce some more notation. For $x \in \mathcal{W}(n_k)$ and $h \in \mathbb{N}_+$, we define $hx \in \mathcal{W}(n_k)$ by

$$(hx)(\mu) := h \cdot x(\mu) \,.$$

In line with this, we put

$$h\mathcal{W}(n_k) := \{\, hx \mid x \in \mathcal{W}(n_k) \,\} \subseteq \mathcal{W}(n_k) \,.$$

Notice that this set has order type ω^{n_k} and is regular. Finally, we lift this notation to tuples of sets by defining

$$h\mathcal{W}(\boldsymbol{n}) := \langle h\mathcal{W}(n_1), \ldots, h\mathcal{W}(n_s) \rangle \,.$$

Lemma 5.3.4 (canonicalization lemma)**.** *Let $\varDelta$ be an automatic partition of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$. There exists $h \in \mathbb{N}_+$ such that the restriction of $\varDelta$ to $[h\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ is canonical.*

Proof. Let η be a morphism into a finite semigroup S simultaneously recognizing all Δ-classes. We show that the exponent h of S has the desired property.

Consider $\boldsymbol{x}, \boldsymbol{y} \in [h\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ which are similar. Let m be the size of their entry sets, $\boldsymbol{z}$ be the least element of their similarity class and $a, b \in \mathcal{W}(m)$ such that $\boldsymbol{x} = a(\boldsymbol{z})$ and $\boldsymbol{y} = a(\boldsymbol{z})$. Recall that $\otimes\boldsymbol{z}$ has length m, say $\otimes\boldsymbol{z} = \boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_m$. Due to the choice of $\boldsymbol{x}$ and $\boldsymbol{y}$, all the $\triangle a(p)$ and $\triangle b(p)$ are divisible by h and hence $t^{\triangle a(p)} = t^{\triangle b(p)}$ for all $t \in S$. Along with lemma 5.3.2, we obtain

$$\begin{aligned}\eta(\otimes\boldsymbol{x}) &= \eta\Big(\boldsymbol{\tau}_1^{\triangle a(1)} \cdots \boldsymbol{\tau}_m^{\triangle a(m)}\Big) \\ &= \eta\Big(\boldsymbol{\tau}_1^{\triangle b(1)} \cdots \boldsymbol{\tau}_m^{\triangle b(m)}\Big) \\ &= \eta(\otimes\boldsymbol{y})\,.\end{aligned}$$

Since η recognizes all Δ-classes, $\boldsymbol{x}$ and $\boldsymbol{y}$ hence belong to the same Δ-class. □

Composing the canonicalization lemma and the standardization lemma 5.2.7 yields the following polarized partition relation, where κ is arbitrary and $S(\boldsymbol{n};\boldsymbol{r})$ the number of similarity classes in $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$:

$$\begin{pmatrix}\omega^{n_1}\\ \vdots \\ \omega^{n_k}\end{pmatrix} \xrightarrow{\mathsf{SA}} \begin{bmatrix}\omega^{n_1}\\ \vdots \\ \omega^{n_k}\end{bmatrix}^{r_1,\ldots,r_s}_{\kappa, S(\boldsymbol{n};\boldsymbol{r})}$$

Along with the positive polarization lemma 5.3.1, we obtain an upper bound on the automatic r-ary Ramsey degree of each ordinal $\alpha < \omega^\omega$, which is again not optimal. In our investigation of the set-theoretic Ramsey degree, we improved this bound by introducing the notion of p-simplicity and demonstrating that non-p-simplicity can be avoided in some sense. Recall that there were three reasons for non-p-simplicity, which are depicted in fig. 4.3

on page 148. Unfortunately, the reason depicted in fig. 4.3(c) cannot be avoided in the context of tuples of regular subsets of $\mathcal{W}(\boldsymbol{n})$ any more, as the following minimal example demonstrates.

Example 5.3.5. Let $X \subseteq \mathcal{W}(2)$ be a regular type ω^2 subset, $\eta\colon \{\mathtt{1}, \mathtt{2}\}^* \to S$ a morphism recognizing X and m the exponent of S. Due to lemma 4.6.3 on page 151, there is $x \in X$ with $\triangle x(1) \geqslant m$ and $\triangle x(2) \geqslant 2m$. Let $y \in \mathcal{W}(2)$ be defined by $y(1) := x(1) + m$ and $y(2) := x(2)$. Notice that $x < y$ and $\langle x, y\rangle$ is *not* p-simple since $x(2) = y(2)$ but $x(1) \neq y(1)$. However, simple pumping arguments show $y \in X$ and hence $\langle x, y\rangle \in [X]^2$. Altogether, $[X]^2$ contains a non-p-simple element. $\square$

In view of this example, we resort to the similar but slightly weaker notion of *b-simplicity*, which still forbids the patterns depicted in figs. 4.3(a) and 4.3(b) but no longer the one in fig. 4.3(c).

Definition 5.3.6. A tuple $\boldsymbol{x} \in [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ is called *b-simple* if, for all indices k, i, μ and ℓ, j, ν, the premise $x_{ki}(\mu) = x_{\ell j}(\nu)$ implies $k = \ell$ and $\mu = \nu$. The number of similarity classes containing a b-simple element is denoted by $B(\boldsymbol{n}; \boldsymbol{r})$.[7]

Obviously, b-simplicity is also a property of similarity classes and p-simplicity implies b-simplicity. One can compute $B(\boldsymbol{n}; \boldsymbol{r})$ from $\boldsymbol{n}$ and $\boldsymbol{r}$ by counting the number of box diagram shapes which exclude the patterns in figs. 4.3(a) and 4.3(b). In particular, we obtain $B(\boldsymbol{n}; \boldsymbol{r}) \geqslant P(\boldsymbol{n}; \boldsymbol{r})$ and this inequality is strict if and only if the pattern in fig. 4.3(c) can be realized, i.e., if there is $k \in [1, s]$ with $n_k \geqslant 2$ and $r_k \geqslant 2$.

[7] The "b" stands for "balanced": As before, the shape of the box diagram for any $\boldsymbol{x} \in [\mathcal{W}(n)]^2$ can be regarded as a string over the alphabet $\{{}_{\square}, {}^{\square}, \boxminus\}$. Then $\boldsymbol{x}$ is b-simple if and only if the $\boxminus$-symbols appear only at positions where the number of ${}_{\square}$-symbols and ${}^{\square}$-symbols to the left is *balanced*.

In the subpartitions constructed in lemma 5.3.4, non-b-simplicity can be avoided in some sense. To make this sense precise, we consider for each $k \in [1, s]$ the set

$$U_{k,s}(n_k) := \left\{ x \in \mathcal{W}(n_k) \;\middle|\; \begin{array}{l} x(\mu) \equiv s\mu + k \pmod{sn_k} \\ \text{for each } \mu \in [1, n_k] \end{array} \right\},$$

which has order type ω^{n_k} and is regular because it is also given by

$$U_{k,s}(n_k) = \mathtt{1}^k \mathtt{1}^s (\mathtt{1}^{ns})^* \mathtt{2}^s (\mathtt{2}^{ns})^* \cdots \mathtt{n}_k^s (\mathtt{n}_k^{ns})^* .$$

Finally, we define a type $\langle \omega^{n_1}, \ldots, \omega^{n_s} \rangle$ tuple of subsets

$$\mathcal{U}(\boldsymbol{n}) := \langle U_{1,s}(n_1), \ldots, U_{k,s}(n_k), \ldots, U_{s,s}(n_s) \rangle \subseteq \mathcal{W}(\boldsymbol{n}) .$$

Lemma 5.3.7 (positive simplification lemma)**.** *For every $h \in \mathbb{N}_+$, all tuples in $[h\mathcal{U}(\boldsymbol{n})]^r$ are b-simple.*

Proof. Observe that any $\boldsymbol{x} \in [\mathcal{W}(\boldsymbol{n})]^r$ is similar to $h\boldsymbol{x}$. Thus, it suffices to prove the claim for $h = 1$. To this end, consider $\boldsymbol{x} \in [\mathcal{U}(\boldsymbol{n})]^r$ and indices k, i, μ and ℓ, j, ν with $x_{ki}(\mu) = x_{\ell j}(\nu)$. Since $x_{ki} \in U_{k,s}(n_k)$ and $x_{\ell j} \in U_{\ell,s}(n_\ell)$, we have

$$k \equiv x_{ki}(\mu) = x_{\ell j}(\nu) \equiv \ell \pmod{s} .$$

Since $1 \leqslant k, \ell \leqslant s$, this implies $k = \ell$. We further conclude

$$s\mu + k \equiv x_{ki}(\mu) = x_{\ell j}(\nu) \equiv s\nu + k \pmod{sn_k}$$

and hence $\mu \equiv \nu \pmod{n_k}$. Since $1 \leqslant \mu, \nu \leqslant n_k$, this finally implies $\mu = \nu$. $\square$

The combination of the standardization, canonicalization and simplification lemmas provides us with a polarized partition relation, whose optimality is established by theorem 5.4.4.

Theorem 5.3.8. *For all $s, \kappa \in \mathbb{N}$ and $\boldsymbol{n}, \boldsymbol{r} \in \mathbb{N}^s$, the following holds:*

$$\begin{pmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{pmatrix} \xrightarrow{\mathsf{SA}} \begin{bmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{bmatrix}^{r_1,\ldots,r_s}_{\kappa, B(\boldsymbol{n};\boldsymbol{r})}$$

Proof. According to lemma 5.2.7, we only need to take automatic partitions $\varDelta$ of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ into account. Applying lemma 5.3.4 to $\varDelta$ yields some $h \in \mathbb{N}_+$ such that the restriction of $\varDelta$ to $[h\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ is canonical. Due to lemma 5.3.7, all tuples in $[h\mathcal{U}(\boldsymbol{n})]^{\boldsymbol{r}}$ are b-simple. Consequently, $h\mathcal{U}(\boldsymbol{n})$ is a type $\langle \omega^{n_1}, \ldots, \omega^{n_s} \rangle$ tuple of subsets of $\mathcal{W}(\boldsymbol{n})$ which is relatively $B(\boldsymbol{n};\boldsymbol{r})$-homogeneous wrt $\varDelta$. □

Similarly to theorem 4.5.4 on page 150, theorem 5.3.9 is an immediate consequence of applying the positive polarization lemma 5.3.1 to the polarized partition relations just shown. For an ordinal $\alpha < \omega^\omega$ with Cantor normal form $\alpha = \omega^{n_1} + \cdots + \omega^{n_s}$, we put

$$\lambda_{\mathsf{SA}}(\alpha; r) := \sum_{\substack{\tilde{r}_1,\ldots,\tilde{r}_s \in \mathbb{N} \\ \tilde{r}_1+\cdots+\tilde{r}_s = r}} B(n_1, \ldots, n_s; \tilde{r}_1, \ldots, \tilde{r}_s)\,.$$

Theorem 5.3.9. *For every ordinal $\alpha < \omega^\omega$ and all $r, \kappa \in \mathbb{N}$, we have*

$$\alpha \xrightarrow{\mathsf{SA}} [\alpha]^r_{\kappa, \lambda_{\mathsf{SA}}(\alpha;r)}\,. \qquad \square$$

5.4 Exact Values of Automatic Ramsey Degrees

In this section, we provide automatic partitions which prove the partition relations from the previous section to be optimal. In this way, we also establish exact values of automatic Ramsey degrees. Similar to sections 4.3 and 4.6, we proceed by proving the negative polarization lemma 5.4.1 and the negative simplification lemma 5.4.3. The former requires more uniform premises than its non-automatic counterpart, but basically allows for the same proof then. In contrast, the latter lemma requires a completely new proof.

Lemma 5.4.1 (negative polarization lemma)**.** *Let $r \in \mathbb{N}$, $\alpha < \omega^\omega$ be an ordinal, $\alpha = \omega^{n_1} + \cdots + \omega^{n_s}$ its Cantor normal form, $\boldsymbol{A}$ a type $\langle \omega^{n_1}, \dots, \omega^{n_s} \rangle$ tuple of string-automatic well-orders and $\ell \colon \mathcal{R}(s, r) \to \mathbb{N}$ a map. If there is, for each $\tilde{\boldsymbol{r}} \in \mathcal{R}(s, r)$, an automatic partition $\Delta_{\tilde{\boldsymbol{r}}}$ of $[\boldsymbol{A}]^{\tilde{\boldsymbol{r}}}$ into $\ell(\tilde{\boldsymbol{r}})$ classes such that every type $\langle \omega^{n_1}, \dots, \omega^{n_s} \rangle$ tuple of regular subsets $\boldsymbol{X} \subseteq \boldsymbol{A}$ is completely inhomogeneous wrt $\Delta_{\tilde{\boldsymbol{r}}}$, then*

$$\alpha \overset{\mathsf{SA}}{\not\to} [\alpha]^r_{|\ell|} .$$

Proof. We assume without loss of generality that the A_k are mutually disjoint and employ the very same construction as in the proof of lemma 4.3.5 on page 138 then. □

Our next goal is to show the automatic negative simplification lemma, which requires some preparation. Again, we fix $s \in \mathbb{N}$ and $\boldsymbol{n}, \boldsymbol{r} \in \mathbb{N}^s$. The next lemma serves the same purpose to the proof of the automatic version of negative simplification lemma as lemma 4.6.3 on page 151 did to the proof of the non-automatic version.

Lemma 5.4.2. *Let $\boldsymbol{U} \subseteq \mathcal{W}(\boldsymbol{n})$ be a type $\langle \omega^{n_1}, \dots, \omega^{n_s} \rangle$ tuple of regular subsets. There are $p \geqslant 1$ and $q_1 \in \mathbb{N}^{n_1}, \dots, q_s \in \mathbb{N}^{n_s}$ such that $px + q_k \in U_k$ for each $k \in [1, s]$ and $x \in \mathcal{W}(n_k)$.*

Proof. According to corollary 5.2.6, there are $p \geqslant 1$ and a tuple of p-uniformly presentable embeddings $\boldsymbol{f} \colon \mathcal{W}(\boldsymbol{n}) \to \boldsymbol{U}$. Fix some $k \in [1, s]$ and put $n := n_k$. Let $\langle u_0, v_1, u_1, \dots, v_n, u_n \rangle$ be a p-uniform presentation of f_k. Since $U \subseteq \mathtt{1}^+ \cdots \mathtt{n}^+$, there are $1 = \mu_0 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_n \leqslant \mu_{n+1} = n$ such that $v_i = \mu_i^p$

for each $i \in [1, n]$ and $u_j \in \mu_j^* \cdots \mu_{j+1}^*$ for each $j \in [0, n]$. If there was some $i \in [1, n-1]$ with $\mu_i = \mu_{i+1}$, we would obtain $v_i^2 u_i v_{i+1} = v_i u_i v_{i+1}^2$, contradicting the injectivity of f_k. Thus, we conclude $\mu_1 < \cdots < \mu_n$ and hence $\mu_i = i$ for each $i \in [1, n]$.

For all $x \in \mathcal{W}(n_k)$ and $\mu \in [1, n]$, the number of μ-symbols in the string representation of $f_k(x)$ is given by

$$|f_k(x)|_\mu = p \triangle x(\mu) + |u_{\mu-1} u_\mu|_\mu \,.$$

Consequently, there is $q_k \in \mathbb{N}^{n_k}$ such that $\triangle q_k(\mu) := |u_{\mu-1} u_\mu|_\mu$ for each $\mu \in [1, n]$. Clearly, this choice does not depend on x but satisfies $f_k(x) = px + q_k$ and hence $px + q_k \in U$. $\square$

The result below is the announced automatic version of the negative simplification lemma.

Lemma 5.4.3 (negative simplification lemma). *For all type $\langle \omega^{n_1}, \ldots, \omega^{n_s} \rangle$ tuples of regular subsets $\boldsymbol{U} \subseteq \mathcal{W}(\boldsymbol{n})$, the set $[\boldsymbol{U}]^{\boldsymbol{r}}$ intersects every b-simple similarity class.*

Proof. Let p and $q_1, \ldots, q_k$ be as in lemma 5.4.2. Consider some b-simple $\boldsymbol{x} \in [\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ and define $\boldsymbol{y} \in [\boldsymbol{U}]^{\boldsymbol{r}}$ by $y_{ki} := p x_{ki} + q_k$ for all indices k, i. We conclude the proof by demonstrating that $\boldsymbol{x}$ and $\boldsymbol{y}$ are similar, i.e., that the equivalence

$$x_{ki}(\mu) < x_{\ell j}(\nu) \quad \iff \quad p x_{ki}(\mu) + q_k(\mu) < p x_{\ell j}(\nu) + q_\ell(\nu)$$

holds for all indices k, i, μ and ℓ, j, ν.

First, suppose that we have $x_{ki}(\nu) < x_{\ell j}(\nu)$ or, equivalently, $x_{ki}(\nu) + 1 \leqslant x_{\ell j}(\nu)$. This implies

$$p x_{ki}(\mu) + q_k(\mu) < p x_{ki}(\mu) + p \leqslant p x_{\ell j}(\nu) \leqslant p x_{\ell j}(\nu) + q_\ell(\nu)\,.$$

The case $x_{ki}(\mu) > x_{\ell j}(\nu)$ is symmetric. Finally, assume that $x_{ki}(\mu) = x_{\ell j}(\nu)$. Since $\boldsymbol{x}$ is b-simple, we obtain $k = \ell$ and $\mu = \nu$. Thus, $p x_{ki}(\mu) + q_k(\mu) = p x_{\ell j}(\nu) + q_\ell(\nu)$. $\square$

Just like theorem 4.6.5 on page 153, its automatic counterpart below is an immediate consequence of the negative simplification lemma.

Theorem 5.4.4. *For all $s \in \mathbb{N}$ and $\boldsymbol{n}, \boldsymbol{r} \in \mathbb{N}^s$, there is an automatic partition Δ of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ into $B(\boldsymbol{n};\boldsymbol{r})$ classes such that every type $\langle \omega^{n_1}, \dots, \omega^{n_s} \rangle$ tuple of regular subsets $\boldsymbol{U} \subseteq \mathcal{W}(\boldsymbol{n})$ is completely inhomogeneous wrt Δ. In particular, the following partition relation holds:*

$$\begin{pmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{pmatrix} \overset{\mathsf{SA}}{\not\longrightarrow} \begin{bmatrix} \omega^{n_1} \\ \vdots \\ \omega^{n_s} \end{bmatrix}^{r_1,\dots,r_s}_{B(\boldsymbol{n};\boldsymbol{r})}$$

Proof. Let Δ be an arbitrary canonical partition of $[\mathcal{W}(\boldsymbol{n})]^{\boldsymbol{r}}$ into $B(\boldsymbol{n};\boldsymbol{r})$ classes such that no two b-simple similarity classes fall into the same Δ-class. Due to corollary 5.3.3, Δ is automatic. Applying lemma 5.4.3 yields that Δ also satisfies the requirement concerning inhomogeneity. □

Applying the negative polarization lemma 5.4.1 to the polarized partition relations from the previous theorem, we obtain that the automatic partition relation in theorem 5.3.9 is optimal.

Theorem 5.4.5. *For all $r \in \mathbb{N}$ and ordinals $\alpha < \omega^\omega$, we have*

$$\alpha \overset{\mathsf{SA}}{\not\longrightarrow} [\alpha]^r_{\lambda_{\mathsf{SA}}(\alpha;r)} \,. \qquad \square$$

Altogether, lemma 5.1.5 and theorems 5.3.9 and 5.4.5 yield the following positive result on the automatic Ramsey degree.

Theorem 5.4.6. *Let $r \in \mathbb{N}$ and $\alpha < \omega^\omega$ be an ordinal. The automatic r-ary Ramsey degree of α is finite and its exact value is given by*

$$\lambda_{\mathsf{SA}}(\alpha;r) = \sum_{\substack{\tilde{r}_1,\dots,\tilde{r}_s \in \mathbb{N} \\ \tilde{r}_1+\dots+\tilde{r}_s = r}} B(n_1,\dots,n_s;\tilde{r}_1,\dots,\tilde{r}_s)\,,$$

provided that $\alpha = \omega^{n_1} + \cdots + \omega^{n_s}$ is the Cantor normal form of α. □

Since the numbers $B(n_1, \ldots, n_s; \tilde{r}_1, \ldots, \tilde{r}_s)$ can be obtained by counting box diagram shapes belonging to b-simple similarity classes, the value of $\lambda_{\mathsf{SA}}(\alpha; r)$ is easily computable from the Cantor normal form of α. Due to the circumstance, that all constructions taken out in this chapter so far are actually effective, one cannot only compute these values but also a relatively $\lambda_{\mathsf{SA}}(\alpha; r)$-homogeneous type α subset of A.

Corollary 5.4.7. *Given $r \in \mathbb{N}$ and presentations of a string-automatic well-order A and an automatic partition Δ of $[A]^r$, one can compute a string-automaton recognizing a relatively $\lambda_{\mathsf{SA}}(\alpha; r)$-homogeneous type α subset $X \subseteq A$, where α is the order type of A.* □

In view of this result, two questions arise immediately: Suppose we are given $r \in \mathbb{N}$, presentations of a string-automatic type α well-order A and an automatic partition Δ of $[A]^r$ as well as some Δ-classes $D_1, \ldots, D_\lambda$.

(1) Is it decidable whether there exists a (regular) type α subset $X \subseteq A$ such that

$$[X]^r \subseteq D_1 \cup \cdots \cup D_\lambda\,?$$

(2) Provided that a regular subset X with these properties does exist, is there a more ingenious way to compute a string-automaton recognizing some such set X other than enumerating all string-automata and taking the first one to match?

Although these questions are definitely worth being answered, we do not address them here but rather keep focused on the automatic Ramsey degree.

5.5 Infinite Automatic Ramsey Degrees

We complete our investigation of the automatic Ramsey degree by proving that, for $r \geqslant 2$, the r-ary Ramsey degree of any ordinal α with $\omega^\omega \leqslant \alpha < \omega^{\omega^\omega}$ is infinite. To this end, we establish the partition relation

$$\alpha \overset{\mathsf{TA}}{\not\longrightarrow} [\alpha]^r_\kappa \tag{5.5}$$

for all $\kappa \in \mathbb{N}$. The first two lemmas imply that we can focus on the case $r = 2$ and $\alpha = \omega^\gamma$ with $\omega \leqslant \gamma < \omega^\omega$. They are tree-automatic analogues of lemma 4.3.4 on page 137 and lemma 4.7.1 on page 156 and can be proved by the very same constructions as these.

Lemma 5.5.1. *Let $r, \kappa \in \mathbb{N}$ and $\alpha < \omega^{\omega^\omega}$ be an ordinal. If the Cantor normal form of α contains a summand ω^γ with*

$$\omega^\gamma \overset{\mathsf{TA}}{\not\longrightarrow} [\omega^\gamma]^r_\kappa\,,$$

then

$$\alpha \overset{\mathsf{TA}}{\not\longrightarrow} [\alpha]^r_\kappa\,. \qquad \square$$

Lemma 5.5.2. *Let $r, \kappa, \lambda \in \mathbb{N}$ and $\alpha, \beta < \omega^{\omega^\omega}$ be infinite ordinals. If $r \geqslant 2$ and*

$$\alpha \overset{\mathsf{TA}}{\longrightarrow} [\beta]^r_{\kappa,\lambda}\,,$$

then

$$\alpha \overset{\mathsf{TA}}{\longrightarrow} [\beta]^2_{\kappa,\lambda}\,. \qquad \square$$

Recall how a tree-automatic type ω^γ well-order was constructed from a string-automatic type γ well-order A with $A \subseteq (\mathtt{1}^*\mathtt{0})^*$ in example 3.3.20 on page 98: The set

$$\mathbb{N}^{(A)} := \left\{ f\colon A \to \mathbb{N} \;\middle|\; f(u) = 0 \text{ for all but finitely many } u \in A \right\}$$

was linearly ordered using $\trianglelefteq$, which was defined by $f \triangleleft g$ if the greatest $u \in A$ with $f(u) \neq g(u)$ satisfies $f(u) < g(u)$. Afterwards, any $f \in \mathbb{N}^{(A)}$ was encoded by the least (wrt inclusion) $t_f \in T_{\{\mathtt{a}\}}$ with $u\mathtt{1}^{f(u)} \in \mathrm{dom}(t_f)$ for all $u \in A$ with $f(u) \neq 0$. The next lemma can be interpreted as follows: Any type ω^γ subset of $\mathbb{N}^{(A)}$ allows for pumping simultaneously in arbitrarily many of the $\mathtt{1}^{f(u)}$-parts.

Lemma 5.5.3. *Let γ be an ordinal with $\omega \leqslant \gamma < \omega^\omega$, A a type γ well-order, $X \subseteq \mathbb{N}^{(A)}$ a type ω^γ subset and $n, \kappa \in \mathbb{N}$. If γ is infinite, there is $f \in X$ such that $f(u) \geqslant n$ for more than κ distinct $u \in A$.*

Proof. Aiming for a contradiction, suppose there is no such f, i.e., X is a subset of

$$T(A, \kappa) := \left\{ f \in \mathbb{N}^{(A)} \;\middle|\; \exists^{\leqslant\kappa} u \in A \colon f(u) \geqslant n \right\}.$$

Let $t(\gamma, \kappa)$ denote the order type of $T(A, \kappa)$. We derive a contradiction by showing that $\omega^d \leqslant \gamma < \omega^{d+1}$ implies

$$t(\gamma, \kappa) < \omega^{\omega^d} \leqslant \omega^\gamma$$

for all $d \geqslant 1$. For any subset $B \subseteq A$, there is a natural way to regard $T(B, \kappa)$ as a subset of $T(A, \kappa)$. Thus, $t(\beta, \kappa) \leqslant t(\gamma, \kappa)$ whenever $\beta \leqslant \gamma$. Accordingly, it suffices to show, for all $d, \ell \geqslant 1$,

$$t(\omega^d \, \ell, \kappa) < \omega^{\omega^d} . \tag{5.6}$$

For this purpose, we proceed by induction on d and ℓ.

Base case: $d = 1$ and $\ell = 1$. For each $m < \omega$, let $B_m \subseteq A$ be the initial segment of size m. Notice that

$$\bigcup_{m<\omega} T(B_m, \kappa) = T(A, \kappa)$$

and hence

$$t(\omega, \kappa) = \sup_{m<\omega} t(m, \kappa)\,. \tag{5.7}$$

If we had $t(m, \kappa) \geqslant \omega^{\kappa+1}$ for some $m < \omega$, lemma 4.6.3 on page 151 would imply that there is $f \in T(B_m, \kappa)$ with $f(u) \geqslant n$ for at least $\kappa + 1$ distinct $u \in B_m$, contradicting the choice of $T(B_m, \kappa)$. Consequently, $t(m, \kappa) < \omega^{\kappa+1}$ for each m and hence

$$t(\omega, \kappa) \leqslant \omega^{\kappa+1} < \omega^{\omega}\,.$$

Inductive step, case 1: $d > 1$ and $\ell = 1$. For each $m < \omega$, let $B_m \subseteq A$ be the initial type $\omega^{d-1} m$ interval. Using the same argument as above and the induction hypothesis, we obtain

$$t(\omega^d) = \sup_{m<\omega} t(\omega^{d-1} m, \kappa) \leqslant \omega^{\omega^{d-1}} < \omega^{\omega^d}\,.$$

Inductive step, case 2: $d \geqslant 1$ and $\ell > 1$. Let $A = A_1 + \cdots + A_\ell$ be the decomposition of A into its type ω^d intervals. We consider the finite set

$$\mathcal{K}(\ell, \kappa) := \left\{\, \tilde{\boldsymbol{\kappa}} \in \mathbb{N}^\ell \;\middle|\; \tilde{\kappa}_1 + \cdots + \tilde{\kappa}_\ell = \kappa \,\right\}.$$

For each $\tilde{\boldsymbol{\kappa}} \in \mathcal{K}(\ell, \kappa)$, let

$$T(A, \tilde{\boldsymbol{\kappa}}) := \left\{\, f \in \mathbb{N}^{(A)} \;\middle|\; \forall i \in [1, \ell]\ \exists^{\leqslant \kappa_i} u \in A_i \colon f(u) \geqslant n \,\right\}.$$

These sets have two useful properties: (1) Each $T(A, \tilde{\boldsymbol{\kappa}})$ is isomorphic to the product well-order $T(A_1, \tilde{\kappa}_1) \cdots T(A_n, \tilde{\kappa}_\ell)$. (2) The union of all the $T(A, \tilde{\boldsymbol{\kappa}})$ is just $T(A, \kappa)$. Theorem 3.2.2 on page 65 hence implies

$$t(\omega^d \ell, \kappa) \leqslant \bigoplus_{\tilde{\boldsymbol{\kappa}} \in \mathcal{K}(\ell, \kappa)} t(\omega^d, \tilde{\kappa}_1) \cdots t(\omega^d, \tilde{\kappa}_\ell)\,. \tag{5.8}$$

If $d = 1$, we obtain

$$
\begin{aligned}
t(\omega\,\ell, \kappa) &\leqslant \bigoplus_{\tilde{\boldsymbol{\kappa}} \in \mathcal{K}(\ell,\kappa)} t(\omega, \tilde{\kappa}_1) \cdots t(\omega, \tilde{\kappa}_\ell) \\
&\leqslant \bigoplus_{\tilde{\boldsymbol{\kappa}} \in \mathcal{K}(\ell,\kappa)} \omega^{\tilde{\kappa}_1+1} \cdots \omega^{\tilde{\kappa}_\ell+1} \\
&\overset{(\star)}{<} \omega^{\kappa+\ell+1} < \omega^{\omega}\,,
\end{aligned}
$$

where $(\star)$ uses that $\mathcal{K}(\ell, k)$ is finite. If $d > 1$, we obtain

$$
\begin{aligned}
t(\omega^d\,\ell, \kappa) &\leqslant \bigoplus_{\tilde{\boldsymbol{\kappa}} \in \mathcal{K}(\ell,\kappa)} t(\omega^d, \tilde{\kappa}_1) \cdots t(\omega^d, \tilde{\kappa}_\ell) \\
&\leqslant \bigoplus_{\tilde{\boldsymbol{\kappa}} \in \mathcal{K}(\ell,\kappa)} \omega^{\omega^{d-1}\,\ell} \\
&< \omega^{\omega^d}\,.
\end{aligned}
$$

This establishes eq. (5.6) and completes the induction. □

The last gap in establishing the partition relation in eq. (5.5) on page 196 is closed by the following theorem.

Theorem 5.5.4. *For all $\kappa \in \mathbb{N}$ and ordinals γ with $\omega \leqslant \gamma < \omega^\omega$, we have*

$$\omega^\gamma \overset{\mathsf{TA}}{\not\to} [\omega^\gamma]^2_\kappa\,.$$

Proof. Let $\mathbb{N}^{(A)}$ be the type ω^γ well-order whose construction we have just recalled before lemma 5.5.3. For the sake of convenience, we identify each $f \in \mathbb{N}^{(A)}$ with its encoding t_f as a tree. In line with this, we regard $\mathbb{N}^{(A)}$ as a tree-automatic linear order itself. We define an automatic partition $\Delta = \{D_1, \ldots, D_\kappa\}$ of $[\mathbb{N}^{(A)}]^2$ as follows:

$$D_\mu := \left\{ \langle f, g\rangle \in [\mathbb{N}^{(A)}]^2 \;\middle|\; \exists^{=\mu} u \in A\colon f(u) < g(u) \right\}$$

for $\mu < \kappa$ and

$$D_\kappa := \left\{ \langle f, g \rangle \in [\mathbb{N}^{(A)}]^2 \;\middle|\; \exists^{\geqslant \kappa} u \in A\colon f(u) < g(u) \right\}.$$

It is easy to see that Δ is indeed automatic.

Now, we consider a regular type ω^γ subset $X \subseteq \mathbb{N}^{(A)}$. Suppose that X is recognized by a tree-automaton with n states. According to lemma 5.5.3 for each $\mu \in [1, \kappa]$, there are $f \in X$ and a subset $U \subseteq A$ of size μ with $f(u) \geqslant n$ for all $u \in U$. Applying a simple pumping argument to each $\mathtt{1}^{f(u)}$-part of t_f with $u \in U$ in the automaton for X, we obtain $g \in X$ with $f(u) < g(u)$ for $u \in U$ and $f(v) = g(v)$ for $v \notin U$. Notice that $\langle f, g \rangle \in D_\mu$. Consequently, X is completely inhomogeneous wrt Δ. □

We summarize the results of this section in terms of the automatic Ramsey degree by composing lemmas 5.5.1 and 5.5.2 with theorem 5.5.4.

Theorem 5.5.5. *For every $r \geqslant 2$ and all ordinals α satisfying $\omega^\omega \leqslant \alpha < \omega^{\omega^\omega}$, the automatic r-ary Ramsey degree of α is infinite.* □

This result completes our investigation of the automatic Ramsey degree. In the remainder of this chapter, we reuse some of the techniques developed in the previous sections in order to contribute a tree-automatic version of Ramsey's theorem.

5.6 Tree-Automatic Versions of Ramsey's Theorem

We conclude this chapter by investigating the effective content of Ramsey's theorem in the context of tree-automatic (hyper)graphs. Recall that every regular language A of strings or of trees admits an automatic linear ordering by virtue of example 2.4.5 on page 36 and example 3.3.1 on page 77, respectively. Accordingly, we assume the node sets of the (hyper)graphs under consideration to be linearly ordered. Moreover, we regard $[A]^r$ as the set of possible hyperedges of an r-ary hypergraph on A. This is a reasonable assumption since a relation $D \subseteq [A]^r$ is automatic if and only if its *symmetric closure*

$$\{ \langle u_{i_1}, \dots, u_{i_r} \rangle \mid \langle u_1, \dots, u_r \rangle \in D, \{i_1, \dots, i_r\} = \{1, \dots, r\} \}$$

is automatic.

Before proving new results, let us recall the current state of knowledge of string-automatic and tree-automatic versions of Ramsey's theorem. Concerning string-automaticity, the picture is quite complete:

Theorem 5.6.1 (Rubin's theorem [Rub08])**.** *Given $r \in \mathbb{N}$, a presentation of a string-automatic linear order A and a string-automaton recognizing a relation $D \subseteq [A]^r$, one can decide whether there is a (possibly non-regular) infinite subset $X \subseteq A$ such that $[X]^r \subseteq D$. In case of a positive answer, one can compute a string-automaton recognizing some* regular *set X with this property.* □

Along with Ramsey's theorem 4.1.3 on page 126 this immediately implies:

Corollary 5.6.2 ([Rub08])**.** *Given $r \in \mathbb{N}$ and presentations of a string-automatic infinite linear order A and an automatic partition Δ of $[A]^r$, one can compute a string-automaton recognizing some regular infinite subset $X \subseteq A$ which is homogeneous wrt Δ.* □

In the context of tree-automatic structures, only the following decidability result is known from the investigation of so-called *Ramsey quantifiers*.

Theorem 5.6.3 ([Kar11])**.** *Given $r \in \mathbb{N}$, a presentation of a tree-automatic linear order A and a tree-automaton recognizing a relation $D \subseteq [A]^r$, one can decide whether there is a (possibly non-regular) infinite subset $X \subseteq A$ such that $[X]^r \subseteq D$.*

With theorem 5.6.1 in mind, one might wonder whether it is possible to compute a tree-automaton recognizing some *regular* set X with this property in case of a positive answer. Unfortunately, this is not possible as the example below shows:[8]

Example 5.6.4. Let $A := \mathbb{N}_+^2$ be ordered lexicographically, i.e., $x <_A y$ if either $x(1) < y(1)$ or both $x(1) = y(1)$ and $x(2) < y(2)$. Moreover, let

$$D := \left\{ \langle x, y \rangle \in [A]^2 \;\middle|\; x(1) < y(1) \text{ and } x(2) < y(2) \right\}.$$

Encoding $x \in A$ by the unique $t_x \in T_{\{\mathtt{a}\}}$ with

$$\operatorname{dom}(t_x) = \mathtt{0}^{\leqslant x(1)} \cup \mathtt{1}^{\leqslant x(2)}$$

turns A into a tree-automatic linear order. Obviously, D is also automatic under this encoding. On the one hand, there is an infinite set $X \subseteq A$ such that $[X]^2 \subseteq D$, e.g.,

$$X = \{\, x \in A \mid x(1) = x(2) \,\}.$$

On the other hand, there is no set X with this property whose encoding is regular.

To see this, we aim for a contradiction and assume there is some such set X. Suppose the encoding of X is recognized by a tree-automaton $\mathcal{T}$ with n states. For distinct $x, y \in X$, the choice of D implies $x(1) \neq y(1)$ and $x(2) \neq y(2)$. Since X is

[8]This example was kindly provided by Alexander Kartzow, the author of [Kar11] himself.

infinite, there hence is $x \in X$ with $x(1), x(2) > n$. Applying a simple pumping argument to t_x in $\mathcal{T}$, we obtain some $y \in X$ with $y(1) > x(1)$ and $y(2) < x(2)$. This implies $x < y$ but $\langle x, y \rangle \notin D$, contradicting $[X]^2 \subseteq D$. $\square$

Intuitively, the crucial property of the set D is that (the encoding of) any infinite subset $X \subseteq A$ with $[X]^2 \subseteq D$ has to grow simultaneously along two infinite branches. Obviously, such behavior cannot be guaranteed by tree-automata. In contrast, every regular infinite language of trees contains a regular infinite subset growing along one branch only. Using this connection, we now show a tree-automatic version of corollary 5.6.2. In addition, we demonstrate a weaker version of theorem 5.6.1 afterwards. Basically, both proofs apply the idea from the proof of lemma 5.2.3 to languages of trees growing along one branch only. In order to make this precise, we need to lift the required concepts of algebraic automata theory from languages of strings to such restricted languages of trees first.

Let Σ be an alphabet and $\bullet \notin \Sigma$ a new symbol. A *Σ-context* is a tree $\alpha \in T_{\Sigma \cup \{\bullet\}}$ satisfying two conditions: (1) there is at most one $u \in \mathrm{dom}(\alpha)$ with $\alpha(u) = \bullet$ and (2) this u is a leaf of α whenever it exists. We refer to this u as the *hole position* of α and call α a *proper context* if it does exist.[9] Otherwise, α is just an ordinary Σ-tree. The set of all Σ-contexts is denoted by C_Σ. Notice that $T_\Sigma \subseteq C_\Sigma$. We turn the set C_Σ into a semigroup by defining

$$\alpha\beta := \begin{cases} \alpha[u/\beta] & \text{if } \alpha \text{ has a hole at position } u, \\ \alpha & \text{if } \alpha \text{ is an ordinary tree.} \end{cases}$$

As a matter of fact, C_Σ contains a neutral element, namely the unique proper context $\alpha \in C_\Sigma$ with $\alpha(\varepsilon) = \bullet$. Using the

[9] As we are not dealing with ordinals in this section, we denote contexts by $\alpha, \beta, \ldots$.

semigroup C_Σ, the idea of pumping in regular languages of trees can be expressed as follows: Let $A \subseteq T_\Sigma$ be a regular language recognized by a tree-automaton with n states. For every $t \in A$ of height $\mathrm{h}(t) \geqslant n$, there are proper contexts $\alpha, \beta \in C_\Sigma$ and a tree $s \in T_\Sigma$ with $t = \alpha\beta s$, $\beta(\varepsilon) \neq \bullet$ and $\alpha\beta^k s \in A$ for all $k \in \mathbb{N}$.

The next step of our algebraization exhibits a relationship between tree-automata over Σ and morphisms from C_Σ into some semigroup. To this end, let $\mathcal{T} = (Q, \iota, \delta, F)$ be a tree-automaton. The *transformation semigroup* of $\mathcal{T}$ is the set Q^Q of maps $f\colon Q \to Q$ together with function composition $(f \circ g)(q) = f(g(q))$. We define a map $\mu_\mathcal{T}\colon C_\Sigma \to Q^Q$ by

$$(\mu_\mathcal{T}(\alpha))(q) := \begin{cases} \delta_{u/q}(\iota, \alpha) & \text{if } \alpha \text{ has a hole at position } u, \\ \delta(\iota, \alpha) & \text{if } \alpha \text{ is an ordinary tree.} \end{cases}$$

Obviously, $\mu_\mathcal{T}(t)$ is a constant map for all $t \in T_\Sigma$. In terms of $\mu_\mathcal{T}$, the language recognized by $\mathcal{T}$ is given as

$$L(\mathcal{T}) = \{\, t \in T_\Sigma \mid \mu_\mathcal{T}(t) \in F^Q \,\}\,.$$

It is a matter of routine to verify that $\mu_\mathcal{T}$ is a morphism of semigroups.[10]

In the following, we need the unsurprising fact that the map $\mu_\mathcal{T}$ can be computed by a tree-automaton over $\Sigma \cup \{\bullet\}$. Clearly, the set C_Σ is easily recognizable by a tree-automaton. For each $f \in Q^Q$, we consider the tree-automaton $\mathcal{T}_f = (Q^Q, \iota', \delta', \{f\})$ whose initial state ι' constantly maps to ι and whose transition

[10] In view of these results, one might think about defining the notion of a language of Σ-trees being recognized by a morphism $\mu\colon C_\Sigma \to S$ into a finite semigroup. In fact, one can show that a language is recognizable in that sense if and only if it is regular. However, this is of no great use here since the semigroup C_Σ is *not* finitely generated.

map δ' is given by

$$(\delta'(g, a, h))(q) := \begin{cases} \delta(g(q), a, h(g)) & \text{if } a \in \Sigma, \\ q & \text{if } a = \bullet. \end{cases}$$

It is another matter of routine to check that $\delta'(\iota', \alpha) = \mu_{\mathcal{T}}(\alpha)$ for all $\alpha \in C_\Sigma$. Consequently, the direct product of $\mathcal{T}_f$ with the tree-automaton recognizing C_Σ accepts a tree $\alpha \in T_{\Sigma \cup \{\bullet\}}$ if and only if it is a context with $\mu_{\mathcal{T}}(\alpha) = f$.

Finally, we need to introduce some technical notation for the convolution of contexts. Let $r \in \mathbb{N}$, $i \in [1, r]$ and $\alpha, \beta \in C_\Sigma$ be two contexts satisfying the following conditions: (1) β is a proper context with hole position u and (2) α is either a proper context with hole position u as well or an ordinary tree with $u \notin \operatorname{dom}(\alpha)$. Let $\alpha \otimes \beta$ denote the convolution of α and β as elements of $T_{\Sigma \cup \{\bullet\}}$. We define a $\Sigma_\diamond^r$-context $\alpha \otimes_i^r \beta$ with hole position u by $\operatorname{dom}(\alpha \otimes_i^r \beta) := \operatorname{dom}(\alpha \otimes \beta)$ and

$$(\alpha \otimes_i^r \beta)(v) := \begin{cases} \langle \diamond, \cdots, \diamond, a, b, \ldots, b \rangle & \text{if } v \neq u \text{ and} \\ & \qquad (\alpha \otimes \beta)(v) = \langle a, b \rangle, \\ \bullet & \text{if } v = u, \end{cases}$$

where the a sits in the i^{th} component, i.e., the number of $\diamond$-symbols and b-symbols are $i-1$ and $r-i$, respectively. Intuitively, $\alpha \otimes_i^r \beta$ is obtained by convolving $i-1$ copies of the "empty tree", one copy of α and $r-i$ copies of β while keeping the hole position the same as in β. Using this notation, we now provide a definition which is fundamental for the remainder of this section.

Definition 5.6.5. Let $r \in \mathbb{N}$, $A \subseteq T_\Sigma$ and $\mathcal{T}$ be a tree-automaton over $\Sigma_\diamond^r$. A *homogenerator for* $\mathcal{T}$ is a triple $\langle \alpha, \beta, s \rangle$ consisting of two proper contexts $\alpha, \beta \in C_\Sigma$ and a tree $s \in T_\Sigma$ satisfying the following conditions:

(1) The hole position of β is not contained in $\mathrm{dom}(s)$.

(2) $\mu_{\mathcal{T}}(\beta \otimes_i^r \beta)$ is idempotent for all $i \in [1, r]$.

The term "homogenerator" is an amalgamation of "homogeneous" and "generator". In fact, the proof of lemma 5.6.7 shows that $\mathcal{T}$ either accepts all elements of $[\alpha(\beta\beta)^+ s]^r$ or none of them. In other words, the set $\alpha(\beta\beta)^+ s$ *generated* by $\langle \alpha, \beta, s \rangle$ is *homogeneous* wrt the relation recognized by $\mathcal{T}$. The proofs of theorems 5.6.8 and 5.6.10 both use a characterization of the existence of homogeneous regular infinite subsets in terms of the existence of homogenerators. This characterization is prepared by the next lemma.

Lemma 5.6.6. *Let $A \subseteq T_\Sigma$ be a regular infinite language and $\mathcal{T}_1, \ldots, \mathcal{T}_\kappa$ tree-automata over $\Sigma_\diamond^r$. There* effectively *exists a triple $\langle \alpha, \beta, s \rangle$ with $\alpha\beta^* s \subseteq A$ which is a homogenerator for all the $\mathcal{T}_\xi$ simultaneously.*

Proof. Since A is infinite, a simple pumping argument provides us with proper contexts $\alpha, \beta \in C_\Sigma$ and a tree $s \in T_\Sigma$ such that β is non-trivial and $\alpha\beta^k s \in A$ for all $k \geqslant 0$. Let $m \geqslant 1$ be a common multiple of the exponents of the transformation semigroups of all $\mathcal{T}_\xi$. Since the hole position u of β is not ε, we may additionally assume $m \cdot |u| > \mathrm{h}(s)$. We show that the triple $\langle \alpha, \beta^m, s \rangle$ is a homogenerator for each $\mathcal{T}_\xi$.

The hole position of β^m is u^m and hence condition (1) of definition 5.6.5 is obviously satisfied. Concerning condition (2), observe that, for each $i \in [1, r]$,

$$\mu_{\mathcal{T}_\xi}(\beta^m \otimes_i^r \beta^m) = \mu_{\mathcal{T}_\xi}((\beta \otimes_i^r \beta)^m) = (\mu_{\mathcal{T}_\xi}(\beta \otimes_i^r \beta))^m .$$

Due to the choice of m, this element of the transformation semigroup of $\mathcal{T}_\xi$ is idempotent. Clearly, all the constructions taken out in this proof are effective. □

The aforementioned characterization of the existence of homogeneous regular infinite subsets is as follows:

Lemma 5.6.7. *Let $A \subseteq T_\Sigma$ be a regular infinite language and $\mathcal{T}$ a tree-automaton recognizing the symmetric closure of a relation $D \subseteq [A]^r$. The following conditions are* effectively *equivalent:*

(1) There is a regular infinite subset $X \subseteq A$ such that $[X]^r \subseteq D$.

(2) There is a homogenerator $\langle \alpha, \beta, s \rangle$ for $\mathcal{T}$ such that $\mathcal{T}$ accepts the tuple

$$\langle \alpha\beta^2 s, \alpha\beta^4 s, \ldots, \alpha\beta^{2r} s \rangle .$$

Proof. First, suppose that condition (1) is satisfied. According to lemma 5.6.6, there is a homogenerator $\langle \alpha, \beta, s \rangle$ for $\mathcal{T}$ with $\alpha\beta^* s \subseteq X$. Since $\mathcal{T}$ recognizes the symmetric closure of D, it particularly accepts the tuple $\langle \alpha\beta^2 s, \alpha\beta^4 s, \ldots, \alpha\beta^{2r} s \rangle$ which is contained therein.

Now, suppose that condition (2) is satisfied. Clearly, the set $X := \alpha(\beta\beta)^+ s$ is regular and infinite. Notice that $[X]^r \subseteq D$ would particularly imply $X \subseteq A$. In order to prove $[X]^r \subseteq D$, it suffices to show that $\mathcal{T}$ accepts, for each $x \in \mathcal{W}(r)$, the tuple

$$\langle \alpha\beta^{2x(1)} s, \alpha\beta^{2x(2)} s, \ldots, \alpha\beta^{2x(r)} s \rangle .$$

According to condition (1) of definition 5.6.5, the hole position of β is not contained in $\mathrm{dom}(s)$. Thus,

$$\begin{aligned} &\otimes\langle \alpha\beta^{2x(1)} s, \ldots, \alpha\beta^{2x(r)} s \rangle \\ &\qquad = (\alpha \otimes_1^r \alpha) \cdot (\beta \otimes_1^r \beta) \cdot \prod_{1 \leqslant i \leqslant r} \Big((\beta \otimes_i^r \beta)^{2\Delta x(i) - 1} \cdot (s \otimes_i^r \beta) \Big) . \end{aligned}$$

Due to condition (2) of definition 5.6.5, we may conclude

$$\begin{aligned}
&\mu_{\mathcal{T}}(\otimes\langle\alpha\beta^{2x(1)}s,\dots,\alpha\beta^{2x(r)}s\rangle)\\
&\quad= \mu_{\mathcal{T}}\Big((\alpha\otimes_1^r\alpha)\cdot(\beta\otimes_1^r\beta)\cdot\prod_{1\leqslant i\leqslant r}\Big((\beta\otimes_i^r\beta)^{2\Delta x(i)-1}\cdot(s\otimes_i^r\beta)\Big)\Big)\\
&\quad\overset{(\star)}{=} \mu_{\mathcal{T}}\Big((\alpha\otimes_1^r\alpha)\cdot(\beta\otimes_1^r\beta)\cdot\prod_{1\leqslant i\leqslant r}\Big((\beta\otimes_i^r\beta)\cdot(s\otimes_i^r\beta)\Big)\Big)\\
&\quad= \mu_{\mathcal{T}}(\otimes\langle\alpha\beta^2 s,\dots,\alpha\beta^{2r}s\rangle)\,,
\end{aligned}$$

where $(\star)$ actually uses that the $\mu_{\mathcal{T}}(\beta\otimes_i^r\beta)$ are idempotent. Since the automaton $\mathcal{T}$ accepts $\langle\alpha\beta^2 s,\dots,\alpha\beta^{2r}s\rangle$, it hence also accepts $\langle\alpha\beta^{2x(1)}s,\dots,\alpha\beta^{2x(r)}s\rangle$. □

Putting together lemmas 5.6.6 and 5.6.7 yields the following tree-automatic version of Ramsey's theorem:

Theorem 5.6.8. *Given $r\in\mathbb{N}$ and presentations of a tree-automatic infinite linear order A and an automatic partition Δ of $[A]^r$, one can compute a tree-automaton recognizing some regular infinite subset $X\subseteq A$ which is homogeneous wrt Δ.*

Proof. For each Δ-class D, let $\mathcal{T}_D$ be a tree-automaton recognizing the symmetric closure of D. According to lemma 5.6.6, there is a triple $\langle\alpha,\beta,s\rangle$ which is a homogenerator for all the $\mathcal{T}_D$ simultaneously. Since Δ forms a partition of $[A]^r$, there is a Δ-class D_0 such that $\mathcal{T}_{D_0}$ accepts the tuple $\langle\alpha\beta^2 s,\alpha\beta^4 s,\dots,\alpha\beta^{2r}s\rangle$. According to lemma 5.6.7, this implies that there is a regular infinite subset $X\subseteq A$ such that $[X]^r\subseteq D_0$. Since all involved constructions are effective and it is decidable whether $\mathcal{T}_D$ accepts a given tuple, the claim follows. □

As already indicated, a quite intricate situation can arise: On the one hand, the algorithm from theorem 5.6.8 yields a tree-automaton recognizing an infinite subset $X\subseteq A$ such that $[X]^r$ is entirely contained in some Δ-class D_1. On the other hand, the decision procedure from theorem 5.6.3 tells us that there is a (possibly non-regular) infinite subset $Y \subseteq A$ such that $[Y]^r$ is entirely contained in a certain other Δ-class D_2. Due to example 5.6.4, there might be no regular set Y with this property *in general.* Two questions arise immediately: Can we find out whether we really are in the general case or rather in a situation where a regular Y does exist? And if we actually find ourselves in this latter situation, can we compute a tree-automaton recognizing some such set Y then? Fortunately, the answer to both question is affirmative. In order to prove this, we show that the characterization in lemma 5.6.7 can be made effective by means of tree-automata:

Lemma 5.6.9. *For every tree-automaton $\mathcal{T}$ over $\Sigma_\diamond^r$, the relation*

$$S_\mathcal{T} := \left\{ \langle \alpha, \beta, s \rangle \;\middle|\; \begin{array}{l} \langle \alpha, \beta, s \rangle \text{ is a homogenerator for } \mathcal{T} \\ \text{and } \mathcal{T} \text{ accepts } \langle \alpha\beta^2 s, \ldots, \alpha\beta^{2r} s \rangle \end{array} \right\}$$

is effectively *automatic.*

Proof. It is easy to see that there is a tree-automaton over $(\Sigma \cup \{\diamond, \bullet\})^3$ recognizing the set of triples $\langle \alpha, \beta, s \rangle$ consisting of two proper contexts $\alpha, \beta \in C_\Sigma$ and a tree $s \in T_\Sigma$ satisfying condition (1) of definition 5.6.5. Hence, it suffices to provide a tree-automaton $\mathcal{T}'$ accepting such triple $\langle \alpha, \beta, s \rangle$ precisely if it satisfies condition (2) of definition 5.6.5 and $\mathcal{T}$ accepts $\langle \alpha\beta^2 s, \ldots, \alpha\beta^{2r} s \rangle$. Put another way, $\mathcal{T}'$ needs to verify that $\mu_\mathcal{T}(\beta \otimes_i^r \beta)$ is idempotent for each $i \in [1, r]$ and that $\mu_\mathcal{T}(\otimes\langle \alpha\beta^2 s, \ldots, \alpha\beta^{2r} s \rangle)$ constantly maps to a final state of $\mathcal{T}$. Just like in the proof of lemma 5.6.7, we have

$$\begin{aligned} &\mu_\mathcal{T}(\otimes\langle \alpha\beta^2 s, \ldots, \alpha\beta^{2r} s \rangle) \\ &\qquad = \mu_\mathcal{T}(\alpha \otimes_1^r \alpha) \cdot \mu_\mathcal{T}(\beta \otimes_1^r \beta) \cdot \prod_{1 \leqslant i \leqslant r} \Big(\mu_\mathcal{T}(\beta \otimes_i^r \beta) \cdot \mu_\mathcal{T}(s \otimes_i^r \beta) \Big) . \end{aligned}$$

Thus, it suffices to demonstrate that $\mathcal{T}'$ can simultaneously compute $\mu_{\mathcal{T}}(\alpha \otimes_1^r \alpha)$, $\mu_{\mathcal{T}}(\beta \otimes_i^r \beta)$ and $\mu_{\mathcal{T}}(s \otimes_i^r \beta)$ for $i \in [1, r]$. In fact, the basic idea behind the construction of such an automaton $\mathcal{T}'$ is the same as for the automaton $\mathcal{T}_f$ on page 204. The missing details are straightforward to add. □

The two announced affirmative answers are given by the theorem below, which is the tree-automatic version of theorem 5.6.1. In fact, it is slightly weaker than its string-automatic counterpart but the best one can expect in view of example 5.6.4.

Theorem 5.6.10. *Given $r \in \mathbb{N}$, a presentation of a tree-automatic linear order A and a tree-automaton recognizing a relation $D \subseteq [A]^r$, one can decide whether there is a* regular *infinite subset $X \subseteq A$ such that $[X]^r \subseteq D$. In case of a positive answer, one can compute a tree-automaton of elementary size which recognizes some set X with this property.*

Proof. Let $\mathcal{T}$ be a tree-automaton recognizing the symmetric closure of D and $\mathcal{T}'$ the tree-automaton recognizing $S_{\mathcal{T}}$, which exists by lemma 5.6.9. According to lemma 5.6.7, there is a regular infinite subset $X \subseteq A$ with $[X]^r \subseteq D$ if and only if $S_{\mathcal{T}}$ is non-empty. Since all involved constructions are effective and non-emptiness of $S_{\mathcal{T}}$ is decidable from $\mathcal{T}'$, the claim on decidability follows. If $S_{\mathcal{T}}$ turns out to be non-empty, one can also compute an element $\langle \alpha, \beta, s \rangle \in S_{\mathcal{T}}$ from $\mathcal{T}'$ and hence a tree-automaton recognizing X. It is a matter of routine to check that the size of this tree-automaton is indeed elementary in the size of the input. □

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List of Figures

Index