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Junjie Huang, Junfeng Sun, Alatancang Chen, Carsten Trunk

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Abstract

In this paper the properties of right invertible row operators, i.e., of $1 \times 2$ surjective operator matrices are studied. This investigation is based on a specific space decomposition. Using this decomposition, we characterize the invertibility of a $2 \times 2$ operator matrix. As an application, the invertibility of Hamiltonian operator matrices is investigated.

Keywords: $2 \times 2$ operator matrix, Hamiltonian operator matrix, invertibility, row operator

MSC 2010: 47A05, 47A10.

1 Introduction

The invertibility of a linear operator is one of the most basic problems in operator theory, and, obviously, appears in the study of the linear equation $Tx = y$ with a linear operator $T$.

This problem becomes even more involved if one considers the invertibility of $2 \times 2$ operator matrices. For this let $A$, $B$, $C$ and $D$ be bounded linear operators on a Hilbert space. If, e.g., they are pairwise commutative, then the operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is invertible if and only if $AD - BC$ is invertible (cf. [3, Problem 70]). If only $C$ and $D$ are commutative, and if, in addition, $D$ is invertible, then the operator matrix $M$ is invertible if and only if $AD - BC$ is invertible (cf. [3, Problem 71]). In fact, the commutativity is essential in the above characterization, see [3, Problem 71]. The situation is even more involved if $A$ and $D$ are not defined on the same space and, hence, the formal expression $AD - BC$ has no meaning.

In general, there is no complete description of the invertibility of operator matrices in the non-commutative case. But if at least one of the entries $A$ or $D$ of
the operator matrix $M$ is invertible, one can describe the invertibility of $M$ in terms of the Schur complement. A similar statement holds also in the case of invertible entries $B$ or $C$. Moreover, the Schur complement method can be effectively used also in the case where the entries of $M$ are unbounded operators under additionally assumptions on the domain of the entries, such as the diagonally (or off-diagonally) dominant or upper (lower) dominant cases, see, e.g., the monograph [7]. We also refer to [5, 8] for sufficient conditions for nonnegative Hamiltonian operators to have bounded inverses.

However, it is easy to see that there are many invertible $2 \times 2$ operator matrices with non invertible entries $A, B, C$ and $D$ (see, e.g., Theorem 2.11 below). Obviously, in such cases, the Schur complement method is not applicable.

It is the aim of the present article to give a full characterization for the invertibility of bounded $2 \times 2$ operator matrices. We do this in the following manner: A necessary condition for the invertibility of a $2 \times 2$ operator matrix $M$ in (1.1) is the fact that the row operator $(A \ B)$ is right invertible (that is, the range $R((A \ B))$ of the operator $(A \ B)$ covers all of the spaces). A further necessary condition is $N((A \ B)) \neq \{0\}$, where $N((A \ B))$ denotes the kernel of $(A \ B)$ (see Corollary 3.3 below). This non-zero kernel $N((A \ B))$ plays a crucial role. Its projection $P_X(N((A \ B)))$ onto the first component is a subset of the kernel of $P_{R(B)} \perp A$, where $P_{R(B)} \perp$ denotes the orthogonal projection onto $R(B) \perp$. Similarly, the projection of $N((A \ B))$ onto the second component is a subset of $N(P_{R(A)} \perp B)$.

Therefore we investigate a right invertible row operator $(A \ B)$ and choose a decomposition of the space into six parts which is built out of the subspaces $N(A), N(B), N(P_{R(B)} \perp A)$ and $N(P_{R(A)} \perp B)$. As a result, we show that the operator $B_2^{-1} \tilde{A}_2$ considered as an operator from $P_X(N((A \ B)))$ to $N(B) \perp \ominus N(P_{R(A)} \perp B) \perp$ is correctly defined. Here $\tilde{A}_2$ ($B_2$) denote the restriction of $A$ ($B$, respectively) to $N(P_{R(B)} \perp A)$ ($N(B) \perp \ominus N(P_{R(A)} \perp B)$, respectively).

The main result of the present article is a full characterization of the invertibility of a $2 \times 2$ matrix operator $M$ in terms of its entries $A, B, C, D$, or to be more precise, in terms of the restrictions $\tilde{A}_2, B_2, C_2$ and $D_2$ which are, in some sense, all related to $N((A \ B))$: A $2 \times 2$ operator matrix $M$ is invertible if and only if the following two statements are satisfied

(i) The restriction $D|_{N(B)}$ is left invertible and

(ii) the operator

$$C_2 - D_2 B_2^{-1} \tilde{A}_2 : P_X(N((A \ B))) \to (R(D|_{N(B)}))^\perp$$

is one-to-one and surjective.

Here $C_2$ ($D_2$) is the restriction of $C$ ($D$, respectively) to $N(P_{R(B)} \perp A)$ ($N(B) \perp \ominus N(P_{R(A)} \perp B)$, respectively) projected onto $(R(D|_{N(B)}))^\perp$.

This characterization is especially helpful if the spaces $N((A \ B)), N(P_{R(B)} \perp A)$ or $N(P_{R(A)} \perp B)$ are known explicitly, see, e.g., Theorem 2.11 in Section 2. Moreover, we use it to derive a characterization for isomorphic row operators in Section 3. Finally, in Section 4 we give an application to Hamiltonian operators.
2 Main result

We always assume that \( \mathcal{X} \) and \( \mathcal{Y} \) are complex separable Hilbert spaces. Let \( T \) be a bounded operator between \( \mathcal{X} \) and \( \mathcal{Y} \). We write \( T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \) and, if \( \mathcal{X} = \mathcal{Y} \), \( T \in \mathcal{B}(\mathcal{X}) \). The range of \( T \) is denoted by \( \mathcal{R}(T) \), the kernel by \( \mathcal{N}(T) \). The term isomorphism is reserved for linear bijections \( T : \mathcal{X} \to \mathcal{Y} \) that are homeomorphisms, i.e., \( T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \) and \( T^{-1} \in \mathcal{B}(\mathcal{Y}, \mathcal{X}) \).

A subspace in \( \mathcal{Y} \) is an operator range if it coincides with the range of some bounded operator \( T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \). The following lemma is from [2, Theorem 2.4].

**Lemma 2.1** Let \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) be operator ranges in \( \mathcal{Y} \) such that \( \mathcal{R}_1 + \mathcal{R}_2 \) is closed.

(i) If \( \mathcal{R}_1 \cap \mathcal{R}_2 \) is closed, then \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are closed.

(ii) If \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are dense in \( \mathcal{Y} \), then \( \mathcal{R}_1 \cap \mathcal{R}_2 \) is dense in \( \mathcal{Y} \).

From [1, Proposition 2.14, Theorem 2.16], we have the following basic facts, which are important in the proofs of our main results.

**Lemma 2.2** Let \( \Omega_1 \) and \( \Omega_2 \) be two closed subspaces in \( \mathcal{X} \). Then

\[
\Omega_1 \cap \Omega_2 = (\Omega_1^\perp + \Omega_2^\perp)^\perp, \quad \Omega_1^\perp \cap \Omega_2^\perp = (\Omega_1 + \Omega_2)^\perp,
\]

and we further have the following equivalent descriptions:

(i) \( \Omega_1 + \Omega_2 \) is closed;

(ii) \( \Omega_1^\perp + \Omega_2^\perp \) is closed;

(iii) \( \Omega_1 + \Omega_2 = (\Omega_1^\perp \cap \Omega_2^\perp)^\perp \);

(iv) \( (\Omega_1 \cap \Omega_2)^\perp = \Omega_1^\perp + \Omega_2^\perp \).

As usual, the symbol \( \oplus \) denotes the orthogonal sum of two closed subspaces in a Hilbert space whereas the symbol \( \dot{+} \) denotes the direct sum of two (not necessarily closed) subspaces in a Hilbert space. If \( \Omega, \Omega_1 \) are closed subspaces, \( \Omega \subset \Omega_1 \), we denote by \( \Omega \ominus \Omega_1 \) the uniquely determined closed subspace \( \Omega_2 \) in \( \Omega \) with \( \Omega = \Omega_1 \oplus \Omega_2 \).

The next lemma is well known, see, e.g., [7, Proposition 1.6.2] or [4, 6].

**Lemma 2.3** Let \( A \in \mathcal{B}(\mathcal{X}), B \in \mathcal{B}(\mathcal{Y}, \mathcal{X}), C \in \mathcal{B}(\mathcal{X}, \mathcal{Y}), \) and \( D \in \mathcal{B}(\mathcal{Y}) \). Let \( A \) \((B)\) be an isomorphism. Then the \( 2 \times 2 \) operator matrix

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y})
\]

is an isomorphism if and only if \( D - CA^{-1}B \) (resp. \( C - DB^{-1}A \)) is an isomorphism.
Recall that an operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is called right invertible if there exists an operator $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ with $TS = I_Y$, where $I_Y$ stands for the identity mapping in $\mathcal{Y}$. Hence, if $T$ is right invertible then it is surjective. Conversely, if $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ then the restriction $T|_{\mathcal{N}(T)^\perp}$ maps $\mathcal{N}(T)^\perp$ onto $\mathcal{R}(T)$ and, if $\mathcal{R}(T) = \mathcal{Y}$, then $T|_{\mathcal{N}(T)^\perp} : \mathcal{N}(T)^\perp \to \mathcal{Y}$ is an isomorphism. Then with

$$S := \begin{pmatrix} 0 \\ (T|_{\mathcal{N}(T)^\perp})^{-1} \end{pmatrix} : \mathcal{Y} \to \mathcal{N}(T) \oplus \mathcal{N}(T)^\perp$$

considered as an operator in $\mathcal{B}(\mathcal{Y}, \mathcal{X})$ we see that $T$ is right invertible. This shows the equivalence of (i)-(iii) in the following (well-known) lemma.

**Lemma 2.4** For $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ the following assertions are equivalent.

(i) The operator $T$ is right invertible.

(ii) $\mathcal{R}(T) = \mathcal{Y}$.

(iii) The operator $T|_{\mathcal{N}(T)^\perp}$ considered as an operator from $\mathcal{N}(T)^\perp$ into $\mathcal{Y}$ is an isomorphism.

(iv) There exists an isomorphism $U \in \mathcal{B}(\mathcal{Y})$ such that $UT$ is a right invertible operator.

**Proof.** It remains to show the equivalence of (iv) with (i)-(iii). Choose $U = I_Y$ and we see that (i) implies (iv). Conversely, let $U \in \mathcal{B}(\mathcal{Y})$ be an isomorphism. If $UT$ is right invertible, then by (ii) $\mathcal{R}(UT) = \mathcal{Y}$. As $\mathcal{R}(T) = \mathcal{R}(UT)$, again (ii) shows that $T$ is right invertible. \(\Box\)

Similarly, $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is called left invertible if there exists an operator $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ with $ST = I_X$. Hence, if $T$ is left invertible then it is injective and for a sequence $(y_n)$ in $\mathcal{R}(T)$ with $y_n \to y$ as $n \to \infty$ we find $(x_n)$ with $Tx_n = y_n$ and

$$x_n = STx_n = Sy_n \to Sy \quad \text{and} \quad y_n = Tx_n \to TSy,$$

which shows the closedness of $\mathcal{R}(T)$.

Conversely, if $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T)$ is closed, then $T$ considered as an operator from $\mathcal{X}$ into $\mathcal{R}(T)$ is an isomorphism and its inverse $T^{-1}$ acts from $\mathcal{R}(T)$ into $\mathcal{X}$. Then with

$$S := \begin{pmatrix} 0 & T^{-1} \end{pmatrix} : \mathcal{R}(T)^\perp \oplus \mathcal{R}(T) \to \mathcal{X},$$

considered as an operator in $\mathcal{B}(\mathcal{Y}, \mathcal{X})$, we see that $T$ is left invertible. We collect these statements in the following lemma, where the equivalence of (i)-(iii) follows from the above considerations and the equivalence of (i)-(iii) with (iv) is obvious.

**Lemma 2.5** For $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ the following assertions are equivalent.
The operator $T$ is left invertible.

The operator $T$ considered as an operator from $X$ into $\mathcal{R}(T)$ is an isomorphism.

There exists an isomorphism $V \in \mathcal{B}(X)$ such that $TV$ is a left invertible operator.

Remark 2.6 The following observation for $T \in \mathcal{B}(X,Y)$ follows immediately from the Lemmas 2.4 and 2.5. If $T$ is right invertible, then there exists a left invertible operator $S \in \mathcal{B}(Y,X)$ (cf. (2.1)) with $TS = I_Y$ and $\mathcal{R}(S) = N(T) \downarrow$. If $T$ is left invertible, then there exists a right invertible operator $S \in \mathcal{B}(Y,X)$ (cf. (2.2)) with $ST = I_X$.

For the orthogonal projection onto a closed subspace $\Omega$ in some Hilbert space we shortly write $P_\Omega$.

Theorem 2.7 Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y,X)$ and assume that the row operator $(A \ B) \in \mathcal{B}(X \oplus Y, X)$ is right invertible. Then $X$ admits the decomposition

$$
X = (\mathcal{R}(A) \downarrow + \mathcal{R}(B) \downarrow) \oplus \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}
$$(2.3)

and the space $X \oplus Y$ admits the decomposition

$$
X \oplus Y = X_1 \oplus X_2 \oplus X_3 \oplus Y_3 \oplus Y_2 \oplus Y_1,
$$(2.4)

where

$$
X_1 := N(A), \hspace{1em} X_2 := N(A) \downarrow \ominus N(P_{\mathcal{R}(B) \downarrow} A) \downarrow, \hspace{1em} X_3 := N(P_{\mathcal{R}(B) \downarrow} A), \hspace{1em} Y_1 := N(B), \hspace{1em} Y_2 := N(B) \downarrow \ominus N(P_{\mathcal{R}(A) \downarrow} B) \downarrow, \hspace{1em} Y_3 := N(P_{\mathcal{R}(A) \downarrow} B) \downarrow.
$$(2.5)

The row operator $(A \ B)$ from $X \oplus Y$ into $X$ admits the following representation with respect to the decompositions (2.3) and (2.4)

$$
\begin{pmatrix}
0 & 0 & 0 & B_3 & 0 & 0 \\
0 & 0 & A_3 & 0 & 0 & 0 \\
0 & A_2 & A_0 & B_0 & B_2 & 0
\end{pmatrix},
$$(2.6)

where

$$
A_0 \in \mathcal{B} \left( X_3, \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} \right), \hspace{1em} A_2 \in \mathcal{B} \left( X_2, \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} \right), \hspace{1em} A_3 \in \mathcal{B} \left( X_3, \mathcal{R}(B) \downarrow \right); \hspace{1em} B_0 \in \mathcal{B} \left( Y_3, \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} \right), \hspace{1em} B_2 \in \mathcal{B} \left( Y_2, \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} \right), \hspace{1em} B_3 \in \mathcal{B} \left( Y_3, \mathcal{R}(A) \downarrow \right).
$$

Then the operators $A_3$ and $B_3$ are isomorphisms and the row operator $(A_2 \ B_2) : X_2 \oplus Y_2 \to \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$ is right invertible and

$$
\mathcal{R}(A_2) = \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} = \overline{\mathcal{R}(B)}.
$$(2.7)
Proof. Step 1. We prove (2.3)–(2.6).

The row operator \((A \ B) : \mathcal{X} + \mathcal{Y} \to \mathcal{X}\) is right invertible and we have with Lemma 2.4
\[
\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{X}.
\] (2.8)
We claim
\[
P_{\mathcal{R}(A)}(\mathcal{R}(B)) = \mathcal{R}(A)^\perp.
\] (2.9)
To see this, it suffices to show the inclusion \(P_{\mathcal{R}(A)}(\mathcal{R}(B)) \supset \mathcal{R}(A)^\perp\). Let \(x \in \mathcal{R}(A)^\perp\). Then there exist \(x_1 \in \mathcal{R}(A)\) and \(x_2 \in \mathcal{R}(B)\) such that \(x = x_1 + x_2\), so \(x = P_{\mathcal{R}(A)}x_2 \in P_{\mathcal{R}(A)}(\mathcal{R}(B))\). This proves the claim. Similarly, we obtain
\[
P_{\mathcal{R}(B)}(\mathcal{R}(A)) = \mathcal{R}(B)^\perp.
\] (2.10)
Moreover, by (2.8), we have
\[
\{0\} = \mathcal{X}^\perp = (\mathcal{R}(A) + \mathcal{R}(B))^\perp = \mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp
\]
and also the sum \(\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}\) is closed. By Lemma 2.2 (iv) it follows that
\[
\left(\overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\right)^\perp = \mathcal{R}(A)^\perp + \mathcal{R}(B)^\perp.
\]
To sum up, we have the space decomposition (2.3). As \(\mathcal{N}(A) \subset \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)\), we have \(\mathcal{N}(P_{\mathcal{R}(B)^\perp}A)^\perp \subset \mathcal{N}(A)^\perp\). Analogously we see \(\mathcal{N}(P_{\mathcal{R}(A)^\perp}B)^\perp \subset \mathcal{N}(B)^\perp\) and, hence, decomposition (2.4) follows.

For \(x \in \mathcal{X}^\perp = \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)\) we have
\[
Ax = \left(I - P_{\mathcal{R}(B)^\perp}\right)Ax = P_{\overline{\mathcal{R}(B)}}Ax.
\]
Hence, \(x \in \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)\) if and only if
\[
Ax \in \overline{\mathcal{R}(B)}.
\] (2.11)
Similarly, \(y \in \mathcal{N}(P_{\mathcal{R}(A)^\perp}B)\) if and only if \(By \in \overline{\mathcal{R}(A)}\). Therefore, if \(x_2 \in \mathcal{X}(y_2 \in \mathcal{Y})\), then it follows that \(x_2 \in \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)\) (resp. \(y_2 \in \mathcal{N}(P_{\mathcal{R}(A)^\perp}B)\)) and, by (2.11)
\[
Ax_2 \in \overline{\mathcal{R}(B)} \quad \text{(resp. } By_2 \in \overline{\mathcal{R}(A)})\). (2.12)
Then the zero entries in (2.6) follow from the fact that \(Ax = 0\) for \(x \in \mathcal{N}(A), By = 0\) for \(y \in \mathcal{N}(B), Ax \in \mathcal{R}(A), By \in \mathcal{R}(B)\), and (2.12).

Step 2. We show that \((A_2 \ B_2)\) is right invertible.

We have \(\mathcal{N}(A) \subset \mathcal{N}(P_{\mathcal{R}(B)^\perp}A), \mathcal{N}(B) \subset \mathcal{N}(P_{\mathcal{R}(A)^\perp}B)\) and by (2.8) and (2.3) we see that \(A_3\) and \(B_3\) are isomorphisms. Thus, there exists an isomorphism \(U \in \mathcal{B}(\mathcal{R}(A)^\perp + \mathcal{R}(B)^\perp) \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\)
\[
U := \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-B_0 B_3^{-1} & -A_0 A_3^{-1} & 1
\end{pmatrix}
\]
such that
\[
U \begin{pmatrix}
0 & 0 & 0 & B_3 & 0 & 0 \\
0 & 0 & A_3 & 0 & 0 & 0 \\
0 & A_2 & A_0 & B_0 & B_2 & 0 \\
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & B_3 & 0 & 0 \\
0 & 0 & A_3 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 & B_2 & 0 \\
\end{pmatrix}.
\]
As \((A, B)\) is right invertible, Lemma 2.4 shows that \((A_2 B_2) : \mathcal{X}_2 \oplus \mathcal{Y}_2 \to \mathcal{R}(A) \cap \mathcal{R}(B)\) is right invertible.

**Step 3. We show (2.7).**

By definition, we have \(\mathcal{R}(A_2) \subset \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}\) and \(\mathcal{R}(B_2) \subset \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}\). We will only show \(\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \subset \overline{\mathcal{R}(A_2)}\) is the same and, hence, we omit this proof.

Let \(z \in \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}\). Then there exists a sequence \((z_n)\) in \(\mathcal{R}(B)\) which converges to \(z\). By the block representation (2.6) for \(B\) we find \(z_{1,n}\) in \(\mathcal{R}(A)^\perp\) and \(z_{3,n} \in \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\) with
\[
\begin{aligned}
z_n &= z_{1,n} + z_{3,n}, \\n\end{aligned}
\] where we have
\[
\begin{aligned}
z_{1,n} &= B_3 y_{3,n} \\ z_{3,n} &= B_0 y_{3,n} + B_2 y_{2,n}
\end{aligned}
\]
for some \(y_{2,n} \in \mathcal{Y}_2\) and \(y_{3,n} \in \mathcal{Y}_3\). The convergence of \((z_n)\) implies the convergence of \((z_{1,n})\) to some \(z_1 \in \mathcal{R}(A)^\perp\) and of \((z_{3,n})\) to some \(z_3 \in \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\),
\[
z = z_1 + z_3.
\]
The vectors \(z\) and \(z_3\) belong to \(\overline{\mathcal{R}(A)}\), thus \(z_1 \in \overline{\mathcal{R}(A)}\) and \(z_1 = 0\) follows. Therefore \((B_3 y_{3,n})\) in (2.14) converges to zero. The fact that \(B_3\) is an isomorphism implies \(y_{3,n} \to 0\) as \(n \to \infty\). We conclude
\[
z = z_3 = \lim_{n \to \infty} z_{3,n} = \lim_{n \to \infty} B_2 y_{2,n}
\]
and \(z \in \overline{\mathcal{R}(B_2)}\) follows. Relation (2.7) is proved.

The following proposition will be used in the proof of the main result.

**Proposition 2.8** Let \(A \in \mathcal{B}(\mathcal{X})\) and \(B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})\) and let the row operator \((A, B) \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y}, \mathcal{X})\) be right invertible. The following assertions are equivalent.

(i) \(\mathcal{R}(B)\) is closed.

(ii) \(P_X(\mathcal{N}((A, B)))\) is a closed subspace in \(\mathcal{X}\).

(iii) \(\mathcal{R}(B_2)\) is closed.
Proof. Let $\mathcal{R}(B)$ be closed. We have

$$P_{\mathcal{X}}(\mathcal{N}((A B))) = \{ x \in \mathcal{X} : Ax \in \mathcal{R}(A) \cap \mathcal{R}(B) \} = \{ x \in \mathcal{X} : Ax \in \mathcal{R}(B) \}$$

and $P_{\mathcal{X}}(\mathcal{N}((A B)))$ is the pre-image of $\mathcal{R}(B)$ under $A$, and, hence, it is a closed subspace and (ii) holds.

If $P_{\mathcal{X}}(\mathcal{N}((A B)))$ is closed, then also

$$\Omega := P_{\mathcal{X}}(\mathcal{N}((A B))) \cap \mathcal{N}(A)^{\perp} = \{ x \in \mathcal{X} : x \in \mathcal{N}(A)^{\perp}, Ax \in \mathcal{R}(A) \cap \mathcal{R}(B) \}$$

is closed. Decompose $x \in \Omega$ with respect to the decomposition, cf. Theorem 2.7, $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ as $x = x_1 + x_2 + x_3$ with $x_j \in \mathcal{X}_j$ for $j = 1, 2, 3$. Then $x_1 = 0$ and for some $y \in \mathcal{X}$ we have $Ax = By$. Decompose $y$ with respect to $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2 \oplus \mathcal{Y}_3$ (cf. Theorem 2.7) as $y = y_1 + y_2 + y_3$ with $y_j \in \mathcal{Y}_j$ for $j = 1, 2, 3$. Relation (2.6) shows

$$Ax = A \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & A_3 x_3 \\ A_2 x_2 + A_0 x_3 \end{pmatrix} = \begin{pmatrix} B_3 y_3 \\ 0 \\ B_0 y_3 + B_2 y_2 \end{pmatrix} = B \begin{pmatrix} y_3 \\ y_2 \\ y_1 \end{pmatrix} = By$$

and, as $A_3$ is an isomorphism, we obtain $x_3 = 0$. Therefore $\Omega \subset \mathcal{X}_2$ and we write

$$\mathcal{X}_2 = \Omega \oplus (\mathcal{X}_2 \oplus \Omega).$$

By Theorem 2.7 $(A_2 B_2)$ is right invertible and we obtain with Lemma 2.4

$$A_2(\mathcal{X}_2 \oplus \Omega) + B_2(\mathcal{Y}_2) = \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}, \quad A_2(\mathcal{X}_2 \oplus \Omega) \cap B_2(\mathcal{Y}_2) = \{ 0 \}.$$

Thus, using Lemma 2.1, we deduce that $A_2(\mathcal{X}_2 \oplus \Omega)$ and $\mathcal{R}(B_2)$ are closed.

Assume that (iii) holds. Then, by (2.7), the operator $B_2$ is an isomorphism. Let $z \in \overline{\mathcal{R}(B)}$. Then there exists a sequence $(z_n)$ in $\mathcal{R}(B)$ which converges to $z$. By the block representation (2.6) for $B$ we find $z_{1,n}$ in $\mathcal{R}(A)^{\perp}$ and $z_{3,n} \in \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$ such that (2.13) and (2.14) hold for some $y_{2,n} \in \mathcal{Y}_2$ and $y_{3,n} \in \mathcal{Y}_3$. The convergence of $(z_n)$ implies the convergence of $(z_{1,n})$ to some $z_1 \in \mathcal{R}(A)^{\perp}$ and of $(z_{3,n})$ to some $z_3 \in \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$, $z = z_1 + z_3$. As the operators $B_2$ and $B_3$ (cf. Theorem 2.7) are isomorphisms, we have

$$y_{3,n} \to B_3^{-1} z_1 \quad y_{2,n} \to -B_2^{-1} B_0 B_3^{-1} z_1 + B_2^{-1} z_3 \quad \text{as } n \to \infty.$$

Thus, with (2.6),

$$B \begin{pmatrix} B_3^{-1} z_1 \\ -B_2^{-1} B_0 B_3^{-1} z_1 + B_2^{-1} z_3 \\ 0 \end{pmatrix} = \begin{pmatrix} z_1 \\ 0 \\ z_3 \end{pmatrix} = z,$$

and $z \in \mathcal{R}(B)$. \qed
Lemma 2.9 Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y, X)$ and assume that the row operator $(A, B) \in \mathcal{B}(X \oplus Y, X)$ is right invertible. Let $A_2$ and $B_2$ be as in Theorem 2.7. Then $B_2$ considered as an operator from $Y_2$ to $\mathcal{R}(B_2)$ is one-to-one and has an inverse $B_2^{-1} : \mathcal{R}(B_2) \rightarrow Y_2$. Define
\[
\tilde{A}_2 := (0 \ A_2) : X_1 \oplus X_2 \rightarrow \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}.
\]
Then $\tilde{A}_2|_{P_X(\mathcal{N}((A, B)))}$ maps to $\mathcal{R}(B_2)$ and the operator
\[
B_2^{-1} \tilde{A}_2|_{P_X(\mathcal{N}((A, B)))} : P_X(\mathcal{N}((A, B))) \rightarrow Y_2
\]
is correctly defined.

If $\mathcal{R}(B)$ is closed, then $B_2$ is an isomorphism and we have
\[
X_1 \oplus X_2 = \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) = P_X(\mathcal{N}((A, B)))
\]
and the operator
\[
B_2^{-1} \tilde{A}_2 : \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) \rightarrow Y_2
\]
is correctly defined.

Proof. As $Y_2 \subset \mathcal{N}(B)^\perp$ the operator $B_2$ is one-to-one, hence its inverse $B_2^{-1} : \mathcal{R}(B_2) \rightarrow Y_2$ exists. From
\[
P_X(\mathcal{N}((A, B))) = \{ x \in X : Ax \in \mathcal{R}(A) \cap \mathcal{R}(B) \} \subset \{ x \in X : Ax \in \overline{\mathcal{R}(B)} \}
\]
we conclude
\[
P_X(\mathcal{N}((A, B))) \subset \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) = X_1 \oplus X_2.
\]
Moreover, we decompose $x \in P_X(\mathcal{N}((A, B)))$ with respect to the decomposition $X = X_1 \oplus X_2 \oplus X_3$ (cf. Theorem 2.7) as $x = x_1 + x_2 + x_3$ with $x_j \in X_j$ for $j = 1, 2, 3$. Then $x_3 = 0$ and for some $y \in Y$ we have $Ax = By$. Decompose $y$ with respect to $Y = Y_1 \oplus Y_2 \oplus Y_3$ (cf. Theorem 2.7) as $y = y_1 + y_2 + y_3$ with $y_j \in Y_j$ for $j = 1, 2, 3$. Relation (2.6) shows
\[
Ax = A \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A_2x_2 \end{pmatrix} = \begin{pmatrix} B_3 y_3 \\ 0 \\ B_0 y_3 + B_2 y_2 \end{pmatrix} = B \begin{pmatrix} y_3 \\ y_2 \\ y_1 \end{pmatrix} = By
\]
and, as $B_3$ is an isomorphism, we obtain $y_3 = 0$ and $A_2 x_2 = B_2 y_2$. Thus $\tilde{A}_2 x \in \mathcal{R}(B_2)$ for $x \in P_X(\mathcal{N}((A, B)))$ and $B_2^{-1} \tilde{A}_2|_{P_X(\mathcal{N}((A, B)))}$ is correctly defined. If $\mathcal{R}(B)$ is closed, then by Proposition 2.8 also $\mathcal{R}(B_2)$ is closed and by (2.7) we see that $B_2$ is an isomorphism. Moreover, from (2.16) we see in this case $X_1 \oplus X_2 = \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) = P_X(\mathcal{N}((A, B)))$ and (2.15) follows. □

The following theorem is the main result. It provides a full characterization of isomorphic $2 \times 2$ operator matrices in terms of their entries.
Theorem 2.10 Let $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Assume that the row operator $(A B) \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y}, \mathcal{X})$ is right invertible and, hence, adopt the notions $A_2$, $B_2$, and $\mathcal{X}_j$, $\mathcal{Y}_j$, $j = 1, 2, 3$, as in Theorem 2.7 and $\tilde{A}_2$ as in Lemma 2.9. Let $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{B}(\mathcal{Y})$. Define the operator matrix $M$ by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$  

Define the operator $B_2^{-1} \tilde{A}_2|\mathcal{P}_X(\mathcal{N}(\mathcal{A} B)))$ as in Lemma 2.9 and define

$$C_2 := P_{(\mathcal{R}(D|_{\mathcal{N}(B)})^{\perp})} C|_{\mathcal{X}_1 \oplus \mathcal{X}_2} : \mathcal{X}_1 \oplus \mathcal{X}_2 \rightarrow (\mathcal{R}(D|_{\mathcal{N}(B)}))^{\perp}$$

and

$$D_2 := P_{(\mathcal{R}(D|_{\mathcal{N}(B)})^{\perp})} D|_{\mathcal{Y}_2} : \mathcal{Y}_2 \rightarrow (\mathcal{R}(D|_{\mathcal{N}(B)}))^{\perp}.$$ 

Then $M$ is an isomorphism if and only if the following two statements are satisfied:

(i) The restriction $D|_{\mathcal{N}(B)} : \mathcal{N}(B) \rightarrow \mathcal{Y}$ is left invertible.

(ii) The operator

$$\left( C_2 - D_2 B_2^{-1} \tilde{A}_2 \right)|_{\mathcal{P}_X(\mathcal{N}(\mathcal{A} B)))} : \mathcal{P}_X(\mathcal{N}(\mathcal{A} B))) \rightarrow (\mathcal{R}(D|_{\mathcal{N}(B)}))^{\perp}$$

is one-to-one and surjective.

Proof. Let $M$ be an isomorphism. Then the row operator $(A B) : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X}$ is right invertible, see Lemma 2.4, and the column operator $\left( \begin{array}{c} B \\ D \end{array} \right) : \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{Y}$ is injective. Moreover, if the range of $\left( \begin{array}{c} B \\ D \end{array} \right)$ is not closed then there exists a sequence $(y_n)$ in $\mathcal{Y}$ with $\|y_n\| = 1$, $n \in \mathbb{N}$, and $\left( \begin{array}{c} B \\ D \end{array} \right) y_n \rightarrow 0$ as $n \rightarrow \infty$. But this implies $M(\left( \begin{array}{c} 0 \\ y_n \end{array} \right)) \rightarrow 0$, a contradiction as $M$ is assumed to be an isomorphism. Therefore the column operator $\left( \begin{array}{c} B \\ D \end{array} \right)$ is left invertible, cf. Lemma 2.5.

Now let $z \in \mathcal{R}(D|_{\mathcal{N}(B)})$. Then, there exists $z_n \in \mathcal{N}(B)$ such that $Dz_n \rightarrow z$ as $n \rightarrow \infty$, and we further have

$$\left( \begin{array}{c} B \\ D \end{array} \right) z_n = \left( \begin{array}{c} 0 \\ Dz_n \end{array} \right) \rightarrow \left( \begin{array}{c} 0 \\ z \end{array} \right),$$

which together with Lemma 2.5 implies

$$\left( \begin{array}{c} B \\ D \end{array} \right) x = \left( \begin{array}{c} 0 \\ z \end{array} \right)$$

for some $x \in \mathcal{N}(B)$, and hence $D|_{\mathcal{N}(B)} x = z$. This proves that $\mathcal{R}(D|_{\mathcal{N}(B)})$ is closed, hence, $D|_{\mathcal{N}(B)}$ is left invertible by Lemma 2.5 and (i) is proved.

As $\mathcal{R}(D|_{\mathcal{N}(B)})$ is a closed subspace in $\mathcal{Y}$, we decompose $\mathcal{Y}$,

$$\mathcal{Y} = (\mathcal{R}(D|_{\mathcal{N}(B)})^{\perp}) \oplus \mathcal{R}(D|_{\mathcal{N}(B)}).$$  \hspace{1cm} (2.17)
Similar to the proof of Theorem 2.7, $M$ as an operator from $\mathcal{N}(P_{\mathcal{R}(B)^\perp}A) \oplus \mathcal{X}_3 \oplus \mathcal{Y}_3 \oplus \mathcal{Y}_2 \oplus \mathcal{Y}_1$ into 

$$(\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp) \oplus \mathcal{R}(A) \cap \mathcal{R}(B) \oplus (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp \oplus \mathcal{R}(D|_{\mathcal{N}(B)})$$

has the following block representation

$$M = \begin{pmatrix}
0 & 0 & B_3 & 0 & 0 \\
0 & A_3 & 0 & 0 & 0 \\
\tilde{A}_2 & A_0 & B_0 & B_2 & 0 \\
C_2 & C_3 & D_1 & D_2 & 0 \\
C_4 & C_5 & D_3 & D_4 & D_5
\end{pmatrix}. \tag{2.18}$$

By Theorem 2.7, $A_3$ and $B_3$ are isomorphisms. Additionally, as $M$ is an isomorphism, $D_5$ is also an isomorphism. Then there exist isomorphisms

$$U \in \mathcal{B}\left((\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp) \oplus \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \oplus (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp \oplus \mathcal{R}(D|_{\mathcal{N}(B)})\right),$$

$$V \in \mathcal{B}\left(\mathcal{N}(P_{\mathcal{R}(B)^\perp}A) \oplus \mathcal{X}_3 \oplus \mathcal{Y}_3 \oplus \mathcal{Y}_2 \oplus \mathcal{Y}_1\right)$$

with

$$U := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
- B_0 B_3^{-1} & - A_0 A_3^{-1} & 1 & 0 & 0 \\
-D_1 B_5^{-1} & - C_3 A_3^{-1} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$V := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-D_5^{-1} C_4 & - D_5^{-1} C_5 & - D_5^{-1} D_3 & - D_5^{-1} D_4 & 1
\end{pmatrix}$$

such that

$$UMV = \begin{pmatrix}
0 & 0 & B_3 & 0 & 0 \\
0 & A_3 & 0 & 0 & 0 \\
\tilde{A}_2 & 0 & 0 & B_2 & 0 \\
C_2 & 0 & 0 & D_2 & 0 \\
0 & 0 & 0 & 0 & D_5
\end{pmatrix}. \tag{2.19}$$

Thus, $M$ is an isomorphism if and only if

$$\Delta := \left(\begin{array}{cc}
\tilde{A}_2 & B_2 \\
C_2 & D_2
\end{array}\right) : \mathcal{N}(P_{\mathcal{R}(B)^\perp}A) \oplus \mathcal{Y}_2 \to (\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}) \oplus (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp \tag{2.20}$$

is an isomorphism.
**Case 1:** $\mathcal{R}(B)$ is closed. In this case, from Lemma 2.9, $B_2 : \mathcal{Y}_2 \to \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ is an isomorphism and $B_2^{-1} \tilde{\mathcal{A}}_2 : \mathcal{N}(P_{\mathcal{R}(B)^\perp A}) \to \mathcal{Y}_2$ is correctly defined, see Lemma 2.9. According to Lemma 2.3, $\Delta$ is an isomorphism if and only if

$$C_2 - D_2 B_2^{-1} \tilde{\mathcal{A}}_2 : \mathcal{N}(P_{\mathcal{R}(B)^\perp A}) \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$$

is an isomorphism. By Lemma 2.9 $\mathcal{N}(P_{\mathcal{R}(B)^\perp A}) = P_A(\mathcal{N}((A \ B)))$ and (ii) is satisfied.

**Case 2:** $\mathcal{R}(B)$ is not closed. By Proposition 2.8 also $\mathcal{R}(B_2)$ is not closed which implies $\dim \mathcal{R}(B_2) = \infty$ and $\dim \mathcal{Y}_2 = \infty$. The dimension does not change when we close a subspace, therefore we conclude from (2.7)

$$\dim \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \dim \overline{\mathcal{R}(B_2)} = \dim \mathcal{R}(B_2) = \infty. \quad (2.21)$$

By Theorem 2.7 $(A_2 \ B_2)$ is right invertible, (2.7) and Lemma 2.1 imply

$$\mathcal{R}(A_2) \cap \mathcal{R}(B_2) = \mathcal{R}(A) \cap \mathcal{R}(B).$$

Obviously, $\mathcal{R}(A_2) \cap \mathcal{R}(B_2) \subset \mathcal{R}(A) \cap \mathcal{R}(B)$ and we obtain $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \subset \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$. Thus

$$\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}.$$ 

From this and from $\mathcal{R}(A) \cap \mathcal{R}(B) \subset \mathcal{R}(A) \cap \overline{\mathcal{R}(B)} \subset \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ we conclude with (2.21)

$$\infty = \dim \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \dim \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} = \dim \mathcal{R}(A) \cap \overline{\mathcal{R}(B)}. \quad (2.22)$$

We will use (2.22) to show

$$\dim \mathcal{N}((A_2 \ B_2)) = \dim \mathcal{N}(P_{\mathcal{R}(B)^\perp A}). \quad (2.23)$$

For this we consider

$$\mathcal{N}((A \ B)) = \{(\begin{smallmatrix} x \\ y \end{smallmatrix}) : x \in \mathcal{N}(A)\} \oplus \{(\begin{smallmatrix} y \\ z \end{smallmatrix}) : y \in \mathcal{N}(A)^\perp, Ay = -Bz \} \quad (2.24)$$

and

$$\mathcal{N}(P_{\mathcal{R}(B)^\perp A}) = \mathcal{N}(A) \oplus \left\{ x : x \in \mathcal{N}(A)^\perp, Ax \in \overline{\mathcal{R}(B)} \right\}.$$ 

As $A$ restricted to $\mathcal{N}(A)^\perp$ is injective, we obtain with (2.22)

$$\dim \left\{ (\begin{smallmatrix} y \\ z \end{smallmatrix}) : y \in \mathcal{N}(A)^\perp, Ay = -Bz \right\} = \dim \mathcal{R}(A) \cap \mathcal{R}(B) = \dim \mathcal{R}(A) \cap \overline{\mathcal{R}(B)} = \dim \left\{ x : x \in \mathcal{N}(A)^\perp, Ax \in \overline{\mathcal{R}(B)} \right\}.$$ 

Therefore

$$\dim \mathcal{N}((A \ B)) = \dim \mathcal{N}(P_{\mathcal{R}(B)^\perp A})$$

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and with (2.19) we obtain \( \dim \mathcal{N}(\tilde{A}_2 B_2) = \dim \mathcal{N}(P_{R(B)} \perp A) \), hence (2.23) is proved. Two separable Hilbert spaces of the same dimension are unitarily equivalent, therefore there exists a left invertible operator

\[
\begin{pmatrix} G \\ H \end{pmatrix} : \mathcal{Y}_2 \to \mathcal{N}(P_{R(B)} \perp A) \oplus \mathcal{Y}_2 \text{ with range } \mathcal{N}(\tilde{A}_2 B_2). \tag{2.25}
\]

Since \( \mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{N}(P_{R(B)} \perp A) \) and by Theorem 2.7 and Lemma 2.9 \((\tilde{A}_2 B_2) : \mathcal{N}(P_{R(B)} \perp A) \oplus \mathcal{Y}_2 \to \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \) is a right invertible operator. Then, see Remark 2.6, there exists a left invertible operator

\[
\begin{pmatrix} E \\ F \end{pmatrix} : \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \to \mathcal{N}(P_{R(B)} \perp A) \oplus \mathcal{Y}_2 \tag{2.26}
\]

such that

\[
\tilde{A}_2 E + B_2 F = I_{\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}} \quad \text{with } \mathcal{R}\left(\begin{pmatrix} E \\ F \end{pmatrix}\right) = (\mathcal{N}(\tilde{A}_2 B_2))^\perp \tag{2.27}
\]

Define

\[
W = \begin{pmatrix} E & G \\ F & H \end{pmatrix} : \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \oplus \mathcal{Y}_2 \to \mathcal{N}(P_{R(B)} \perp A) \oplus \mathcal{Y}_2. \tag{2.28}
\]

As \( \begin{pmatrix} G \\ H \end{pmatrix} \) and \( \begin{pmatrix} E \\ F \end{pmatrix} \) are left invertible and from (2.25) and (2.27) we obtain easily that \( W \) is an isomorphism. We have

\[
\Delta W = \begin{pmatrix} I_{\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}} & 0 \\ C_2 E + D_2 F & C_2 G + D_2 H \end{pmatrix}, \tag{2.29}
\]

As \( M \) is an isomorphism, \( \Delta \) is an isomorphism (see (2.20)) and the operator \( C_2 G + D_2 H : \mathcal{Y}_2 \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp \) is an isomorphism. Moreover, the operator \( B_2 \) considered as an operator from \( \mathcal{Y}_2 \) to \( \mathcal{R}(B_2) \) is one-to-one and has an inverse, see Lemma 2.9. From \( \tilde{A}_2 G + B_2 H = 0 \) we conclude \(- B_2^{-1} \tilde{A}_2 G = H \) and

\[
C_2 G + D_2 H = (C_2 - D_2 B_2^{-1} \tilde{A}_2) G. \tag{2.30}
\]

Therefore, \( C_2 - D_2 B_2^{-1} \tilde{A}_2 : \mathcal{R}(G) \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp \) is one-to-one with range equal to \((\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp \). From

\[
\mathcal{R}\left(\begin{pmatrix} G \\ H \end{pmatrix}\right) = \mathcal{N}(\tilde{A}_2 B_2) = \left( \begin{pmatrix} \mathcal{N}(A) \\ 0 \end{pmatrix} \oplus \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \in \mathcal{N}(A)^\perp, y \in \mathcal{N}(B)^\perp, Ax = -By \right\} \right) = \mathcal{N}(\begin{pmatrix} A \\ B \end{pmatrix}), \tag{2.31}
\]

see (2.24), it follows that \( \mathcal{R}(G) = P_{\mathcal{X}}(\mathcal{N}(\begin{pmatrix} A \\ B \end{pmatrix})) \) and (ii) is shown.
Now let us assume that (i) and (ii) hold. Then $\mathcal{R}(D|_{\mathcal{N}(B)})$ is a closed subspace and $\mathcal{Y}$ admits a decomposition as in (2.17) and we obtain the representation of $M$ as in (2.18), where $A_3, B_3$ and $D_5$ are isomorphisms. Then, taking the same $U$ and $V$ as above, we obtain the relation (2.19). Moreover, if $\Delta$ in (2.20) is an isomorphism, then $M$ is an isomorphism.

If $\mathcal{R}(B)$ is closed, then from Lemma 2.9, $B_2 : \mathcal{Y}_2 \to \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ is an isomorphism and $B_2^{-1} \tilde{A}_2 : \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) \to \mathcal{Y}_2$ is correctly defined. Moreover, Lemma 2.9, $\mathcal{N}(P_{\mathcal{R}(B)^\perp} A) = P_{\mathcal{X}}(\mathcal{N}((A B)))$. Then, by (ii),

$$C_2 - D_2 B_2^{-1} \tilde{A}_2 : \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$$

is an isomorphism and according to Lemma 2.3, $\Delta$ is an isomorphism and, hence, $M$ is an isomorphism.

If $\mathcal{R}(B)$ is not closed, then as above, we define the operators $G, H, E, F,$ and $W$ as in (2.25), (2.26), (2.27), and (2.28). Moreover, the operator $W$ in (2.28) is an isomorphism and also (2.30) and (2.31) hold. By (2.31) $\mathcal{R}(G) = P_{\mathcal{X}}(\mathcal{N}((A B)))$ and as $B_2$ is one-to-one, we see that the operator $G$ in (2.25) is one-to-one. Hence, together with (ii), the operator $(C_2 - D_2 B_2^{-1} \tilde{A}_2)G : \mathcal{Y}_2 \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$ is one-to-one with range equal to $(\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$. Therefore, by (2.30), $C_2 G + D_2 H$ is an isomorphism and, by (2.29) and as $W$ is an isomorphism, also $\Delta$ is an isomorphism. Therefore, see (2.20), $M$ is an isomorphism. $\square$

Finally, we consider the following special case.

**Theorem 2.11** Let $A, B, C, D \in \mathcal{B}(\mathcal{X})$ and let $\mathcal{X}', \mathcal{X}''$ be closed subspaces of $\mathcal{X}$ with

$$\mathcal{X} = \mathcal{X}' \oplus \mathcal{X}''$$

such that

$$\mathcal{R}(A) = \mathcal{X}', \quad \mathcal{N}(A) = \mathcal{X}'', \quad \mathcal{R}(B) = \mathcal{X}'', \quad \text{and} \quad \mathcal{N}(B) = \mathcal{X}'. $$

Moreover assume that the restriction $D|_{\mathcal{X}'} : \mathcal{X}' \to \mathcal{X}$ is left invertible. Then the $2 \times 2$ operator matrix $M$,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

is an isomorphism if and only if

$$C_2 := P_{(\mathcal{R}(D|_{\mathcal{X}'})^\perp)} : \mathcal{X}'' \to (\mathcal{R}(D|_{\mathcal{X}'})^\perp)$$

is an isomorphism.

In particular, if, in addition, $\mathcal{R}(B) \neq \{0\}$ and the operator $D|_{\mathcal{X}'} : \mathcal{X}' \to \mathcal{X}$ is an isomorphism, then for every operator $C \in \mathcal{B}(\mathcal{X})$ the $2 \times 2$ operator matrix $M$ is not an isomorphism.
Proof. Denote by $P_X$ the orthogonal projection in $X \oplus X$ onto the first component. Then

$$P_X(\mathcal{N}((A \ 0))) = \mathcal{N}(A) = X''.$$

Moreover, we have

$$\mathcal{N}((P_{\mathcal{R}(B)^\perp} \cdot A)) = \mathcal{N}(P_{X'} A) = \mathcal{N}(A) = X'' \cap X'' = \{0\}.$$ Then the space $X_2$ in Theorem 2.7 equals zero and the operators $A_2$ and $A_2$ in Theorem 2.10 are zero. Then the statements of Theorem 2.11 follow from Theorem 2.10.

3 A characterization of isomorphic row operators

In this section let $A, B, C, D$ and $M$ be as in Theorem 2.10. In the following we use Theorems 2.7 and 2.10 to characterize the case of an isomorphic row operator $(A \ 0)$ and to derive a necessary condition for $M$ to be an isomorphism.

**Proposition 3.1** Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y, X)$. The row operator $(A \ 0) \in \mathcal{B}(X \oplus Y, X)$ is an isomorphism (i.e. $(A \ 0)$ is left and right invertible) if and only if the following two statements are satisfied:

(i) $\mathcal{N}(A) = \mathcal{N}(0) = \{0\}$.

(ii) $\mathcal{R}(A) = \mathcal{R}(B) = \mathcal{R}(A) = \mathcal{R}(B)$.

Proof. If (i) and (ii) hold, then $Ax + By = 0$ for some $x \in X, y \in Y$ implies $Ax = -By \in \mathcal{R}(B)$. By (ii), $Ax = 0$ and, hence, $By = 0$ follows. Then (i) implies $x = y = 0$ and $\mathcal{N}((A \ 0)) = \{0\}$. Moreover, we have with (ii)

$$\mathcal{R}((A \ 0)) \subset \mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A) + \mathcal{R}(A) = X$$

and the row operator $(A \ 0)$ is an isomorphism.

For the contrary let the row operator $(A \ 0)$ be an isomorphism. If for some $x \in X$ we have $Ax = 0$ then $(A \ 0)(x) = 0$ and, as $\mathcal{N}(A \ 0) = \{0\}$, $x = 0$ follows. That is, $\mathcal{N}(A) = \{0\}$ and, similarly, we see $\mathcal{N}(A \ 0) = \{0\}$. This shows (i). In order to show (ii) let $x \in \mathcal{R}(A) \cap \mathcal{R}(B)$ and assume $x \neq 0$. Then there exists sequences $(x_n)$ in $X$ and $(y_n)$ in $Y$ such that $(Ax_n)$ and $(By_n)$ converge both to $x$ with $\lim \inf_{n \to \infty} \|x_n\| > 0$ and $\lim \inf_{n \to \infty} \|y_n\| > 0$. But then $((A \ 0))(x_n, -y_n) = Ax_n - By_n$ tends to zero and $\mathcal{R}((A \ 0))$ is not closed, a contradiction. This shows

$$\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}. \quad (3.1)$$

As $x \in \mathcal{N}(P_{\mathcal{R}(B)^\perp} \cdot A)$ if and only if $Ax \in \mathcal{R}(B)$ (see also (2.11)), we conclude with $\mathcal{N}(A) = \{0\}$ and (3.1)

$$\mathcal{N}(P_{\mathcal{R}(B)^\perp} \cdot A) = \{0\}.$$
In the same way we obtain from (3.1) and \( \mathcal{N}(B) = \{0\} \) that \( \mathcal{N}(P_{\mathcal{R}(A)} B) = \{0\} \). Then for the spaces \( \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3 \) from Theorem 2.7 we conclude

\[
\mathcal{X}_1 = \{0\}, \quad \mathcal{X}_2 = \{0\}, \quad \mathcal{X}_3 = \mathcal{X}, \quad \mathcal{Y}_1 = \{0\}, \quad \mathcal{Y}_2 = \{0\}, \quad \text{and} \quad \mathcal{Y}_3 = \mathcal{Y}
\]

and the row operator \((A B)\) admits a representation according to Theorem 2.7 with respect to the decompositions \( \mathcal{X} \oplus \mathcal{Y} \) and \( \mathcal{X} = \mathcal{R}(A)^\perp \mathcal{R}(B)^\perp \) of the form

\[
\begin{pmatrix}
0 & B_3 \\
A_3 & 0
\end{pmatrix},
\]

where \( A_3 \in \mathcal{B}(\mathcal{X}, \mathcal{R}(B)^\perp) \) and \( B_3 \in \mathcal{B}(\mathcal{Y}, \mathcal{R}(A)^\perp) \) are isomorphisms. This shows (ii).

**Example 3.2** Let \( \mathcal{X} = \mathcal{Y} = \ell^2(\mathbb{N}) \) and consider the following operators \( A \) and \( B \) in \( \mathcal{X} \):

\[
A(x_n)_{n \in \mathbb{N}} := (x_1, 0, x_2, 0 \ldots) \quad \text{and} \quad B(x_n)_{n \in \mathbb{N}} := (0, x_1, 0, x_2 \ldots).
\]

Then the row operator \((A B)\) satisfies (i) and (ii) of Proposition 3.1 and, hence, \((A B)\) is an isomorphism.

As a consequence, we derive the following condition for \( M \) to be an isomorphism.

**Corollary 3.3** Let \( A \in \mathcal{B}(\mathcal{X}), \ B \in \mathcal{B}(\mathcal{Y}, \mathcal{X}), \ C \in \mathcal{B}(\mathcal{X}), \mathcal{Y}) \) and \( D \in \mathcal{B}(\mathcal{Y}) \). If

\[
\mathcal{Y} \neq \{0\} \quad \text{and} \quad \mathcal{N}((A B)) = \{0\}
\]

then the operator matrix \( M \)

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

is not a isomorphism.

**Proof.** If \( M \) is an isomorphism, then as noted in the proof of Theorem 2.10, the row operator \((A B)\) is right invertible. Assume \( \mathcal{N}((A B)) = \{0\} \). Then \((A B)\) is an isomorphism, and, by Proposition 3.1, \( \mathcal{N}(B) = \{0\} \). Hence, we obtain \( (\mathcal{R}(D)_{\mathcal{N}(B)})^\perp = \mathcal{Y} \) and (ii) in Theorem 2.10 cannot be true unless \( \mathcal{Y} = \{0\} \). Therefore, either \( \mathcal{Y} = \{0\} \) or \( \mathcal{N}((A B)) \neq \{0\} \) holds.

\( \square \)

### 4 Application to Hamiltonian operators

In this section we consider the special case of Hamiltonian operators, i.e., in the situation of Theorem 2.10, \( \mathcal{X} = \mathcal{Y} \), the operators \( B, C \) are self-adjoint and \( D = -A^* \). Under these assumptions, Theorem 2.10 takes the following simple form.
**Theorem 4.1** Let $A, B, C \in \mathcal{B}(X)$. Assume that the row operator $(A \ B) \in \mathcal{B}(X \oplus X, X)$ is right invertible and that $B$ and $C$ are self-adjoint operators in $X$, i.e. $B = B^*$ and $C = C^*$. Adopt the notions $A_2, B_2,$ and $X_j$, $j = 1, 2, 3$, as in Theorem 2.7 and $\tilde{A}_2$ as in Lemma 2.9. Define the operator $B_2^{-1}\tilde{A}_2|_{\mathcal{N}(\mathcal{N}((A \ B)))}$ as in Lemma 2.9 and define

$$C_2 := P_{\mathcal{N}(P_{R(B)} \perp A)}C|_{X_1 \oplus X_2} : X_1 \oplus X_2 \to \mathcal{N}(P_{R(B)} \perp A)$$

and

$$(-A^*)_2 := -P_{\mathcal{N}(P_{R(B)} \perp A)}A^*|_{Y_2} : Y_2 \to \mathcal{N}(P_{R(B)} \perp A).$$

Then the Hamiltonian operator

$$H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$$

is an isomorphism if and only if

(i) the operator

$$\left( C_2 - (-A^*)_2B_2^{-1}\tilde{A}_2 \right)|_{\mathcal{N}(\mathcal{N}((A \ B)))} : \mathcal{N}(\mathcal{N}((A \ B))) \to \mathcal{N}(P_{R(B)} \perp A)$$

is one-to-one and surjective.

If in this case we have, in addition, that $\mathcal{R}(B)$ is closed, then $C_2 - (-A^*)_2B_2^{-1}\tilde{A}_2 \in \mathcal{B}(\mathcal{N}(P_{R(B)} \perp A))$ is an isomorphism.

**Proof.** By assumption, the row operator $(A \ B)$ is right invertible, hence (see Lemma 2.4) its range is closed and $\mathcal{R}(A) + \mathcal{R}(B) = X$. The same applies to $(B - A)$ and thus its adjoint,

$$(B - A)^* = \begin{pmatrix} B \\ -A^* \end{pmatrix},$$

has a closed range and is one-to-one. Let $z \in \overline{\mathcal{R}(-A^*|_{\mathcal{N}(B)})}$. Then, there exists $z_n \in \mathcal{N}(B)$ such that $-A^*z_n \to z$ as $n \to \infty$, and we further have

$$\begin{pmatrix} B \\ -A^* \end{pmatrix} z_n = \begin{pmatrix} 0 \\ -A^*z_n \end{pmatrix} \to \begin{pmatrix} 0 \\ z \end{pmatrix},$$

which together with the closedness of the range of $(B - A)^*$ implies

$$\begin{pmatrix} B \\ -A^* \end{pmatrix} x = \begin{pmatrix} 0 \\ z \end{pmatrix}$$

for some $x \in \mathcal{N}(B)$, and hence $-A^*|_{\mathcal{N}(B)}x = z$. This proves that $\mathcal{R}(-A^*|_{\mathcal{N}(B)})$ is closed and (i) in Theorem 2.10 is satisfied for $D = -A^*$. 

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Next, we verify
\[ (\mathcal{R}(-A^*|_{\mathcal{N}(B)}))^\perp = \mathcal{N}(P_{\mathcal{R}(B)^\perp} A). \] (4.1)

Indeed, if \( x \in (\mathcal{R}(-A^*|_{\mathcal{N}(B)}))^\perp \), we have \( (-Ax, y) = (x, -A^*y) = 0 \) for every \( y \in \mathcal{N}(B) \), hence \( -Ax \in \mathcal{N}(B)^\perp \), which together with the self-adjointness of \( B \) deduces \( Ax \in \mathcal{R}(B) \), and hence \( x \in \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) \); while if \( x \in \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) \), then \( Ax \in \mathcal{R}(B) \), and hence we have for \( y \in \mathcal{N}(B) \) that \( (x, -A^*y) = (-Ax, y) = 0 \), i.e., \( x \in (\mathcal{R}(-A^*|_{\mathcal{N}(B)}))^\perp \).

Now the equivalence of (i) and the fact that \( H \) is an isomorphism follows from (4.1) and Theorem 2.10. The additional statement in the case of a closed range of \( B \) follows from Lemma 2.9.

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References

Contact information

Junjie Huang
School of Mathematical Sciences, Inner Mongolia University
010021 Hohhot, P.R. China
huangjunjie@imu.edu.cn

Junfeng Sun
School of Mathematical Sciences, Inner Mongolia University
010021 Hohhot, P.R. China
sjfmss@163.com

Alatancang Chen
School of Mathematical Sciences, Inner Mongolia University
010021 Hohhot, P.R. China
alatanca@imu.edu.cn

Carsten Trunk
Institut für Mathematik, Technische Universität Ilmenau
Postfach 100565, D-98684 Ilmenau, Germany
carsten.trunk@tu-ilmenau.de