Entropy of absolutely convex hulls

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Contents

Zusammenfassung

Es sei $A \subset X$ eine präkompakte Teilmenge eines Banachraums X. Dann ist bekanntlich auch die reelle absolutkonvexe Hülle $\operatorname{aco}(A)$ von A, gegeben durch

$$\operatorname{aco}(A) = \left\{ \sum_{i=1}^{n} \lambda_i \, a_i \,\middle|\, n \in \mathbb{N}, a_i \in A, \lambda_i \in \mathbb{R}, \sum_{i=1}^{n} |\lambda_i| \leq 1 \right\},$$

präkompakt, das heißt

$$A \text{ präkompakt} \quad \text{impliziert} \quad \operatorname{aco}(A) \text{ präkompakt.} \tag{0.0.1}$$

Gegenstand dieser Arbeit ist es zu untersuchen, wie sich die Präkompaktheit beim Übergang von A zur reellen absolutkonvexen Hülle $\operatorname{aco}(A)$ verändert. Dazu benötigt man ein Maß für Präkompaktheit. An diese Stelle treten die Entropiezahlen einer Menge; die Konvergenzrate der Entropiezahlenfolge kann als ein Maß für Präkompaktheit aufgefasst werden. In der Sprache der Entropiezahlen liest sich Implikation (0.0.1) als

$$\lim_{n\to\infty} \varepsilon_n(A) = 0 \quad \text{impliziert} \quad \lim_{n\to\infty} \varepsilon_n(\operatorname{aco}(A)) = 0.$$

Es soll erforscht werden, auf welche Weise die Konvergenzrate der Entropiezahlen $\varepsilon_n(A)$ von A die Konvergenzrate der dyadischen Entropiezahlen $e_n(\operatorname{aco}(A))$ von $\operatorname{aco}(A)$ beeinflusst. Dieses Problem wurde in allgemeiner Form erstmals von Dudley [D87] studiert; seine Forschung war durch Anwendungen im Bereich der empirischen Prozesse motiviert. Allerdings führte die Untersuchung von Operatoren, die von einem l_1–Raum in einen Banachraum abbilden, auch schon früher zu solchen Problemen. In den vergangenen Jahren wurde die Entropie absolutkonvexer Hüllen unter verschiedenen Aspekten intensiv studiert. Die Resultate hängen wesentlich vom

- zugrunde liegenden Banachraum X und
- dem Grad der Präkompaktheit von A, ausgedrückt durch die Konvergenzrate der Entropiezahlen von A,

ab. Der Hilbertraumfall ist bereits weitgehend erforscht. Unser Interesse gilt vorwiegend dem Fall, dass X ein Banachraum vom Typ p ist. Einige Autoren haben die Entropie absolutkonvexer Hüllen auch für beliebige Banachräume studiert. Was die Konvergenzrate der Entropiezahlen von A betrifft, interessieren wir uns für die folgenden typischen Fälle:

- polynomiell: $\varepsilon_n(A) \preccurlyeq n^{-1/r} (\log(n+1))^{-\beta}$ mit $r > 0$, $\beta \in \mathbb{R}$,
- logarithmisch: $e_n(A) \preccurlyeq n^{-1/r} (\log(n+1))^{-\beta}$ mit $r > 0$, $\beta \in \mathbb{R}$.

Zusätzlich betrachten wir den Fall, dass die (dyadischen) Entropiezahlen von A zu einem Lorentz-Folgenraum $l_{r,s}$ mit $0 < r, s < \infty$ gehören. Wenn es sich anbietet, werden wir von Entropiezahlen zu den eng verwandten Überdeckungszahlen wechseln. Allerdings muss auch die Struktur der Menge A beachtet werden. Wir sprechen vom sogenannten *diagonalen Fall*, wenn A von der Form

$$A = \{x_n | \, n \in \mathbb{N}\} \subset X$$

mit $\|x_n\| \leq \sigma_n$ für alle $n \in \mathbb{N}$ ist, wobei $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ und $\lim_{n\to\infty} \sigma_n = 0$. Es liegt hier also eine Folge von Vektoren abnehmender Länge vor. Wir beschäftigen uns in dieser Arbeit hauptsächlich mit dem *nicht-diagonalen Fall*, indem A eine beliebige präkompakte Teilmenge von X ist und keine Informationen zur konkreten Struktur von A vorliegen. Natürlich ist der diagonale Fall im nicht-diagonalen Fall enthalten. Jedoch kann es vorkommen, dass sich die Entropie der absolutkonvexen Hülle $\mathrm{aco}(A)$ im diagonalen Fall von der im nicht-diagonalen Fall unterscheidet, obwohl in beiden Fällen die Entropiezahlen von A das gleiche asymptotische Verhalten aufweisen.

Die vorliegende Arbeit gliedert sich wie folgt. In Kapitel 1 geben wir einen detaillierten Überblick über bereits bekannte Resultate, neue Erkenntnisse und offene Fragen. Das erste Kapitel kann in diesem Sinne als erweiterte Einleitung verstanden werden. Trotzdem wollen wir im Folgenden die Hauptresultate dieser Arbeit kurz zusammentragen. Wir verwenden dabei die verallgemeinerten Lorentz-Folgenräume $l_{p,q,\varphi}$ aus Abschnitt 2.3.

Für präkompakte Teilmengen $A \subset X$ eines Banachraums X vom Typ p, $1 < p \leq 2$, gelten die folgenden Aussagen:

- (Theorem 1.3.4) Wenn $\big(e_n(A)\big)_n \in l_{r,s}$ für $0 < r < p'$ und $0 < s < \infty$, dann gilt $\big(e_n(\mathrm{aco}(A))\big)_n \in l_{p',s,\varphi}$ mit $\varphi(n) = (\log(n+1))^{1/r-1/p'-1/s}$.

  Dieses Resultat erweitert das von Steinwart aus [St04, Th. 1.3]. Der Fall $s = \infty$ wurde bereits in [CKP99] betrachtet.

- (Theorem 1.3.5) Sei $p' < r < \infty$, $0 < s \leq \infty$ und φ eine langsam variierende Funktion. Dann gilt $\big(e_n(A)\big)_n \in l_{r,s,\varphi}$ genau dann, wenn $\big(e_n(\mathrm{aco}(A))\big)_n \in l_{r,s,\varphi}$.

  Der Fall $s = \infty$ ist implizit in [St00, Cor. 3] enthalten.

- (Theorem 1.3.6) Wenn $\big(e_n(A)\big)_n \in l_{p',s}$ für $p \leq s < \infty$, dann gilt $e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'} (\log(n+1))^{1-1/s}$.

Der Fall $0 < s < p$ ist ein offenes Problem. Unter einer zusätzlichen Regularitätsannahme kann ein Ergebnis für $1 < s < p$ erzielt werden (Proposition 1.3.7). Der Fall $s = \infty$ wurde bereits in [CrSt02] betrachtet.

- (Theorem 1.3.8) Wenn $e_n(A) \preccurlyeq n^{-1/p'} (\log(n+1))^{-1}$, dann gilt $e_n(\text{aco}(A)) \preccurlyeq n^{-1/p'} \log\log(n+3)$.

  In Theorem 1.3.9 zeigen wir, dass diese Abschätzung für eine gewisse Teilmenge von l_p asymptotisch optimal ist.

Um einen Überblick über den derzeitigen Stand der Forschung zur Entropie absolutkonvexer Hüllen zu geben, wurden die wichtigsten Resultate in Form von Übersichten zusammengefasst. Diese befinden sich auf Seite 17 (diagonaler Fall), Seite 18 und 21 (nicht-diagonaler Fall, Hilbertraum) und auf Seite 30 (nicht-diagonaler Fall, Banachraum vom Typ p).

Im zweiten Kapitel definieren wir relevante Begriffe und Größen und erläutern diese näher. Im Mittelpunkt stehen verständlicherweise Entropiezahlen von Mengen und Operatoren, Banachräume vom Typ p und absolutkonvexe Hüllen. Darüber hinaus betrachten wir langsam variierende Funktionen und benutzen sie zur Einführung von verallgemeinerten Lorentz-Folgenräumen $l_{p,q,\varphi}$.

In Kapitel 3 beweisen wir einige technische Ungleichungen für langsam variierende Funktionen. Diese Ungleichungen werden im Rahmen der verallgemeinerten Lorentz-Folgenräume, aber auch für die Beweise im vierten Kapitel, eine wichtige Rolle spielen.

Das fünfte Kapitel fällt nur scheinbar aus dem Rahmen, denn hier beschäftigen wir uns mit den Entropiezahlen eines Operators $TD_\sigma : l_u \to Y$, wobei $D_\sigma : l_u \to l_v$ ein Diagonaloperator und $T : l_v \to Y$ ein linearer, beschränkter Operator ist. Als Spezialfall ergeben sich jedoch neue Erkenntnisse zur Entropie absolutkonvexer Hüllen im diagonalen Fall (Korollar 5.0.7).

Schließlich befasst sich das sechste und letzte Kapitel mit einer Anwendung der Ergebnisse zur Entropie absolutkonvexer Hüllen. Wir führen zunächst wichtige Begriffe und Kenngrößen für $C(M)$–wertige Operatoren ein, wobei $C(M)$ der Raum der stetigen Funktionen auf einem kompakten metrischen Raum (M, d) ist. Das Studium der Entropiezahlen von $C(M)$–wertigen Operatoren ist insofern interessant, da die Entropiezahlen eines kompakten Operators $T : X \to Y$ zwischen Banachräumen X und Y bis auf Konstanten mit denen eines Operators $S : X \to C(M)$ zusammenfallen (siehe [CS90, S. 159]). In diesem Sinne können $C(M)$–wertige Operatoren stellvertretend für alle kompakten Operatoren betrachtet werden. Wir untersuchen zunächst Operatoren $T_K : X \to C(M)$ definiert durch abstrakte Kerne $K \in C(M, X')$ und zeigen, wie die Entropiezahlen des Bildes des Kerns K die

Entropie- und Kolmogorovzahlen des Operators T_K beeinflussen (Abschnitt 6.1). Die Ergebnisse lassen sich auch für abstrakt definierte Operatoren $T_K : X \to l_\infty(M)$ anwenden (Abschnitt 6.2). Anschließend betrachten wir den Spezialfall eines schwach singulären Integraloperators von $L_p[0,1]$ nach $C[0,1]$ näher. Für typische Kernfunktionen $k : (0,1] \to \mathbb{R}$ der Form

$$k(x) = x^{-\tau} \, (c_0 - \ln x)^{-\beta} \Big(c_0 + \ln(c_0 - \ln x)\Big)^{-\gamma}$$

mit $0 < \tau \leq 1/p'$, $\beta, \gamma \in \mathbb{R}$ und $c_0 > 0$ erhalten wir für $2 \leq p < \infty$ Entropieabschätzungen von $T_K : L_p[0,1] \to C[0,1]$, die in allen nicht-kritischen Fällen scharf sind (Proposition 6.3.6). Im Hilbertraumfall $p = 2$ können wir sogar die Kolmogorovzahlen des schwach singulären Integraloperators $T_K : L_2[0,1] \to C[0,1]$ scharf abschätzen (Theorem 6.4.1). Darüber hinaus gehen wir in Abschnitt 6.4 auch auf den Fall ein, dass das Bild von T_K nicht in $C[0,1]$, sondern in $L_q[0,1]$ mit $1 \leq q < \infty$ liegt. Wir enden mit Abschätzungen der Entropie des klassischen Riemann-Liouville Operators.

Teile dieser Dissertation werden in [CHR12] und [CR13] veröffentlicht.

Introduction

Let us consider a precompact subset $A \subset X$ of a Banach space X. Then it is common knowledge that also the real absolutely convex hull $\operatorname{aco}(A)$ of A, given by

$$\operatorname{aco}(A) = \left\{ \sum_{i=1}^{n} \lambda_i \, a_i \,\middle|\, n \in \mathbb{N}, a_i \in A, \lambda_i \in \mathbb{R}, \sum_{i=1}^{n} |\lambda_i| \leq 1 \right\},$$

is precompact, i.e.

$$A \text{ precompact} \quad \text{implies} \quad \operatorname{aco}(A) \text{ precompact.} \tag{0.0.2}$$

The aim of this dissertation to investigate how the precompactness of A changes when passing from A to the real absolutely convex hull $\operatorname{aco}(A)$. For this purpose we need a measure of precompactness. This place is taken by the entropy numbers of a set; the rate of decay of the entropy numbers of a set can be considered as a measure of precompactness of the set. Implication (0.0.2) can be reformulated in the language of entropy numbers as

$$\lim_{n\to\infty} \varepsilon_n(A) = 0 \quad \text{implies} \quad \lim_{n\to\infty} \varepsilon_n(\operatorname{aco}(A)) = 0.$$

We want to investigate this implication in more detail and ask, how the rate of decay of the entropy numbers $\varepsilon_n(A)$ of A affects the rate of decay of the dyadic entropy numbers $e_n(\operatorname{aco}(A))$ of $\operatorname{aco}(A)$. This problem was first treated in a general form by Dudley [D87]; his research was motivated by applications in the field of empirical processes. However, it should be noted that the study of operators acting from an l_1–space into a Banach space led to such a problem much earlier. In recent years, the entropy of absolutely convex hulls has been intensively studied in different settings. Results essentially depend on

- the underlying Banach space X and
- the degree of precompactness of A, expressed by the rate of decay of the entropy number sequence $(\varepsilon_n(A))_n$.

The problem of estimating the entropy of the absolutely convex hull $\operatorname{aco}(A)$ has already been examined thoroughly in the Hilbert space case. This work mainly deals with the case that X is a Banach space of type p. Some authors also studied the setting where X is an arbitrary Banach space. As far as the decay of the entropy numbers of A is concerned, we are interested in the following common cases:

- polynomial decay: $\varepsilon_n(A) \preccurlyeq n^{-1/r} (\log(n+1))^{-\beta}$ with $r > 0$, $\beta \in \mathbb{R}$,
- logarithmic decay: $e_n(A) \preccurlyeq n^{-1/r} (\log(n+1))^{-\beta}$ with $r > 0$, $\beta \in \mathbb{R}$.

In addition, we consider the case where the (dyadic) entropy numbers of A belong to some Lorentz sequence space $l_{r,s}$ for $0 < r, s < \infty$. Due to technical reasons, we will sometimes switch from entropy to covering numbers and vice versa. However, we must also take account of the structure of the set A. We speak of the so-called *diagonal case* if A is of the form

$$A = \{x_n | \, n \in \mathbb{N}\} \subset X$$

with $\|x_n\| \leq \sigma_n$ for all $n \in \mathbb{N}$, where $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ and $\lim_{n\to\infty} \sigma_n = 0$. Hence, in this setting we have a sequence of vectors of decreasing length. This work mainly deals with the *non-diagonal case* where A is any precompact subset of X and no further information about the structure of A are given. Of course, the diagonal case is contained in the non-diagonal case. However, it may appear that the entropy of $\operatorname{aco}(A)$ in the diagonal case differs from the entropy of $\operatorname{aco}(A)$ in the non-diagonal case, although the entropy numbers of A have the same asymptotic behavior.

This work is structured as follows: In chapter 1 we give a detailed overview on already known results, new insights and open questions. Hence, the first chapter can be considered as an extended introduction. However, in the following we briefly summarize the main results of this work using the generalized Lorentz sequence spaces $l_{p,q,\varphi}$ from section 2.3.

If $A \subset X$ is a precompact subset of a Banach space X of type p, $1 < p \leq 2$, then the following statements hold:

- (Theorem 1.3.4) If $\big(e_n(A)\big)_n \in l_{r,s}$ for $0 < r < p'$ and $0 < s < \infty$, then we have $\big(e_n(\operatorname{aco}(A))\big)_n \in l_{p',s,\varphi}$ with $\varphi(n) = (\log(n+1))^{1/r-1/p'-1/s}$.

  This result extends Theorem 1.3 of Steinwart [St04]. The case $s = \infty$ has been considered in [CKP99].

- (Theorem 1.3.5) Let $p' < r < \infty$, $0 < s \leq \infty$ and φ be a slowly varying function. Then we have $\big(e_n(A)\big)_n \in l_{r,s,\varphi}$ if and only if $\big(e_n(\operatorname{aco}(A))\big)_n \in l_{r,s,\varphi}$.

  The case $s = \infty$ is implicitly contained in [St00, Cor. 3].

- (Theorem 1.3.6) If $\big(e_n(A)\big)_n \in l_{p',s}$ for $p \leq s < \infty$, then we have $e_n(\operatorname{aco}(A)) \preccurlyeq n^{-1/p'} (\log(n+1))^{1-1/s}$.

  The case $0 < s < p$ is an open problem. However, under an additional regularity assumption we can prove a result for $1 < s < p$ (Proposition 1.3.7). The case $s = \infty$ has been considered in [CrSt02].

- (Theorem 1.3.8) If $e_n(A) \preccurlyeq n^{-1/p'} (\log(n+1))^{-1}$, then we have $e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'} \log\log(n+3)$.

  In Theorem 1.3.9 we show that this estimate is asymptotically optimal for a certain subset of the sequence space l_p.

In order to give an overview of the present state of research on the entropy of absolutely convex hulls, the most important results are summarized in tables. The latter can be found on page 17 (diagonal case), page 18 and 21 (non-diagonal case, Hilbert space) and on page 30 (non-diagonal case, Banach space of type p).

In the second chapter we define relevant notions and concepts and explain them. Of course, entropy numbers of sets and operators, Banach spaces of type p and absolutely convex hulls are of particular interest. In addition, we consider slowly varying functions and use them to introduce generalized Lorentz sequence spaces $l_{p,q,\varphi}$.

In chapter 3 we prove some technical inequalities for slowly varying functions. These inequalities will become helpful not only when dealing with generalized Lorentz sequence spaces, but also for the proofs in the fourth chapter.

The fifth chapter seems to get out of line, because here we investigate the entropy numbers of a composition operator $TD_\sigma : l_u \to Y$, where $D_\sigma : l_u \to l_v$ is a diagonal operator and $T : l_v \to Y$ is a linear bounded operator. However, as a special case of this setting we get new insights into the entropy of absolutely convex hulls in the diagonal case (Corollary 5.0.7).

Finally, in the sixth and last chapter, we deal with an application of entropy and Gelfand numbers of absolutely convex hulls. We start with recalling important notions and concepts for $C(M)$–valued operators, where $C(M)$ is the space of continuous functions on a compact metric space (M, d). Studying entropy numbers of $C(M)$–valued operators is interesting in so far as the entropy numbers of a compact operator $T : X \to Y$ between Banach spaces X and Y are always shared by the entropy numbers of a compact operator $S : X \to C(M)$ (cf. [CS90, p. 159]). In this sense, $C(M)$–valued operators can be considered representative for all compact operators. First, we study operators $T_K : X \to C(M)$ defined by abstract kernels $K \in C(M, X')$ and show how the entropy of the image of the abstract kernel K affects the entropy and Kolmogorov numbers of such an abstract kernel operator T_K (section 6.1). The results can also be applied to operators $T_K : X \to l_\infty(M)$ defined by abstract kernels (section 6.2). Afterwards we consider the special case of a weakly singular integral operator from $L_p[0,1]$ in $C[0,1]$. For typical kernel functions $k : (0,1] \to \mathbb{R}$ of the form

$$k(x) = x^{-\tau} (c_0 - \ln x)^{-\beta} \Big(c_0 + \ln(c_0 - \ln x)\Big)^{-\gamma}$$

with $0 < \tau \leq 1/p'$, $\beta, \gamma \in \mathbb{R}$ and $c_0 > 0$ we obtain, for $2 \leq p < \infty$, entropy estimates of $T_K : L_p[0,1] \to C[0,1]$. These estimates are sharp in all non-critical cases (Proposition 6.3.6). In the Hilbert space setting $p = 2$ we can even give sharp estimates of the Kolmogorov numbers of the weakly singular integral operator $T_K : L_2[0,1] \to C[0,1]$ (Theorem 6.4.1). In addition, section 6.4 deals with the case that the image of T_K belongs to $L_q[0,1]$ with $1 \leq q < \infty$. We end with entropy estimates of the classical Riemann-Liouville operator.

Parts of this dissertation will be published in [CHR12] and [CR13].

1 History and results

In the following, we are going to give an overview of the present state of research on the entropy of real absolutely convex hulls. This will not only provide us with a historical background but also reveal gaps and open questions which this work will be investigating in more detail. Before we start, let us recall some well-known facts.

The entropy numbers of the real absolutely convex hull of a bounded subset A of a Banach space X can be expressed in terms of entropy numbers of operators. For this purpose, let $T_A : l_1(A) \to X$ be the operator defined by

$$T_A(\xi) := \sum_{t\in A} \xi_t \, t, \quad \xi = (\xi_t)_{t\in A},$$

where $l_1(I)$ denotes the Banach space of all summable families $(\xi_t)_{t\in I}$ of real numbers over the index set I, equipped with the norm

$$\|(\xi_t)_{t\in I}\| = \sum_{t\in I} |\xi_t|.$$

Then it holds that $\mathrm{aco}(A) \subset T_A(B_{l_1(A)}) \subset \mathrm{cl}\Big(\mathrm{aco}(A)\Big)$ and, consequently,

$$e_n(\mathrm{aco}(A)) = e_n(T_A : l_1(A) \to X), \quad n \in \mathbb{N}.$$

This is an easy but important step, because now we can take advantage of the useful properties of entropy numbers of operators (cf. section 2.4). In this context, we also define the *n-th Gelfand number of* $\mathrm{aco}(A)$ by

$$c_n(\mathrm{aco}(A)) := c_n(T_A : l_1(A) \to X), \quad n \in \mathbb{N}. \tag{1.0.1}$$

For a geometrical interpretation of Gelfand numbers (of operators) and a comparison with Gelfand widths we refer to [CHR12]. Considering the dual operator of $T_A : l_1(A) \to X$, we observe that $(T_A' x')(t) = \langle t, x' \rangle$ for $x' \in X'$ and $t \in A$. Consequently, $T_A' : X' \to l_\infty(A)$ is a 1-Hölder-continuous operator, which in fact maps into $C(A)$ (cf. section 6.1). Hence, by duality of entropy numbers (cf. section 2.5), the problem of estimating the entropy of absolutely convex hulls is embedded in the problem of estimating the entropy of 1-Hölder-continuous operators $T : E \to C(K)$.

1.1 The diagonal case

As mentioned above, in the diagonal case the set A is given by

$$A = \{x_1, x_2, x_3, \ldots\} \subset X \tag{1.1.1}$$

with $\|x_n\| \le \sigma_n$ for all $n \in \mathbb{N}$, where $(\sigma_n)_n$ is a monotone decreasing null sequence. In this setting, we have that $\varepsilon_n(A) \le \sigma_n$ for all $n = 1, 2, 3, \ldots$. Furthermore, the entropy numbers of the absolutely convex hull of A coincide with the entropy numbers of the operator $SD_\sigma : l_1 \to X$ (cf. [CE03, p. 400]),

$$\varepsilon_n(\mathrm{aco}(A)) = \varepsilon_n(SD_\sigma : l_1 \to X), \quad n \in \mathbb{N}, \tag{1.1.2}$$

where $D_\sigma : l_1 \to l_1$ is the diagonal operator generated by the sequence $(\sigma_n)_n$ and S is defined on the canonical unit vector basis $\{e_n\}_n$ of l_1 by

$$S : l_1 \to X,\ e_n \mapsto \begin{cases} x_n/\sigma_n, & \sigma_n > 0, \\ 0, & \sigma_n = 0. \end{cases}$$

Hence, the problem of estimating entropy numbers of the absolutely convex hull in the diagonal case is embedded into the more general problem of estimating the entropy numbers of a composition operator $SD_\sigma : l_1 \to X$, where $D_\sigma : l_1 \to l_1$ is a diagonal operator generated by a sequence $(\sigma_n)_n$ and $S : l_1 \to X$ is an arbitrary operator. In Chapter 5 we will have a closer look at this and similar problems.

Apart from the papers by Marcus [M74] and Oloff [O78], the first result in the diagonal case has been given by Carl [C81b] for the sequence space l_p. Note that a generalization to symmetric Banach spaces was given by Schütt [Sch84]. Curiously enough, a result in the field of function spaces was obtained by Birman and Solomjak [BS67] even much earlier (see also [Tr75, Hö80, C81c]). Consider the set

$$A = \{\sigma_n\, e_n |\, n \in \mathbb{N}\} \subset l_p, \quad 1 \le p \le \infty, \tag{1.1.3}$$

where $\{e_n\}_n$ denotes the canonical unit vector basis of l_p. In this setting, we have that

$$\varepsilon_n(\mathrm{aco}(A)) = \varepsilon_n(D_\sigma : l_1 \to l_p), \quad n \in \mathbb{N},$$

with D_σ being the diagonal operator generated by the sequence $(\sigma_n)_n$. Hence, according to [C81b, Th. 2], the statement

$$(\sigma_n)_n \in l_{r,s} \quad \text{implies} \quad \Big(e_n(\mathrm{aco}(A))\Big)_n \in l_{q,s}$$

holds true for all $0 < r < \infty$ and all $0 < s \le \infty$, where q is given by $1/q = 1/r + 1/p'$. In this context, every estimate on entropy numbers of diagonal operators $D_\sigma : l_1 \to l_p$

can also be considered as an estimate of entropy numbers of absolutely convex hulls in the diagonal case (1.1.3). Among others, Kühn intensively studied entropy numbers of diagonal operators between l_p spaces (cf. [Kü01, Kü05]).

Using quasi-norm techniques and interpolation, Carl [C82] studied entropy numbers of composition operators $SD : l_1 \to X$ with $D : l_1 \to l_1$ being a diagonal operator and $S : l_1 \to X$ being an arbitrary operator with image in a Banach space X of type p. One of the main results of [C82] reads as follows:

Theorem 1.1.1. [C82] *Let $S \in \mathcal{L}(l_1, X)$ be an operator with image in a Banach space X of type p. Suppose that $D_\sigma \in \mathcal{L}(l_1, l_1)$ is a diagonal operator generated by a sequence $(\sigma_n)_n \in l_{r,s}$ for $0 < r < \infty$, $0 < s \leq \infty$. Then for the composition operator $SD_\sigma : l_1 \to X$ we have that*

$$\Big(e_n(SD_\sigma : l_1 \to X)\Big)_n \in l_{q,s}$$

for $1/q = 1/r + 1/p'$.

As mentioned above, from Theorem 1.1.1 we get results for the general diagonal case (1.1.1) with X being a Banach space of type p; we see that the implication

$$(\sigma_n)_n \in l_{r,s} \quad \text{implies} \quad \Big(e_n(\mathrm{aco}(A))\Big)_n \in l_{q,s} \tag{1.1.4}$$

holds true for all $0 < r < \infty$ and all $0 < s \leq \infty$, where q is given by $1/q = 1/r + 1/p'$. In particular, in the case of polynomial decay we have that

$$\|x_n\| \preccurlyeq n^{-1/r} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/r-1/p'} \tag{1.1.5}$$

for all $r > 0$. A direct proof of (1.1.5) has been given by Ball and Pajor [BP90, Th. 1] (see also [Ta93, p. 522]) in the Hilbert space case, where $p = p' = 2$. We remark that (1.1.4) also holds in a more general context: If $0 < r < \infty$, $0 < s \leq \infty$ and φ is slowly varying function, then

$$(\sigma_n)_n \in l_{r,s,\varphi} \quad \text{implies} \quad \Big(e_n(\mathrm{aco}(A))\Big)_n \in l_{q,s,\varphi} \tag{1.1.6}$$

for $1/q = 1/r + 1/p'$. More detailed information and a proof of this fact can be found in Chapter 5.

Next we focus on the case of *logarithmic decay*, i.e. $\sigma_n = (\log(n+1))^{-1/r}$ for some positive r. It turns out that this case is more complicated then the case (1.1.5) of polynomial decay. For X being a Hilbert space, Ball and Pajor [BP90] showed that for all $0 < r < 2$ it holds that

$$\|x_n\| \preccurlyeq (\log(n+1))^{-1/r} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/2}\,(\log(n+1))^{1/2-1/r}.$$

Due to Talagrand [Ta87], this estimate remains true also for $r = 2$, i.e.

$$\|x_n\| \preccurlyeq (\log(n+1))^{-1/2} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/2}.$$

A very general result was obtained by Li and Linde [LL00, Th. 5.1] in the Hilbert space case. They proved that if $(\log(n+1))^{1/2}\,\sigma_n$ is increasing, then

$$e_n(\mathrm{aco}(A)) \preccurlyeq \sigma_{2^n}. \tag{1.1.7}$$

Furthermore, if $(\log(n+1))^{1/2}\,\sigma_n$ is decreasing and $(\sigma_n)_n$ satisfies the doubling condition $\sigma_n \preccurlyeq \sigma_{2n}$, then

$$e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/2}\,(\log(n+1))^{1/2}\,\sigma_n. \tag{1.1.8}$$

Both estimates remain true if the dyadic entropy numbers $e_n(\mathrm{aco}(A))$ of $\mathrm{aco}(A)$ are replaced by the Gelfand numbers $c_n(\mathrm{aco}(A))$ of $\mathrm{aco}(A)$. As a consequence of these estimates, we have complete knowledge about the behavior of entropy and Gelfand numbers of $\mathrm{aco}(A)$ in the case of logarithmic decay with X being a Hilbert space.

Hilbert space, logarithmic decay: Let (s_n) stand either for the Gelfand numbers (c_n) or for the dyadic entropy numbers (e_n). Then it holds that

$$\|x_n\| \preccurlyeq (\log(n+1))^{-1/r}\,(\log\log(n+3))^{-\beta} \quad \text{implies} \quad s_n(\mathrm{aco}(A)) \preccurlyeq f(n,r,\beta),$$

where $f(n,r,\beta) =$

$$\begin{cases} n^{-1/2}\,(\log(n+1))^{1/2-1/r}\,(\log\log(n+3))^{-\beta}, & 0<r<2,\ \beta\in\mathbb{R}, \quad (1.1.9)\\ n^{-1/2}\,(\log(n+1))^{-\beta}, & r=2,\ \beta\le 0, \quad (1.1.10)\\ n^{-1/2}\,(\log\log(n+3))^{-\beta}, & r=2,\ \beta>0, \quad (1.1.11)\\ n^{-1/r}\,(\log(n+1))^{-\beta}, & 2<r<\infty,\ \beta\in\mathbb{R}, \quad (1.1.12)\end{cases}$$

All these estimates are asymptotically optimal (cf. [LL00, p. 44]). This complements the results from Talagrand and Ball/Pajor given above. We observe that the asymptotic behavior of Gelfand as well as dyadic entropy numbers of the absolutely convex hull significantly changes if the parameter r crosses the point $r = 2$. Furthermore, for fixed $r = 2$, a sudden jump occurs if the parameter β crosses the point $\beta = 0$. An alternative proof of Li and Linde's results (1.1.7) and (1.1.8) can be found in [CE03, Prop. 4]. Until now, no results for the diagonal case have been published in the case that X is a Banach space of type p and $(\sigma_n)_n$ decreases logarithmically.

1.2 The non-diagonal case – Hilbert space

In the non-diagonal case, we study the entropy of the real absolutely convex hull aco(A) of a precompact set A knowing only the entropy behavior of A but nothing on the norms of the extremal points. The first results have been obtained for X being a Hilbert space. In 1987, Dudley proved that in the case of polynomial decay the implication

$$\varepsilon_n(A) \preccurlyeq n^{-1/r} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/r-1/2+\delta}$$

holds true for every $r > 0$, where $\delta > 0$ is arbitrary (cf. [D87, Th. 5.1]). Due to Carl [C97, Th. 3], this result remains true even for $\delta = 0$, i.e.

$$\varepsilon_n(A) \preccurlyeq n^{-1/r} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/r-1/2} \tag{1.2.1}$$

for all $r > 0$ and this estimate is asymptotically optimal (see [CKP99, Prop. 5.1]). However, it should be noted that, from today's perspective, the result (1.2.1) was already contained in a dual version in [CHK88, Th. 1]. By establishing a sharp inequality for the case of polynomial decay, Steinwart was able to extend (1.2.1). His result [St00, Th. 6] implies that

$$\varepsilon_n(A) \preccurlyeq n^{-1/r}\,(\log(n+1))^{-\beta} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/r-1/2}\,(\log(n+1))^{-\beta}$$

for all $r > 0$ and all $\beta \in \mathbb{R}$. The estimate is best possible and remains true also for the Gelfand numbers $c_n(\mathrm{aco}(A))$ of the absolutely convex hull. Finally, in the year 2004, it was also Steinwart [St04, Th. 1.2] who proved that

$$\Big(\varepsilon_n(A)\Big)_n \in l_{r,s} \quad \text{implies} \quad \Big(e_n(\mathrm{aco}(A))\Big)_n \in l_{q,s} \tag{1.2.2}$$

for all $0 < r, s < \infty$, where q is given by $1/q = 1/r + 1/2$. We remark that if

$$2^{1/r}\, e_{n+1}(A) \le e_n(A) \quad \text{for all } n \in \mathbb{N}, \tag{1.2.3}$$

then implication (1.2.2) remains true for the Gelfand numbers $c_n(\mathrm{aco}(A))$ of the absolutely convex hull (cf. [St99, Remark 3.7, 4.17]). We conjecture that condition (1.2.3) is superfluous, but cannot prove it. However, it should be noted that Steinwart's techniques also work in the more general case of a Banach space of type p. We will deal with this setting in more detail in the next section. In the following, we summarize the results for the case of polynomial decay in the Hilbert space setting.

Hilbert space, polynomial decay: Let (s_n) stand either for the Gelfand numbers (c_n) or for the dyadic entropy numbers (e_n). Furthermore, let $A \subset H$ be a precompact subset of a Hilbert space H. Then for all $0 < r < \infty$ and all $\beta \in \mathbb{R}$ it holds that

$$\varepsilon_n(A) \preccurlyeq n^{-1/r}\,(\log(n+1))^{-\beta} \quad \text{implies} \quad s_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/r-1/2}\,(\log(n+1))^{-\beta}.$$

Furthermore, we have that

$$\Big(\varepsilon_n(A)\Big)_n \in l_{r,s} \quad \text{implies} \quad \Big(e_n(\mathrm{aco}(A))\Big)_n \in l_{q,s}$$

for all $0 < r, s < \infty$, where q is given by $1/q = 1/r + 1/2$. All these estimates are asymptotically optimal.

The paper [CKP99] has been the starting point of a comprehensive and systematic study of the entropy behavior of absolutely convex hulls in Hilbert and Banach spaces. Using a refined version of a Sudakov-type inequality, the authors proved striking results in the Hilbert space case. The following inequalities connect the covering numbers of a subset A of a Hilbert space with the Gelfand and covering numbers of its absolutely convex hull.

Theorem 1.2.1. [CKP99]

(i) There is a universal constant $c > 0$ such that for each precompact subset A of the unit ball of a Hilbert space H and for all natural numbers $n \in \mathbb{N}$ we have

$$n^{1/2}\, c_n(\mathrm{aco}(A)) \leq c \inf_{\varepsilon>0} \left(\int_{\varepsilon/4}^{1} (\log N(A,s))^{1/2}\, \mathrm{d}s + n^{1/2}\, \varepsilon \right). \tag{1.2.4}$$

Furthermore, the inequality

$$\log N(\mathrm{aco}(A), \varepsilon) \leq c \left(\frac{1}{\varepsilon} \int_{\varepsilon/4}^{1} (\log N(A,s))^{1/2}\, \mathrm{d}s \right)^2 \tag{1.2.5}$$

holds true.

(ii) There is a universal constant $c > 0$ such that for each precompact subset A of a Hilbert space H and for all $k, n \in \mathbb{N}$ we have

$$k^{1/2}\, c_{k+n}(\mathrm{aco}(A)) \leq c \int_0^{\varepsilon_n(A)} (\log N(A,s))^{1/2}\, \mathrm{d}s. \tag{1.2.6}$$

A probabilistic proof of (1.2.5), which is due to Lifshits, can be found in [Kl12b, Prop. 8]. Using (1.2.4) and (1.2.6), Carl et. al showed that (cf. [CKP99, Prop. 5.5])

$$e_n(A) \preccurlyeq n^{-1/r} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq \begin{cases} n^{-1/2}\, (\log(n+1))^{1/2-1/r}, & 0 < r < 2, \\ n^{-1/r}, & 2 < r < \infty. \end{cases}$$

Both estimates are asymptotically optimal and remain true if $e_n(\mathrm{aco}(A))$ is replaced by $c_n(\mathrm{aco}(A))$. The critical case $r = 2$ was left open; however, it should be noted that, from today's point of view, the answer was implicitly contained in (1.2.5) (cf.

[CE03], [Kl12b]). Indeed, if $e_n(A) \preccurlyeq n^{-1/2}$ then $\log N(A,\varepsilon) \preccurlyeq \varepsilon^{-2}$ for $\varepsilon \to 0+$. Applying inequality (1.2.5) gives $\log N(\operatorname{aco}(A),\varepsilon) \preccurlyeq \varepsilon^{-2}(\log 1/\varepsilon)^2$ for $\varepsilon \to 0+$ and this means that $e_n(\operatorname{aco}(A)) \preccurlyeq n^{-1/2}\log(n+1)$. Hence, we see that in the Hilbert space setting the implication

$$e_n(A) \preccurlyeq n^{-1/2} \quad \text{implies} \quad e_n(\operatorname{aco}(A)) \preccurlyeq n^{-1/2}\log(n+1) \tag{1.2.7}$$

holds true. It was Gao [G01] who first obtained this result using, however, a different approach as outlined above. He also invented an ingenious example of a set to prove that (1.2.7) is the best possible result. Remarkably, comparing the upper estimate of $e_n(\operatorname{aco}(A))$ from (1.2.7) to that from (1.1.10) for $\beta = 0$ reveals an additional logarithmic term. Hence, in the diagonal case we get a better upper estimate than in the non-diagonal case, although in both cases the dyadic entropy numbers of A have the same asymptotic behavior: $e_n(A) \preccurlyeq n^{-1/2}$. Interestingly, such a phenomena does not appear when $e_n(A) \preccurlyeq n^{-1/r}$ for $r \neq 2$. This is one reason why $\varepsilon_n(A) \preccurlyeq (\log(n+1))^{-1/2}$ can be considered as the critical case of logarithmic decay in the Hilbert space setting.

Since covering numbers and entropy numbers are strongly related, it is not very surprising that an integral of covering numbers can be transformed into a sum of entropy numbers. Indeed, Carl and Edmunds [CE03] reformulated (1.2.4) and (1.2.6) as follows.

Theorem 1.2.2. [CE03] *Let A be a precompact subset of a Hilbert space H. Then the following inequalities hold:*

(i) For all $n \in \mathbb{N}$ we have that

$$n^{1/2}\, c_n(\operatorname{aco}(A)) \le c\,(1+\|A\|)\left(1+\sum_{k=1}^{n} k^{-1/2}\, e_k(A)\right), \tag{1.2.8}$$

where $c > 0$ is an absolute constant.

(ii) For all $k, n \in \mathbb{N}$ we have that

$$k^{1/2}\, c_{k+n}(\operatorname{aco}(A)) \le c\left((\log(n+1))^{1/2}\,\varepsilon_n(A) + \sum_{j=n+1}^{\infty} \frac{\varepsilon_j(A)}{j\,(\log(j+1))^{1/2}}\right), \tag{1.2.9}$$

where $c > 0$ is an absolute constant. In particular, it holds that

$$2^{n/2}\, c_{2^n}(\operatorname{aco}(A)) \le c\left(n^{1/2}\, e_n(A) + \sum_{j=n}^{\infty} j^{-1/2}\, e_j(A)\right) \tag{1.2.10}$$

for $n \in \mathbb{N}$, where $c > 0$ is an absolute constant.

The inequalities (1.2.8)-(1.2.10) are very strong and can be seen as the key to the Hilbert space case. Precompactness of the set A, expressed in terms of (dyadic) entropy numbers of A, is transformed into upper estimates of Gelfand numbers of the absolutely convex hull $\mathrm{aco}(A)$. By using well-known connections between Gelfand numbers and dyadic entropy numbers of operators (cf. Theorem 2.1.1, see also [CKP99, Lemma 5.6] [CHP11, Th. B]), this leads to the desired estimates of the entropy of $\mathrm{aco}(A)$.

With the help of (1.2.8), Carl and Edmunds established sharp inequalities for the case of logarithmic decay (cf. [CE03, Prop. 1, 2], see also [St00, Th. 4], [CrSt02, Th. 1.3]). The Gelfand numbers of the absolutely convex hull of a precompact subset of a Hilbert space are estimated in terms of finitely many entropy numbers of A. Their results imply that

$$c_n(\mathrm{aco}(A)) \preccurlyeq \begin{cases} n^{-1/2}\,(\log(n+1))^{1/2-1/r}\,(\log\log(n+3))^{-\beta}, & 0 < r < 2,\ \beta \in \mathbb{R}, \\ n^{-1/2}\,(\log(n+1))^{1-\beta}, & r = 2,\ \beta < 1, \\ n^{-1/r}\,(\log(n+1))^{-\beta}, & 2 < r < \infty,\ \beta \in \mathbb{R}, \end{cases}$$

for $e_n(A) \preccurlyeq n^{-1/r}\,(\log(n+1))^{-\beta}$ (cf. [CE03, Prop. 3]). The estimates are asymptotically optimal. The case $r = 2$, $1 \le \beta < \infty$ was left open; from today's perspective, (1.2.8) gives the optimal estimate for $\beta = 1$ and the case $\beta > 1$ is contained in (1.2.9).

The paper [CHP11] filled the remaining gaps in the case of logarithmic decay. Based on (1.2.8) and (1.2.10), the authors established sharp inequalities which imply complete knowledge about the behavior of both entropy and Gelfand numbers of the absolutely convex hull $\mathrm{aco}(A)$ in the case of logarithmic decay as well as in the case that $\big(e_n(A)\big)_n \in l_{r,s}$ for $0 < r, s < \infty$ (cf. [CHP11, Th. 1.1, Th. 1.4]). Very recent results of Gao show that all these estimates are asymptotically optimal (cf. [G12]). In the following, we summarize these results.

Hilbert space, logarithmic decay: Let (s_n) stand either for the Gelfand numbers (c_n) or for the dyadic entropy numbers (e_n). Furthermore, let $A \subset H$ be a precompact subset of a Hilbert space H. Then it holds that

$$e_n(A) \preccurlyeq n^{-1/r}\,(\log(n+1))^{-\beta} \quad \text{implies} \qquad s_n(\mathrm{aco}(A)) \preccurlyeq f(n, r, \beta),$$

where $f(n,r,\beta) =$

$$\begin{cases} n^{-1/2}\,(\log(n+1))^{1/2-1/r}\,(\log\log(n+3))^{-\beta}, & 0<r<2,\ \beta\in\mathbb{R}, \quad (1.2.11)\\ n^{-1/2}\,(\log(n+1))^{1-\beta}, & r=2,\ \beta<1, \quad (1.2.12)\\ n^{-1/2}\,\log\log(n+3), & r=2,\ \beta=1, \quad (1.2.13)\\ n^{-1/2}\,(\log\log(n+3))^{1-\beta}, & r=2,\ \beta>1, \quad (1.2.14)\\ n^{-1/r}\,(\log(n+1))^{-\beta}, & 2<r<\infty,\ \beta\in\mathbb{R}. \quad (1.2.15)\end{cases}$$

Furthermore, we have that $\big(e_n(A)\big)_n \in l_{r,s}$ implies

$$\big(s_n(\mathrm{aco}(A))\big)_n \in \begin{cases} l_{2,s,\varphi_1}, & 0<r<2,\ 0<s<\infty, \quad (1.2.16)\\ l_{2,\infty,\varphi_2}, & r=2,\ 0<s<\infty, \quad (1.2.17)\\ l_{r,s}, & 2<r<\infty,\ 0<s<\infty, \quad (1.2.18)\end{cases}$$

where $\varphi_1(n) = (\log(n+1))^{1/r-1/2-1/s}$ and $\varphi_2(n) = (\log(n+1))^{\min\{0;1/s-1\}}$. All these estimates are asymptotically optimal.

As in the diagonal case, we see that the behavior of Gelfand and entropy numbers of $\mathrm{aco}(A)$ suddenly changes if the parameter r crosses the point $r=2$. This is why we call $\varepsilon_n(A) \preccurlyeq (\log(n+1))^{-1/2}$ the critical case of logarithmic decay. In addition, if $r=2$ is fixed, then a sudden jump occurs if the parameter β crosses the point $\beta=1$. Consequently, we can consider $\varepsilon_n(A) \preccurlyeq (\log(n+1))^{-1/2}\,(\log\log(n+3))^{-1}$ as the super-critical case of logarithmic decay. However, if we compare the results above with the results (1.1.9)–(1.1.12) from the diagonal case, we observe the same behavior in the case $r \neq 2$, but a drastically different behavior in the critical case $r=2$. Hence, the difference between the diagonal and the non-diagonal case can be quite large.

We remark that also Kley [Kl12a, Kl12b] intensively studied the Hilbert space setting using a purely probabilistic approach. He dealt with polynomial as well as logarithmic decay. Kley's methods are based both on the theory of small deviations of Gaussian processes and inequality (1.2.5). In the cases (1.2.11)–(1.2.15), Kley computed the same sharp upper bounds for the entropy of $\mathrm{aco}(A)$ as [CHP11] did.

1.3 The non-diagonal case – Banach space of type p

In the non-diagonal case, the setting of a Banach space of type p was first studied in the seminal paper [CKP99]. The authors developed techniques to handle both the case of polynomial and logarithmic decay. They obtained asymptotically optimal results except in the so-called critical case of logarithmic decay, that is $\varepsilon_n(A) \preccurlyeq (\log(n+1))^{-1/p'}$. We start with recalling that in the case of polynomial decay the implication

$$\varepsilon_n(A) \preccurlyeq n^{-1/r} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/r-1/p'} \tag{1.3.1}$$

holds true for all $r > 0$ (cf. [CKP99, Prop. 6.1]). The result is asymptotically optimal for certain subsets of l_p. Furthermore, Carl et al. [CKP99, Prop. 6.2, 6.4] proved that $e_n(A) \preccurlyeq n^{-1/r}$ implies

$$e_n(\mathrm{aco}(A)) \preccurlyeq \begin{cases} n^{-1/p'} (\log(n+1))^{1/p'-1/r}, & 0 < r < p', \qquad (1.3.2) \\ n^{-1/r}, & p' < r < \infty, \qquad (1.3.3) \end{cases}$$

The critical case $r = p'$ of logarithmic decay was left open.

In the following, it was Steinwart [St00] who published further results. Building on the ideas of [HK85] and [CHK88], Steinwart developed a delicate decomposition technique for 1-Hölder-continuous operators $T : E \to C(K)$. The latter was used to investigate how the entropy numbers of the underlying compact metric space K and the geometry of the Banach space E affect the entropy behavior of such an operator T. As an application, Steinwart established universal inequalities relating finitely many (dyadic) entropy numbers of A with finitely many dyadic entropy numbers of $\mathrm{aco}(A)$. Steinwart's methods are very strong and lead, consequently, to sharp results extending (1.3.1), (1.3.2) and (1.3.3). In the following, we give an overview of his results.

Theorem 1.3.1. [St00] *Let X be a Banach space of type p, $1 < p \leq 2$. Define*

$$c_A := \frac{\sup\{\|x\| : x \in A\}}{\varepsilon_1(A)}.$$

Then the following statements hold true.

(i) For all $r > 0$ and all $\beta \in \mathbb{R}$ there exists a constant $c = c(p, r, \beta, X) > 0$ such that for all $n \in \mathbb{N}$ and all precompact subsets $A \subset X$ the inequality

$$\sup_{1 \leq k \leq n} k^{1/r+1/p'} (\log(k+1))^{\beta} e_k(\mathrm{aco}(A)) \leq c\, c_A \sup_{1 \leq k \leq a_n} k^{1/r} (\log(k+1))^{\beta} \varepsilon_k(A)$$

holds, where $a_n = n^{1+\frac{r}{p'}}$. If X is a Hilbert space and $p = p' = 2$, then this is also true for the Gelfand numbers $c_k(\mathrm{aco}(A))$ of the absolutely convex hull.

(ii) For $0 < r < p'$ let $0 \le \sigma < 1/r - 1/p'$ and $f : [0,\infty) \to (0,\infty)$ be a function satisfying

$$a^{-\sigma} f(x) \le f(ax) \le a^{\sigma} f(x)$$

for all $a, x \ge 1$. Then there exists a constant $c = c(p, r, f, X) > 0$ such that for all $n \in \mathbb{N}$ and all precompact subsets $A \subset X$ the inequality

$$\sup_{1\le k\le n} k^{1/p'} (\log(k+1))^{1/r-1/p'} f(\log(k+1))\, e_k(\mathrm{aco}(A)) \le c\, c_A \sup_{1\le k\le a_n} k^{1/r} f(k)\, e_k(A)$$

holds, where $a_n = n^{\frac{r}{p'(1-r\sigma)}} \log(n+1)$. If X is a Hilbert space and $p = p' = 2$, then this is also true for the Gelfand numbers $c_k(\mathrm{aco}(A))$ of the absolutely convex hull.

(iii) For all r with $p' < r < \infty$ there exists a constant $c = c(p, r, X) > 0$ such that for all $n \in \mathbb{N}$ and all precompact subsets $A \subset X$ the inequality

$$\sup_{1\le k\le n} k^{1/r} e_k(\mathrm{aco}(A)) \le c\, c_A \sup_{1\le k\le n} k^{1/r} e_k(A)$$

holds. If X is a Hilbert space and $p = p' = 2$, then this is also true for the Gelfand numbers $c_k(\mathrm{aco}(A))$ of the absolutely convex hull.

The inequalities of Steinwart listed above imply the following asymptotically optimal estimates (cf. [St00, Cor. 3, 4, 5]). In the case of polynomial decay, we have that

$$\varepsilon_n(A) \preccurlyeq n^{-1/r} (\log(n+1))^{-\beta} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/r-1/p'} (\log(n+1))^{-\beta} \tag{1.3.4}$$

for all $r > 0$ and all $\beta \in \mathbb{R}$. Furthermore, in the case of logarithmic decay it holds that $e_n(A) \preccurlyeq n^{-1/r} (\log(n+1))^{-\beta}$ implies

$$e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'} (\log(n+1))^{1/p'-1/r} (\log\log(n+3))^{-\beta}$$

for $0 < r < p'$ and $\beta \in \mathbb{R}$. Moreover, it turns out that in B-convex Banach spaces the subsets A and $\mathrm{aco}(A)$ surprisingly have the same entropy behavior, whenever $\big(e_n(A)\big)_n$ or $\big(e_n(\mathrm{aco}(A))\big)_n$ decrease slow enough. More precisely, if $p' < r < \infty$ and $(a_n)_n$ is a positive sequence such that $(n^{1/r} a_n)_n$ is monotone increasing, then

$$e_n(A) \preccurlyeq a_n \quad \text{if and only if} \quad e_n(\mathrm{aco}(A)) \preccurlyeq a_n$$

and

$$e_n(A) \sim a_n \quad \text{if and only if} \quad e_n(\mathrm{aco}(A)) \sim a_n.$$

Steinwart also considered the critical case of logarithmic decay. He proved that (cf. [St00, p. 327])

$$e_n(A) \preccurlyeq n^{-1/p'} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'} (\log(n+1))^{1+1/p'}.$$

We would like to remark that this result can also be obtained by adapting the proof of Prop. 6.4 in [CKP99]. However, it is not the best possible estimate; the latter is due to Creutzig and Steinwart [CrSt02]. Extending Gao's ideas [G01] from Hilbert spaces to B-convex Banach spaces, they proved the following inequality.

Theorem 1.3.2. [CrSt02] *Let X be a Banach space of type p, $1 < p \leq 2$, and let $-\infty < \beta < 1$. Then there is a constant $c = c(p, \beta, X) > 0$ such that for every precompact $A \subset X$ and all $n \in \mathbb{N}$ we have*

$$\sup_{1 \leq k \leq n} k^{1/p'} (\log(k+1))^{\beta - 1} e_k(\mathrm{aco}(A)) \leq c\, c_A \sup_{1 \leq k \leq a_n} k^{1/p'} (\log(k+1))^{\beta} e_k(A),$$

where $a_n := \frac{n}{(\log(n+1))^{p'}} + 1$ and $c_A := \frac{\sup\{\|x\| : x \in A\}}{\varepsilon_1(A)}$.

As a consequence of this result, we see that in the critical case of logarithmic decay the implication

$$e_n(A) \preccurlyeq n^{-1/p'} (\log(n+1))^{-\beta} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'} (\log(n+1))^{1-\beta} \tag{1.3.5}$$

holds true for all $-\infty < \beta < 1$. Creutzig and Steinwart also showed, that this result is asymptotically optimal whenever X is an infinite dimensional Banach space of optimal type p (cf. [CrSt02, Th. 1.5]).

The next major step forward was achieved in [St04]. By refining the decomposition techniques of [CKP99, St00], Steinwart established an inequality which estimates the entropy numbers of $\mathrm{aco}(A)$ in terms of finitely many entropy numbers of A.

Theorem 1.3.3. [St04] *Let X be a Banach space of type p, $1 < p \leq 2$, and let $0 < t < \infty$. Then there exists a constant $c(t) > 0$ such that for all integers $n \geq 2$, all integers $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ and all bounded symmetric subsets $A \subset X$ we have*

$$\begin{aligned} e_{2m}(\mathrm{aco}(A)) \leq c(t)\, m^{-1/t - 1/p'} & \sup_{i \leq \min\{m^{1+t/p'}; \alpha_1\}} i^{1/t} \varepsilon_i(A) \\ & + 23\, \tau_p(X)\, 2^{-n/p'} \left(\sum_{k=1}^{n} \left(2^{k/p'} \sum_{i=k}^{n} \varepsilon_{\alpha_i}(A) \right)^p \right)^{1/p}, \end{aligned}$$

where $m := \left\lfloor 2^{n+2} \sum_{k=2}^{n} 2^{-k} \log\left(\frac{2^{k+2}\alpha_k}{2^n} + 3 \right) \right\rfloor + 2.$

Steinwart's inequality can be used to prove various known results on entropy numbers of absolutely convex hulls in Banach spaces of type p (cf. [St04]). The main problem is to find a suitable choice of integers α_k; the latter has to be adapted to the decay of entropy numbers of A. In the following, we will heavily use Theorem 1.3.3 to prove new results.

We start with recalling the following result of Steinwart [St04, Th. 1.2]: For all $0 < r, s < \infty$ the implication

$$\Big(\varepsilon_n(A)\Big)_n \in l_{r,s} \quad \text{implies} \quad \Big(e_n(\mathrm{aco}(A))\Big)_n \in l_{q,s}$$

holds true, where q is given by $1/q = 1/r + 1/p'$. The result is the best possible one whenever X has optimal type p. Moreover, Steinwart proved a similar implication for subsets A with logarithmically decreasing entropy numbers (cf. [St04, Th. 1.3]): Let $0 < s < \infty$ and define r by $1/r = 1/p' + 1/s$, then

$$\Big(e_n(A)\Big)_n \in l_{r,s} \quad \text{implies} \quad \Big(e_n(\mathrm{aco}(A))\Big)_n \in l_{p',s}. \tag{1.3.6}$$

However, since the choice of s fixes the parameter r, this result does not have the desired generality. The following theorem closes this gap.

Theorem 1.3.4. *Let X be a Banach space of type p, $1 < p \leq 2$. Suppose that $0 < r < p'$ and $0 < s < \infty$. Then for all precompact subsets $A \subset X$ we have that*

$$\Big(e_n(A)\Big)_n \in l_{r,s} \quad \textit{implies} \quad \Big(e_n(\mathrm{aco}(A))\Big)_n \in l_{p',s,\varphi}$$

where $\varphi(n) = (\log(n+1))^{1/r-1/p'-1/s}$. The result is optimal in the following sense: If $\psi(n) = (\log(n+1))^{\beta}$ with $\beta > 1/r - 1/p' - 1/s$, then there exists a precompact subset $A \subset l_p$ such that $\Big(e_n(A)\Big)_n \in l_{r,s}$ and $\Big(e_n(\mathrm{aco}(A))\Big)_n \notin l_{p',s,\psi}$.

Observe that Steinwart's result (1.3.6) is contained in Theorem 1.3.4 as a special case. Moreover, (1.3.2) can be considered as the limit case $s = \infty$ of Theorem 1.3.4.

As already mentioned above, in the case of slow logarithmic decay of $\varepsilon_n(A)$ the subsets A and $\mathrm{aco}(A)$ surprisingly have the same entropy behavior. Using Theorem 1.3.1 *(iii)* and some inequalities of Hardy-type, we will show that this is also true when considering summability properties.

Theorem 1.3.5. *Let X be a Banach space of type p, $1 < p \leq 2$. Suppose that $p' < r < \infty$, $0 < s < \infty$ and φ is a slowly varying function. Then there exists a constant $c = c(p, r, s, \varphi, X) > 0$ such that for all $N \in \mathbb{N}$ and all precompact subsets $A \subset X$ we have that*

$$\sum_{n=1}^{N} (\varphi(n))^s \, n^{s/r-1} \Big(e_n(\mathrm{aco}(A))\Big)^s \leq c\, c_A^s \sum_{n=1}^{N} (\varphi(n))^s \, n^{s/r-1} \Big(e_n(A)\Big)^s$$

and

$$\sup_{1\le n\le N} \varphi(n)\, n^{1/r}\, e_n(\mathrm{aco}(A)) \le c\, c_A \sup_{1\le n\le N} \varphi(n)\, n^{1/r}\, e_n(A),$$

where $c_A := \frac{\sup\{\|x\|:\, x\in A\}}{\varepsilon_1(A)}$. *In the context of Lorentz sequence spaces, these inequalities imply that*

$$\Big(e_n(A)\Big)_n \in l_{r,s,\varphi} \quad \textit{if and only if} \quad \Big(e_n(\mathrm{aco}(A))\Big)_n \in l_{r,s,\varphi} \tag{1.3.7}$$

for $p' < r < \infty$, $0 < s \le \infty$ *and* φ *being a slowly varying function.*

We would like to remark that [St00, Cor. 3] gives (1.3.7) for $s = \infty$.

Also in the critical case of logarithmic decay, Steinwart's inequality from Theorem 1.3.3 turns out to be a useful tool. We will use the latter to give an alternative proof of (1.3.5) for $-\infty < \beta < 1/p$ (cf. also [St04, Ex. 1.6]). Furthermore, we prove a partial result in the critical case that $\Big(e_n(A)\Big)_n \in l_{p',s}$ for $0 < s < \infty$.

Theorem 1.3.6. *Let* X *be a Banach space of type* p, $1 < p \le 2$, *and let* $A \subset X$ *be a precompact subsets of* X. *Then for all* $-\infty < \beta < 1/p$ *we have that*

$$e_n(A) \preccurlyeq n^{-1/p'} (\log(n+1))^{-\beta} \quad \textit{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'} (\log(n+1))^{1-\beta}.$$

Furthermore, if $p \le s < \infty$ *then*

$$\Big(e_n(A)\Big)_n \in l_{p',s} \quad \textit{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'} (\log(n+1))^{1-1/s}.$$

The results are optimal in the following sense: If X *is an infinite dimensional Banach space of optimal type* $p \in (1,2]$ *and* $-\infty < \beta < 1$, *then there exists a subset* A *of* X *satisfying both*

$$e_n(A) \preccurlyeq n^{-1/p'} (\log(n+1))^{-\beta} \quad \textit{and} \quad e_n(\mathrm{aco}(A)) \succcurlyeq n^{-1/p'} (\log(n+1))^{1-\beta}. \tag{1.3.8}$$

Furthermore, if $1 < s < \infty$ *and* $\beta > -1 + 1/s$ *then there exists a subset* A *of* X *with*

$$\Big(e_n(A)\Big)_n \in l_{p',s} \quad \textit{and} \quad \sup_{n\in\mathbb{N}} n^{1/p'} (\log(n+1))^{\beta}\, e_n(\mathrm{aco}(A)) = \infty.$$

Moreover, for $0 < s < \infty$ *there exists a subset* $A \subset l_p$ *of the sequence space* l_p, $1 < p \le 2$, *satisfying both*

$$\Big(e_n(A)\Big)_n \in l_{p',s} \quad \textit{and} \quad \Big(e_n(\mathrm{aco}(A))\Big)_n \notin l_{p',t}$$

for $0 < t < \infty$.

What is left open is the case that $\big(e_n(A)\big)_n \in l_{p',s}$ for $0 < s < p$. Here we conjecture that

$$\big(e_n(\mathrm{aco}(A))\big)_n \in \begin{cases} l_{p',\infty}, & 0 < s \le 1, \qquad (1.3.9)\\ l_{p',\infty,\varphi}, & 1 < s < p, \qquad (1.3.10)\end{cases}$$

with $\varphi(n) = (\log(n+1))^{-1+1/s}$. Theorem 1.3.6 shows that both results would be the best possible ones. We remark that in order to prove (1.3.9) it is enough to show the implication

$$\big(e_n(A)\big)_n \in l_{p',1} \quad \text{implies} \quad \big(e_n(\mathrm{aco}(A))\big)_n \in l_{p',\infty}.$$

Observe that, in the Hilbert space case, this implication is a consequence of inequality (1.2.8). Hence, it would be sufficient to prove an analogon of (1.2.8) for Banach spaces of type p. Of course, such an analogon would be of great help also in other cases; however, we cannot prove it. As far as (1.3.10) is concerned, we can give a proof under an additional regularity condition. This is the subject of the next proposition.

Proposition 1.3.7. *Let $A \subset X$ be a precompact subset of a Banach space X of type p, $1 < p \le 2$. Suppose that $1 < s < \infty$ and $\big(n^{1/p'} e_n(A)\big)_n$ is monotonically decreasing. Then there exists a constant $c = c(p,s,X) > 0$ such that*

$$\sup_{1\le k\le n} (\log(k+1))^{1/s-1}\, k^{1/p'}\, e_k(\mathrm{aco}(A)) \le c\, c_A \left(\sum_{i=1}^{n} i^{s/p'-1}\big(e_i(A)\big)^s\right)^{1/s},$$

where $c_A := \frac{\sup\{\|x\|:\, x\in A\}}{\varepsilon_1(A)}$. In particular, we have that

$$\big(e_n(A)\big)_n \in l_{p',s} \quad \textit{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'}\,(\log(n+1))^{1-1/s}$$

for $1 < s < \infty$.

Finally, let us deal with the remaining case

$$e_n(A) \preccurlyeq n^{-1/p'}\,(\log(n+1))^{-\beta} \quad \text{for } \beta \ge 1.$$

So far, no results have been obtained. However, we would like to remark that Theorem 1.3.2 implies the (weak) estimate

$$e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'}\,(\log(n+1))^{\delta}, \tag{1.3.11}$$

where δ is an arbitrary positive number. In view of the Hilbert space results (1.2.13) and (1.2.14), it is reasonable to conjecture that the asymptotically optimal estimates are given by

$$e_n(\mathrm{aco}(A)) \preccurlyeq \begin{cases} n^{-1/p'}\,\log\log(n+3), & \beta = 1, \qquad (1.3.12)\\ n^{-1/p'}\,(\log\log(n+3))^{1-\beta}, & \beta > 1. \qquad (1.3.13)\end{cases}$$

Pushing the technique used for the proof of [CrSt02, Prop. 2.1] to its limit, Kley was able to show (1.3.12) in the Hilbert space setting (cf. [Kl12b, Sec. 4.2]). It turns out, that Kley's proof can be modified to work also in our more general framework, leading to the following result.

Theorem 1.3.8. *Let X be a Banach space of type p, $1 < p \leq 2$. Furthermore, let $A \subset X$ be a precompact subset of X satisfying*

$$\log N(A, \varepsilon) \preccurlyeq \varepsilon^{-p'} \left(\log \frac{1}{\varepsilon}\right)^{-p'}, \quad 0 < \varepsilon \leq 1/2.$$

Then for all $0 < \varepsilon \leq 1/5$ it holds that

$$\log N(\mathrm{aco}(A), \varepsilon) \preccurlyeq \varepsilon^{-p'} \left(\log \log \frac{1}{\varepsilon}\right)^{p'}.$$

In the language of entropy numbers this means that

$$e_n(A) \preccurlyeq n^{-1/p'} \, (\log(n+1))^{-1} \quad \textit{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'} \, \log\log(n+3). \tag{1.3.14}$$

The next theorem shows that (1.3.14) is asymptotically optimal for certain subsets of the sequence space l_p. The proof builds on ideas of Gao [G12] originally given in the Hilbert space setting.

Theorem 1.3.9. *For every $1 < p \leq 2$ there exists a subset $A \subset l_p$ of the sequence space l_p satisfying both*

$$\log N(A, \varepsilon) \leq 2^{3p'+2} \, \varepsilon^{-p'} \left(\log \frac{1}{\varepsilon}\right)^{-p'} \quad \textit{for all } 0 < \varepsilon < 1/2 \tag{1.3.15}$$

and

$$\log N(\mathrm{conv}(A), \varepsilon) \geq c(p) \, \varepsilon^{-p'} \left(\log \log \frac{1}{\varepsilon}\right)^{p'} \quad \textit{for all } 0 < \varepsilon \leq \frac{1}{2} \left(\frac{15}{2^{12}}\right)^{1/p} 2^{-2^{17}}, \tag{1.3.16}$$

where $c(p) = 15^{p'} \, 2^{-16p'/p - 4p' - 11}$. In terms of entropy numbers this means that

$$e_n(A) \preccurlyeq n^{-1/p'} \, (\log(n+1))^{-1} \quad \textit{and} \quad e_n(\mathrm{conv}(A)) \succcurlyeq n^{-1/p'} \, \log\log(n+3).$$

Unfortunately, we cannot give a proof of (1.3.13). Hence, if A is a precompact subset of a Banach space of type p satisfying $e_n(A) \preccurlyeq n^{-1/p'} \, (\log(n+1))^{-\beta}$ with $1 < \beta < \infty$, then the exact entropy behavior of $\mathrm{aco}(A)$ remains an open problem. To handle this case we would need, for example, an analogon of (1.2.9) or (1.2.10). However, we can at least prove that there is no better estimate than (1.3.13).

Theorem 1.3.10. *Let $1 < p \leq 2$. Then for every $1 < \beta < \infty$ there exists a subset $A^\beta \subset l_p$ of the sequence space l_p satisfying both*

$$\log N\Big(A^\beta, \varepsilon\Big) \leq 2^{2+p'} (\beta+1)^{\beta p'} \varepsilon^{-p'} \left(\log \frac{1}{\varepsilon}\right)^{-\beta p'} \quad \text{for all } 0 < \varepsilon \leq 1/2 \qquad (1.3.17)$$

and

$$\log N\Big(\operatorname{conv}(A^\beta), \varepsilon\Big) \geq 2^{-12p'-5} \varepsilon^{-p'} \left(\log\log \frac{1}{\varepsilon}\right)^{p'(1-\beta)} \quad \text{for all } 0 < \varepsilon \leq c(\beta), \qquad (1.3.18)$$

where $c(\beta) \leq 2^{-2^{17}}$ is a positive constant depending on β. In terms of entropy numbers this means that

$$e_n(A^\beta) \preccurlyeq n^{-1/p'} (\log(n+1))^{-\beta} \quad \text{and} \quad e_n\Big(\operatorname{conv}(A^\beta)\Big) \succcurlyeq n^{-1/p'} (\log\log(n+3))^{1-\beta}.$$

Finally, let us summarize the present state of research on the entropy of absolutely convex hulls in Banach spaces of type p.

Banach space of type p, polynomial decay: Let $A \subset X$ be a precompact subset of a Banach space X of type p, $1 < p \leq 2$. Then for all $0 < r < \infty$ and all $\beta \in \mathbb{R}$ it holds that

$$\varepsilon_n(A) \preccurlyeq n^{-1/r} (\log(n+1))^{-\beta} \quad \text{implies} \quad e_n(\operatorname{aco}(A)) \preccurlyeq n^{-1/r-1/p'} (\log(n+1))^{-\beta}. \qquad (1.3.19)$$

Furthermore, we have that

$$\Big(\varepsilon_n(A)\Big)_n \in l_{r,s} \quad \text{implies} \quad \Big(e_n(\operatorname{aco}(A))\Big)_n \in l_{q,s} \qquad (1.3.20)$$

for all $0 < r, s < \infty$, where q is given by $1/q = 1/p' + 1/r$.

Banach space of type p, logarithmic decay: It holds that

$$e_n(A) \preccurlyeq n^{-1/r} (\log(n+1))^{-\beta} \quad \text{implies} \quad e_n(\operatorname{aco}(A)) \preccurlyeq f(n,p,r,\beta),$$

where $f(n,p,r,\beta) =$

$$\begin{cases} n^{-1/p'} (\log(n+1))^{1/p'-1/r} (\log\log(n+3))^{-\beta}, & 0 < r < p',\ \beta \in \mathbb{R}, \quad (1.3.21) \\ n^{-1/p'} (\log(n+1))^{1-\beta}, & r = p',\ \beta < 1, \quad (1.3.22) \\ n^{-1/p'} \log\log(n+3), & r = p',\ \beta = 1, \quad (1.3.23) \\ \text{open problem}, & r = p',\ \beta > 1, \quad (1.3.24) \\ n^{-1/r} (\log(n+1))^{-\beta}, & p' < r < \infty,\ \beta \in \mathbb{R}. \quad (1.3.25) \end{cases}$$

Furthermore, we have that $\big(e_n(A)\big)_n \in l_{r,s}$ implies

$$\big(e_n(\mathrm{aco}(A))\big)_n \in \begin{cases} l_{p',s,\varphi_1}, & 0 < r < p',\ 0 < s < \infty, & (1.3.26)\\ \text{open problem}, & r = p',\ 0 < s < p, & (1.3.27)\\ l_{p',\infty,\varphi_2}, & r = p',\ p \le s < \infty, & (1.3.28)\\ l_{r,s}, & p' < r < \infty,\ 0 < s < \infty, & (1.3.29)\end{cases}$$

where $\varphi_1(n) = (\log(n+1))^{1/r-1/p'-1/s}$ and $\varphi_2(n) = (\log(n+1))^{1/s-1}$.

1.4 Remarks and addenda

We want to point out that some authors studied the entropy of absolutely convex hulls in arbitrary Banach spaces. Carl et al. dealt with the case of very fast decreasing entropy numbers of A, i.e. $\varepsilon_n(A) \le 2^{-\gamma(n+1)^\sigma}$ for some $\gamma, \sigma > 0$; they obtained asymptotically optimal estimates of the entropy of $\mathrm{aco}(A)$ (cf. [CKP99, Cor. 4.2]). Furthermore, the authors gave optimal results for the case that the entropy numbers of A form a non-rapidly decreasing sequence. More precisely (cf. [CKP99, Prop. 4.5]), if $(a_n)_n$ is a positive decreasing sequence satisfying the doubling condition $a_n \preccurlyeq a_{2n}$ and if $A \subset X$ is a precompact subset of a Banach space X then

$$\varepsilon_n(A) \preccurlyeq a_n \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq a_n.$$

The estimate is asymptotically optimal for certain subsets of l_1. In particular, in the setting of a general Banach space we have that

$$\varepsilon_n(A) \preccurlyeq n^{-1/r}\,(\log(n+1))^{-\beta} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/r}\,(\log(n+1))^{-\beta}$$

for all $r > 0$, $\beta \in \mathbb{R}$. Now comparing this result to (1.3.4) where X is a Banach space of type p, we observe a difference which is on the polynomial scale. However, both results are the best possible ones. Note that the difference is caused by the fact that l_1 is not B-convex. This shows how strong the underlying Banach space X affects the entropy of the absolutely convex hull. Considering an arbitrary Banach space and logarithmic decay of the entropy numbers of A, it turns out that $e_n(A) \preccurlyeq n^{-1/r}\,(\log(n+1))^{-\beta}$ implies

$$e_n(\mathrm{aco}(A)) \preccurlyeq (\log(n+1))^{-1/r}\,(\log\log(n+3))^{-\beta}$$

for all $r > 0$, $\beta \in \mathbb{R}$. Hence, the entropy numbers of the absolutely convex hull of A show a drastically different behavior than in the setting of a B-convex Banach space (1.3.21)-(1.3.25), where we have critical and super-critical cases.

Until now, in the setting of an arbitrary Banach space no results have been published for the case that the entropy numbers of A belong to some (generalized) Lorentz sequence space. The following theorem, which is based on [CKP99, Prop. 4.4], closes this gap.

Theorem 1.4.1. *Suppose that $0 < r, s < \infty$ and φ is a slowly varying function. Then there exists a constant $c = c(r,s,\varphi) > 0$ such that for all Banach spaces X, all precompact subsets $A \subset X$ and all integers $N \in \mathbb{N}$ the inequalities*

$$\sum_{n=1}^{N} (\varphi(n))^s \, n^{s/r-1} \Big(e_n(\mathrm{aco}(A))\Big)^s \le c\, c_A^s \sum_{n=1}^{N} (\varphi(n))^s \, n^{s/r-1} \Big(\varepsilon_n(A)\Big)^s$$

and

$$\sup_{1\le n\le N} \varphi(n)\, n^{1/r}\, e_n(\mathrm{aco}(A)) \le c\, c_A \sup_{1\le n\le N} \varphi(n)\, n^{1/r}\, \varepsilon_n(A)$$

hold, where $c_A := \frac{\sup\{\|x\| : x\in A\}}{\varepsilon_1(A)}$. In the context of Lorentz sequence spaces, these inequalities imply that

$$\Big(\varepsilon_n(A)\Big)_n \in l_{r,s,\varphi} \quad \textit{implies} \quad \Big(e_n(\mathrm{aco}(A))\Big)_n \in l_{r,s,\varphi} \tag{1.4.1}$$

for $0 < r < \infty$, $0 < s \le \infty$ and φ being a slowly varying function.

Consider the set $A = \{\sigma_n e_n \,|\, n \in \mathbb{N}\} \subset l_1$, where $\{e_n\}_n$ denotes the canonical unit vector basis of l_1 and $(\sigma_n)_n$ is a positive non-increasing sequence satisfying the doubling condition $s_n \preccurlyeq s_{2n}$. Then from [CKP99, Remark 4.6] we know that

$$\varepsilon_n(A) \le \sigma_n \qquad \text{and} \qquad e_n(\mathrm{aco}(A)) \sim \sigma_n.$$

This shows that (1.4.1) is the best possible result.

In his thesis, Steinwart studied the case of very slow decreasing entropy numbers of A. Recall that a positive null sequence $(a_n)_n$ is said to be *regular*, if there is a constant $c \ge 1$ such that $a_n \le c\, a_{2n}$ and $a_m \le c\, a_n$ for all $1 \le n \le m$. Steinwart proved the following result (cf. [St99, Prop. 4.7]): Let X be an arbitrary Banach space and $(a_n)_n$ be a regular sequence with $a_n \sim a_{2^n}$, then for every precompact subset A of X we have that

$$e_n(A) \preccurlyeq a_n \quad \text{if and only if} \quad e_n(\mathrm{aco}(A)) \preccurlyeq a_n$$

and

$$e_n(A) \sim a_n \quad \text{if and only if} \quad e_n(\mathrm{aco}(A)) \sim a_n.$$

As we can see, it turns out that A and $\mathrm{aco}(A)$ surprisingly have the same entropy behavior. An example of a regular sequence $(a_n)_n$ satisfying $a_n \sim a_{2^n}$ can be found, for instance, in [St99, Ex. 1.3].

Finally, we remark that Hildebrandt [Hi03] studied p-convex hulls of countable point sets in Hilbert spaces. The entropy of absolutely convex hulls of finite sets was studied by Kyrezi [Ky00]. Furthermore, Gao [G04] established some general inequalities to estimate the metric entropy of the absolutely convex hull of a set in terms of covering numbers of the set.

1.5 Open problems

This section is devoted to open problems which, in our opinion, are of great interest. Of course, the following list does not claim to be exhaustive.

Problem 1: Consider the diagonal case with X being a Banach space of type p. Is there an analogon of (1.1.7) and (1.1.8)? In particular, does the implication

$$\sigma_n \preccurlyeq (\log(n+1))^{-1/p'} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'}$$

holds true?

Problem 2: Let $A \subset H$ be a precompact subset of a Hilbert space H. For $0 < r < \infty$ define q by $1/q = 1/r + 1/2$. Does for all $0 < s < \infty$ the implication

$$\Big(\varepsilon_n(A)\Big)_n \in l_{r,s} \quad \text{implies} \quad \Big(c_n(\mathrm{aco}(A))\Big)_n \in l_{q,s}$$

holds true?

Problem 3: Are there analoga of (1.2.4) - (1.2.6) for Banach spaces of type p?

Problem 4: Let $A \subset X$ be a precompact subset of a Banach space X of type p. Is it true that

$$e_n(A) \preccurlyeq n^{-1/p'} (\log(n+1))^{-\beta} \quad \text{implies} \quad e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'} (\log\log(n+3))^{1-\beta}$$

for $1 < \beta < \infty$?

Problem 5: Let $A \subset X$ be a precompact subset of a Banach space X of type p. Does for all $0 < s < p$ the implication

$$\Big(e_n(A)\Big)_n \in l_{p',s} \quad \text{implies} \quad \Big(e_n(\mathrm{aco}(A))\Big)_n \in l_{p',\infty,\varphi}$$

with $\varphi(n) = (\log(n+1))^{\min\{0;1/s-1\}}$ holds true?

Problem 6: Let X be an infinite dimensional Banach space of optimal type p. Is there a subset A of X satisfying both

$$e_n(A) \preccurlyeq n^{-1/p'} (\log(n+1))^{-1} \quad \text{and} \quad e_n(\mathrm{aco}(A)) \succcurlyeq n^{-1/p'} \log\log(n+3) \ ?$$

Problem 7: Let X be an infinite dimensional Banach space of optimal type p and let $1 < \beta < \infty$. Is there a subset A of X satisfying both

$$e_n(A) \preccurlyeq n^{-1/p'} (\log(n+1))^{-\beta} \quad \text{and} \quad e_n(\mathrm{aco}(A)) \succcurlyeq n^{-1/p'} (\log\log(n+3))^{1-\beta} \ ?$$

2 Preliminaries

Before proving the results stated in the previous chapter, let us fix some notations and conventions. After that, using slowly varying functions we generalize the classical Lorentz sequence spaces. Finally, since we want to deal with the entropy of absolutely convex hulls in B-convex Banach spaces, it seems to be reasonable to take a closer look at entropy numbers, Banach spaces of type p and the absolutely convex hull of a set.

2.1 Basic notations

Unless otherwise stated, in the following X_0, X, Y and Y_0 denote real or complex Banach spaces. The *closed unit ball* of X is denoted by B_X and $\mathring{B}_X$ stands for the open unit ball of X. The dual Banach space of X is denoted by X'. Furthermore, $\mathcal{L}(X,Y)$ denotes the Banach space of all linear bounded operators acting between X and Y, equipped with the operator norm

$$\|T : X \to Y\| = \|T\| := \sup\{\|Tx\| : x \in B_X\}.$$

Moreover, $\mathcal{K}(X,Y)$ stands for the closed subspace of all *compact* operators from X to Y. Given a linear bounded operator $T \in \mathcal{L}(X,Y)$, the dual operator $T' \in \mathcal{L}(Y',X')$ is defined by

$$\langle x, T'y' \rangle := \langle Tx, y' \rangle\,, \quad x \in X,\, y' \in Y'.$$

A linear and surjective operator $T : X \to Y$ is said to be a *metric isomorphism* if it preserves the norm, i.e. $\|Tx\| = \|x\|$ for all $x \in X$. This concept can be weakened by dropping the surjectivity or the preservation of the norm. In the first case, a linear operator $J : X \to Y$ satisfying $\|Jx\| = \|x\|$ for all $x \in X$ is called *metric injection.* An important example of a metric injection is given by the canonical embedding $\mathcal{K}_X : X \to X''$ of a Banach space X into its bidual space X'',

$$\langle x', \mathcal{K}_X x \rangle := \langle x, x' \rangle\,, \quad x \in X,\, x' \in X'.$$

In the second case, a linear operator Q mapping X onto Y is called a *surjection.* We speak of a *metric surjection* if the open unit ball of X is mapped onto the open unit ball of Y, i.e. $Q(\mathring{B}_X) = \mathring{B}_Y$. For every closed subspace N of Y, the quotient map $Q_N^Y : Y \to Y/N$ is a metric surjection. Moreover, I_X or $id : X \to X$ stands for the identity operator on a Banach space X.

We are interested in the following *s-numbers* associated with an operator $T \in \mathcal{L}(X,Y)$ (cf. [P87, CS90]):

- the *n-th approximation number of* T, defined by

$$a_n(T) := \inf \{\|T - A\| : A \in \mathcal{L}(X,Y) \text{ with } \operatorname{rank} A < n\},$$

- the *n-th Gelfand number of* T, defined by

$$c_n(T) := \inf \left\{ \left\| T I_E^X \right\| : E \text{ subspace of } X \text{ with } \operatorname{codim}(E) < n \right\},$$

  where I_E^X is the natural embedding of E into X,

- the *n-th Kolmogorov number of* T, defined by

$$d_n(T) := \inf \left\{ \left\| Q_F^Y T \right\| : F \text{ subspace of } Y \text{ with } \dim(F) < n \right\},$$

  where $Q_F^Y : Y \to Y/F$ is the usual quotient map,

- the *n-th symmetrized approximation number of* T, defined by

$$t_n(T) := a_n(J_Y T Q_X),$$

  where Q_X is the canonical quotient map from $l_1(B_X)$ onto X and J_Y is the canonical embedding of Y into $l_\infty(B_{Y'})$ (cf. [CS90, p. 52, 60]).

Recall that $t_n(T) = c_n(TQ_X) = d_n(J_Y T)$, $t_n(T) \leq c_n(T), d_n(T) \leq a_n(T)$ and $t_n(T) = t_n(T')$ holds true for all operators $T \in \mathcal{L}(X,Y)$ (cf. [P74, CS90]). The following inequality relates the entropy numbers of an operator (see Section 2.4) to the above-mentioned s-numbers (cf. [C81a, Th. 1], [CS90, Th. 3.1.1], see also [CKP99, Th. 1.3]).

Theorem 2.1.1. [C81a] *For every $0 < \alpha < \infty$ there exists a constant $c(\alpha) \geq 1$ such that for every operator $T \in \mathcal{L}(X,Y)$ between arbitrary Banach spaces X and Y and all $n \in \mathbb{N}$ we have*

$$\sup_{1 \leq k \leq n} k^\alpha e_k(T) \leq c(\alpha) \sup_{1 \leq k \leq n} k^\alpha t_k(T).$$

Let (X, d) be a metric space, then

$$B_X(c, r) := \{x \in X : d(x, c) \leq r\}$$

denotes the closed ball in X with radius $r \geq 0$ and center $c \in X$. A subset A of X is called *bounded* if the *diameter* of A,

$$\mathrm{diam}(A) := \sup\{d(x, y) : x, y \in A\},$$

is finite. Furthermore, $cl(A)$ denotes the *closure* of the set A. For a subset $A \subset X$ of a normed space X we define

$$\|A\| := \sup\{\|x\| : x \in A\}.$$

It is clear that A is bounded if and only if $\|A\| < \infty$. The subset A is called *symmetric*, if $A = -A$. Moreover, the convex hull of A is denoted by $\mathrm{conv}(A)$ and $\mathrm{aco}(A)$ stands for the real absolutely convex hull of A. In section 2.7 we will consider the latter in more detail.

For $1 \leq p \leq \infty$ we define the so-called *conjugated exponent* p' by the Hölder condition $1/p + 1/p' = 1$. By $\log(x)$ we denote the binary logarithm, i.e. the logarithm with base 2, and $\ln(x)$ stands for the natural logarithm with base e. Furthermore, for a real number $x \in \mathbb{R}$ the largest integer not greater than x is given by $\lfloor x \rfloor := \max\{z \in \mathbb{Z} \mid z \leq x\}$. Note that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ holds true for all $x \in \mathbb{R}$. Moreover, we define $(x)_+ := \max\{x; 0\}$.

In order to compare positive sequences $(x_n)_n$, $(y_n)_n$ we introduce the following notations: We write $x_n \preccurlyeq y_n$, if there exists a constant $c > 0$ such that $x_n \leq c\, y_n$ for all natural numbers $n \in \mathbb{N}$. Furthermore, $x_n \sim y_n$ means that both $x_n \preccurlyeq y_n$ and $y_n \preccurlyeq x_n$.

2.2 Slowly varying functions

In 1930, Karamata introduced the class of slowly varying functions (cf. [Ka30]). In the following section we will use such functions to generalize the classical Lorentz sequence spaces. This is reason enough to have a closer look at the subject. We start with the definition.

Definition. A positive and continuous function φ defined on some neighborhood $[D, \infty)$ of infinity is said to be *slowly varying (at infinity)* if for all $\lambda > 0$ we have that

$$\lim_{x \to \infty} \frac{\varphi(\lambda x)}{\varphi(x)} = 1.$$

In the following, we may assume that a slowly varying function φ is defined on the whole interval $(0,\infty)$, for instance, by taking $\varphi(x) = \varphi(D)$ on $(0, D)$. Trivial examples of slowly varying functions are given by positive constant functions. More general, a positive and continuous function φ satisfying $\lim_{x\to\infty} \varphi(x) = c$ with $0 < c < \infty$ is slowly varying. An important non-trivial example is given by the logarithm function $\varphi(x) = \log(x)$. Next, let us prove some elementary properties of slowly varying functions.

Lemma 2.2.1. *For slowly varying functions, the following results hold:*

(i) If φ varies slowly and $\alpha \in \mathbb{R}$, then $\big(\varphi(x)\big)^\alpha$ varies slowly.

(ii) If φ_1, φ_2 vary slowly, then $\varphi_1 + \varphi_2$ and $\varphi_1 \cdot \varphi_2$ vary slowly.

(iii) If $\varphi_1, \dots, \varphi_n$ vary slowly and $r(x_1, \dots, x_n)$ is a rational function with positive coefficients, then $r\big(\varphi_1(x), \dots, \varphi_n(x)\big)$ varies slowly.

Proof. The proof of (i) is obvious. To prove (ii), we just observe that

$$\frac{\varphi_1(\lambda x) + \varphi_2(\lambda x)}{\varphi_1(x) + \varphi_2(x)} = \frac{\varphi_1(\lambda x)}{\varphi_1(x)} + \left(\frac{\varphi_2(\lambda x)}{\varphi_2(x)} - \frac{\varphi_1(\lambda x)}{\varphi_1(x)}\right)\frac{\varphi_2(x)}{\varphi_1(x) + \varphi_2(x)}.$$

Finally, (iii) is a consequence of (i) and (ii). ■

Next, we recall Karamata's uniform convergence theorem for slowly varying functions (cf. [Ka30], [BGT87, Section 1.2]). This fundamental result states that we have uniform convergence in λ on every compact subset of $(0, \infty)$.

Theorem 2.2.2. [Ka30] *If the function φ is slowly varying and $0 < a < b < \infty$, then*

$$\lim_{x\to\infty} \frac{\varphi(\lambda x)}{\varphi(x)} = 1$$

uniformly for all $\lambda \in [a, b]$.

In Lemma 2.2.1 we have seen that slowly varying functions behave well under addition, multiplication and exponentiation. But what can be said about the composition of slowly varying functions? This is the subject of the next lemma.

Lemma 2.2.3. *If φ_1, φ_2 vary slowly and $\lim_{x\to\infty} \varphi_2(x) = \infty$, then $\varphi_1 \circ \varphi_2$ varies slowly.*

Proof. Let $\lambda > 0$ be arbitrary. We have to show that

$$\lim_{x\to\infty} \frac{\varphi_1\big(\varphi_2(\lambda x)\big)}{\varphi_1\big(\varphi_2(x)\big)} = 1.$$

To this end, choose an arbitrary $\varepsilon \in (0,1)$. Since φ_2 varies slowly, there exists a positive number x_0 such that for all $x \geq x_0$ the estimate

$$(1-\varepsilon)\,\varphi_2(x) \leq \varphi_2(\lambda x) \leq (1+\varepsilon)\,\varphi_2(x)$$

holds. Hence, for a fixed $x \geq x_0$, we have that

$$\varphi_2(\lambda x) = \mu\,\varphi_2(x) \quad \text{with } \mu \in [1-\varepsilon, 1+\varepsilon].$$

According to the uniform convergence theorem, we know that

$$\lim_{y\to\infty} \frac{\varphi_1(\mu y)}{\varphi_1(y)} = 1$$

uniformly for all $\mu \in [1-\varepsilon, 1+\varepsilon]$. Consequently, we can find a positive number y_0 such that for all $y \geq y_0$ and all $\mu \in [1-\varepsilon, 1+\varepsilon]$ we have that

$$(1-\varepsilon)\,\varphi_1(y) \leq \varphi_1(\mu y) \leq (1+\varepsilon)\,\varphi_1(y). \tag{2.2.1}$$

Since $\varphi_2(x) \to \infty$ for $x \to \infty$, we can find a number $x_1 \geq x_0$ such that $\varphi_2(x) \geq y_0$ for all $x \geq x_1$. Hence, applying (2.2.1) with $y = \varphi_2(x)$ leads to

$$(1-\varepsilon)\,\varphi_1\big(\varphi_2(x)\big) \leq \varphi_1\big(\varphi_2(\lambda x)\big) \leq (1+\varepsilon)\,\varphi_1\big(\varphi_2(x)\big)$$

for all $x \geq x_1$. This finishes the proof. ■

Using Lemma 2.2.1 and 2.2.3 we can give various examples of slowly varying functions. For instance, for every choice of $\alpha, \beta \in \mathbb{R}$, the function

$$\varphi(x) = (\log(x+1))^{\alpha}\,(\log\log(x+3))^{\beta}$$

varies slowly. However, slowly varying functions can behave quite surprisingly; there exist slowly varying functions satisfying both

$$\liminf_{x\to\infty} \varphi(x) = 0 \quad \text{and} \quad \limsup_{x\to\infty} \varphi(x) = \infty.$$

An example is given in [BGT87, p. 16].

The following lemma can be found in [BGT87, Prop. 1.3.6]. Roughly speaking, it states that a slowly varying function $\varphi(x)$ is dominated by x^{α}, $\alpha > 0$.

Lemma 2.2.4. *Let φ be a slowly varying function. Suppose that $\alpha > 0$ is arbitrary, then it holds that*

$$\lim_{x\to\infty} x^{-\alpha}\,\varphi(x) = 0 \quad \textit{and} \quad \lim_{x\to\infty} x^{\alpha}\,\varphi(x) = \infty.$$

In this context, we also recall the following fact (cf. [BGT87, Th. 1.5.3, 1.5.4]).

Lemma 2.2.5. *If φ is a slowly varying function, then for every $\alpha > 0$ there exists a non-decreasing function Φ and a non-increasing function Ψ with*

$$\lim_{x\to\infty} \frac{x^{\alpha}\varphi(x)}{\Phi(x)} = 1 \quad \textit{and} \quad \lim_{x\to\infty} \frac{x^{-\alpha}\varphi(x)}{\Psi(x)} = 1.$$

Finally, we deal with slowly varying functions and integration up to infinity. The following result, which can be found in [BGT87, Prop. 1.5.10], will be of some importance for our later work. Roughly speaking, it states that $\varphi(t)$ can be taken out of the integral as if it were $\varphi(x)$.

Lemma 2.2.6. *If φ is a slowly varying function and $\alpha < -1$ then*

$$\lim_{x\to\infty} \frac{x^{a+1}\varphi(x)}{\int_x^\infty t^{\alpha}\varphi(t)\,\mathrm{d}t} = -(\alpha+1).$$

Remark 1. For further information about slowly varying functions we refer to the monograph [BGT87].

2.3 Spaces of vectors and sequences

First, we want to recall the classical spaces of vectors and sequences. Let the symbol $\mathbb{K}$ stand for the field of real or complex numbers, then for $1 \le p < \infty$ the expression

$$\|x\|_p := \left(\sum_{k=1}^{n} |\xi_k|^p\right)^{1/p}, \quad x = (\xi_1, \dots, \xi_n) \in \mathbb{K}^n, \tag{2.3.1}$$

is a norm on the vector space $\mathbb{K}^n$. The normed space $[\mathbb{K}^n, \|\cdot\|_p]$ resulting from this will be denoted by $l_p^n(\mathbb{K})$ or simply l_p^n. The expression in (2.3.1) makes also sense for $0 < p < 1$; however, in this case it is a p-norm. Moreover, the symbol l_∞^n stands for $\mathbb{K}^n$ equipped with the norm

$$\|x\|_\infty := \max_{1\le k\le n} |\xi_k|, \quad x = (\xi_1, \dots, \xi_n) \in \mathbb{K}^n.$$

Note that for $0 < p \le q \le \infty$ and $x \in \mathbb{K}^n$ the inequalities $\|x\|_q \le \|x\|_p \le n^{1/p-1/q}\,\|x\|_q$ hold true. For the unit balls of these spaces this means that

$$B_{l_p^n} \subset B_{l_q^n} \subset n^{1/p-1/q} B_{l_p^n} \quad \text{for} \quad 0 < p \le q \le \infty.$$

Furthermore, it turns out that

$$\|x\|_\infty = \lim_{p\to\infty} \|x\|_p \quad \text{for} \quad x \in \mathbb{K}^n.$$

Next we have a look at sequence spaces. For $0 < p < \infty$ we define the vector space l_p of p-summable sequences by

$$l_p = l_p(\mathbb{K}) := \left\{ (\xi_k)_k : \xi_k \in \mathbb{K}, \sum_{k=1}^{\infty} |\xi_k|^p < \infty \right\},$$

where addition and scalar multiplication are given coordinate-wise. For $1 \leq p < \infty$, the expression

$$\|x\|_p := \left(\sum_{k=1}^{\infty} |\xi_k|^p \right)^{1/p}, \quad x = (\xi_k)_k,$$

is a norm on l_p and it turns out that l_p is complete with respect to this norm. For $0 < p < 1$, $[l_p, \|\cdot\|_p]$ is a p-Banach space. Moreover, the Banach space of all bounded scalar sequences equipped with the supremum norm

$$\|x\|_\infty := \sup_{k \in \mathbb{N}} |\xi_k|, \quad x = (\xi_k)_k,$$

is denoted by l_∞.

In the following we recall the classical *Lorentz sequence spaces*, which can be seen as a generalization of the above mentioned l_p spaces. To this end, we will need the non-increasing rearrangement of a bounded sequence. The idea is to rearrange the sequence of absolute values such that this rearrangement is monotonically decreasing.

Definition. For a bounded sequence $x = (\xi_k)_k \in l_\infty$ of real or complex numbers we define the so-called *non-increasing rearrangement* $s(x) = \big(s_n(x)\big)_n$ of x by

$$s_n(x) := \inf_{I \subset \mathbb{N}, |I| < n} \sup_{k \in \mathbb{N} \setminus I} |\xi_k|.$$

If $x = (\xi_k)_k$ is a sequence such that $|\xi_1| \geq |\xi_2| \geq \ldots \geq 0$, then $s_n(x) = |\xi_n|$ for all natural numbers n. By the very definition, the non-increasing rearrangement of a bounded sequence is monotonically decreasing, but we point out that, in general, it is no rearrangement of the sequence of absolute values. For example, consider the sequence x given by $\xi_k = 1 - 1/k$, then $s_n(x) = 1$ for all natural numbers n. However, if x is a null sequence, then $s(x)$ is a rearrangement of $\big(|\xi_n|\big)_n$. In addition, we also have an additivity and multiplicativity in the following sense: Let $n, m \in \mathbb{N}$ be natural numbers and $x, y \in l_\infty$ be bounded sequences, then

$$s_{n+m-1}(x+y) \leq s_n(x) + s_m(y) \quad \text{and} \quad s_{n+m-1}(xy) \leq s_n(x)\, s_m(y).$$

Here, $xy = (\xi_k \eta_k)_k$ for $x = (\xi_k)_k$, $y = (\eta_k)_k$.

Now, for $0 < p < \infty$ and $0 < q \leq \infty$, the *classical Lorentz sequence space* $l_{p,q}$ consists of all bounded sequences $x \in l_\infty$ which satisfy $\big(n^{1/p-1/q}\, s_n(x)\big)_n \in l_q$. Clearly, we have that $l_{p,p} = l_p$ for all $0 < p < \infty$. A detailed explanation of Lorentz sequence spaces can be found, for instance, in Section 2.1 of Pietsch's monograph [P87].

However, for our purposes, classical Lorentz sequence spaces are not sufficient. As pointed out in the introduction, given a precompact set A, we want to study the entropy behavior of the absolutely convex hull $\operatorname{aco}(A)$ with respect to the entropy behavior of A. In this context, it is obvious to ask for the entropy behavior of $\operatorname{aco}(A)$ under the assumption that the dyadic entropy numbers of A belong to a certain l_p space. To give an example, if A is a precompact subset of a Hilbert space with $\big(e_n(A)\big)_n \in l_1$, then Carl et al. proved that (cf. [CHP11, Th. 1.1])

$$\sum_{n=1}^{\infty} (\log(n+1))^{-1/2}\, n^{-1/2}\, e_n(\operatorname{aco}(A)) < \infty.$$

As we can see, it may happen that the sequence $\big(e_n(\operatorname{aco}(A))\big)_n$ is beyond the scope of classical Lorentz sequence spaces. Therefore, our next step is to generalize the concept of classical Lorentz sequence spaces.

Definition. Let $0 < p < \infty$, $0 < q \leq \infty$ and let φ be a slowly varying function. Then the bounded sequence $x \in l_\infty$ belongs to the *generalized Lorentz sequence space* $l_{p,q,\varphi}$ if and only if $\big(\varphi(n)\, n^{1/p-1/q}\, s_n(x)\big)_n \in l_q$, i.e.

$$x \in l_{p,q,\varphi} \quad \text{if and only if} \quad \begin{cases} \sum\limits_{n=1}^{\infty} (\varphi(n))^q\, n^{q/p-1}\, (s_n(x))^q < \infty, & 0 < q < \infty, \\ \sup\limits_{n\in\mathbb{N}} \varphi(n)\, n^{1/p}\, s_n(x) < \infty, & q = \infty. \end{cases}$$

Note that we get the classical Lorentz sequence spaces for $\varphi \equiv 1$. Following Pietsch's monograph [P87], we now take a closer look on generalized Lorentz sequence spaces. It turns out that well-known results for classical Lorentz sequence spaces can be carried over to generalized Lorentz sequence spaces in the above-mentioned sense. Hence, slowly varying functions seem to be a good choice to generalize the classical $l_{p,q}$–spaces. Unless otherwise stated, we assume in the following that $0 < p < \infty$, $0 < q \leq \infty$ and φ is a slowly varying function. We start to show that $l_{p,q,\varphi}$ has a vector space structure.

Proposition 2.3.1. *The generalized Lorentz sequence space $l_{p,q,\varphi}$ becomes a vector space under coordinate-wise addition and scalar multiplication.*

Proof. Let $x, y \in l_{p,q,\varphi}$ and $\alpha \in \mathbb{K}$. Since for all natural numbers $n \in \mathbb{N}$ we have that

$$s_n(\alpha x) = |\alpha|\, s_n(x),$$

we see that $\alpha x \in l_{p,q,\varphi}$. Now let n be an arbitrary natural number. To show that $x+y$ also belongs to $l_{p,q,\varphi}$, we first observe that

$$s_{2n}(x+y) \le s_{2n-1}(x+y) \le s_n(x) + s_n(y).$$

Furthermore, if $p \ge q$ then

$$(2n)^{1/p-1/q} \le (2n-1)^{1/p-1/q} \le n^{1/p-1/q}$$

and in the case $p \le q$ we obtain

$$(2n-1)^{1/p-1/q} \le (2n)^{1/p-1/q} = 2^{1/p-1/q} n^{1/p-1/q}.$$

In addition, due to Lemma 3.0.5 from Chapter 3, we can find a constant $c(\varphi) > 0$ such that $\varphi(2n-1) \le c(\varphi)\,\varphi(n)$ and $\varphi(2n) \le c(\varphi)\,\varphi(n)$ for all $n \in \mathbb{N}$. Hence, for $0 < q < \infty$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \Big[\varphi(n)\, n^{1/p-1/q}\, s_n(x+y)\Big]^q \\
&= \sum_{n=1}^{\infty} \Big[\varphi(2n-1)\,(2n-1)^{1/p-1/q}\, s_{2n-1}(x+y)\Big]^q + \sum_{n=1}^{\infty} \Big[\varphi(2n)\,(2n)^{1/p-1/q}\, s_{2n}(x+y)\Big]^q \\
&\le 2\,(c(\varphi))^q \Big(\max\Big\{2^{1/p-1/q}, 1\Big\}\Big)^q \sum_{n=1}^{\infty} \Big[\varphi(n)\, n^{1/p-1/q} \Big(s_n(x)+s_n(y)\Big)\Big]^q < \infty.
\end{aligned}$$

The case $q = \infty$ can be proved analogously. ■

In order to decide whether a given sequence belongs to some generalized Lorentz sequence space it is enough to consider the odd indices. This is the subject of the next lemma.

Lemma 2.3.2. *For generalized Lorentz sequence spaces the following equivalence holds:*

$$x \in l_{p,q,\varphi} \quad \text{if and only if} \quad \Big(s_{2n-1}(x)\Big)_n \in l_{p,q,\varphi}.$$

Proof. First, observe that

$$s_n\Big((s_{2k-1}(x))_k\Big) = s_{2n-1}(x).$$

Hence, the only-if-part is a direct consequence of $s_{2n-1}(x) \le s_n(x)$. The proof of the if-part is analogous to that of Proposition 2.3.1. For $0 < q < \infty$ we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \Big[\varphi(n)\, n^{1/p-1/q}\, s_n(x)\Big]^q \\
&= \sum_{n=1}^{\infty} \Big[\varphi(2n-1)\,(2n-1)^{1/p-1/q}\, s_{2n-1}(x)\Big]^q + \sum_{n=1}^{\infty} \Big[\varphi(2n)\,(2n)^{1/p-1/q}\, s_{2n}(x)\Big]^q \\
&\le 2\,(c(\varphi))^q \Big(\max\Big\{2^{1/p-1/q}, 1\Big\}\Big)^q \sum_{n=1}^{\infty} \Big[\varphi(n)\, n^{1/p-1/q}\, s_{2n-1}(x)\Big]^q,
\end{aligned}$$

which yields the assertion. The case $q = \infty$ can be proved analogously. ■

Next we give a simple but important dyadic characterization.

Lemma 2.3.3. *For generalized Lorentz sequence spaces the following equivalence holds:*

$$x \in l_{p,q,\varphi} \quad \textit{if and only if} \quad \Big(\varphi(2^n)\, 2^{n/p}\, s_{2^n}(x)\Big)_n \in l_q.$$

Proof. Let $0 < q < \infty$. For $m = 0, 1, 2, \dots$ define

$$U_m := \{n \in \mathbb{N} : 2^m \le n < 2^{m+1}\}.$$

Then, $|U_m| = 2^m$ and for all natural numbers $n \in U_m$ we have that $s_n(x) \le s_{2^m}(x)$ and $n^{q/p-1} \le c_1(p,q)\, 2^{m(q/p-1)}$. Furthermore, due to Lemma 3.0.5 from Chapter 3, we can find a constant $c_2(\varphi) > 0$ such that $\varphi(n) \le c_2(\varphi)\, \varphi(2^m)$ for all $m = 0, 1, 2, \dots$ and all $n \in U_m$. Hence, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} (\varphi(n))^q\, n^{q/p-1} \Big(s_n(x)\Big)^q &= \sum_{m=0}^{\infty} \sum_{n \in U_m} (\varphi(n))^q\, n^{q/p-1} \Big(s_n(x)\Big)^q \\ &\le c_3(p,q,\varphi) \sum_{m=0}^{\infty} (\varphi(2^m))^q\, 2^{mq/p} \Big(s_{2^m}(x)\Big)^q. \qquad (2.3.2)\end{aligned}$$

Consequently,

$$\Big(\varphi(2^n)\, 2^{n/p}\, s_{2^n}(x)\Big)_n \in l_q \quad \text{implies} \quad x \in l_{p,q,\varphi}.$$

Conversely, for $m = 0, 1, 2, \dots$ we consider $V_m := \{n \in \mathbb{N} : 2^m < n \le 2^{m+1}\}$. Analogously to above we find that

$$\begin{aligned}\sum_{n=2}^{\infty} (\varphi(n))^q\, n^{q/p-1} \Big(s_n(x)\Big)^q &= \sum_{m=0}^{\infty} \sum_{n \in V_m} (\varphi(n))^q\, n^{q/p-1} \Big(s_n(x)\Big)^q \\ &\ge c_4(p,q,\varphi) \sum_{m=0}^{\infty} (\varphi(2^{m+1}))^q\, 2^{(m+1)q/p} \Big(s_{2^{m+1}}(x)\Big)^q \\ &= c_4(p,q,\varphi) \sum_{m=1}^{\infty} (\varphi(2^m))^q\, 2^{mq/p} \Big(s_{2^m}(x)\Big)^q. \qquad (2.3.3)\end{aligned}$$

Hence,

$$x \in l_{p,q,\varphi} \quad \text{implies} \quad \Big(\varphi(2^n)\, 2^{n/p}\, s_{2^n}(x)\Big)_n \in l_q.$$

Now let us deal with the case $q = \infty$. On the one hand, we trivially have

$$\sup_{n \in \mathbb{N}} \varphi(n)\, n^{1/p}\, s_n(x) \ge \sup_{m \in \mathbb{N}_0} \varphi(2^m)\, 2^{m/p}\, s_{2^m}(x).$$

On the other hand, it holds that

$$\sup_{n\in\mathbb{N}} \varphi(n)\, n^{1/p}\, s_n(x) = \sup_{m\in\mathbb{N}_0} \max_{2^m\le n<2^{m+1}} \varphi(n)\, n^{1/p}\, s_n(x) \\ \le c_5(p,\varphi) \sup_{m\in\mathbb{N}_0} \varphi(2^m)\, 2^{m/p}\, s_{2^m}(x).$$

This finishes the proof. ∎

The next result is a consequence of the Hardy-type inequalities given in Chapter 3 (cf. Lemma 3.0.8, 3.0.9).

Lemma 2.3.4. *Let $0 < t < p < \infty$, $0 < q \le \infty$ and let φ be a slowly varying function. Then*

$$x \in l_{p,q,\varphi} \quad \textit{if and only if} \quad \left(\left(\frac{1}{n} \sum_{k=1}^{n} (s_k(x))^t \right)^{1/t} \right)_n \in l_{p,q,\varphi}.$$

Proof. First, observe that the sequence $\left(\left(\frac{1}{n} \sum_{k=1}^{n} (s_k(x))^t \right)^{1/t} \right)_n$ is non-increasing and non-negative. The if-part is a direct consequence of the monotonicity of $\left(s_n(x)\right)_n$: For all natural numbers $n \in \mathbb{N}$ we have that

$$s_n(x) \le \left(\frac{1}{n} \sum_{k=1}^{n} (s_k(x))^t \right)^{1/t}.$$

To prove the only-if-part we apply Lemma 3.0.8 and 3.0.9 with $\sigma_n = s_n(x)$. ∎

As a simple conclusion we get the following result.

Lemma 2.3.5. *For generalized Lorentz sequence spaces the following equivalence holds:*

$$x \in l_{p,q,\varphi} \quad \textit{if and only if} \quad \left(\left(\prod_{k=1}^{n} s_k(x) \right)^{1/n} \right)_n \in l_{p,q,\varphi}.$$

Proof. First, observe that $\left((\prod_{k=1}^{n} s_k(x))^{1/n} \right)_n$ is a non-increasing and non-negative sequence. Again, the if-part is a consequence of the monotonicity: We have that

$$s_n(x) \le \left(\prod_{k=1}^{n} s_k(x) \right)^{1/n}$$

for all natural numbers $n \in \mathbb{N}$. The only-if-part follows from the inequality of means. Indeed, fix $t > 0$ with $t < p$ to obtain

$$\left(\prod_{k=1}^{n} s_k(x) \right)^{1/n} \le \left(\frac{1}{n} \sum_{k=1}^{n} (s_k(x))^t \right)^{1/t},$$

which yields the assertion due to Lemma 2.3.4. ■

As a consequence of Lemma 3.0.10 and 3.0.11 from Chapter 3 we get further insights into generalized Lorentz sequence spaces.

Lemma 2.3.6. *Let $0 < p < 1$, $0 < q \leq \infty$ and let φ be a slowly varying function. Then*

$$x \in l_{p,q,\varphi} \quad \textit{implies} \quad \left(\sum_{k=n}^{\infty} s_k(x)\right)_n \in l_{t,q,\varphi}$$

with $1/t = 1/p - 1$.

Now, we ask what can be said about the product of two sequences belonging to a generalized Lorentz sequence space. In this context, the above-mentioned dyadic characterization will turn out to be helpful.

Lemma 2.3.7. *Let $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$ and let φ_1, φ_2 be slowly varying functions. If $x \in l_{p_1,q_1,\varphi_1}$ and $y \in l_{p_2,q_2,\varphi_2}$, then $xy \in l_{p,q,\varphi}$ where*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \quad \textit{and} \quad \varphi = \varphi_1 \cdot \varphi_2.$$

Proof. According to Lemma 2.3.3 it holds that

$$\left(\varphi_1(2^n)\, 2^{n/p_1}\, s_{2^n}(x)\right)_n \in l_{q_1} \quad \text{and} \quad \left(\varphi_2(2^n)\, 2^{n/p_2}\, s_{2^n}(y)\right)_n \in l_{q_2}.$$

Hence, from Hölder's inequality we conclude that

$$\left(\varphi_1(2^n)\, 2^{n/p_1}\, s_{2^n}(x)\, \varphi_2(2^n)\, 2^{n/p_2}\, s_{2^n}(y)\right)_n \in l_q.$$

Moreover, from the multiplicativity and monotonicity of the non-increasing rearrangement we obtain the estimate

$$s_{2^n}(xy) = s_{2^{n-1}+2^{n-1}}(xy) \leq s_{2^{n-1}}(x)\, s_{2^{n-1}}(y)$$

for all natural numbers $n \in \mathbb{N}$. Since $\varphi = \varphi_1 \cdot \varphi_2$ varies slowly, we can find a constant $c_1(\varphi) > 0$ such that for all $n \in \mathbb{N}$ we have that

$$\varphi(2^n) \leq c_1(\varphi)\, \varphi(2^{n-1}).$$

Consequently, we have that

$$\begin{aligned}
\varphi(2^n)\, 2^{n/p}\, s_{2^n}(xy) &\leq \varphi(2^n)\, 2^{n/p}\, s_{2^{n-1}}(x)\, s_{2^{n-1}}(y) \\
&\leq c_1(\varphi)\, 2^{1/p}\, \varphi(2^{n-1})\, 2^{(n-1)/p}\, s_{2^{n-1}}(x)\, s_{2^{n-1}}(y) \\
&= c_2(p,\varphi)\, \varphi_1(2^{n-1})\, 2^{(n-1)/p_1}\, s_{2^{n-1}}(x)\, \varphi_2(2^{n-1})\, 2^{(n-1)/p_2}\, s_{2^{n-1}}(y)
\end{aligned}$$

for all $n \in \mathbb{N}$. This yields

$$\Big(\varphi(2^n)\, 2^{n/p}\, s_{2^n}(xy)\Big)_n \in l_q$$

and finishes the proof due to Lemma 2.3.3. ■

Finally, we study inclusions within the 3–parameter scale of generalized Lorentz sequence spaces.

Lemma 2.3.8. *For generalized Lorentz sequence spaces the following inclusions hold:*

(i) If $0 < p_1 < p_2 < \infty$, then for any $0 < q_1, q_2 \leq \infty$ and any slowly varying functions φ_1, φ_2 we have that

$$l_{p_1,q_1,\varphi_1} \subset l_{p_2,q_2,\varphi_2}.$$

(ii) If $0 < q_1 < q_2 \leq \infty$, then for any $0 < p < \infty$ and any slowly varying function φ we have that

$$l_{p,q_1,\varphi} \subset l_{p,q_2,\varphi}.$$

(iii) If φ_1, φ_2 are slowly varying functions satisfying $\varphi_2(2^n) \preccurlyeq \varphi_1(2^n)$, then for any $0 < p < \infty$ and any $0 < q \leq \infty$ we have that

$$l_{p,q,\varphi_1} \subset l_{p,q,\varphi_2}.$$

Proof. Again, we use the dyadic characterization given in Lemma 2.3.3 for the proof of this result. To prove (i), let $x \in l_{p_1,q_1,\varphi_1}$. Then

$$\Big(\varphi_1(2^n)\, 2^{n/p_1}\, s_{2^n}(x)\Big)_n \in l_{q_1} \subset l_\infty.$$

Hence, we can find a constant $c > 0$ such that for all natural numbers $n \in \mathbb{N}$ the estimate

$$s_{2^n}(x) \leq c\, (\varphi_1(2^n))^{-1}\, 2^{-n/p_1}$$

holds true. Consequently, we obtain

$$\varphi_2(2^n)\, 2^{n/p_2}\, s_{2^n}(x) \leq c\, \frac{\varphi_2(2^n)}{\varphi_1(2^n)}\, 2^{n/p_2 - n/p_1}$$

for all $n \in \mathbb{N}$. Define $\alpha := (1/p_1 - 1/p_2)/3 > 0$. Due to Lemma 2.3.3 it is enough to show that

$$\left(\frac{\varphi_2(2^n)}{\varphi_1(2^n)}\, 2^{-3\alpha n}\right)_n \in l_{q_2}.$$

According to Lemma 2.2.4, we have

$$\lim_{x\to\infty} x^{\alpha}\,\varphi_1(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} x^{-\alpha}\,\varphi_2(x) = 0.$$

Consequently, there exists a natural number n_0 such that

$$2^{\alpha n}\,\varphi_1(2^n) \geq 1 \quad \text{and} \quad 2^{-\alpha n}\,\varphi_2(2^n) \leq 1$$

for all $n \geq n_0$. This leads to

$$\frac{\varphi_2(2^n)}{\varphi_1(2^n)}\,2^{-3\alpha n} \leq 2^{2\alpha n}\,2^{-3\alpha n} = 2^{-\alpha n}$$

for all $n \geq n_0$. Since $\alpha > 0$, we see that $(2^{-\alpha n})_n \in l_{q_2}$ and this shows $x \in l_{p_2,q_2,\varphi_2}$. The second assertion is a consequence of the fact that $l_{q_1} \subset l_{q_2}$ for $0 < q_1 < q_2 \leq \infty$. Finally, the proof of (iii) is obvious. ■

Next, we want to give some examples. To this end, we prove the following useful lemma.

Lemma 2.3.9. *Let $\alpha > 0$ and let ψ be a slowly varying function. Suppose that the sequence x is given by $x := \big(n^{-\alpha}\,\psi(n)\big)_n$. Then it holds that*

$$s_k(x) \sim k^{-\alpha}\,\psi(k).$$

Proof. According to Lemma 2.2.5, there exists a non-increasing function Ψ with

$$\lim_{n\to\infty} \frac{n^{-\alpha}\,\psi(n)}{\Psi(n)} = 1.$$

This yields

$$n^{-\alpha}\,\psi(n) \sim \Psi(n)$$

and, therefore,

$$s_k(x) = s_k\Big((n^{-\alpha}\,\psi(n))_n\Big) \sim s_k\Big((\Psi(n))_n\Big) = \Psi(k).$$

This finishes the proof. ■

Example 1. Let $0 < p < \infty$ and let ψ be a slowly varying function. Consider the sequence $x = (\xi_n)_n$ given by

$$\xi_n = n^{-1/p}\,\psi(n).$$

Then, for any $0 < q \leq \infty$ and any slowly varying function φ, we have that $x \notin l_{p_1,q,\varphi}$ for $0 < p_1 < p$ and $x \in l_{p_2,q,\varphi}$ for $p < p_2 < \infty$. In the critical case, we can say that if $\big(\varphi(2^n)\,\psi(2^n)\big)_n \in l_q$ then $x \in l_{p,q,\varphi}$.

This example shows that the inclusion in Lemma 2.3.8 (i) is strict.

Proof. Let $0 < p_1 < p$. From Lemma 2.3.9 we obtain that

$$\varphi(2^n)\, 2^{n/p_1}\, s_{2^n}(x) \sim \varphi(2^n)\, \psi(2^n)\, 2^{n/p_1 - n/p}.$$

Define $\alpha := (1/p_1 - 1/p)/2 > 0$. Since $\varphi \cdot \psi$ is a slowly varying function, Lemma 2.2.4 tells us that

$$\lim_{x\to\infty} x^\alpha\, \varphi(x)\, \psi(x) = \infty.$$

Hence, there exists a natural number n_0 such that

$$2^{\alpha n}\, \varphi(2^n)\, \psi(2^n) \geq 1$$

for all $n \geq n_0$, which gives

$$\varphi(2^n)\, \psi(2^n)\, 2^{n/p_1 - n/p} = \varphi(2^n)\, \psi(2^n)\, 2^{2\alpha n} \geq 2^{\alpha n}$$

for all $n \geq n_0$. Since $(2^{\alpha n})_n \notin l_\infty$, we conclude that $\Big(\varphi(2^n)\, 2^{n/p_1}\, s_{2^n}(x)\Big)_n \notin l_\infty$. Hence, $x \notin l_{p_1,q,\varphi}$. Now let $p < p_2 < \infty$. This time, we define $\alpha := (1/p - 1/p_2)/2 > 0$. Then

$$\lim_{x\to\infty} x^{-\alpha}\, \varphi(x)\, \psi(x) = 0.$$

Consequently, there exists a natural number n_0 such that

$$2^{-\alpha n}\, \varphi(2^n)\, \psi(2^n) \leq 1$$

for all $n \geq n_0$. This yields

$$\varphi(2^n)\, \psi(2^n)\, 2^{n/p_2 - n/p} = \varphi(2^n)\, \psi(2^n)\, 2^{-2\alpha n} \leq 2^{-\alpha n}$$

for all $n \geq n_0$. Since $a > 0$, we have that $(2^{-\alpha n})_n \in l_q$. From

$$\varphi(2^n)\, 2^{n/p_2}\, s_{2^n}(x) \sim \varphi(2^n)\, \psi(2^n)\, 2^{n/p_2 - n/p}$$

we conclude that also $x \in l_{p_2,q,\varphi}$. Finally, from

$$\varphi(2^n)\, 2^{n/p}\, s_{2^n}(x) \sim \varphi(2^n)\, \psi(2^n)$$

we see that $(\varphi(2^n)\, \psi(2^n))_n \in l_q$ implies $x \in l_{p,q,\varphi}$. ■

2.4 Entropy numbers

Let us start with recalling a definition: A subset $A \subset X$ of a metric space (X,d) is called *precompact* if and only if for every $\varepsilon > 0$ there exist $x_1, \dots, x_n \in X$ such that

$$A \subset \bigcup_{i=1}^{n} B_X(x_i, \varepsilon).$$

The set $\{x_1, x_2, \dots, x_n\} \subset X$ is called *ε-net for A*. If the underlying metric space X is complete, then A is precompact if and only if $\mathrm{cl}(A)$ is compact. When intending to quantify precompactness the following two questions naturally arise:

(i) Given a fixed radius $\varepsilon > 0$, what is the least natural number n such that n balls in X with radius ε cover the set A?

or, vice versa,

(ii) Given a fixed natural number n, what is the least radius ε such that n balls in X with radius ε cover the set A?

These questions lead to the following definitions.

Definition. Let (X,d) be a metric space and $A \subset X$ a precompact set. For $\varepsilon > 0$, the *ε-covering number of A* is defined by

$$N(A,\varepsilon) := \min\left\{ n \in \mathbb{N} \,\middle|\, \exists\, x_1, \dots, x_n \in X : A \subset \bigcup_{i=1}^{n} B_X(x_i, \varepsilon) \right\}.$$

The concept of covering numbers goes back until 1932, see Pontrjagin and Schnirelman [PS32, p. 156]. Given a precompact set, we are interested in the asymptotic behavior of $N(A,\varepsilon)$ as $\varepsilon \to 0+$. To handle the case when covering numbers increases exponentially as ε tends to zero, Kolmogorov suggested (cf. [K56], [KTi61]) to consider the logarithm of the covering number with respect to the base 2, called the *metric entropy of A*:

$$H(A,\varepsilon) := \log N(A,\varepsilon), \quad \varepsilon > 0.$$

Definition. Let (X,d) be a metric space and $A \subset X$ a bounded set. For a natural number n, the *n-th entropy number of A* is defined by

$$\varepsilon_n(A;X) := \inf\left\{ \varepsilon \geq 0 \,\middle|\, \exists\, x_1, \dots, x_n \in X : A \subset \bigcup_{i=1}^{n} B_X(x_i, \varepsilon) \right\}.$$

Moreover, the *n-th dyadic entropy number* of A is given by

$$e_n(A;X) := \varepsilon_{2^{n-1}}(A;X), \quad n = 1, 2, 3, \dots.$$

If there is no risk of confusion, we write $\varepsilon_n(A)$ instead of $\varepsilon_n(A;X)$.

Remark 2. Given a precompact subset A of a metric space (X,d) it is common knowledge that there is a finite ε-net for A consisting of elements of X if and only if there is a finite ε-net for A consisting of elements of A. In general however, the cardinality of the ε-net depends on whether the elements of the net are taken from X or from A. As a consequence, the entropy numbers $\varepsilon_n(A;X)$ and $\varepsilon_n(A;A)$ may differ. However, we remark that if A is a precompact subset of a metric space (X,d) then

$$\varepsilon_n(A;X) \leq \varepsilon_n(A;A) \leq 2\,\varepsilon_n(A;X), \quad n = 1,2,\dots.$$

Covering numbers and entropy numbers are, in a sense, inverse functions. It follows directly from the definitions that

(i) $\varepsilon_n(A) < \varepsilon \quad \text{implies} \quad N(A,\varepsilon) \leq n$,

(ii) $N(A,\varepsilon) \leq n \quad \text{implies} \quad \varepsilon_n(A) \leq \varepsilon$,

(iii) $e_{n+1}(A) < \varepsilon \quad \text{implies} \quad \log N(A,\varepsilon) \leq n$,

(iv) $\log N(A,\varepsilon) \leq n \quad \text{implies} \quad e_{n+1}(A) \leq \varepsilon$.

Hence, it is possible to switch between both quantities, for example it holds that

$$N(A,\varepsilon) \preccurlyeq \varepsilon^{-\alpha} \quad \text{if and only if} \quad \varepsilon_n(A) \preccurlyeq n^{-1/\alpha}.$$

The following lemma deals with this topic in more detail. The proof is inspired by the proof of Proposition 5 in [Kl12a].

Lemma 2.4.1. *Let (A,d) be a precompact metric space. Let $F : (0,c_F] \to (0,\infty)$ be a decreasing and continuous function with $\lim_{\varepsilon\to 0+} F(\varepsilon) = \infty$. Furthermore, let $G : [c_G,\infty) \to (0,\infty)$ be a decreasing and continuous function with $\lim_{n\to\infty} G(n) = 0$. Then the following statements hold:*

(i) If for every natural number k there exists a constant $C_k = C(k,G) > 0$ such that

$$G(n) \leq C_k\, G(kn) \quad \textit{for all large } n \in \mathbb{N}$$

and if

$$\limsup_{\varepsilon\to 0+} \frac{\varepsilon}{G\big(F(\varepsilon)\big)} < \infty$$

then

$$\log N(A,\varepsilon) \preccurlyeq F(\varepsilon) \quad \textit{implies} \quad e_n(A) \preccurlyeq G(n).$$

(ii) If for every $\gamma > 0$ there exists a constant $C_\gamma = C(\gamma, F) > 0$ such that

$$F(\gamma\delta) \leq C_\gamma \, F(\delta) \quad \textit{for all small } \delta > 0$$

and if

$$\limsup_{n\to\infty} \frac{n}{F\big(G(n)\big)} < \infty$$

then

$$e_n(A) \preccurlyeq G(n) \quad \textit{implies} \quad \log N(A,\varepsilon) \preccurlyeq F(\varepsilon).$$

Both statements remain true if $e_n(A)$ and $\log N(A,\varepsilon)$ is replaced by $\varepsilon_n(A)$ and $N(A,\varepsilon)$, respectively.

Proof. Let us start with the proof of (i). According to the assumption there exists a constant $c_1 > 0$ such that

$$\log N(A,\varepsilon) \leq c_1 \, F(\varepsilon) \quad \text{for all } 0 < \varepsilon \leq c_F.$$

Let $n \geq \max\{c_G; F(c_F)\}$ be a natural number. Then there is an $\varepsilon(n) > 0$ satisfying

$$n \leq F(\varepsilon(n)) < n+1.$$

From the monotonicity of G we conclude that

$$G\Big(F(\varepsilon(n))\Big) \leq G(n).$$

Furthermore, it holds that

$$\log N(A,\varepsilon(n)) \leq c_1 \, F(\varepsilon(n)) \leq c_1\,(n+1) \leq mn,$$

where $m = m(c_1)$ is a suitable natural number. Hence, we have that

$$e_{mn+1}(A) \leq \varepsilon(n) = \frac{\varepsilon(n)}{G(n)}\, G(n) \leq \frac{\varepsilon(n)}{G\Big(F(\varepsilon(n))\Big)}\, G(n).$$

Observe that $\varepsilon(n) \to 0+$ as $n \to \infty$. Taking into account that

$$\limsup_{\varepsilon\to 0+} \frac{\varepsilon}{G\big(F(\varepsilon)\big)} < \infty$$

we finally get

$$e_{mn+1}(A) \leq c_2 \, G(n)$$

for all $n \geq \max\{c_G; F(c_F)\}$, where $m = m(c_1)$ is a natural number and $c_2 = c_2(F, G) > 0$ is a constant. Now let j be a (sufficiently large) natural number. Then there is an $n \geq \max\{c_G; F(c_F)\}$ with

$$mn + 1 \leq j \leq m(n+1) + 1 \leq 2mn.$$

We conclude that

$$e_j(A) \leq e_{mn+1}(A) \leq c_2\, G(n) \leq c_2\, C_{2m}\, G(2mn) \leq c_2\, C_{2m}\, G(j).$$

This shows that

$$e_n(A) \preccurlyeq G(n).$$

Now let us deal with (ii). Since $\lim_{n\to\infty} G(n) = 0$ there exists a natural number $n_G \geq c_G$ such that $G(n) \leq c_F$ for all $n \geq n_G$. Furthermore, according to the assumption there is a constant $c_3 > 0$ with

$$e_n(A) \leq c_3\, G(n) \quad \text{for all } n \geq c_G.$$

Now let $\varepsilon > 0$ with $\varepsilon \leq G(n_G)$. Then we can find a natural number $n(\varepsilon) \geq n_G$ satisfying

$$G(n(\varepsilon) + 1) < \varepsilon \leq G(n(\varepsilon)).$$

Since $G(n(\varepsilon)) \leq G(n_G) \leq c_F$, we can conclude that

$$F\Big(G(n(\varepsilon))\Big) \leq F(\varepsilon).$$

Furthermore, it holds that

$$e_{n(\varepsilon)+1}(A) \leq c_3\, G(n(\varepsilon) + 1) < c_3\, \varepsilon$$

and consequently we obtain

$$\log N(A, c_3\, \varepsilon) \leq n(\varepsilon) = \frac{n(\varepsilon)}{F(\varepsilon)}\, F(\varepsilon) \leq \frac{n(\varepsilon)}{F\Big(G(n(\varepsilon))\Big)}\, F(\varepsilon).$$

But $\varepsilon \to 0+$ implies $n(\varepsilon) \to \infty$ and therefore it holds that

$$\log N(A, c_3\, \varepsilon) \leq c_4\, F(\varepsilon)$$

for all $0 < \varepsilon \leq G(n_G)$, where c_3 and $c_4 = c_4(F, G)$ are positive constants. Now let $\delta > 0$ be sufficiently small. Then, with $\varepsilon = \delta / c_3$, we have that

$$\log N(A, \delta) = \log(A, c_3\, \varepsilon) \leq c_4\, F(\delta / c_3) \leq c_4\, C_{c_3^{-1}}\, F(\delta).$$

This shows that

$$\log N(A,\varepsilon) \preccurlyeq F(\varepsilon)$$

and finishes the proof. The case of $\varepsilon_n(A)$ and $N(A,\varepsilon)$ can be proved analogously. ■

Usually, the functions F and G mentioned above are given by

$$F(\varepsilon) = \varepsilon^{-\alpha} f(\varepsilon^{-1}) \quad \text{and} \quad G(n) = n^{-\beta} g(n),$$

where $\alpha, \beta > 0$ and f, g are powers of the logarithm function or other slowly varying functions. In the following we give connections between covering numbers and entropy numbers in the most common cases (cf. [Kl12a, Prop. 5]).

Corollary 2.4.2. *Let (A,d) be a precompact metric space and let $\alpha > 0$. Furthermore, suppose that $\beta, \gamma \in \mathbb{R}$. Then the following statements hold true:*

(i) $N(A,\varepsilon) \preccurlyeq \varepsilon^{-\alpha} (\log \frac{1}{\varepsilon})^{\alpha\beta} (\log\log \frac{1}{\varepsilon})^{\alpha\gamma}$ *if and only if* $\varepsilon_n(A) \preccurlyeq n^{-1/\alpha} (\log(n+1))^{\beta} (\log\log(n+3))^{\gamma}$.

(ii) $\log N(A,\varepsilon) \preccurlyeq \varepsilon^{-\alpha} (\log \frac{1}{\varepsilon})^{\alpha\beta} (\log\log \frac{1}{\varepsilon})^{\alpha\gamma}$ *if and only if* $e_n(A) \preccurlyeq n^{-1/\alpha} (\log(n+1))^{\beta} (\log\log(n+3))^{\gamma}$.

Now let us take a closer look on entropy numbers. The following properties concerning the entropy numbers of a bounded subset A of a metric space are easy to verify:

(i) Monotonicity: $\varepsilon_1(A) \geq \varepsilon_2(A) \geq \ldots \geq 0$ and $\varepsilon_1(A) \geq \frac{1}{2}\operatorname{diam}(A)$.

(ii) Characterization of precompactness: A is precompact if and only if $\varepsilon_n(A) \to 0$ for $n \to \infty$.

(iii) If $B \subset A$ then $\varepsilon_n(B) \leq \varepsilon_n(A)$ for all $n \in \mathbb{N}$.

(iv) For all natural numbers $n \in \mathbb{N}$ we have that $\varepsilon_n(A) = \varepsilon_n\Big(\operatorname{cl}(A)\Big)$.

Furthermore, if A and B are bounded subsets of a normed space, we can say the following:

(v) The inequality $\varepsilon_1(A) \leq \|A\|$ holds and if A is symmetric then $\varepsilon_1(A) = \|A\|$.

(vi) Homogeneity: For all scalars $\alpha \in \mathbb{K}$ and all natural numbers $n \in \mathbb{N}$ we have that $\varepsilon_n(\alpha A) = |\alpha|\, \varepsilon_n(A)$.

(vii) Additivity: For all natural numbers $n, m \in \mathbb{N}$ it holds

$$\varepsilon_{nm}(A+B) \leq \varepsilon_n(A) + \varepsilon_n(B),$$

where $A+B$ denotes the Minkowski sum of A and B.

The n-th entropy number of a bounded set has been defined as an infimum and in general this infimum is not attained. However, for a bounded subset of a finite-dimensional Banach space it is attained.

Lemma 2.4.3. *Let A be a bounded subset of a finite-dimensional Banach space X. Then there exist vectors $x_1, \dots, x_n \in X$ such that*

$$A \subset \bigcup_{i=1}^{n} B_X\Big(x_i, \varepsilon_n(A)\Big).$$

Proof. The proof rests on the well-known Bolzano-Weierstrass theorem. By the definition of entropy numbers, for every natural number $k \in \mathbb{N}$ we can find vectors $x_1^{(k)}, \dots, x_n^{(k)} \in X$ such that

$$A \subset \bigcup_{i=1}^{n} B_X\left(x_i^{(k)}, \varepsilon_n(A) + 1/k\right).$$

Fix $i \in \{1, 2, \dots, n\}$. First we show that the sequence $\left(x_i^{(k)}\right)_k$ is bounded. To this end, for $k \in \mathbb{N}$ choose a vector $a_i^{(k)} \in A \cap B_X\left(x_i^{(k)}, \varepsilon_n(A) + 1/k\right)$. We may assume that this intersection is not empty, otherwise the ball $B_X\left(x_i^{(k)}, \varepsilon_n(A) + 1/k\right)$ would not help to cover the set A. Then, for all $k \in \mathbb{N}$, we obtain the estimate

$$\left\|x_i^{(k)}\right\| \le \left\|x_i^{(k)} - a_i^{(k)}\right\| + \left\|a_i^{(k)}\right\| \le \varepsilon_n(A) + 1/k + \|A\| \le 2\,\|A\| + 1,$$

which proves the boundedness of the sequence. The Bolzano-Weierstrass theorem provides us with a convergent subsequence with limit $x_i \in X$. For the sake of simplicity, we denote this subsequence again by $\left(x_i^{(k)}\right)_k$. In this way we obtain vectors $x_1, \dots, x_n \in X$ and we claim that they form an $\varepsilon_n(A)$-net for A. To see this, choose $a \in A$ arbitrarily and observe that, for all $k \in \mathbb{N}$,

$$\min_{1 \le i \le n} \left\|a - x_i^{(k)}\right\| \le \varepsilon_n(A) + 1/k.$$

Letting $k \to \infty$ leads to

$$\min_{1 \le i \le n} \|a - x_i\| \le \varepsilon_n(A)$$

due to continuity. This finishes the proof. ■

Note that the entropy numbers allow us to quantify precompactness; the rate of decay of $\left(\varepsilon_n(A)\right)_n$ can be interpreted as a degree of precompactness of the set A. Hence, it is possible to classify the precompactness of sets by classifying the rate of decay of their entropy numbers. The next step is defining entropy numbers of operators in a reasonable way. To this end, observe that linear bounded operators acting between Banach spaces are closely connected with bounded sets. Indeed, a linear bounded operator $T \in \mathcal{L}(X,Y)$ maps bounded sets in X into bounded sets in Y. In particular, $T(B_X)$ is a bounded subset of Y and, therefore, has well-defined entropy numbers. Recalling the fact that an operator $T \in \mathcal{L}(X,Y)$ is called *compact* if and only if $T(B_X)$ is precompact in Y, motivates the following definition.

Definition. Let X, Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. Then, for $n \in \mathbb{N}$, the *n-th entropy number of the operator* T is defined by

$$\varepsilon_n(T : X \to Y) := \varepsilon_n(T(B_X), Y).$$

Furthermore, the *n-th dyadic entropy number* of T is given by

$$e_n(T : X \to Y) := \varepsilon_{2^{n-1}}(T : X \to Y), \quad n = 1, 2, 3, \ldots.$$

If there is no risk of confusion, we write $\varepsilon_n(T)$ or $e_n(T)$ for short.

According to the definition, the operator T is compact if and only if $\varepsilon_n(T) \to 0$ for $n \to \infty$. Again, we can measure the compactness of an operator by considering the rate of decay of its entropy numbers. In addition, the entropy numbers of operators have some useful properties listed below. For a proof we refer to [CP76], [P78], [CS90] and [ETr96].

(i) Monotonicity: For all operators $T \in \mathcal{L}(X,Y)$ it holds that

$$\|T\| = \varepsilon_1(T) \geq \varepsilon_2(T) \geq \ldots \geq 0.$$

(ii) Additivity: For all operators $S, T \in \mathcal{L}(X,Y)$ and all natural numbers $m, n \in \mathbb{N}$ the inequality

$$\varepsilon_{nm}(T + S) \leq \varepsilon_n(T) + \varepsilon_m(S)$$

holds. In particular, for $n \in \mathbb{N}$ we have that

$$\varepsilon_n(T + S) \leq \varepsilon_n(T) + \|S\|.$$

(iii) Multiplicativity: For all operators $S \in \mathcal{L}(X_0, X)$, $T \in \mathcal{L}(X,Y)$ and $R \in \mathcal{L}(Y, Y_0)$ and all natural numbers $m, n \in \mathbb{N}$ it holds that

$$\varepsilon_{nm}(TS) \leq \varepsilon_n(T)\, \varepsilon_m(S).$$

In particular, for $n \in \mathbb{N}$ we have that

$$\varepsilon_n(RTS) \leq \|R\|\ \varepsilon_n(T)\ \|S\|.$$

(iv) Surjectivity and injectivity: For all operators $T \in \mathcal{L}(X,Y)$ and all metric surjections $Q : X_0 \to X$ it holds that

$$\varepsilon_n(TQ) = \varepsilon_n(T), \quad n = 1, 2, 3, \ldots .$$

Furthermore, if $J : Y \to Y_0$ is a metric injection, then we have

$$\varepsilon_n(JT) \le \varepsilon_n(T) \le 2\,\varepsilon_n(JT), \quad n = 1, 2, 3, \ldots .$$

Note that both inequalities are sharp (cf. [CS90, p. 125]).

(v) Continuity: For all operators $S, T \in \mathcal{L}(X,Y)$ and all natural numbers $n \in \mathbb{N}$ it holds that

$$|\varepsilon_n(S) - \varepsilon_n(T)| \le \|S - T\| .$$

In terms of dyadic entropy numbers, we obtain the following results for natural numbers $n, m \in \mathbb{N}$ and linear bounded operators S, T:

(i) Monotonicity: $\|T\| = e_1(T) \ge e_2(T) \ge \ldots \ge 0$

(ii) Additivity: $e_{n+m-1}(T + S) \le e_n(T) + e_m(S)$

(iii) Multiplicativity: $e_{n+m-1}(TS) \le e_n(T)\, e_m(S)$

(iv) Surjectivity: $e_n(TQ) = e_n(T)$, if Q is a metric surjection

(v) Injectivity: $e_n(JT) \le e_n(T) \le 2\, e_n(JT)$, if J is a metric injection

Note that the dyadic entropy numbers of an operator behave very much like an additive and multiplicative s-number sequence in the sense of Pietsch (cf. [P87]). Next, we consider the entropy numbers of the identity operator on a Banach space, i.e. the entropy numbers of the closed unit ball.

Lemma 2.4.4. *For the unit ball B_X of a normed space X the following statements hold:*

(i) If $\dim(X) = m < \infty$, then $\varepsilon_1(B_X) = \cdots = \varepsilon_m(B_X) = 1$.

(ii) Let $\dim(X) = m < \infty$. If X is real, then it holds

$$n^{-\frac{1}{m}} \le \varepsilon_n(B_X) \le 3\,(n+1)^{-\frac{1}{m}}, \quad n = 1, 2, 3, \ldots ,$$

and, if X is complex, then we have

$$n^{-\frac{1}{2m}} \le \varepsilon_n(B_X) \le 3\,(n+1)^{-\frac{1}{2m}}, \quad n = 1, 2, 3, \ldots .$$

(iii) If X is an infinite-dimensional normed space, then

$$\varepsilon_n(B_X) = 1 \quad \textit{for all } n \in \mathbb{N}.$$

Proof. The proof of *(i)* rests on Riesz's Lemma. Trivially, we have that $\varepsilon_1(B_X) = 1$. Due to the monotonicity of entropy numbers, it is enough to show that $\varepsilon_m(B_X) \geq 1$. Given $\varepsilon > \varepsilon_m(B_X)$ arbitrary, we can find vectors $x_1, \ldots, x_m \in X$ such that

$$B_X \subset \bigcup_{i=1}^{m} B_X(x_i, \varepsilon).$$

We assume that $0 < \varepsilon < 1$ and show that this leads to a contradiction. To this end, consider $\operatorname{span}\{x_1, \ldots, x_{m-1}\}$ as a proper subspace of X. According to Riesz's lemma, there exists a vector $x^* \in X$ such that $\|x^*\| = 1$ and $\|x - x^*\| \geq 1$ for all $x \in \operatorname{span}\{x_1, \ldots, x_{m-1}\}$. In particular, for all $i = 1, \ldots, m-1$ we have that

$$\|x_i - x^*\| \geq 1 \quad \text{and} \quad \|x_i - (-x^*)\| \geq 1.$$

Hence, x^* and $-x^*$ do not belong to $\bigcup_{i=1}^{m-1} B_X(x_i, \varepsilon)$, but both x^* and $-x^*$ are elements of the unit ball of X. Consequently, we see that

$$x^*, -x^* \in B_X(x_m, \varepsilon)$$

and this is a contradiction since $\|x^* - (-x^*)\| = 2$. Hence, we obtain $\varepsilon \geq 1$ and, therefore, $\varepsilon_m(B_X) \geq 1$.

The proof of *(ii)* is based on volume arguments. In [CS90, pp. 8-10] one can find a proof which shows that *(ii)* holds true with constant 4 instead of 3. It goes back to a lecture of Carl (see also [Ru10, Prop. 1]) that the constant 4 can be improved to 3. Finally, *(iii)* is again a consequence of Riesz's lemma. ■

Lemma 2.4.4 has some immediate consequences: First of all, it shows that the identity operator on a Banach space X is compact if and only if X is finite-dimensional. Furthermore, we see that every bounded subset of a finite-dimensional Banach space is precompact. Moreover, it turns out that covering numbers of a bounded set can be infinite. For example, if X is an infinite-dimensional Banach space, then $N(B_X, 1/2) = \infty$. In contrast to that, entropy numbers of a bounded set are always finite.

In view of Lemma 2.4.4 *(i)*, we remark that $\varepsilon_{m+1}(B_X)$ can be the first entropy number which is strictly smaller than 1. To give an example, it is easy to see that $\varepsilon_3(B_{l_2^2(\mathbb{R})}) \leq \frac{\sqrt{3}}{2} < 1$. In contrast to that, $\varepsilon_3(B_{l_\infty^2(\mathbb{R})}) = 1$. More precisely, Richter and Börner showed that

$$\varepsilon_{n^m}(B_{l_\infty^m(\mathbb{R})}) = \varepsilon_{n^m+1}(B_{l_\infty^m(\mathbb{R})}) = \cdots = \varepsilon_{(n+1)^m-1}(B_{l_\infty^m(\mathbb{R})}) = \frac{1}{n}$$

for arbitrary natural number $n, m \in \mathbb{N}$ (cf. [RB97, Th. 3]). In particular, we have that

$$\varepsilon_{2^m-1}(B_{l_\infty^m(\mathbb{R})}) = 1 \quad \text{and} \quad \varepsilon_{2^m}(B_{l_\infty^m(\mathbb{R})}) = \frac{1}{2},$$

which shows that the estimate from below given in Lemma 2.4.4 *(ii)* is best possible. We do not know, whether the constant 3 of the estimate from above can be improved to 2.

The next lemma gives lower estimates of the entropy of certain subsets of l_p (cf. [CKP99], [CHP11, Lemma 2.4]). Such estimates will become important when dealing with the optimality of results.

Lemma 2.4.5. *Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ be a non-increasing sequence of non-negative real numbers. For $1 < p \leq 2$, let*

$$A = \{\sigma_n u_n \mid n \in \mathbb{N}\} \subset l_p,$$

where $\{u_1, u_2, \ldots\}$ denotes the canonical unit vector basis of the sequence space l_p. Then for all $n \in \mathbb{N}$ we have

$$\varepsilon_n(A) \leq \sigma_n \quad \text{and} \quad e_n(\text{aco}(A)) \geq c \, \max\{n^{-1/p'}(\log(n+1))^{1/p'}\sigma_{n^2}; \sigma_{2^n}\},$$

where $c = c(p) > 0$ is a constant only depending on p.

Proof. Since $\|\sigma_k u_k \mid l_p\| = \sigma_k \leq \sigma_n$ for all $k \geq n$, it is obvious that $\varepsilon_n(A) \leq \sigma_n$. To estimate the entropy numbers of the absolutely convex hull of A, we consider the sections

$$\Delta_{n,m} := \text{aco}\{\sigma_k u_k \mid n \leq k \leq m\}, \quad m, n \in \mathbb{N}, m > n.$$

A monotonicity argument shows that

$$e_n(\text{aco}(A)) \geq e_n(\Delta_{n,m}) \geq \sigma_m \, e_n(id : l_1^{m-n} \to l_p^{m-n})$$

and by a result of Schütt [Sch84] (see [CPa88] for a generalization) it holds

$$e_n(id : l_1^{m-n} \to l_p^{m-n}) \geq c \left(\frac{\log(m/n)}{n}\right)^{1/p'},$$

where $c > 0$ is an absolute constant. By putting $m = n^2$ and $m = 2^n$, respectively, the assertion follows. ■

Finally, we want to point out that entropy numbers are known to be very helpful for eigenvalue estimates. For example, due to Carl's observation as a special case of the *Carl-Triebel-inequality*, we have that

$$|\lambda_n(T)| \leq \sqrt{2}\, e_n(T).$$

Here the eigenvalues $\lambda_n(T)$ of the compact operator $T : X \to X$ are ordered by non-increasing absolute values and counted according to their algebraic multiplicities. This is one example illustrating that it is worthwhile studying the concept of entropy numbers, further examples will follow. We will not look at the Carl-Triebel-inequality in more detail. Interested readers are recommended to consult [C81a], [CTr80] or Chapter 4 in [CS90].

2.5 Duality of entropy numbers

A fundamental theorem of Schauder says that a linear bounded operator $T : X \to Y$ is compact if and only if its dual operator $T' : Y' \to X'$ is compact (cf. [Sch30]). In the language of entropy numbers this means that

$$\lim_{n\to\infty} e_n(T) = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} e_n(T') = 0.$$

Now it is only natural to ask whether T and T' share the same asymptotic entropy behavior. A strong interpretation of this question would be the following: Does there exist a natural number k and a constant $c \geq 1$ such that for all linear bounded operators $T : X \to Y$ between Banach spaces X, Y and all natural numbers n the estimate

$$e_{kn}(T') \leq c\, e_n(T) \tag{2.5.1}$$

holds? Note that this would imply $e_{kn}(T) \leq 2c\, e_n(T')$ since

$$e_m(T) \leq 2\, e_m(\mathcal{K}_Y T) = 2\, e_m(T''\mathcal{K}_X) \leq 2\, e_m(T''), \quad m = 1, 2, 3, \ldots .$$

It has become one of the most interesting problems of operator theory whether (2.5.1), or maybe a weaker duality relation, is true. Although a full solution is still missing, some excellent results have been already obtained in the cases that one of the Banach spaces X and Y is a Hilbert space (cf. [To87], [AMS03], [AMS04]) or B-convex (cf. [BPST89], [AMST04], see also [M07] for a review article). In addition, König, Milman and Tomczak-Jaegermann considered the entropy duality problem for an operator with fixed finite rank (cf. [KöM86], [KöMT86], [Pi89, Cor. 8.11]). Furthermore, we would like to highlight the case of $T : H \to K$ being a linear bounded operator between Hilbert spaces H and K. Here we have $e_n(T) = e_n(T')$ as a consequence of the polar decomposition theorem (cf. review of [EE86], [ETr96, Section 1.3.]).

For our purpose it is sufficient to work with the following striking result of Bourgain, Pajor, Szarek and Tomczak-Jaegermann, which can be found in [BPST89, Th. 3, Remark (2) after Th. 1].

Theorem 2.5.1. [BPST89] *Let X and Y be Banach spaces such that one of them is B-convex. Then for every $0 < \alpha < \infty$ there exists a constant $c > 0$ depending on α and the type constant of X (resp. Y) such that for all compact operators $T : X \to Y$ and all $n \in \mathbb{N}$ we have*

$$c^{-1} \sup_{1\le k\le n} k^{\alpha} e_k(T) \le \sup_{1\le k\le n} k^{\alpha} e_k(T') \le c \sup_{1\le k\le n} k^{\alpha} e_k(T).$$

As a consequence of Theorem 2.5.1 and some Hardy type inequalities given in Chapter 3 (cf. Lemma 3.0.8, 3.0.9), we get insights into the duality of entropy numbers in the context of Lorentz sequence spaces.

Proposition 2.5.2. *Let X and Y be Banach spaces such that one of them is B-convex and let $T \in \mathcal{K}(X,Y)$. Then for all $0 < p < \infty$, $0 < q \le \infty$ and any slowly varying function φ we have that*

$$\Big(e_n(T)\Big)_n \in l_{p,q,\varphi} \quad \textit{if and only if} \quad \Big(e_n(T')\Big)_n \in l_{p,q,\varphi}.$$

Proof. Choose a constant t with $0 < t < p$. Using Theorem 2.5.1 and the monotonicity of entropy numbers gives

$$n^{1/t} e_n(T) \le \sup_{1\le k\le n} k^{1/t} e_k(T) \le c_1(t,X) \sup_{1\le k\le n} k^{1/t}\, e_k(T') \le c_1(t,X) \left(\sum_{j=1}^{n} \Big(e_j(T')\Big)^t \right)^{1/t}$$

Hence, we have that

$$e_n(T) \le c_1(t,X) \left(\frac{1}{n} \sum_{j=1}^{n} \Big(e_j(T')\Big)^t \right)^{1/t}.$$

For $0 < q < \infty$ this yields

$$\sum_{n=1}^{N} (\varphi(n))^q\, n^{q/p-1} \Big(e_n(T)\Big)^q \le c_2(t,q,X) \sum_{n=1}^{N} (\varphi(n))^q\, n^{q/p-1} \left(\frac{1}{n} \sum_{j=1}^{n} \Big(e_j(T')\Big)^t \right)^{q/t}.$$

Consequently, applying Lemma 3.0.8 from Chapter 3, we see that

$$\Big(e_n(T')\Big)_n \in l_{p,q,\varphi} \quad \text{implies} \quad \Big(e_n(T)\Big)_n \in l_{p,q,\varphi}.$$

The remaining implication can be proved analogously. In the case $q = \infty$ we apply Lemma 3.0.9 instead of Lemma 3.0.8. ∎

We state another consequence of Theorem 2.5.1. The proof uses a trick due to Carl (cf. [C85, p. 106]) and can be found in [St99, Cor. 1.19].

Proposition 2.5.3. [St99] *Let X and Y be Banach spaces such that one of them is B-convex and let $T \in \mathcal{K}(X,Y)$. If $(a_n)_n$ is a regular sequence (see p. 32 for a definition), then we have that*

$$e_n(T) \preccurlyeq a_n \quad \text{if and only if} \quad e_n(T') \preccurlyeq a_n$$

and

$$e_n(T) \sim a_n \quad \text{if and only if} \quad e_n(T') \sim a_n.$$

2.6 Banach spaces of type p

For a natural number $i \in \mathbb{N}$, the i-th *Rademacher function* $r_i : [0,1] \to \mathbb{R}$ is given by

$$r_i(t) := \operatorname{sign}(\sin(2^i \pi t)).$$

In order to get an understanding of these functions we draw the graphs of the Rademacher functions r_1, r_2 and r_3.

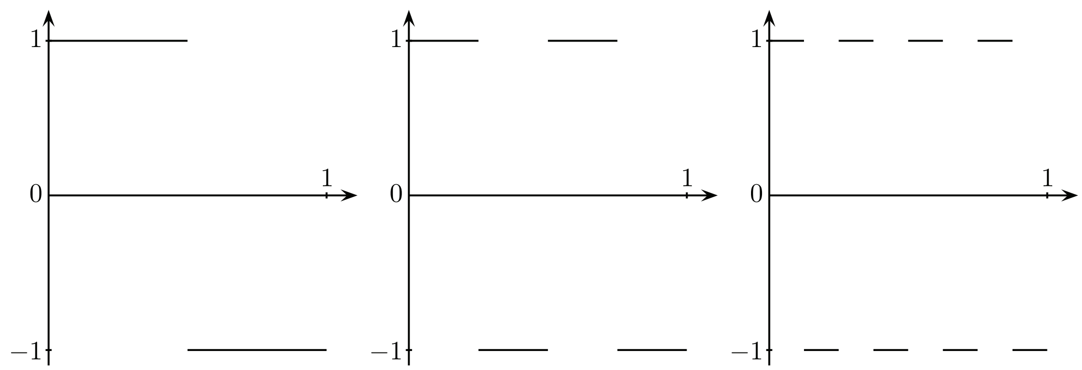


Figure 2.1: Graphs of the Rademacher functions r_1, r_2 and r_3

The Rademacher functions play a key role in the following definition of Banach spaces of type p (cf. [DPR72, Ho74, MaPi76]).

Definition. A Banach space X is said to be *of type p*, $p > 0$, if there exists a constant $\varrho > 0$ such that for all $k \in \mathbb{N}$ and all $x_1, \ldots, x_k \in X$

$$\int_0^1 \left\| \sum_{i=1}^k r_i(t)\, x_i \right\| \mathrm{d}t \le \varrho \left(\sum_{i=1}^k \|x_i\|^p \right)^{\frac{1}{p}}.$$

The *type p constant* $\tau_p(X)$ is the smallest constant ϱ satisfying the above inequality.

We recall that for any p, q with $0 < p, q < \infty$ there exists a constant $K(p,q) > 0$ for which

$$\left(\int_0^1 \left\| \sum_{i=1}^k r_i(t)\, x_i \right\|^p \mathrm{d}t \right)^{1/p} \le K(p,q) \left(\int_0^1 \left\| \sum_{i=1}^k r_i(t)\, x_i \right\|^q \mathrm{d}t \right)^{1/q},$$

regardless of the choice of a Banach space X and of finitely many vectors $x_1, \ldots, x_k$ from X. This result is known as *Kahane's Inequality* (cf. [Ka68], [DJT95, 11.1]). Obviously, we have $K(p,q) = 1$ for $p \le q$. However, if $p > q$ then we only know that $K(2,1) = \sqrt{2}$ is the best possible constant (cf. [LO94]). In the special case, if $[X, \|\cdot\|] = [\mathbb{R}, |\cdot|]$ and $p = 2$, $0 < q < \infty$ or $0 < p < \infty$, $q = 2$ then Kahane's Inequality is also known as *Khintchine's Inequality.* Here Haagerup found the best constants $K(2,q)$ and $K(p,2)$ (cf. [Ha82]).

We would like to point out that

$$\int_0^1 \left\| \sum_{i=1}^k r_i(t)\, x_i \right\| \mathrm{d}t = \frac{1}{2^k} \sum_{(\varepsilon_1,\ldots,\varepsilon_k) \in \{-1,1\}^k} \left\| \sum_{i=1}^k \varepsilon_i x_i \right\|.$$

As a consequence of Kahane's Inequality, the L_1–Rademacher average can be replaced, at the expense of a constant, by the corresponding L_r–average for any choice of $0 < r < \infty$.

Let us now collect some well-known facts about Banach spaces of type p. As a consequence of the triangle inequality of the norm, every Banach space is of type 1. Furthermore, a Banach space of type p is also of type q for any $0 < q < p$. Moreover, if $X \supsetneqq \{0\}$ is a Banach space of type p, then $p \le 2$. These facts explain why type p of Banach spaces is only considered for $1 \le p \le 2$. For X being a Banach space let

$$p_X := \sup \{p : X \text{ is of type } p\}.$$

We remark that the supremum need not be attained so that X is not necessarily of type p_X. In this context we say that X has *optimal type* q, if X is of type q but of no greater type than q. Furthermore, a Banach space is said to have *no type* if it is of optimal type 1. Moreover, we say that a Banach space is *B-convex*, if it is of some type $p > 1$. Hence, X is not B-convex if and only if X is of optimal type 1.

Due to Lindenstrauss and Rosenthal's principle of local reflexivity (cf. [LR69]), a Banach space and its bidual space have the same type. Moreover, a Banach space is B-convex if and only if its dual is (cf. [Pi73a, Pi73b], [DJT95, 13.7]), but note that this does not mean that both spaces have the same type. As an immediate consequence of the parallelogram identity, Hilbert spaces are of type 2. In order to give further examples of Banach spaces of type p, we introduce $\mathcal{L}_p$-spaces. Roughly speaking, a Banach space is an $\mathcal{L}_p$-space if its finite dimensional subspaces are contained in slightly distorted copies of l_p^n's.

Definition. Let $1 \leq p \leq \infty$ and $\lambda > 1$. The Banach space X is said to be an $\mathcal{L}_{p,\lambda}$-*space* if every finite dimensional subspace E of X is contained in a finite dimensional subspace F of X for which there is an isomorphism $T : F \to l_p^{\dim F}$ with $\|T\| \cdot \|T^{-1}\| < \lambda$. We say that X is an $\mathcal{L}_p$-space if it is an $\mathcal{L}_{p,\lambda}$-space for some $\lambda > 1$.

We would like to recall that an infinite dimensional $\mathcal{L}_p$-space has optimal type $\min\{p; 2\}$ for $1 \leq p < \infty$. An $\mathcal{L}_\infty$-space has no type, provided it is infinite dimensional. A proof of these facts can be found in [DJT95, Cor. 11.7]. We point out that the classical Lebesque function and sequence spaces are $\mathcal{L}_p$-spaces: If (Ω, Σ, μ) is any measure space and $1 \leq p \leq \infty$, then $L_p(\mu)$ is an $\mathcal{L}_{p,\lambda}$-space for all $\lambda > 1$. It is commonly known that (cf. e.g. [MaPi76], [C85, Lemma 3])

$$\tau_{\min\{p;2\}}\Big(L_p(\mu)\Big) \leq K(p,2) \leq \sqrt{p}$$

for $1 \leq p < \infty$, where $K(p,2)$ denotes the constant in Khintchin's inequality. Furthermore, if K is a compact Hausdorff space, then $C(K)$ is an $\mathcal{L}_{\infty,\lambda}$-space for all $\lambda > 1$. For a proof we refer to [DJT95, Th. 3.2]. Finally, we recall a deep theorem due to Maurey and Pisier (cf. [MaPi76, Th. 2.1], see also [MS86, Th. 13.2]) which gives insights into the local structure of infinite dimensional Banach spaces.

Theorem 2.6.1. [MaPi76] *Let X be an infinite dimensional Banach space. Then the sequence space l_{p_X} is finitely representable in X. This means that for every $\varepsilon > 0$ and for every finite dimensional subspace E of l_{p_X} there is a subspace F of X and an isomorphism $T : E \to F$ with $\|T\| \cdot \|T^{-1}\| \leq 1 + \varepsilon$.*

Corollary 2.6.2. *Let X be an infinite dimensional Banach space of optimal type p, $1 \leq p \leq 2$. Suppose that $\varepsilon > 0$ is arbitrary. Then for all natural numbers $n \in \mathbb{N}$ there are subspaces X_n of X and isomorphisms $T_n : l_p^n \to X_n$ with $\|T_n\| \cdot \|T_n^{-1}\| \leq 1 + \varepsilon$.*

For what follows we need so-called *local estimates* of entropy numbers. In this context, we recall an unpublished result of B. Maurey (cf. [Pi81]). We use the formulation given in [CKP99, Th. 1.7].

Theorem 2.6.3. (Maurey) *Let X be a Banach space of type p, $1 < p \leq 2$. Then for all integers k, n with $1 \leq k \leq n$ and all operators $S \in \mathcal{L}(l_1^n, X)$ the estimate*

$$e_k(S : l_1^n \to X) \leq c(p)\, \tau_p(X) \left(\frac{\log\left(\frac{n}{k} + 1\right)}{k} \right)^{1-1/p} \|S\|$$

is satisfied, where $c(p) \geq 1$ is a constant depending only on p.

We also need the following inequality due to Maurey and Pisier [MaPi76] (see also [Ho74]).

Theorem 2.6.4. [MaPi76] *If X is a Banach space of type p and $Y_1, \ldots, Y_n$ are independent X-valued random variables with finite p-th moment then the inequality*

$$\mathbb{E}\left\|\sum_{i=1}^{n}(Y_i - \mathbb{E}Y_i)\right\| \leq 4\,\tau_p(X)\left(\sum_{i=1}^{n}\mathbb{E}\left\|Y_i\right\|^p\right)^{1/p}$$

holds.

2.7 The absolutely convex hull

The real absolutely convex hull $\operatorname{aco}(A)$ of a precompact subset A of a real or complex Banach space plays a key role in this work. However, we would like to point out that it is also feasible and reasonable to consider the *complex* absolutely convex hull of a subset of a complex vector space. Therefore, in the following we introduce $\operatorname{aco}_{\mathbb{K}}(A)$ as the absolutely convex hull of A with respect to the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. In this sense we have that $\operatorname{aco}(A) = \operatorname{aco}_{\mathbb{R}}(A)$. The aim of this section is to present elementary properties of the absolutely convex hull. We start with recalling some basic definitions.

Definition. Let $\mathbb{K}$ stand for the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. A subset $A \subset V$ of a (real or complex) vector space V is called

(i) *convex*, if for all vectors $a, b \in A$ the line segment connecting a and b is in A, i.e. for all $\lambda \in [0,1]$ one has that $\lambda a + (1-\lambda)b \in A$.

(ii) $\mathbb{K}$-*balanced*, if for all vectors $a \in A$ and all scalars $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ it holds that $\lambda a \in A$.

(iii) $\mathbb{K}$-*symmetric*, if $A = \lambda A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| = 1$.

Furthermore, the smallest convex set that contains A is called the *convex hull of A* and is given by

$$\operatorname{conv}(A) := \bigcap_{\substack{B \supset A,\\ B \text{ convex}}} B = \left\{\sum_{i=1}^{n}\lambda_i\, a_i \,\middle|\, n \in \mathbb{N}, a_i \in A, \lambda_i \geq 0, \sum_{i=1}^{n}\lambda_i = 1\right\}.$$

Moreover, $\operatorname{ba}_{\mathbb{K}}(A)$ stands for the $\mathbb{K}$-*balanced hull of A* which is defined by

$$\operatorname{ba}_{\mathbb{K}}(A) := \bigcap_{\substack{B \supset A,\\ B\ \mathbb{K}\text{-balanced}}} B = \left\{\lambda a \,|\, a \in A, \lambda \in \mathbb{K}, |\lambda| \leq 1\right\}.$$

Obviously, a $\mathbb{K}$-balanced set A is $\mathbb{K}$-symmetric and for every $a \in A$ the line segment connecting a and $-a$ is in A. Furthermore, we would like to point out that

$$A \text{ is } \mathbb{K}\text{-symmetric} \quad \text{if and only if} \quad A = \bigcup_{\lambda\in\mathbb{K}\,:\,|\lambda|=1} \lambda A.$$

If a set is both convex and $\mathbb{K}$-balanced, then it is called $\mathbb{K}$-*absolutely convex*. In the following, we give an equivalent definition.

Definition. A subset $A \subset V$ of a (real or complex) vector space V is called $\mathbb{K}$-*absolutely convex* if for any vectors $a, b \in A$ and any scalars $\lambda, \mu \in \mathbb{K}$ with $|\lambda|+|\mu| \leq 1$ the absolutely convex combination $\lambda a + \mu b$ belongs to A.

The intersection of any collection of $\mathbb{K}$-absolutely convex sets is again a $\mathbb{K}$-absolutely convex set. Hence, every subset A of V is contained in a smallest $\mathbb{K}$-absolutely convex set, called the $\mathbb{K}$-*absolutely convex hull of* A, which is given by the intersection of all $\mathbb{K}$-absolutely convex supersets of A, i.e.

$$\operatorname{aco}_{\mathbb{K}}(A) := \bigcap_{\substack{B\supset A,\\ B\ \mathbb{K}\text{-absolutely convex}}} B.$$

This concept is well-known from introducing the convex hull and the balanced hull of a set. First, let us collect some alternative representations of the $\mathbb{K}$-absolutely convex hull. By mathematical induction one can shows that $\operatorname{aco}_{\mathbb{K}}(A)$ is the set of all finite absolutely convex combinations, i.e.

$$\operatorname{aco}_{\mathbb{K}}(A) = \left\{ \sum_{i=1}^{n} \lambda_i\, a_i \,\middle|\, n \in \mathbb{N}, a_i \in A, \lambda_i \in \mathbb{K}, \sum_{i=1}^{n} |\lambda_i| \leq 1 \right\}.$$

Furthermore, the $\mathbb{K}$-absolutely convex hull of A is the convex hull of the $\mathbb{K}$-balanced hull of A, i.e.

$$\operatorname{aco}_{\mathbb{K}}(A) = \operatorname{conv}\Big(\operatorname{ba}_{\mathbb{K}}(A)\Big).$$

In this context, note that

$$\operatorname{ba}_{\mathbb{K}}\Big(\operatorname{conv}(A)\Big) \subset \operatorname{aco}_{\mathbb{K}}(A)$$

and, in general, this inclusion is strict. Another representation is given by

$$\operatorname{aco}_{\mathbb{K}}(A) = \operatorname{conv}\left(\bigcup_{\lambda\in\mathbb{K}\,:\,|\lambda|=1} \lambda A \right). \tag{2.7.1}$$

In particular, for $\mathbb{K}$-symmetric sets the $\mathbb{K}$-absolutely convex hull and the convex hull coincide. Next, we ask what can be said about the $\mathbb{K}$-absolutely convex hull of the Minkowski sum of sets. This is the subject of the following lemma.

Lemma 2.7.1. *Let A, B be subsets of a (real or complex) vector space V. Then*

$$\mathrm{aco}_{\mathbb{K}}(A+B) \subset \mathrm{aco}_{\mathbb{K}}(A) + \mathrm{aco}_{\mathbb{K}}(B)$$

and this inclusion can be strict. However, if one of the sets A and B is $\mathbb{K}$-symmetric, then

$$\mathrm{aco}_{\mathbb{K}}(A+B) = \mathrm{aco}_{\mathbb{K}}(A) + \mathrm{aco}_{\mathbb{K}}(B).$$

Proof. Since the Minkowski sum of two $\mathbb{K}$-absolutely convex sets is again an $\mathbb{K}$-absolutely convex set, we have that $\mathrm{aco}_{\mathbb{K}}(A) + \mathrm{aco}_{\mathbb{K}}(B)$ is an $\mathbb{K}$-absolutely convex superset of $A + B$. Hence, it follows from the very definition that $\mathrm{aco}_{\mathbb{K}}(A+B) \subset \mathrm{aco}_{\mathbb{K}}(A) + \mathrm{aco}_{\mathbb{K}}(B)$. Finally, let A be a $\mathbb{K}$-symmetric set and B an arbitrary set. It is enough to show that, in this setting, $\mathrm{aco}_{\mathbb{K}}(A) + \mathrm{aco}_{\mathbb{K}}(B) \subset \mathrm{aco}_{\mathbb{K}}(A+B)$. To this end, we use the the connections between the convex hull and the $\mathbb{K}$-absolutely convex hull mentioned in (2.7.1) and the well-known additivity of the convex hull. First, observe that

$$\mathrm{aco}_{\mathbb{K}}(A) + \mathrm{aco}_{\mathbb{K}}(B) = \mathrm{conv}(A) + \mathrm{conv}\left(\bigcup_{\lambda\in\mathbb{K}\,:\,|\lambda|=1} \lambda B\right) = \mathrm{conv}\left(A + \bigcup_{\lambda\in\mathbb{K}\,:\,|\lambda|=1} \lambda B\right).$$

Furthermore, for every scalar $\lambda \in \mathbb{K}$ with $|\lambda| = 1$ we have that

$$A + \lambda B = \lambda A + \lambda B = \lambda(A+B) \subset \mathrm{aco}_{\mathbb{K}}(A+B).$$

Hence, we see that

$$\mathrm{conv}\left(A + \bigcup_{\lambda\in\mathbb{K}\,:\,|\lambda|=1} \lambda B\right) \subset \mathrm{aco}_{\mathbb{K}}(A+B),$$

which proves the statement. ■

As already mentioned in the introduction, a subset of a normed space is precompact if and only if its $\mathbb{K}$-absolutely convex hull is precompact. At first glance, this may surprise since, in general, the $\mathbb{K}$-absolutely convex hull of a set is much larger than the set itself. In the following, we give an elementary proof of this fact.

Lemma 2.7.2. *Let X be a (real or complex) normed space and let A be a subset of X. Then A is precompact if and only if $\mathrm{aco}_{\mathbb{K}}(A)$ is precompact.*

Proof. Choose $\varepsilon > 0$ arbitrary. Since A is precompact we can find a finite $\varepsilon/2$-net $\{x_1, x_2, \ldots, x_n\} \subset X$ for A, hence we have that

$$A \subset \{x_1, x_2, \ldots, x_n\} + B_X(0, \varepsilon/2) \subset \mathrm{aco}_{\mathbb{K}}\{x_1, x_2, \ldots, x_n\} + B_X(0, \varepsilon/2).$$

From the very definition of the $\mathbb{K}$-absolutely convex hull of A we conclude that

$$\mathrm{aco}_{\mathbb{K}}(A) \subset \mathrm{aco}_{\mathbb{K}}\{x_1, x_2, \dots, x_n\} + B_X(0, \varepsilon/2).$$

The next step is to show that $\mathrm{aco}_{\mathbb{K}}\{x_1, x_2, \dots, x_n\}$ is a compact set. To this end, consider

$$B_{l_1^n(\mathbb{K})} = \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \;\middle|\; \sum_{i=1}^{n} |\lambda_i| \le 1 \right\}$$

and the mapping $f : B_{l_1^n(\mathbb{K})} \to X$ given by $(\lambda_1, \dots, \lambda_n) \mapsto \sum_{i=1}^n \lambda_i x_i$. Observe that $B_{l_1^n(\mathbb{K})}$ is compact and f is a continuous mapping, therefore also

$$f(B_{l_1^n(\mathbb{K})}) = \mathrm{aco}_{\mathbb{K}}\{x_1, x_2, \dots, x_n\}$$

is compact. Consequently we can find vectors $y_1, y_2, \dots, y_m \in X$ such that

$$\mathrm{aco}_{\mathbb{K}}\{x_1, x_2, \dots, x_n\} \subset \{y_1, y_2, \dots, y_m\} + B_X(0, \varepsilon/2),$$

which leads to

$$\mathrm{aco}_{\mathbb{K}}(A) \subset \{y_1, y_2, \dots, y_m\} + B_X(0, \varepsilon/2) + B_X(0, \varepsilon/2) = \{y_1, y_2, \dots, y_m\} + B_X(0, \varepsilon)$$

and finishes the proof. ∎

Things change dramatically when considering quasi-normed spaces instead of normed spaces. Indeed, even the convex hull of a precompact subset of a quasi-normed space is not bounded in general. To give an example, we consider the sequence space l_p with $0 < p < 1$ and the subset

$$A := \{\sigma_n e_n \mid n \in \mathbb{N}\} \cup \{0\},$$

where $\{e_1, e_2, \dots\}$ denotes the canonical unit vector basis of l_p and $(\sigma_n)_n$ is a non-increasing null sequence. Since $\varepsilon_n(A) \le \sigma_n$ and $\sigma_n \to 0$ for $n \to \infty$, we have that A is precompact. Now we show that the convex hull $\mathrm{conv}(A)$ of A is not bounded if the sequence $(\sigma_n)_n$ decreases slow enough. To this end, consider the convex combination

$$x = \sum_{k=1}^{n} \lambda_k \, \sigma_k e_k$$

with $\lambda_k \ge 0$ and $\sum_{k=1}^n \lambda_k \le 1$. The norm of x is computed as

$$\|x \mid l_p\|^p = \sum_{k=1}^{n} \lambda_k^p \, \sigma_k^p,$$

hence it suffices to find a summable sequence $(\lambda_k)_k$ such that $\sum_{k=1}^{\infty} \lambda_k^p \, \sigma_k^p = \infty$. In order to do this, choose r with $p < r < 1$ and put $\lambda_k = \left(\sum_{j=1}^{\infty} j^{-1/r} \right)^{-1} k^{-1/r}$. Then we obtain

$$\|x \mid l_p\|^p = \left(\sum_{j=1}^{\infty} j^{-1/r} \right)^{-p} \sum_{k=1}^{n} k^{-p/r} \, \sigma_k^p,$$

which tends to infinity if, for instance, $\sigma_k = (\log(k+1))^{\beta}$ with $\beta < 0$.

This consideration shows that in quasi-normed spaces it may happen that

$$\varepsilon_n(A) \to 0 \qquad \text{but} \qquad \varepsilon_n(\mathrm{aco}_{\mathbb{K}}(A)) \not\to 0$$

for $n \to \infty$. Hence, in the setting of a quasi-normed space the problem of estimating $e_n(\mathrm{aco}_{\mathbb{K}}(A))$ in terms of $\varepsilon_n(A)$ makes only sense with some additional information. We will not deal with this topic in more detail.

3 Inequalities of Hardy-type

In order to prove some generalized inequalities of Hardy-type, we first collect some technical lemmata dealing with slowly varying functions.

Lemma 3.0.3. *Given a slowly varying function φ and a real number $q > 1$, there exists a constant $c = c(\varphi, q) > 0$ such that for all natural numbers $n \in \mathbb{N}$ it holds that*

$$\sum_{k=0}^{n} \varphi(2^k)\, q^k \le c \cdot \varphi(2^n)\, q^n.$$

Proof. Define $\alpha := \frac{1}{2}\log(q) > 0$. According to Lemma 2.2.5, there exists a non-decreasing function Φ with

$$\lim_{x\to\infty} \frac{x^\alpha \varphi(x)}{\Phi(x)} = 1.$$

Consequently, we can find positive constants c_1, c_2 depending on φ and q such that for all $k = 0, 1, 2, \dots$ we have that

$$c_1(\varphi, q)\, 2^{-\alpha k}\, \Phi(2^k) \le \varphi(2^k) \le c_2(\varphi, q)\, 2^{-\alpha k}\, \Phi(2^k).$$

Hence, the estimate

$$\sum_{k=0}^{n} \varphi(2^k)\, q^k \le c_2(\varphi, q) \sum_{k=0}^{n} 2^{-\alpha k}\, \Phi(2^k)\, q^k \le c_2(\varphi, q)\, \Phi(2^n) \sum_{k=0}^{n} (2^{-\alpha} q)^k$$

holds. From $2^{-\alpha} q > 1$ we conclude that

$$\sum_{k=0}^{n} (2^{-\alpha} q)^k \le \frac{2^{-\alpha} q}{2^{-\alpha} q - 1}\, (2^{-\alpha} q)^n = c_3(q)\, 2^{-\alpha n}\, q^n,$$

which gives

$$\sum_{k=0}^{n} \varphi(2^k)\, q^k \le c_4(\varphi, q)\, 2^{-\alpha n}\, \Phi(2^n)\, q^n.$$

Finally, for all $n = 0, 1, 2, \dots$ we have that

$$2^{-\alpha n}\, \Phi(2^n) \le \frac{1}{c_1(\varphi, q)}\, \varphi(2^n),$$

which yields the assertion. ∎

Lemma 3.0.4. *Given a slowly varying function φ and a real number q with $0 < q < 1$, there exists a constant $c = c(\varphi, q) > 0$ such that for all natural numbers $n \in \mathbb{N}_0$ it holds that*

$$\sum_{k=n}^{\infty} \varphi(2^k)\, q^k \le c \cdot \varphi(2^n)\, q^n.$$

Proof. The proof is analogous to that of Lemma 3.0.3. This time, we define $\alpha := \frac{1}{2}\log(1/q) > 0$. According to Lemma 2.2.5 there exists a non-increasing function Ψ with

$$\lim_{x\to\infty} \frac{x^{-\alpha}\,\varphi(x)}{\Psi(x)} = 1.$$

Hence, there exist constants $c_1, c_2 > 0$ depending on φ and q such that for all $k = 0, 1, 2, \ldots$ we have that

$$c_1(\varphi, q)\Psi(2^k)\, 2^{k\alpha} \le \varphi(2^k) \le c_2(\varphi, q)\Psi(2^k)\, 2^{k\alpha}.$$

This leads to the estimate

$$\sum_{k=n}^{\infty} \varphi(2^k)\, q^k \le c_2(\varphi, q) \sum_{k=n}^{\infty} \Psi(2^k)\, (2^\alpha q)^k \le c_2(\varphi, q)\, \Psi(2^n) \sum_{k=n}^{\infty} (2^\alpha q)^k.$$

Since $0 < 2^\alpha q < 1$, the summation formula for the geometric series gives

$$\sum_{k=n}^{\infty} \varphi(2^k)\, q^k \le \frac{c_2}{1 - 2^\alpha q}\, \Psi(2^n)\, (2^\alpha q)^n = c_3(\varphi, q)\, \Psi(2^n)\, 2^{\alpha n}\, q^n.$$

Finally, the estimate

$$\Psi(2^n)\, 2^{\alpha n} \le \frac{1}{c_1(\varphi, q)}\, \varphi(2^n)$$

finishes the proof. ■

As a consequence of the uniform convergence theorem of slowly varying functions (cf. Theorem 2.2.2) we get the following useful lemma.

Lemma 3.0.5. *Let φ be a slowly varying function and $0 < a < b < \infty$. Then there exist positive constants c_1, c_2 depending on φ, a and b such that for all $x \ge 1$ and all y with $ax \le y \le bx$ we have that*

$$c_1\, \varphi(x) \le \varphi(y) \le c_2\, \varphi(x).$$

Proof. Since

$$\lim_{x\to\infty} \frac{\varphi(\lambda x)}{\varphi(x)} = 1$$

uniformly for all $\lambda \in [a, b]$, we can find a positive number x_0 such that for all $x \ge x_0$ and all $\lambda \in [a, b]$ it holds that $1/2 \le \varphi(\lambda x)/\varphi(x) \le 3/2$. Furthermore, since φ is positive and continuous, we can find positive constants c_1, c_2 depending on φ, a and b such that for all x with $1 \le x \le x_0$ and all $\lambda \in [a,b]$ the estimate $c_1 \le \varphi(\lambda x)/\varphi(x) \le c_2$ holds. Hence, for all $x \ge 1$ and all $\lambda \in [a,b]$ we have that

$$\min\{1/2; c_1\} \le \frac{\varphi(\lambda x)}{\varphi(x)} \le \max\{3/2; c_2\}.$$

The assertion follows. ∎

Next, we state a corollary of Lemma 3.0.3.

Corollary 3.0.6. *Let φ be a slowly varying function and $\alpha > -1$. Then there exists a positive constant $c = c(\varphi, \alpha)$ such that for all natural numbers $n \in \mathbb{N}$ we have that*

$$\sum_{k=1}^{n} \varphi(k)\, k^{\alpha} \le c \cdot \varphi(n)\, n^{\alpha+1}.$$

Proof. For $m = 0, 1, \ldots, \lfloor \log(n) \rfloor$ we define $U_m := \{k \in \mathbb{N} \,|\, 2^m \le k < 2^{m+1}\}$. Then $|U_m| = 2^m$ and for all natural numbers $k \in U_m$ we have that $k^{\alpha} \le c_1(\alpha)\, 2^{m\alpha}$. Furthermore, due to Lemma 3.0.5, we can find a constant $c_2(\varphi) > 0$ such that $\varphi(k) \le c_2(\varphi)\, \varphi(2^m)$ for all $m = 0, 1, 2, \ldots$ and all $k \in U_m$. Hence, we obtain

$$\sum_{k=1}^{n} \varphi(k)\, k^{\alpha} \le \sum_{m=0}^{\lfloor \log(n) \rfloor} \sum_{k \in U_m} \varphi(k)\, k^{\alpha} \le c_3(\varphi, \alpha) \sum_{m=0}^{\lfloor \log(n) \rfloor} \varphi(2^m) \left(2^{\alpha+1}\right)^m.$$

Now applying Lemma 3.0.3 with $q = 2^{\alpha+1} > 1$ yields

$$\sum_{k=1}^{n} \varphi(k)\, k^{\alpha} \le c_4(\varphi, \alpha)\, \varphi\left(2^{\lfloor \log(n) \rfloor}\right) \left(2^{\alpha+1}\right)^{\lfloor \log(n) \rfloor}.$$

Observe that for all natural numbers n it holds that

$$2^{\lfloor \log(n) \rfloor} \le n \le 2 \cdot 2^{\lfloor \log(n) \rfloor}.$$

Hence, according to Lemma 3.0.5, we can find a constant $c_5(\varphi) > 0$ such that

$$\varphi\left(2^{\lfloor \log(n) \rfloor}\right) \le c_5(\varphi)\, \varphi(n)$$

for all $n \in \mathbb{N}$. Consequently, we obtain

$$\sum_{k=1}^{n} \varphi(k)\, k^{\alpha} \le c_6(\varphi, \alpha)\, \varphi(n) \left(2^{\alpha+1}\right)^{\lfloor \log(n) \rfloor} \le c_6(\varphi, \alpha)\, \varphi(n)\, n^{\alpha+1},$$

which finishes the proof. ∎

In view of Lemma 3.0.4, we can state the following corollary.

Corollary 3.0.7. *Let φ be a slowly varying function and $\alpha < -1$. Then there exists a positive constant $c = c(\varphi, \alpha)$ such that for all natural numbers $n \in \mathbb{N}$ we have that*

$$\sum_{k=n}^{\infty} \varphi(k)\, k^{\alpha} \leq c \cdot \varphi(n)\, n^{\alpha+1}.$$

Proof. For $m = \lfloor \log(n) \rfloor, \lfloor \log(n) \rfloor + 1, \ldots$ we define $U_m := \{k \in \mathbb{N} \,|\, 2^m \leq k < 2^{m+1}\}$. Clearly, $|U_m| = 2^m$ and for all natural numbers $k \in U_m$ we have that $k^{\alpha} \leq 2^{\alpha m}$. Furthermore, due to Lemma 3.0.5, we can find a constant $c_1(\varphi) > 0$ such that $\varphi(k) \leq c_1(\varphi)\, \varphi(2^m)$ for all $m = \lfloor \log(n) \rfloor, \lfloor \log(n) \rfloor + 1, \ldots$ and all $k \in U_m$. This leads to the estimate

$$\sum_{k=n}^{\infty} \varphi(k)\, k^{\alpha} \leq \sum_{m=\lfloor \log(n) \rfloor}^{\infty} \sum_{k \in U_m} \varphi(k)\, k^{\alpha} \leq c_1(\varphi) \sum_{m=\lfloor \log(n) \rfloor}^{\infty} \varphi(2^m) \left(2^{\alpha+1}\right)^m.$$

Since $0 < 2^{\alpha+1} < 1$, we can apply Lemma 3.0.4 to obtain

$$\sum_{k=n}^{\infty} \varphi(k)\, k^{\alpha} \leq c_2(\varphi, \alpha)\, \varphi(2^{\lfloor \log(n) \rfloor}) \left(2^{\alpha+1}\right)^{\lfloor \log(n) \rfloor} \leq c_3(\varphi, \alpha)\, \varphi(2^{\lfloor \log(n) \rfloor})\, n^{\alpha+1}.$$

Finally, according to Lemma 3.0.5 we can find a constant $c_4(\varphi) > 0$ such that

$$\varphi\left(2^{\lfloor \log(n) \rfloor}\right) \leq c_4(\varphi)\, \varphi(n)$$

for all $n \in \mathbb{N}$. This finishes the proof. ∎

Now we are well prepared to prove some generalized Hardy-type inequalities, which will become useful in the context of generalized Lorentz sequence spaces.

Lemma 3.0.8. *Let $0 < t < r < \infty$, $0 < s < \infty$ and φ be a slowly varying function. If $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ is a non-increasing sequence of non-negative real numbers, then*

$$\sum_{n=1}^{N} \varphi(n)\, n^{s/r-1} \left(\frac{1}{n} \sum_{k=1}^{n} \sigma_k^t\right)^{s/t} \leq c \sum_{n=1}^{N} \varphi(n)\, n^{s/r-1}\, \sigma_n^s$$

for all $N \in \mathbb{N}$, where $c = c(t, r, s, \varphi) > 0$ is a constant depending on t, r, s and φ.

Proof. The idea of the proof goes back to the paper [CHP11, Lemma 2.3], where the result of Lemma 3.0.8 was proved for the special case of $\varphi(x) = (\log(x+1))^{\alpha}$, $\alpha \in \mathbb{R}$.

Define q by the equality $1/q := 1/t + 1/s$ and choose a number u with $1/r < u < 1/t$. Due to the monotonicity of the sequence $(\sigma_n)_n$ we have that

$$k\, \sigma_k^t \leq \sum_{j=1}^{n} \sigma_j^t$$

for all $k = 1, 2, \dots, n$ and using Hölder's inequality we get

$$\begin{aligned}\left(\sum_{k=1}^n \sigma_k^t\right)^{1/q} &= \left(\sum_{k=1}^n \left[(\varphi(k))^{-1/s} k^{-u} (\varphi(k))^{1/s} k^{u-1/s} \sigma_k (k\,\sigma_k^t)^{1/s}\right]^q\right)^{1/q} \\ &\le \left(\sum_{k=1}^n (\varphi(k))^{-t/s} k^{-ut}\right)^{1/t} \left(\sum_{k=1}^n \left[(\varphi(k))^{1/s} k^{u-1/s} \sigma_k\right]^s k\,\sigma_k^t\right)^{1/s} \\ &\le \left(\sum_{k=1}^n (\varphi(k))^{-t/s} k^{-ut}\right)^{1/t} \left(\sum_{k=1}^n \left[(\varphi(k))^{1/s} k^{u-1/s} \sigma_k\right]^s\right)^{1/s} \left(\sum_{j=1}^n \sigma_j^t\right)^{1/s}.\end{aligned}$$

Hence, the estimate

$$\left(\sum_{k=1}^n \sigma_k^t\right)^{1/t} \le \left(\sum_{k=1}^n (\varphi(k))^{-t/s} k^{-ut}\right)^{1/t} \left(\sum_{k=1}^n \varphi(k) k^{us-1} \sigma_k^s\right)^{1/s}$$

holds. Since $-ut > -1$, Corollary 3.0.6 yields

$$\left(\sum_{k=1}^n (\varphi(k))^{-t/s} k^{-ut}\right)^{1/t} \le c_1(u,t,\varphi,s)\,(\varphi(n))^{-1/s}\, n^{-u+1/t}$$

and, consequently, we obtain

$$\left(\frac{1}{n}\sum_{k=1}^n \sigma_k^t\right)^{1/t} \le c_1(u,t,\varphi,s)\,(\varphi(n))^{-1/s}\, n^{-u} \left(\sum_{k=1}^n \varphi(k) k^{us-1}\,\sigma_k^s\right)^{1/s}.$$

This leads to the estimate

$$\sum_{n=1}^N \varphi(n)\, n^{s/r-1} \left(\frac{1}{n}\sum_{k=1}^n \sigma_k^t\right)^{s/t} \le\ (c_1(u,t,\varphi,s))^s \sum_{n=1}^N n^{s/r-1-us} \sum_{k=1}^n \varphi(k)\, k^{us-1} \sigma_k^s$$

and changing the order of summation on the right-hand side gives

$$\sum_{n=1}^N \varphi(n)\, n^{s/r-1} \left(\frac{1}{n}\sum_{k=1}^n \sigma_k^t\right)^{s/t} \le (c_1(u,t,\varphi,s))^s \sum_{k=1}^N \varphi(k)\, k^{us-1} \sigma_k^s \sum_{n=k}^N n^{s/r-1-us}.$$

Since $s/r - 1 - us < -1$, we can continue with

$$\begin{aligned}\sum_{n=k}^N n^{s/r-1-us} \le \sum_{n=k}^\infty n^{s/r-1-us} &\le k^{s/r-1-us} + \int_k^\infty x^{s/r-1-us}\,dx \\ &= k^{s/r-1-us} + \frac{k^{s/r-us}}{us - s/r} \le k^{s/r-us}\left(1 + \frac{1}{us - s/r}\right) \\ &= c_2(r,s,u)\, k^{s/r-us},\end{aligned}$$

which gives the assertion. ∎

Lemma 3.0.9. *Let* $0 < t < r < \infty$ *and let* φ *be a slowly varying function. If* $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ *is a non-increasing sequence of non-negative real numbers, then*

$$\sup_{1\leq n\leq N} \varphi(n)\, n^{1/r} \left(\frac{1}{n}\sum_{k=1}^{n} \sigma_k^t\right)^{1/t} \leq c \sup_{1\leq n\leq N} \varphi(n)\, n^{1/r}\, \sigma_n$$

for all $N \in \mathbb{N}$*, where* $c = c(t, r, \varphi) > 0$ *is a constant depending on* t, r *and* φ*.*

Proof. Clearly, we have that

$$\sum_{k=1}^{n} \sigma_k^t = \sum_{k=1}^{n} (\varphi(k))^{-t}\, k^{-t/r}\, (\varphi(k))^t\, k^{t/r}\, \sigma_k^t \leq \sup_{1\leq k\leq n} (\varphi(k))^t\, k^{t/r}\, \sigma_k^t \sum_{k=1}^{n} (\varphi(k))^{-t}\, k^{-t/r}.$$

Since $-t/r > -1$, we can apply Corollary 3.0.6 to obtain

$$\sum_{k=1}^{n} \sigma_k^t \leq c_1(t, r, \varphi)\, (\varphi(n))^{-t}\, n^{-t/r+1} \sup_{1\leq k\leq n} (\varphi(k))^t\, k^{t/r}\, \sigma_k^t,$$

which yields

$$(\varphi(n))^t\, n^{t/r}\, \frac{1}{n}\sum_{k=1}^{n} \sigma_k^t \leq c_1(t, r, \varphi) \sup_{1\leq k\leq n} (\varphi(k))^t\, k^{t/r}\, \sigma_k^t.$$

Hence, we get

$$\begin{aligned}\varphi(n)\, n^{1/r} \left(\frac{1}{n}\sum_{k=1}^{n} \sigma_k^t\right)^{1/t} &\leq (c_1(t, r, \varphi))^{1/t} \left(\sup_{1\leq k\leq n} (\varphi(k))^t\, k^{t/r}\, \sigma_k^t\right)^{1/t} \\ &= c_2(t, r, \varphi) \sup_{1\leq k\leq n} \varphi(k)\, k^{1/r}\, \sigma_k\end{aligned}$$

for all natural numbers $n \in \mathbb{N}$. Taking the supremum with respect to $1 \leq n \leq N$ completes the proof. ■

Lemma 3.0.10. *Let* $0 < r < 1$ *and* $0 < s < \infty$*. Furthermore, let* φ *be a slowly varying function. If* $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ *is a non-increasing sequence of non-negative real numbers, then there exists a constant* $c = c(r, s, \varphi) > 0$ *such that*

$$\sum_{n=1}^{\infty} \varphi(n)\, n^{s(1/r-1)-1} \left(\sum_{k=n}^{\infty} \sigma_k\right)^s \leq c \sum_{n=1}^{\infty} \varphi(n)\, n^{s/r-1}\, \sigma_n^s.$$

Proof. From (2.3.2) and the monotonicity of the sequence $(\sigma_n)_n$ we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \varphi(n)\, n^{s(1/r-1)-1} \left(\sum_{k=n}^{\infty} \sigma_k\right)^s &\leq c_1(r, s, \varphi) \sum_{m=0}^{\infty} \varphi(2^m)\, 2^{ms(1/r-1)} \left(\sum_{k=2^m}^{\infty} \sigma_k\right)^s \\ &\leq c_1(r, s, \varphi) \sum_{m=0}^{\infty} \varphi(2^m)\, 2^{ms(1/r-1)} \left(\sum_{k=m}^{\infty} 2^k\, \sigma_{2^k}\right)^s.\end{aligned} \tag{3.0.1}$$

Now choose a constant α with $0 < \alpha < 1/r - 1$. First we show that there exists a constant $c_2(\alpha, s) > 0$ such that

$$\left(\sum_{k=m}^{\infty} 2^k \sigma_{2^k}\right)^s \le c_2(\alpha, s)\, 2^{-\alpha s m} \sum_{k=m}^{\infty} 2^{ks(\alpha+1)} \sigma_{2^k}^s. \tag{3.0.2}$$

In the case $0 < s \le 1$ the desired estimate follows immediately from $\|\cdot\|_1^s \le \|\cdot\|_s^s$:

$$\left(\sum_{k=m}^{\infty} 2^k \sigma_{2^k}\right)^s \le \sum_{k=m}^{\infty} 2^{-\alpha s k}\, 2^{ks(\alpha+1)} \sigma_{2^k}^s \le 2^{-\alpha s m} \sum_{k=m}^{\infty} 2^{ks(\alpha+1)} \sigma_{2^k}^s.$$

For $1 < s < \infty$ we apply Hölder's inequality with the pair of conjugated exponents s and s' to obtain

$$\left(\sum_{k=m}^{\infty} 2^k \sigma_{2^k}\right)^s = \left(\sum_{k=m}^{\infty} 2^{-\alpha k}\, 2^{k(\alpha+1)} \sigma_{2^k}\right)^s \le \left(\sum_{k=m}^{\infty} \left(2^{-\alpha k}\right)^{s'}\right)^{s/s'} \sum_{k=m}^{\infty} \left(2^{k(\alpha+1)} \sigma_{2^k}\right)^s.$$

Using the summation formula for the geometric series results in

$$\begin{aligned}\left(\sum_{k=m}^{\infty} 2^k \sigma_{2^k}\right)^s &\le \left(\frac{2^{-\alpha s' m}}{1 - 2^{-\alpha s'}}\right)^{s/s'} \sum_{k=m}^{\infty} 2^{ks(\alpha+1)} \sigma_{2^k}^s \\ &= \left(1 - 2^{-\alpha s'}\right)^{-s/s'} 2^{-\alpha s m} \sum_{k=m}^{\infty} 2^{ks(\alpha+1)} \sigma_{2^k}^s.\end{aligned}$$

Combining (3.0.1) and (3.0.2) we get

$$\sum_{n=1}^{\infty} \varphi(n)\, n^{s(1/r-1)-1} \left(\sum_{k=n}^{\infty} \sigma_k\right)^s \le c_3(r, s, \varphi, \alpha) \sum_{m=0}^{\infty} \varphi(2^m)\, 2^{ms(1/r-1-\alpha)} \sum_{k=m}^{\infty} 2^{ks(\alpha+1)} \sigma_{2^k}^s.$$

Changing the order of summation on the right hand sight and applying Lemma 3.0.3 yields

$$\begin{aligned}\sum_{n=1}^{\infty} \varphi(n)\, n^{s(1/r-1)-1} \left(\sum_{k=n}^{\infty} \sigma_k\right)^s &\le c_3(r, s, \varphi, \alpha) \sum_{k=0}^{\infty} 2^{ks(\alpha+1)} \sigma_{2^k}^s \sum_{m=0}^{k} \varphi(2^m)\, 2^{ms(1/r-1-\alpha)} \\ &\le c_4(r, s, \varphi, \alpha) \sum_{k=0}^{\infty} \varphi(2^k)\, 2^{ks/r} \sigma_{2^k}^s.\end{aligned}$$

Finally, using (2.3.3) from the proof of Lemma 2.3.3, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \varphi(2^k)\, 2^{ks/r} \sigma_{2^k}^s &\le \varphi(1)\, \sigma_1^s + c_5(r, s, \varphi) \sum_{n=2}^{\infty} \varphi(n)\, n^{s/r-1} \sigma_n^s \\ &\le (1 + c_5(r, s, \varphi)) \sum_{n=1}^{\infty} \varphi(n)\, n^{s/r-1} \sigma_n^s.\end{aligned}$$

This completes the proof. ■

Lemma 3.0.11. *Let* $0 < r < 1$ *and let* φ *be a slowly varying function. If* $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ *is a non-increasing sequence of non-negative real numbers, then for all natural numbers* $n \in \mathbb{N}$ *the estimate*

$$\varphi(n)\, n^{1/r-1} \sum_{k=n}^{\infty} \sigma_k \leq c \sup_{k \geq n} \varphi(k)\, k^{1/r}\, \sigma_k$$

holds true, where $c = c(r, \varphi) > 0$ *is a constant depending on* r *and* φ*.*

Proof. The proof is straightforward. First observe that

$$\varphi(n)\, n^{1/r-1} \sum_{k=n}^{\infty} \sigma_k \leq \left(\sup_{k \geq n} \varphi(k)\, k^{1/r}\, \sigma_k \right) \varphi(n)\, n^{1/r-1} \sum_{k=n}^{\infty} (\varphi(k))^{-1}\, k^{-1/r}.$$

Since $-1/r < -1$, Corollary 3.0.7 implies that

$$\sum_{k=n}^{\infty} (\varphi(k))^{-1}\, k^{-1/r} \leq c_1(r, \varphi)\, (\varphi(n))^{-1}\, n^{-1/r+1}$$

This finishes the proof. ∎

4 Proof of the results

4.1 Proof of Theorem 1.3.4

The proof uses techniques and ideas from the proof of [St04, Th. 1.3]. Let us first assume that A is symmetric. We choose a constant a with $\frac{1}{2} < a < 1$ and define $\alpha_k := \left\lfloor 2^{n2^{a(k-1)}} \right\rfloor$ for $k = 1, 2, \ldots, n$, then $2^n = \alpha_1 < \alpha_2 < \ldots < \alpha_n$. Furthermore with

$$m = \left\lfloor 2^{n+2} \sum_{k=2}^{n} 2^{-k} \log \left(\frac{2^{k+2} \left\lfloor 2^{n2^{a(k-1)}} \right\rfloor}{2^n} + 3 \right) \right\rfloor + 2$$

it holds

$$\begin{aligned} m &\geq 2^{n+2} \sum_{k=2}^{n} 2^{-k} \log \left(2^{k+2-n} \left(2^{n2^{a(k-1)}} - 1 \right) \right) \geq 2^{n+2} \sum_{k=2}^{n} 2^{-k} \log \left(2^{k+2-n}\, 2^{n2^{a(k-1)}-1} \right) \\ &= 2^{n+2} \sum_{k=2}^{n} 2^{-k} \left(k + 1 + n \left(2^{a(k-1)} - 1 \right) \right) \geq 2^n \left(3 + n(2^a - 1) \right) \geq c_1\, n 2^n \end{aligned}$$

on the one hand and

$$\begin{aligned} m &\leq 2 + 2^{n+2} \sum_{k=2}^{n} 2^{-k} \log \left(2^{k+2-n}\, 2^{n2^{a(k-1)}} + 3 \right) \leq 2 + 2^{n+2} \sum_{k=2}^{n} 2^{-k} \log \left(2^{k+3}\, 2^{n2^{ak}} \right) \\ &\leq 2 + 2^{n+2} \sum_{k=2}^{n} 2^{-k} (k + 3 + n2^{ak}) \leq 2 + 2^{n+2} \sum_{k=2}^{n} 3n (2^{a-1})^k \leq c_2\, n 2^n \end{aligned}$$

on the other hand, where $c_1, c_2 > 0$ are constants depending on a. We may assume that c_2 is a natural number. Since $2^{\left\lfloor n2^{a(i-1)} \right\rfloor} \leq \left\lfloor 2^{n2^{a(i-1)}} \right\rfloor$ for $i \in \mathbb{N}$, we have that $\varepsilon_{\alpha_i}(A) \leq e_{\left\lfloor n2^{a(i-1)} \right\rfloor + 1}(A)$. Consequently, we obtain the estimate

$$\begin{aligned} \left(\sum_{k=1}^{n} \left(2^{k/p'} \sum_{i=k}^{n} \varepsilon_{\alpha_i}(A) \right)^p \right)^{1/p} &\leq \sum_{k=1}^{n} 2^{k/p'} \sum_{i=k}^{n} e_{\left\lfloor n2^{a(i-1)} \right\rfloor + 1}(A) \\ &= \sum_{i=1}^{n} e_{\left\lfloor n2^{a(i-1)} \right\rfloor + 1}(A) \sum_{k=1}^{i} 2^{k/p'} \\ &\leq c_3 \sum_{i=1}^{n} 2^{i/p'} e_{\left\lfloor n2^{a(i-1)} \right\rfloor + 1}(A), \end{aligned}$$

where $c_3 > 0$ only depends on p. Hence, Theorem 1.3.3 yields

$$\begin{aligned} e_{2c_2 n 2^n}(\mathrm{aco}(A)) &\le e_{2m}(\mathrm{aco}(A)) \\ &\le c_4\,(n2^n)^{-1/t-1/p'} \sup_{1\le i\le 2^n} i^{1/t}\varepsilon_i(A) + c_5\, 2^{-n/p'} \sum_{i=1}^{n} 2^{i/p'} e_{\lfloor n2^{a(i-1)}\rfloor+1}(A) \end{aligned}$$

for all $n \ge 2$ and all $t \in (0,\infty)$, where $c_4, c_5 > 0$ do not depend on n and A. The next step is to obtain an estimate for $e_{2c_2 2^n}(\mathrm{aco}(A))$. Given $n \ge 2$, choose $l \in \mathbb{N}$ such that $2^l \le 2n2^n < 2^{l+1}$. Then, clearly, it holds that $n \le l \le 2n$ and consequently $2^n \ge \frac{2^l}{2n} \ge \frac{2^l}{2l}$. Furthermore, we observe that $e_{\lfloor n2^{a(i-1)}\rfloor+1}(A) \le e_{\lfloor l2^{a(i-1)-1}\rfloor+1}(A)$. Hence we get the estimate

$$\begin{aligned} e_{c_2 2^{l+1}}(\mathrm{aco}(A)) \le e_{2c_2 n2^n}(\mathrm{aco}(A)) \le c_4 \left(\frac{2^l}{2}\right)^{-1/t-1/p'} &\sup_{i\le \frac{2^{l+1}}{l}} i^{1/t}\varepsilon_i(A) \\ + c_5 \left(\frac{2^l}{2l}\right)^{-1/p'} &\sum_{i=1}^{l} 2^{i/p'} e_{\lfloor l2^{a(i-1)-1}\rfloor+1}(A). \end{aligned}$$

Note that if $2^l \le 2n2^n < 2^{l+1}$, then $2^{l+1} \le 2(n+1)2^{n+1} < 2^{l+3}$. Hence, there are constants $c_6, c_7 > 0$ independent of n and A such that

$$e_{2c_2 2^n}(\mathrm{aco}(A)) \le c_6\, 2^{-n/t-n/p'} \sup_{1\le i\le 2^n} i^{1/t}\varepsilon_i(A) + c_7\, n^{1/p'} 2^{-n/p'} \sum_{i=1}^{n} 2^{i/p'} e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)$$

for all $n \ge 2$ and all $t \in (0,\infty)$. According to Lemma 2.3.3 the assertion is equivalent to

$$\Big(n^\alpha\, 2^{n/p'}\, e_{2^n}(\mathrm{aco}(A))\Big)_n \in l_s, \quad \alpha := \frac{1}{r} - \frac{1}{p'} - \frac{1}{s},$$

and it suffices to show $\Big(n^\alpha\, 2^{n/p'} e_{2c_2 2^n}(\mathrm{aco}(A))\Big)_n \in l_s$. To this end we check that

(1) $\left(n^\alpha\, 2^{-n/t} \sup_{1\le i\le 2^n} i^{1/t}\varepsilon_i(A)\right)_n \in l_s$

(2) $\left(n^{\alpha+1/p'} \sum_{i=1}^{n} 2^{i/p'} e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)_n \in l_s$

for a suitable $t \in (0,\infty)$.

First let us deal with (1). Since $\big(e_n(A)\big)_n \in l_{r,s} \subset l_{r,\infty}$, there exists a constant $c_8 > 0$ such that $\varepsilon_n(A) \le c_8\,(\log(n+1))^{-1/r}$ for all $n \in \mathbb{N}$. We fix $t \in (0, r]$ and obtain

$$\sup_{1\le i\le 2^n} i^{1/t}\varepsilon_i(A) \le c_8 \sup_{1\le i\le 2^n} \frac{i^{1/t}}{(\log(i+1))^{1/r}} \le c_9\, n^{-1/r}\, 2^{n/t}.$$

Hence we have that

$$n^\alpha\, 2^{-n/t} \sup_{1\le i\le 2^n} i^{1/t}\varepsilon_i(A) \le c_9\, n^{\alpha-1/r} = c_9\, n^{-1/p'-1/s}$$

which yields

$$\sum_n \left(n^\alpha\, 2^{-n/t} \sup_{1\le i\le 2^n} i^{1/t}\varepsilon_i(A)\right)^s \le c_9^s \sum_n n^{-s/p'-1} < \infty.$$

The proof of (2) is more technical. Let $\varepsilon > 0$ be arbitrary. First of all we show that

$$\left(\sum_{i=1}^n 2^{i/p'} e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)^s \le c_{10} \sum_{i=1}^n 2^{ib} \left(e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)^s,$$

where $b := (1+\varepsilon)\frac{s}{p'}$ and $c_{10} > 0$ only depends on s and p. In the case $0 < s \le 1$ this is obvious since

$$\left(\sum_{i=1}^n 2^{i/p'} e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)^s \le \sum_{i=1}^n 2^{is/p'} \left(e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)^s.$$

If $1 < s < \infty$ we use Hölder's inequality to see that

$$\begin{aligned}\left(\sum_{i=1}^n 2^{i/p'} e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)^s &= \left(\sum_{i=1}^n \left(2^{is/p'-ib}\, 2^{ib} \left(e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)^s\right)^{1/s}\right)^s \\ &\le \left(\sum_{i=1}^n 2^{(is/p'-ib)/(s-1)}\right)^{s-1} \sum_{i=1}^n 2^{ib} \left(e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)^s.\end{aligned}$$

Since $(s/p'-b)/(s-1) < 0$, we obtain the desired result. Consequently, we get

$$\begin{aligned}\sum_{n=1}^N \left(n^{\alpha+1/p'} \sum_{i=1}^n 2^{i/p'} e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)^s &\le c_{10} \sum_{n=1}^N n^{s/r-1} \sum_{i=1}^n 2^{ib} \left(e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)^s \\ &= c_{10} \sum_{i=1}^N 2^{ib} \sum_{n=i}^N n^{s/r-1} \left(e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)^s.\end{aligned}$$

In a last step we check that

$$\sum_{i=1}^N 2^{ib} \sum_{n=i}^N n^{s/r-1} \left(e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)^s \le c_{11} \sum_{i=1}^N 2^{ib-ias/r} \sum_{n=1}^\infty n^{s/r-1}\big(e_n(A)\big)^s,$$

where $c_{11} > 0$ is a constant independent of A and N. For the sake of simplicity we introduce the notation $a_i := a(i-1)-1$. First observe that there exists a constant $C_1(r,s) > 0$ such that for all $n, i \in \mathbb{N}$ with $n \geq i$ it holds $(n2^{a_i})^{s/r-1} \leq C_1(r,s)\,(\lfloor n2^{a_i} \rfloor + 1)^{s/r-1}$. Hence, we have that

$$\sum_{n=i}^{N} n^{s/r-1} \left(e_{\lfloor n2^{a_i}\rfloor+1}(A)\right)^s = 2^{-a_i(s/r-1)} \sum_{n=i}^{N} (n2^{a_i})^{s/r-1} \left(e_{\lfloor n2^{a_i}\rfloor+1}(A)\right)^s$$
$$\leq C_2(r,s)\, 2^{-ia(s/r-1)} \sum_{n=i}^{N} (\lfloor n2^{a_i}\rfloor + 1)^{s/r-1} \left(e_{\lfloor n2^{a_i}\rfloor+1}(A)\right)^s.$$

Next, due to the monotonicity of the entropy numbers, we have that

$$\left(e_{\lfloor n2^{a_i}\rfloor+1}(A)\right)^s \leq \frac{\left(e_{\lfloor (n-1)2^{a_i}\rfloor+1}(A)\right)^s + \left(e_{\lfloor (n-1)2^{a_i}\rfloor+2}(A)\right)^s + \ldots + \left(e_{\lfloor n2^{a_i}\rfloor+1}(A)\right)^s}{\lfloor n2^{a_i}\rfloor - \lfloor (n-1)2^{a_i}\rfloor + 1}$$

and from

$$\lfloor n2^{a_i}\rfloor - \lfloor (n-1)2^{a_i}\rfloor + 1 \geq n2^{a_i} - (n-1)2^{a_i} = 2^{a_i}$$

we conclude

$$\left(e_{\lfloor n2^{a_i}\rfloor+1}(A)\right)^s \leq 2^{-a_i} \left(\left(e_{\lfloor n2^{a_i}\rfloor+1}(A)\right)^s + \sum_{k=\lfloor (n-1)2^{a_i}\rfloor+1}^{\lfloor n2^{a_i}\rfloor} \left(e_k(A)\right)^s \right).$$

In addition we trivially have

$$\left(e_{\lfloor n2^{a_i}\rfloor+1}(A)\right)^s \leq \sum_{k=\lfloor (n-1)2^{a_i}\rfloor+1}^{\lfloor n2^{a_i}\rfloor} \left(e_k(A)\right)^s$$

and putting both estimates together leads to

$$\left(e_{\lfloor n2^{a_i}\rfloor+1}(A)\right)^s \leq 2^{1-a_i} \sum_{k=\lfloor (n-1)2^{a_i}\rfloor+1}^{\lfloor n2^{a_i}\rfloor} \left(e_k(A)\right)^s$$
$$\leq C_3\, 2^{-ia} \sum_{k=\lfloor (n-1)2^{a_i}\rfloor+1}^{\lfloor n2^{a_i}\rfloor} \left(e_k(A)\right)^s,$$

where $C_3 > 0$ is an absolute constant. Consequently, we get

$$\sum_{i=1}^{N} 2^{ib} \sum_{n=i}^{N} n^{s/r-1} \left(e_{\left\lfloor n2^{a(i-1)-1}\right\rfloor+1}(A)\right)^s$$
$$\leq C_2(r,s) \sum_{i=1}^{N} 2^{ib-ia(s/r-1)} \sum_{n=i}^{N} (\lfloor n2^{a_i}\rfloor + 1)^{s/r-1} \left(e_{\lfloor n2^{a_i}\rfloor+1}(A)\right)^s$$
$$\leq C_4(r,s) \sum_{i=1}^{N} 2^{ib-ias/r} \sum_{n=i}^{N} \sum_{k=\lfloor (n-1)2^{a_i}\rfloor+1}^{\lfloor n2^{a_i}\rfloor} (\lfloor n2^{a_i}\rfloor + 1)^{s/r-1} \left(e_k(A)\right)^s.$$

Now we claim that there exists a constant $C_5(r,s) > 0$ such that for all $n, i \in \mathbb{N}$ with $n \geq i$ it holds $(\lfloor n2^{a_i} \rfloor + 1)^{s/r-1} \leq C_5(r,s)\, k^{s/r-1}$. If $s/r - 1 < 0$ then this is obvious since $k \leq \lfloor n2^{a_i} \rfloor + 1$. To handle the case $s/r - 1 \geq 0$ we observe that

$$\frac{\lfloor n2^{a_i} \rfloor + 1}{\lfloor (n-1)2^{a_i} \rfloor + 1} \leq \frac{n2^{a_i} + 1}{(n-1)2^{a_i}} = \frac{n}{n-1} + \frac{1}{(n-1)2^{a_i}} \leq 4$$

for all $n \geq 2$ and all $i \in \mathbb{N}$. Therefore, we obtain

$$4k \geq 4\left(\lfloor (n-1)2^{a_i} \rfloor + 1\right) \geq \lfloor n2^{a_i} \rfloor + 1 \tag{4.1.1}$$

and hence the assertion for all $n \geq i \geq 2$. Thus, left open is the case $i = 1$. Since $2^{a_1} = \frac{1}{2}$, we have to show that

$$\left(\lfloor n/2 \rfloor + 1\right)^{s/r-1} \leq C_6(r,s)\, k^{s/r-1}, \quad s/r - 1 \geq 0,$$

for all $n \in \mathbb{N}$, where $\lfloor (n-1)/2 \rfloor + 1 \leq k \leq \lfloor n/2 \rfloor$. However, this is clear since $k \geq \lfloor (n-1)/2 \rfloor + 1 \geq \frac{1}{2}\left(\lfloor n/2 \rfloor + 1\right)$ for all $n \in \mathbb{N}$. Hence, we conclude that

$$\begin{aligned}
&\sum_{i=1}^{N} 2^{ib} \sum_{n=i}^{N} n^{s/r-1} \Big(e_{\lfloor n2^{a(i-1)-1} \rfloor + 1}(A)\Big)^s \\
&\qquad \leq C_7(r,s) \sum_{i=1}^{N} 2^{ib-ias/r} \sum_{n=i}^{N} \sum_{k=\lfloor (n-1)2^{a_i} \rfloor + 1}^{\lfloor n2^{a_i} \rfloor} k^{s/r-1} \Big(e_k(A)\Big)^s \\
&\qquad \leq C_7(r,s) \sum_{i=1}^{N} 2^{ib-ias/r} \sum_{m=1}^{\infty} m^{s/r-1} \Big(e_m(A)\Big)^s,
\end{aligned}$$

which is the desired estimate.

Therefore, putting all these estimates together we find that

$$\sum_{n=1}^{N} \left(n^{\alpha+1/p'} \sum_{i=1}^{n} 2^{i/p'} e_{\lfloor n2^{a(i-1)-1} \rfloor + 1}(A) \right)^s \leq c_{12} \sum_{i=1}^{N} 2^{ib-ias/r} \sum_{n=1}^{\infty} n^{s/r-1} \Big(e_n(A)\Big)^s$$

with a constant $c_{12} > 0$ independent of A and N. Recall that $\frac{1}{2} < a < 1$ and $b = (1+\varepsilon)\frac{s}{p'}$ with $\varepsilon > 0$ arbitrary. Now fix $\varepsilon > 0$ such that $(1+\varepsilon)\frac{r}{p'} < 1$. Then we can choose a as a constant satisfying both $\frac{1}{2} < a < 1$ and $a > (1+\varepsilon)\frac{r}{p'}$. Consequently, we have $b - as/r < 0$, which yields the assertion.

For an arbitrary precompact subset A of X we define the symmetrization of A by $A_S := A \cup (-A)$. Obviously, A_S is a precompact and symmetric subset of X satisfying $e_{n+1}(A_S) \leq e_n(A)$ for all $n \in \mathbb{N}$ and $\operatorname{aco}(A_S) = \operatorname{aco}(A)$. Hence, we get

$$\Big(e_n(A)\Big)_n \in l_{r,s} \quad \Rightarrow \quad \Big(e_n(A_S)\Big)_n \in l_{r,s} \quad \Rightarrow \quad \Big(e_n(\operatorname{aco}(A))\Big)_n \in l_{p',s,\varphi}.$$

This finishes the proof.

Finally we deal with the optimality of the result. Let $\psi(n) = (\log(n+1))^\beta$ with $\beta > \alpha = \frac{1}{r} - \frac{1}{p'} - \frac{1}{s}$. Choose γ with $\gamma s > 1$ and consider

$$A = \{\sigma_n u_n \mid n \in \mathbb{N}\} \subset l_p$$

with $\sigma_n = (\log(n+1))^{-1/r}(\log\log(n+3))^{-\gamma}$. Taking Lemma 2.4.5 into account, we obtain

$$e_n(A) \le \sigma_{2^{n-1}} \le C_8(r,s)\, n^{-1/r}\, (\log(n+1))^{-\gamma}$$

and hence it holds

$$\sum_n n^{s/r-1}\Big(e_n(A)\Big)^s \le C_8^s(r,s) \sum_n n^{-1}\, (\log(n+1))^{-\gamma s}.$$

Since the latter series is convergent, we see that $\Big(e_n(A)\Big)_n \in l_{r,s}$. Furthermore, Lemma 2.4.5 yields

$$e_n(\mathrm{aco}(A)) \ge C_9(p,r,s)\, n^{-1/p'}\, (\log(n+1))^{1/p'-1/r}\, (\log\log(n+3))^{-\gamma}$$

and consequently we have

$$\begin{aligned} &\sum_n (\log(n+1))^{\beta s}\, n^{s/p'-1}\, \Big(e_n(\mathrm{aco}(A))\Big)^s \\ &\qquad \ge C_9^s(p,r,s) \sum_n n^{-1}\, (\log(n+1))^{s(\beta+1/p'-1/r)}\, (\log\log(n+3))^{-\gamma s} \\ &\qquad = C_9^s(p,r,s) \sum_n n^{-1}\, (\log(n+1))^{s(\beta-\alpha)-1}\, (\log\log(n+3))^{-\gamma s} = \infty, \end{aligned}$$

which means $\Big(e_n(\mathrm{aco}(A))\Big)_n \notin l_{p',s,\psi}$. ■

4.2 Proof of Theorem 1.3.5

The proof is based on Theorem 1.3.1 *(iii)* in combination with the Hardy-type inequalities from Chapter 3. Choose a constant t with $p' < t < r$. Then according to Theorem 1.3.1 *(iii)* it holds

$$n^{1/t} e_n(\mathrm{aco}(A)) \le \sup_{1 \le k \le n} k^{1/t} e_k(\mathrm{aco}(A)) \le c_1\, c_A \sup_{1 \le k \le n} k^{1/t} e_k(A),$$

where $c_1 > 0$ only depends on p, t and X. In addition, due to the monotonicity of the entropy numbers we have that

$$\sup_{1 \le k \le n} k^{1/t} e_k(A) \le \left(\sum_{k=1}^n \Big(e_k(A)\Big)^t \right)^{1/t}$$

and this gives the estimate

$$e_n(\mathrm{aco}(A)) \le c_1\, c_A \left(\frac{1}{n}\sum_{k=1}^{n}\Big(e_k(A)\Big)^t\right)^{1/t}.$$

Consequently, in the case $0 < s < \infty$ we obtain

$$\sum_{n=1}^{N}(\varphi(n))^s\, n^{s/r-1}\Big(e_n(\mathrm{aco}(A))\Big)^s \le c_1^s\, c_A^s \sum_{n=1}^{N}(\varphi(n))^s\, n^{s/r-1}\left(\frac{1}{n}\sum_{k=1}^{n}\Big(e_k(A)\Big)^t\right)^{s/t}$$

and since $t < r$, we can use Lemma 3.0.8 to get

$$\sum_{n=1}^{N}(\varphi(n))^s\, n^{s/r-1}\Big(e_n(\mathrm{aco}(A))\Big)^s \le c_2\, c_A^s \sum_{n=1}^{N}(\varphi(n))^s\, n^{s/r-1}\Big(e_n(A)\Big)^s,$$

where $c_2 > 0$ is a constant depending on p, r, s, φ and X. The second inequality can be treated analogously using Lemma 3.0.9. ■

4.3 Proof of Theorem 1.3.6

Let $A \subset X$ be a precompact subset of a Banach space X of type p satisfying $e_n(A) \preccurlyeq n^{-1/p'}(\log(n+1))^{-\beta}$ with $-\infty < \beta < 1/p$. We may assume that A is symmetric. For $n \ge 2$ and $k = 1, 2, \dots, n$ we define $\alpha_k := 2^{2^k}$ and

$$m := \left\lfloor 2^{n+2}\sum_{k=2}^{n} 2^{-k}\log\left(\frac{2^{k+2}\alpha_k}{2^n} + 3\right)\right\rfloor + 2.$$

Then the estimates

$$\begin{aligned} m &\ge 2^{n+2}\sum_{k=2}^{n} 2^{-k}\,\log\left(2^{k+2+2^k-n}\right) = 2^{n+2}\left(\frac{n}{2} + \frac{3}{2} - \frac{4}{2^n}\right)\\ &= 2n2^n + 6\cdot 2^n - 16 \ge 2n2^n \end{aligned}$$

and

$$\begin{aligned} m &\le 2 + 2^{n+2}\sum_{k=2}^{n} 2^{-k}\,\log\left(2^{k+2+2^k}\right) = 2 + 2^{n+2}\left(n + \frac{3}{2} - \frac{n+4}{2^n}\right)\\ &= 4n2^n + 6\cdot 2^n - 4n - 14 \le 6n2^n \end{aligned}$$

hold. Consequently, applying Steinwart's inequality from Theorem 1.3.3 with $t = p'$ yields

$$\begin{aligned} e_{12n2^n}(\mathrm{aco}(A)) &\le e_{2m}(\mathrm{aco}(A))\\ &\le c_1(p,\beta)\,(n2^n)^{-2/p'} + c_2(X,p)\,2^{-n/p'}\left(\sum_{k=1}^{n}\left(2^{k/p'}\sum_{i=k}^{n}\varepsilon_{\alpha_i}(A)\right)^p\right)^{1/p} \end{aligned} \tag{4.3.1}$$

for all $n \geq 2$. Observe that

$$\varepsilon_{\alpha_i}(A) = e_{2^i+1}(A) \preccurlyeq 2^{-i/p'}\, i^{-\beta}.$$

Hence, since $\beta < 1/p$, applying Lemma 3.0.4 and Corollary 3.0.6 gives

$$\begin{aligned}\left(\sum_{k=1}^{n}\left(2^{k/p'}\sum_{i=k}^{n}\varepsilon_{\alpha_i}(A)\right)^{p}\right)^{1/p} &\preccurlyeq \left(\sum_{k=1}^{n}\left(2^{k/p'}\sum_{i=k}^{\infty}2^{-i/p'}\, i^{-\beta}\right)^{p}\right)^{1/p} \\ &\preccurlyeq \left(\sum_{k=1}^{n}k^{-\beta p}\right)^{1/p} \preccurlyeq n^{1/p-\beta}.\end{aligned}$$

Consequently, we get

$$e_{12n2^n}(\mathrm{aco}(A)) \leq c_1(p,\beta)\,(n2^n)^{-2/p'} + c_3(X,p,\beta)\,2^{-n/p'}\,n^{1/p-\beta}$$

for all $n \geq 2$. Changing from $12n2^n$ to 2^n gives

$$\begin{aligned}e_{2^n}(\mathrm{aco}(A)) &\leq c_4(p,\beta)\,(2^n)^{-2/p'} + c_5(X,p,\beta)\,\left(\frac{2^n}{n}\right)^{-1/p'}\,n^{1/p-\beta} \\ &= c_4(p,\beta)\,2^{-2n/p'} + c_5(X,p,\beta)\,2^{-n/p'}\,n^{1-\beta} \\ &\leq c_6(X,p,\beta)\,2^{-n/p'}\,n^{1-\beta}\end{aligned}$$

for all $n \geq 7$. This shows that

$$e_n(\mathrm{aco}(A)) \preccurlyeq n^{-1/p'}\,(\log(n+1))^{1-\beta}.$$

The optimality statement (1.3.8) was proved by Creutzig and Steinwart in [CrSt02, Th. 1.5].

Now let us deal with the case $\left(e_n(A)\right)_n \in l_{p',s}$ with $p \leq s < \infty$. According to Lemma 2.3.3 it is enough to prove that

$$e_{2^n}(\mathrm{aco}(A)) \preccurlyeq 2^{-n/p'}\,n^{1-1/s}.$$

From (4.3.1) we know that

$$\begin{aligned}n^{1/s-1}\,(n2^n)^{1/p'}\,e_{12n2^n}(\mathrm{aco}(A)) \leq{}& c_1(p,\beta)\,n^{1/s-1-1/p'}\,2^{-n/p'} \\ &+ c_2(X,p)\,n^{1/s-1/p}\left(\sum_{k=1}^{n}\left(2^{k/p'}\sum_{i=k}^{n}\varepsilon_{\alpha_i}(A)\right)^{p}\right)^{1/p}\end{aligned} \tag{4.3.2}$$

for all $n \geq 2$. It is clear that the sequence $\left(n^{1/s-1-1/p'}\,2^{-n/p'}\right)_n$ tends to zero and thus is bounded. Now let us consider the second summand. Since $p \leq s$ it holds that

$$n^{1/s-1/p}\left(\sum_{k=1}^{n}\left(2^{k/p'}\sum_{i=k}^{n}\varepsilon_{\alpha_i}(A)\right)^{p}\right)^{1/p} \leq \left(\sum_{k=1}^{n}\left(2^{k/p'}\sum_{i=k}^{n}\varepsilon_{\alpha_i}(A)\right)^{s}\right)^{1/s}.$$

Let α be a positive constant which will be specified later. Using Hölder's inequality we obtain

$$\left(\sum_{i=k}^{n} \varepsilon_{\alpha_i}(A)\right)^s \le \left(\sum_{i=k}^{n} e_{2^i}(A)\right)^s \le \sum_{i=k}^{n} \left(2^{i\alpha}\, e_{2^i}(A)\right)^s \left(\sum_{i=k}^{n} (2^{-i\alpha})^{s'}\right)^{s/s'}$$
$$\le c_1(\alpha, s)\, 2^{-ks\alpha} \sum_{i=k}^{n} 2^{is\alpha} \left(e_{2^i}(A)\right)^s$$

and therefore

$$\left(\sum_{k=1}^{n} \left(2^{k/p'} \sum_{i=k}^{n} \varepsilon_{\alpha_i}(A)\right)^s\right)^{1/s} \preccurlyeq \left(\sum_{k=1}^{n} 2^{ks(1/p'-\alpha)} \sum_{i=k}^{n} 2^{is\alpha} \left(e_{2^i}(A)\right)^s\right)^{1/s}.$$

Changing the order of summation gives

$$\left(\sum_{k=1}^{n} 2^{ks(1/p'-\alpha)} \sum_{i=k}^{n} 2^{is\alpha} \left(e_{2^i}(A)\right)^s\right)^{1/s} = \left(\sum_{i=1}^{n} 2^{is\alpha} \left(e_{2^i}(A)\right)^s \sum_{k=1}^{i} 2^{ks(1/p'-\alpha)}\right)^{1/s}.$$

Now choose α as a constant satisfying $0 < \alpha < 1/p'$. Then

$$\sum_{k=1}^{i} 2^{ks(1/p'-\alpha)} \le c_2(\alpha, s, p)\, 2^{is(1/p'-\alpha)}$$

and, consequently, we obtain

$$\left(\sum_{i=1}^{n} 2^{is\alpha} \left(e_{2^i}(A)\right)^s \sum_{k=1}^{i} 2^{ks(1/p'-\alpha)}\right)^{1/s} \preccurlyeq \left(\sum_{i=1}^{n} 2^{is/p'} \left(e_{2^i}(A)\right)^s\right)^{1/s}.$$

Hence, we have found out that

$$n^{1/s-1/p} \left(\sum_{k=1}^{n} \left(2^{k/p'} \sum_{i=k}^{n} \varepsilon_{\alpha_i}(A)\right)^p\right)^{1/p} \preccurlyeq \left(\sum_{i=1}^{n} 2^{is/p'} \left(e_{2^i}(A)\right)^s\right)^{1/s}$$

and $\left(e_n(A)\right)_n \in l_{p',s}$ implies the boundedness of the sequence on the left hand side (cf. Lemma 2.3.3). In view of (4.3.2) we see that the sequence

$$\left(n^{1/s-1} \,(n2^n)^{1/p'}\, e_{12n2^n}(\mathrm{aco}(A))\right)_n$$

is bounded, i.e.

$$e_{12n2^n}(\mathrm{aco}(A)) \le c(\alpha, s, p, X)\, (n2^n)^{-1/p'}\, n^{1-1/s}$$

for all $n \in \mathbb{N}$. Finally, choosing $\alpha = 1/(2p')$, for example, and changing from $12n2^n$ to 2^n gives

$$e_{2^n}(\mathrm{aco}(A)) \leq \tilde{c}(s,p,X)\, 2^{-n/p'}\, n^{1-1/s}$$

and finishes the proof.

Now let us deal with the remaining optimality statements. Let $1 < s < \infty$ and $\beta > -1 + 1/s$. Choose a constant γ satisfying $1/s < \gamma < \min\{1; \beta + 1\}$. According to (1.3.8) we can find a precompact subset A of X satisfying both

$$(e_n(A))_n \preccurlyeq n^{-1/p'}\, (\log(n+1))^{-\gamma} \quad \text{and} \quad e_n(\mathrm{aco}(A)) \succcurlyeq n^{-1/p'}\, (\log(n+1))^{1-\gamma}.$$

Since $\gamma s > 1$, we conclude that $\Big(e_n(A)\Big)_n \in l_{p',s}$. Furthermore, since $\beta + 1 - \gamma > 0$, it holds that

$$\sup_{n\in\mathbb{N}} n^{1/p'}\, (\log(n+1))^{\beta}\, e_n(\mathrm{aco}(A)) \succcurlyeq \sup_{n\in\mathbb{N}} (\log(n+1))^{\beta+1-\gamma} = \infty.$$

Finally, we deal with the last statement. Let $0 < s < \infty$ and choose γ such that $s\gamma > 1$. For $1 < p \leq 2$, consider the subset

$$A = \{\sigma_n u_n \,|\, n \in \mathbb{N}\} \subset l_p$$

with $\sigma_n = (\log(n+1))^{-1/p'}\, (\log\log(n+3))^{-\gamma}$. Then applying Lemma 2.4.5 yields

$$e_n(A) \leq \sigma_{2^{n-1}} \preccurlyeq n^{-1/p'}\, (\log(n+1))^{-\gamma}$$

and

$$e_n(\mathrm{aco}(A)) \succcurlyeq n^{-1/p'}\, (\log(n+1))^{1/p'} \sigma_{n^2} \succcurlyeq n^{-1/p'}\, (\log\log(n+3))^{-\gamma}.$$

Hence, we compute that

$$\sum_n n^{s/p'-1} \Big(e_n(A)\Big)^s \preccurlyeq \sum_n n^{-1}\, (\log(n+1))^{-\gamma s} < \infty$$

and, for $0 < t < \infty$,

$$\sum_n n^{t/p'-1} \Big(e_n(\mathrm{aco}(A))\Big)^t \succcurlyeq \sum_n n^{-1}\, (\log\log(n+3))^{-\gamma t} = \infty.$$

This shows $\Big(e_n(A)\Big)_n \in l_{p',s}$ but $\Big(e_n(\mathrm{aco}(A))\Big)_n \notin l_{p',t}$ for $0 < t < \infty$. ■

Remark 3. Unfortunately, the proof presented above does not work in the remaining case $\Big(e_n(A)\Big)_n \in l_{p',s}$ for $s < p$. Looking back, the crucial step of the proof was to show the boundedness of the sequence

$$\left(n^{1/s-1/p} \left(\sum_{k=1}^{n} \left(2^{k/p'} \sum_{i=k}^{n} \varepsilon_{\alpha_i}(A) \right)^p \right)^{1/p} \right)_n,$$

where $a_i = 2^{2^i}$. However, for $s < p$ it appears that, in general, this sequence is not bounded. To give an example, we consider a precompact subset A with $e_n(A) \sim n^{-2/p'}$. Then we have that

$$\sum_n n^{s/p'-1} \left(e_n(A)\right)^s \sim \sum_n n^{-s/p'-1} < \infty$$

and this shows $\left(e_n(A)\right)_n \in l_{p',s}$. Furthermore, from

$$\varepsilon_{\alpha_i}(A) = e_{2^i+1}(A) \sim 2^{-2i/p'}$$

we conclude that

$$\begin{aligned}\left(\sum_{k=1}^n \left(2^{k/p'} \sum_{i=k}^n \varepsilon_{\alpha_i}(A)\right)^p\right)^{1/p} &\sim \left(\sum_{k=1}^n \left(2^{k/p'} \sum_{i=k}^n 2^{-2i/p'}\right)^p\right)^{1/p} \\ &\sim \left(\sum_{k=1}^n 2^{-kp/p'}\right)^{1/p} \sim 1.\end{aligned}$$

This yields

$$n^{1/s-1/p} \left(\sum_{k=1}^n \left(2^{k/p'} \sum_{i=k}^n \varepsilon_{\alpha_i}(A)\right)^p\right)^{1/p} \sim n^{1/s-1/p}$$

and shows that the sequence is not bounded for $s < p$.

4.4 Proof of Proposition 1.3.7

Since $1 < s < \infty$, we can apply Theorem 1.3.2 with $\beta = 1/s$ to obtain

$$\begin{aligned}&\sup_{1\le k\le n} k^{1/p'} \left(\log(k+1)\right)^{1/s-1} e_k(\mathrm{aco}(A)) \\ &\qquad \le c_1(p,s,X)\, c_A \sup_{1\le k\le n} k^{1/p'} \left(\log(k+1)\right)^{1/s} e_k(A).\end{aligned}$$

Due to the monotonicity of $\left(n^{1/p'} e_n(A)\right)_n$ we have that

$$\begin{aligned}\left(\sum_{i=1}^k i^{s/p'-1}\left(e_i(A)\right)^s\right)^{1/s} &= \left(\sum_{i=1}^k i^{-1} \left(i^{1/p'} e_i(A)\right)^s\right)^{1/s} \ge k^{1/p'} e_k(A) \left(\sum_{i=1}^k i^{-1}\right)^{1/s} \\ &\ge c_2(s)\, k^{1/p'} \left(\log(k+1)\right)^{1/s} e_k(A).\end{aligned}$$

This gives

$$\sup_{1\le k\le n} \left(\log(k+1)\right)^{1/s-1} k^{1/p'} e_k(\mathrm{aco}(A)) \le c_3(p,s,X)\, c_A \left(\sum_{i=1}^n i^{s/p'-1}\left(e_i(A)\right)^s\right)^{1/s}$$

and finishes the proof. ■

4.5 Proof of Theorem 1.3.8

For short we write $f(\varepsilon) := \varepsilon^{-p'} \left(\log \frac{1}{\varepsilon}\right)^{-p'}$. Let $c_0 > 0$ be a constant such that

$$\ln N(A, \varepsilon) \leq c_0 \, f(\varepsilon)$$

for all $0 < \varepsilon \leq 1/2$. For fixed $\gamma \in [1/2, 2/3]$ and $n \in \mathbb{N}$ we define $\varepsilon_0 := \gamma 2^{-n}$. By the definition of covering numbers there exist $\gamma 2^{-k}$-nets $N_k \subset X$ of A with cardinality

$$|N_k| \leq \exp\left[c_0 f(\gamma 2^{-k})\right], \quad k = 1, 2, \ldots, n.$$

We define sets $D_1 := N_1$ and

$$D_k := \left\{z \in N_k - N_{k-1} : \|z\| \leq 3\gamma 2^{-k}\right\}, \quad k = 2, 3, \ldots, n.$$

Then

$$|D_k| \leq |N_k||N_{k-1}| \leq \exp\left[2c_0 f(\gamma 2^{-k})\right], \quad k = 2, 3, \ldots, n,$$

and for $D'_k := D_k \cup (-D_k) \cup \{0\}$ we obtain

$$|D'_k| \leq 3|D_k| \leq 3 \exp\left[2c_0 f(\gamma 2^{-k})\right], \quad k = 1, 2, \ldots, n.$$

Observe that there is a constant $c_1 = c_1(p) > 2c_0$ with $\log(3) + 2c_0 f(\gamma 2^{-k}) \leq c_1 f(\gamma 2^{-k})$ for all $k = 1, 2, \ldots, n$. Consequently, we have that

$$|D'_k| \leq \exp\left[c_1 f(\gamma 2^{-k})\right], \quad k = 1, 2, \ldots, n. \tag{4.5.1}$$

Now we define sets

$$C_k := \operatorname{conv}(D'_k) = \operatorname{aco}(D_k) \quad \text{and} \quad E_n := \sum_{k=1}^{n} C_k.$$

The next step is to show that N_n is a subset of E_n. To this end, let $k \geq 2$ and $\eta_k \in N_k$. Choose an element $a \in A$ with $\|\eta_k - a\| \leq \gamma 2^{-k}$. Since N_{k-1} is a $\gamma 2^{-(k-1)}$-net of A, there is an element $\eta_{k-1} \in N_{k-1}$ with $\|\eta_{k-1} - a\| \leq \gamma 2^{-(k-1)}$. By triangle inequality we get

$$\|\eta_k - \eta_{k-1}\| \leq \gamma 2^{-k} + \gamma 2^{-(k-1)} = 3\gamma 2^{-k}.$$

Hence, $\eta_k - \eta_{k-1} \in D_k$. Now let $\eta_n \in N_n$. According to the above-mentioned considerations we can find elements $\eta_{n-1}, \eta_{n-2}, \ldots, \eta_1$ such that $\eta_k - \eta_{k-1} \in D_k$ for $k = n, n-1, \ldots, 2$. Hence we have that

$$\eta_n = \eta_n - \eta_{n-1} + \eta_{n-1} - \ldots + \eta_2 - \eta_1 + \eta_1 \in D_n + D_{n-1} + \ldots + D_1$$

and this shows $N_n \subset E_n$. Since N_n is an ε_0-net of A, we also have that E_n is an ε_0-net of A. Since E_n is absolutely convex, it is also an ε_0-net of $\mathrm{aco}(A)$. Hence, any ε_0-net of E_n is also an $2\varepsilon_0$-net of $\mathrm{aco}(A)$. We conclude that

$$\log N\Big(\mathrm{aco}(A), 2\varepsilon_0\Big) \le \log N(E_n, \varepsilon_0).$$

Define the set

$$M := \left\{ \sum_{k=1}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} d_{k,i} : d_{k,i} \in D'_k \right\},$$

where $m_1, \dots, m_n$ are natural numbers which will be specified later. Using an probabilistic argument due to Maurey (cf. [Pi81]) we will show that M is an ε_0-net of E_n. Denote the elements of $D'_k \setminus \{0\}$ with $x_1^k, \dots, x_{d_k}^k$. Fix $z \in E_n$ and write $z = \sum_{k=1}^n z_k$ with $z_k \in C_k$. Then each z_k can be represented by

$$z_k = \sum_{i=1}^{d_k} a_i^k x_i^k, \quad \text{where } a_i^k \ge 0 \text{ and } \sum_{i=1}^{d_k} a_i^k \le 1.$$

Define a random variable Z_k with values in D'_k by

$$\mathbb{P}(Z_k = x_i^k) = a_i^k \quad \text{for} \quad i = 1, \dots, d_k \quad \text{and} \quad \mathbb{P}(Z_k = 0) = 1 - \sum_{i=1}^{d_k} a_i^k.$$

We compute that $\mathbb{E}Z_k = \sum_{i=1}^{d_k} a_i^k x_i^k = z_k$. Moreover, take independent random variables $Z_{1,1}, \dots, Z_{1,m_1}, \dots, Z_{n,1}, \dots, Z_{n,m_n}$ where $Z_{k,1}, \dots, Z_{k,m_k}$ is distributed like Z_k for $k = 1, \dots, n$. With $Y_{k,i} := \frac{1}{m_k} Z_{k,i}$ and Theorem 2.6.4 we obtain

$$\begin{aligned}
\mathbb{E}\left\| z - \sum_{k=1}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} Z_{k,i} \right\| &= \mathbb{E}\left\| \sum_{k=1}^{n} \mathbb{E}Z_k - \sum_{k=1}^{n}\sum_{i=1}^{m_k} Y_{k,i} \right\| = \mathbb{E}\left\| \sum_{k=1}^{n}\sum_{i=1}^{m_k} (\mathbb{E}Y_{k,i} - Y_{k,i}) \right\| \\
&\le 4\tau_p(X) \left(\sum_{k=1}^{n}\sum_{i=1}^{m_k} \mathbb{E}\left\|Y_{k,i}\right\|^p \right)^{1/p} \qquad (4.5.2) \\
&\le 12\gamma\, \tau_p(X) \left(\sum_{k=1}^{n} \frac{1}{m_k^{p-1}} 2^{-kp} \right)^{1/p}.
\end{aligned}$$

Now the crucial step is the choice of the natural numbers m_k. From $\varepsilon_0 \le 2^{-n}$ we conclude that

$$\frac{k^{p'-1}\,(\log(n+1))^{p'/p}}{\varepsilon_0^{p'}\,2^{kp'}} \ge 1$$

for all $k = 1, \dots, n$. Hence, for each $k = 1, 2, \dots, n$ there is a natural number m_k satisfying

$$c_2 \frac{k^{p'-1}\,(\log(n+1))^{p'/p}}{\varepsilon_0^{p'}\,2^{kp'}} \le m_k < 2c_2 \frac{k^{p'-1}\,(\log(n+1))^{p'/p}}{\varepsilon_0^{p'}\,2^{kp'}},$$

where $c_2 := (8\,\tau_p(X))^{p'}$. Using $\frac{p-1}{p} = \frac{1}{p'}$ and $(p'-1)(p-1) = 1$ we can continue at (4.5.2) with

$$\mathbb{E}\left\| z - \sum_{k=1}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} Z_{k,i} \right\| \leq \varepsilon_0 \, \frac{12\gamma\,\tau_p(X)}{c_2^{1/p'}} \,(\log(n+1))^{-1/p} \left(\sum_{k=1}^{n} k^{-1} \right)^{1/p}$$
$$\leq \varepsilon_0 \, \frac{3\gamma}{2} \leq \varepsilon_0 .$$

Here we used that $\sum_{k=1}^{n} k^{-1} \leq \log(n+1)$ for $n \in \mathbb{N}$. Since $\sum_{k=1}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} Z_{k,i}$ only takes values in the set M, there exists an element $m \in M$ with $\|z - m\| \leq \varepsilon_0$. This shows that M is an ε_0-net of E_n. Consequently, by the very definition of covering numbers we obtain

$$\log N(E_n, \varepsilon_0) \leq \log |M|.$$

It remains to estimate the cardinality of M. In view of (4.5.1) we get

$$\log |M| \leq \log \left(\prod_{k=1}^{n} |D_k'|^{m_k} \right) = \sum_{k=1}^{n} m_k \, \log(|D_k'|) \leq \frac{c_1}{\ln 2} \sum_{k=1}^{n} m_k f(\gamma 2^{-k})$$
$$\leq c_3(p,X)\, \varepsilon_0^{-p'} \,(\log(n+1))^{p'/p} \sum_{k=1}^{n} k^{-1}$$
$$\leq c_3(p,X)\, \varepsilon_0^{-p'} \,(\log(n+1))^{p'}$$
$$\leq c_4(p,X)\, \varepsilon_0^{-p'} \left(\log\log \frac{1}{\varepsilon_0} \right)^{p'} .$$

This shows

$$\log N\Big(\operatorname{aco}(A), 2\varepsilon_0\Big) \leq c(p,X)\, \varepsilon_0^{-p'} \left(\log\log \frac{1}{\varepsilon_0} \right)^{p'}$$

for $\varepsilon_0 = \gamma 2^{-n}$ with $1/2 \leq \gamma \leq 2/3$ fix and $n \in \mathbb{N}$.

Now let $0 < \varepsilon \leq 1/10$ be arbitrary. Then there exists a natural number n with $\gamma 2^{-n} \leq \varepsilon < \gamma 2^{-n+1}$. We conclude that

$$\log N\Big(\operatorname{aco}(A), 2\varepsilon\Big) \leq \log N\Big(\operatorname{aco}(A), 2\gamma 2^{-n}\Big) \leq c(p,X)\,(\gamma 2^{-n})^{-p'} \left(\log\log \frac{1}{\gamma 2^{-n}} \right)^{p'}$$
$$\leq c_5(p,X)\,(2\varepsilon)^{-p'} \left(\log\log \frac{1}{2\varepsilon} \right)^{p'} .$$

This finishes the proof of Theorem 1.3.8. ■

Remark 4. Working with the *real* absolutely convex hull is crucial for the proof of Theorem 1.3.8. Indeed, the presented proof does not work for the complex absolutely convex hull. Let us discuss this in more detail: For the finite set D_k we defined

$D'_k := D_k \cup (-D_k) \cup \{0\}$. It is clear that also D'_k is finite and $|D'_k| \leq 3|D_k|$. We concluded that

$$\operatorname{aco}(D_k) = \operatorname{conv}(D'_k).$$

Note that this is a purely real argument, see (2.7.1) in Section 2.7. When dealing with the *complex* absolutely convex hull, the set D'_k has to be modified as follows:

$$D'_k = \bigcup_{\lambda \in \mathbb{C}:\, |\lambda|=1} \lambda D_k.$$

However, this leads to a massive problem: It is not clear how to estimate the number of elements in D'_k. We do not know how to treat the complex case.

4.6 Proof of Theorem 1.3.9

We cannot help the fact that the proof is very technical. We start with the definition of disjoint index sets

$$I_k := \left\{ i \in \mathbb{N} : \exp\left(2^{p'(k-1)}\right) \leq i < \exp\left(2^{p'k}\right) \right\}, \quad k = 1, 2, \ldots.$$

Furthermore, we define subset A_k of l_p by

$$A_k := \left\{ \left(k 2^k\right)^{-1} e_i : i \in I_k \right\} \cup \{0\}, \quad k = 1, 2, \ldots,$$

where $\{e_i\}_{i=1}^{\infty}$ denotes the canonical unit vector basis of the sequence space l_p. Finally, we define

$$A := A_1 + A_2 + \cdots + A_k + \cdots \subset l_p.$$

First, we prove that A satisfies (1.3.15). For any $0 < \varepsilon < 1/2$ there exists a natural number $n \geq 2$ with

$$(n 2^n)^{-1} \leq \varepsilon < \left[(n-1) 2^{n-1}\right]^{-1}. \tag{4.6.1}$$

For $S_n := A_1 + A_2 + \cdots + A_n$ we compute that

$$\begin{aligned} |S_n| &\leq \prod_{k=1}^{n} |A_k| \leq \prod_{k=1}^{n} \exp\left(2^{p'k}\right) = \exp\left(\sum_{k=1}^{n} 2^{p'k}\right) = \exp\left(\frac{2^{p'(n+1)} - 2^{p'}}{2^{p'} - 1}\right) \\ &\leq \exp\left(\frac{2^{p'(n+1)}}{2^{p'-1}}\right) = \exp\left(2^{p'n+1}\right). \end{aligned}$$

Furthermore, we claim that S_n is a $(n2^n)^{-1}$-net for the set A. To see this, let $a = a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots \in A$ with $a_k \in A_k$. For $s_n := a_1 + a_2 + \cdots + a_n \in S_n$ we compute that

$$\|a - s_n\|_p \leq \sum_{j=1}^{\infty} \|a_{n+j}\|_p \leq \sum_{j=1}^{\infty} \left[(n+j) 2^{n+j}\right]^{-1} \leq \sum_{j=1}^{\infty} \left(n 2^{n+j}\right)^{-1} = (n 2^n)^{-1}.$$

Hence, we have that S_n is a $(n2^n)^{-1}$-net for A consisting of at most $\exp\Big(2^{p'n+1}\Big)$ elements. In view of (4.6.1), we conclude that

$$\log N(A,\varepsilon) \le \log N\Big(A,(n2^n)^{-1}\Big) \le \frac{2^{p'n+1}}{\ln 2} \le 2^{p'n+2}.$$

Finally, since $\varepsilon < [(n-1)2^{n-1}]^{-1} \le 4\,(n2^n)^{-1}$ and $\frac{1}{\varepsilon} \le n2^n \le 2^{2n}$, we obtain

$$\varepsilon^{p'}\left(\log\frac{1}{\varepsilon}\right)^{p'}\log N(A,\varepsilon) \le 4^{p'}\,(n2^n)^{-p'}\,(2n)^{p'}\,2^{p'n+2} = 2^{3p'+2}.$$

This finishes the proof of (1.3.15). Now let us deal with (1.3.16). The first step is to prove the following statement.

Lemma 4.6.1. *For every natural number $k \ge 9$ and any δ with*

$$\exp\Big(-2^{p'(k-1)-1}\Big) \le \delta \le k^{-1/p'}\,2^{-k-4/p'} \tag{4.6.2}$$

there exists a subset $U_k \subset \operatorname{conv}(A_k)$ with

$$|U_k| \ge \exp\left(\frac{15}{128}\,k^{-1}\,\delta^{-p'}\right) \quad \textit{and} \quad \|u-u'\|_p > \left(\frac{15}{256}\right)^{1/p} k^{-1/p}\,\delta$$

for all $u,u' \in U_k$, $u \ne u'$.

Proof of Lemma 4.6.1. At first we estimate the number of elements in the index set I_k. Observe that $x - x^{2^{-p'}} \ge x^{1/2}$ for $x \ge 4$. Using this estimate with $x = \exp\Big(2^{p'k}\Big)$ we conclude that

$$|I_k| = \exp\Big(2^{p'k}\Big) - \exp\Big(2^{p'(k-1)}\Big) \ge \exp\Big(2^{p'k-1}\Big). \tag{4.6.3}$$

Now define

$$r := \Big\lfloor (k\,2^{p'k}\,\delta^{p'})^{-1}\Big\rfloor\,. \tag{4.6.4}$$

In view of (4.6.2) we compute that

$$k\,2^{p'k}\,\delta^{p'} \le k\,2^{p'k}\,k^{-1}\,2^{-p'k-4} = 2^{-4} \tag{4.6.5}$$

and this yields $\Big(k2^{p'k}\delta^{p'}\Big)^{-1} \ge 16$. Hence, we have that $r \ge 16$. Furthermore, the estimate

$$k\,2^{p'k}\,\delta^{p'} \ge \delta^{p'} \ge \exp\Big(-2^{p'(k-1)-1}\,p'\Big) \ge \exp\Big(-2^{p'k-2}\Big)$$

holds true. Here we used that $p' \le 2^{p'-1}$. Taking (4.6.3) into account, we obtain

$$r \le \left(k\, 2^{p'k}\, \delta^{p'}\right)^{-1} \le \exp\left(2^{p'k-2}\right) \le |I_k|^{1/2}. \tag{4.6.6}$$

Now define the set

$$V_k := \left\{ 2^{(p'-1)k}\, \delta^{p'} \sum_{i \in I_k} v_i e_i : v_i \in \mathbb{N}_0, \sum_{i \in I_k} v_i \le r \right\}.$$

First, we show that $V_k \subset \mathrm{conv}(A_k)$. Let $v = 2^{(p'-1)k}\, \delta^{p'} \sum_{i \in I_k} v_i e_i \in V_k$ be an element of V_k. Then

$$v = \sum_{i \in I_k} \left[k\, 2^{p'k}\, \delta^{p'}\, v_i\right] \left(k 2^k\right)^{-1} e_i$$

and

$$\sum_{i \in I_k} k\, 2^{p'k}\, \delta^{p'}\, v_i \le k\, 2^{p'k}\, \delta^{p'}\, r \le 1.$$

From $0 \in A_k$ we conclude that $v \in \mathrm{conv}(A_k)$. Now we estimate the number of elements in V_k. A combinatorial argument shows that

$$|V_k| = \binom{|I_k| + r}{|I_k|} = \frac{(|I_k| + 1) \cdots (|I_k| + r)}{r!} \ge \frac{|I_k|^r}{r!}.$$

Furthermore, in view of (4.6.6) we have that $r! \le r^r \le |I_k|^{r/2}$. Consequently, we arrive at $|V_k| \ge |I_k|^{r/2}$. Choose a natural number l with $r/16 \le l \le r/8$. For fixed $v' \in V_k$, consider the neighborhood

$$B_l(v') := \left\{ v \in V_k : \|v - v'\|_1 \le l\, 2^{(p'-1)k}\, \delta^{p'} \right\}.$$

We claim that

$$|B_l(v')| \le |I_k|^{2l}.$$

To see this, let $v' = 2^{(p'-1)k}\, \delta^{p'} \sum_{i \in I_k} v'_i e_i \in V_k$ and $v = 2^{(p'-1)k}\, \delta^{p'} \sum_{i \in I_k} v_i e_i \in B_l(v')$. Since $\|v - v'\|_1 \le l\, 2^{(p'-1)k}\, \delta^{p'}$ we have that $\sum_{i \in I_k} |v_i - v'_i| \le l$. With $d_i := |v_i - v'_i| \in \mathbb{N}_0$, there are $\binom{|I_k|+l}{|I_k|}$ possible choices of d_i's with $\sum_{i \in I_k} d_i \le l$. Of course, the number of nonzero d_i's is less or equal to l. The corresponding v_i's can take at most two values, namely $v_i = v'_i + d_i$ or $v_i = v'_i - d_i$. Hence, we see that

$$|B_l(v')| \le \binom{|I_k| + l}{|I_k|} 2^l = \frac{(|I_k| + 1) \cdots (|I_k| + l)}{l!}\, 2^l \le (|I_k| + l)^l\, 2^l \le (4|I_k|)^l \le |I_k|^{2l}.$$

Consequently, for each $v' \in V_k$, the neighborhood $B_l(v') \subset V_k$ contains no more than $|I_k|^{2l}$ elements. But this means that there exists a subset $U_k \subset V_k$ of V_k with

$$|U_k| \ge \frac{|I_k|^{r/2}}{|I_k|^{2l}} \ge \frac{|I_k|^{r/2}}{|I_k|^{r/4}} = |I_k|^{r/4} \tag{4.6.7}$$

and

$$\|u - u'\|_1 > l\, 2^{(p'-1)k}\, \delta^{p'} \quad \text{for all } u, u' \in U_k, u \neq u'.$$

Let $u = 2^{(p'-1)k}\, \delta^{p'} \sum_{i \in I_k} u_i e_i$ and $u' = 2^{(p'-1)k}\, \delta^{p'} \sum_{i \in I_k} u'_i e_i \in U_k$ with $u \neq u'$. Then it holds that $\sum_{i \in I_k} |u_i - u'_i| > l$ and from $|u_i - u'_i| \in \mathbb{N}_0$ we conclude that

$$\sum_{i \in I_k} |u_i - u'_i|^p \geq \sum_{i \in I_k} |u_i - u'_i| > l.$$

Consequently, we get

$$\|u - u'\|_p = 2^{(p'-1)k}\, \delta^{p'} \left(\sum_{i \in I_k} |u_i - u'_i|^p \right)^{1/p} > 2^{(p'-1)k}\, \delta^{p'}\, l^{1/p}. \tag{4.6.8}$$

In order to estimate $l^{1/p}$ we compute with (4.6.5) that

$$r \geq \left(k\, 2^{p'k}\, \delta^{p'} \right)^{-1} - 1 \geq \frac{15}{16} \left(k\, 2^{p'k}\, \delta^{p'} \right)^{-1} \tag{4.6.9}$$

and this yields

$$l^{1/p} \geq \left(\frac{r}{16} \right)^{1/p} \geq \left(\frac{15}{256} \right)^{1/p} \left(k\, 2^{p'k}\, \delta^{p'} \right)^{-1/p}.$$

Consequently, we can continue (4.6.8) with

$$\begin{aligned} \|u - u'\|_p &> 2^{(p'-1)k}\, \delta^{p'}\, l^{1/p} \geq \left(\frac{15}{256} \right)^{1/p} k^{-1/p}\, 2^{(p'-1-p'/p)k}\, \delta^{p'-p'/p} \\ &= \left(\frac{15}{256} \right)^{1/p} k^{-1/p}\, \delta. \end{aligned}$$

The last step is to estimate the number of elements in U_k. In view of (4.6.3), (4.6.7) and (4.6.9) we get

$$\ln |U_k| \geq \frac{r}{4} \ln |I_k| \geq \frac{15}{64} (k\, 2^{p'k}\, \delta^{p'})^{-1}\, 2^{p'k-1} = \frac{15}{128}\, k^{-1}\, \delta^{-p'}.$$

This finishes the proof of Lemma 4.6.1. ■

Next, for $m \in \mathbb{N}$ we define

$$E_m := \operatorname{conv}(A_{2^{m-1}+1}) + \operatorname{conv}(A_{2^{m-1}+2}) + \cdots + \operatorname{conv}(A_{2^m})$$

and prove the following statement.

Lemma 4.6.2. *For every natural number $m \geq 4$ and any δ with*

$$\exp\left(-2^{p'2^{m-1}-1}\right) \leq \delta \leq 2^{-2^{m+1}}$$

there exists a subset $Q_m \subset E_m$ with

$$|Q_m| \geq \exp\left(\frac{15}{512}\,\delta^{-p'}\right) \quad \textit{and} \quad \|q-q'\|_p > \left(\frac{15}{4096}\right)^{1/p}\,\delta$$

for all $q, q' \in Q_m$, $q \neq q'$.

Proof of Lemma 4.6.2. Fix a natural number $m \geq 4$ and δ in the range described above. In what follows, let $k \in \{2^{m-1}+1, 2^{m-1}+2, \dots, 2^m\}$. We want to apply Lemma 4.6.1 for all these k's but with the same δ given above. Consequently, the first step is to check that

$$\max_k \exp\left(-2^{p'(k-1)-1}\right) \leq \delta \leq \min_k k^{-1/p'}\,2^{-k-4/p'}.$$

To this end, we compute that

$$\max_k \exp\left(-2^{p'(k-1)-1}\right) = \exp\left(-2^{p'2^{m-1}-1}\right) \leq \delta$$

and

$$\min_k k^{-1/p'}\,2^{-k-4/p'} = 2^{-m/p'}\,2^{-2^m-4/p'} = 2^{-(m/p'+4/p'+2^m)} \geq 2^{-2^{m+1}} \geq \delta.$$

Consequently, for each $k \in \{2^{m-1}+1, 2^{m-1}+2, \dots, 2^m\}$, by Lemma 4.6.1 there exists a subset $U_k \subset \operatorname{conv}(A_k)$ with $|U_k| \geq \exp\left(\frac{15}{128}k^{-1}\,\delta^{-p'}\right)$ and

$$\|u-u'\|_p > \left(\frac{15}{256}\right)^{1/p} k^{-1/p}\,\delta \quad \text{for all } u, u' \in U_k, u \neq u'. \tag{4.6.10}$$

We assume that all these subsets U_k have the same cardinality

$$C_m := \min_k \exp\left(\frac{15}{128}k^{-1}\,\delta^{-p'}\right) = \exp\left(\frac{15}{128}\,2^{-m}\,\delta^{-p'}\right).$$

Define

$$R_m := U_{2^{m-1}+1} + U_{2^{m-1}+2} + \cdots + U_{2^m} \subset E_m.$$

Let $r = r_{2^{m-1}+1} + r_{2^{m-1}+2} + \cdots + r_{2^m}$ and $r' = r'_{2^{m-1}+1} + r'_{2^{m-1}+2} + \cdots + r'_{2^m}$ with $r_k, r'_k \in U_k$ be elements of R_m. Let the Hamming-distance of r and r' be given by

$$h(r,r') := \left|\{k : r_k \neq r'_k\}\right|.$$

For fixed $r' \in R_m$ we consider the neighborhood

$$B_h(r') := \left\{ r \in R_m : h(r, r') \le 2^{m-4} \right\}.$$

We estimate the cardinality of $B_h(r')$ by

$$|B_h(r')| = \sum_{k=0}^{2^{m-4}} |\{r \in R_m : h(r,r') = k\}| \le \sum_{k=0}^{2^{m-4}} \binom{2^{m-1}}{k} (C_m)^k.$$

From the elementary inequalities

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{e\, n}{k}\right)^k$$

it follows that

$$\binom{2^{m-1}}{k} \le \left(\frac{e\, 2^{m-1}}{k}\right)^k = (8e)^k \left(\frac{2^{m-4}}{k}\right)^k \le (8e)^k \binom{2^{m-4}}{k}.$$

Hence, we obtain the estimate

$$\begin{aligned} |B_h(r')| &\le \sum_{k=0}^{2^{m-4}} \binom{2^{m-4}}{k} (8e\, C_m)^k \le (8e\, C_m)^{2^{m-4}} \sum_{k=0}^{2^{m-4}} \binom{2^{m-4}}{k} \\ &= (16e\, C_m)^{2^{m-4}} \le (C_m)^{2^{m-3}}. \end{aligned}$$

Consequently, for each $r' \in R_m$ the neighborhood $B_h(r') \subset R_m$ contains no more than $(C_m)^{2^{m-3}}$ elements. But the set R_m contains exactly $(C_m)^{2^{m-1}}$ elements, because all subsets U_k have the same cardinality C_m and the index sets I_k are disjoint. We conclude that there exists a subset $Q_m \subset R_m$ of R_m with

$$|Q_m| \ge \frac{(C_m)^{2^{m-1}}}{(C_m)^{2^{m-3}}} \ge (C_m)^{2^{m-2}} = \exp\left(\frac{15}{512}\, \delta^{-p'}\right)$$

and $h(q, q') > 2^{m-4}$ for all $q, q' \in Q_m$, $q \ne q'$. Let $q = q_{2^{m-1}+1} + q_{2^{m-1}+2} + \cdots + q_{2^m}$ and $q' = q'_{2^{m-1}+1} + q'_{2^{m-1}+2} + \cdots + q'_{2^m} \in Q_m$ be elements of Q_m with $q \ne q'$. Since q_k and q'_k are elements of U_k, we obtain with (4.6.10) that

$$\|q - q'\|_p^p = \left\| \sum_{k=2^{m-1}+1}^{2^m} (q_k - q'_k) \right\|_p^p = \sum_{k=2^{m-1}+1}^{2^m} \|q_k - q'_k\|_p^p > 2^{m-4}\, \frac{15}{256}\, 2^{-m}\, \delta^p = \frac{15}{4096}\, \delta^p.$$

This finishes the proof of Lemma 4.6.2. ■

Finally, for a natural number $n \ge 4$ we define $m_1 := \left\lfloor \frac{1}{2} \log n \right\rfloor$ and $m_2 := \lfloor \log n \rfloor$ and consider the set

$$E_{m_1} + E_{m_1+1} + \cdots + E_{m_2} \subset \operatorname{conv}(A).$$

The last step is to show the following statement.

Lemma 4.6.3. *For every natural number $n \geq 2^{16}$ it holds that*

$$\log N\left(E_{m_1}+E_{m_1+1}+\cdots+E_{m_2}, \frac{1}{2}\left(\frac{15}{2^{16}}\right)^{1/p}(\log n)^{1/p}\, 2^{-2n}\right) \geq \frac{15}{2048}\, 2^{2p'n}\, \log n.$$

Proof of Lemma 4.6.3. Fix a natural number $n \geq 2^{16}$. In what follows, m is a natural number with $m_1 \leq m \leq m_2$. We want to apply Lemma 4.6.2 for all these m's with $\delta = 2^{-2n}$. We check that

$$\delta = 2^{-2n} = 2^{-2^{\log(n)+1}} \leq 2^{-2^{m_2+1}} = \min_m 2^{-2^{m+1}}.$$

Furthermore, we have that $m \geq \log(\sqrt{n})-1$ and, therefore, we obtain

$$\max_m \exp\left(-2^{p'2^{m-1}-1}\right) \leq \exp\left(-2^{p'2^{\log(\sqrt{n})-2}-1}\right) = \exp\left(-2^{p'\,2^{-2}\sqrt{n}-1}\right) \leq 2^{-2n} = \delta.$$

Here we used that

$$2^{p'\,2^{-2}\sqrt{n}-1} \geq 2^{2^{-1}\sqrt{n}-1} \geq \ln(2)\, 2n$$

holds true for $n \geq 2^9$. According to Lemma 4.6.2 with $\delta = 2^{-2n}$, for each m with $m_1 \leq m \leq m_2$ there exists a subsets $Q_m \subset E_m$ with

$$|Q_m| = \exp\left(\frac{15}{512} 2^{2p'n}\right) \quad \text{and} \quad \|q-q'\|_p > \left(\frac{15}{4096}\right)^{1/p} 2^{-2n} \tag{4.6.11}$$

for all $q, q' \in Q_m$, $q \neq q'$. Now define

$$D := Q_{m_1}+Q_{m_1+1}+\cdots+Q_{m_2} \subset E_{m_1}+E_{m_1+1}+\cdots+E_{m_2}.$$

At first we estimate the cardinality of D. This this end, observe that

$$m_2-m_1+1 \geq \log(n)-\left\lfloor \frac{1}{2}\log n\right\rfloor \geq \frac{1}{2}\log n.$$

Consequently, we obtain

$$|D| = \left[\exp\left(\frac{15}{512}2^{2p'n}\right)\right]^{m_2-m_1+1} \geq \exp\left(\frac{15}{512}2^{2p'n-1}\,\log n\right).$$

Now let $d = d_{m_1}+d_{m_1+1}+\cdots+d_{m_2}$ and $d' = d'_{m_1}+d'_{m_1+1}+\cdots+d'_{m_2}$ with $d_m, d'_m \in Q_m$ be elements of D. For fixed $d' \in D$, consider the Hamming-neighborhood

$$B_h(d') := \left\{d \in D : h(d,d') \leq \left\lfloor 2^{-4}\log n\right\rfloor + 1\right\}.$$

Then, analogously to the above, the cardinality of $B_h(d')$ can be estimated by

$$\begin{aligned}|B_h(d')| &= \sum_{k=0}^{\lfloor 2^{-4}\log n\rfloor+1} |\{d \in D : h(d,d') = k\}| \\ &\leq \sum_{k=0}^{\lfloor 2^{-4}\log n\rfloor+1} \binom{m_2 - m_1 + 1}{k} \left[\exp\left(\frac{15}{512} 2^{2p'n}\right)\right]^k \\ &\leq \left[\exp\left(\frac{15}{512} 2^{2p'n}\right)\right]^{\lfloor 2^{-4}\log n\rfloor+1} \sum_{k=0}^{\lfloor 2^{-4}\log n\rfloor+1} \binom{m_2 - m_1 + 1}{k}.\end{aligned}$$

Furthermore, from

$$\begin{aligned}\binom{m_2 - m_1 + 1}{k} &\leq \left(\frac{e\,(m_2 - m_1 + 1)}{k}\right)^k \leq \left(\frac{e\,\log n}{k}\right)^k \leq (16e)^k \left(\frac{\lfloor 2^{-4}\log n\rfloor + 1}{k}\right)^k \\ &\leq (16e)^k \binom{\lfloor 2^{-4}\log n\rfloor + 1}{k}\end{aligned}$$

we conclude that

$$\sum_{k=0}^{\lfloor 2^{-4}\log n\rfloor+1} \binom{m_2 - m_1 + 1}{k} \leq (32e)^{\lfloor 2^{-4}\log n\rfloor+1}.$$

Hence, we obtain the estimate

$$\begin{aligned}|B_h(d')| &\leq \left[32e\,\exp\left(\frac{15}{512} 2^{2p'n}\right)\right]^{\lfloor 2^{-4}\log n\rfloor+1} \leq \left[\exp\left(\frac{15}{512} 2^{2p'n+1}\right)\right]^{\lfloor 2^{-4}\log n\rfloor+1} \\ &\leq \left[\exp\left(\frac{15}{512} 2^{2p'n+1}\right)\right]^{2^{-3}\log n} = \exp\left(\frac{15}{512} 2^{2p'n-2}\,\log n\right),\end{aligned}$$

where we used that $\lfloor 2^{-4}\log n\rfloor + 1 \leq 2^{-3}\log n$ for $n \geq 2^{16}$. Consequently, for each $d' \in D$ the neighborhood $B_h(d') \subset D$ contains no more than $\exp\left(\frac{15}{512} 2^{2p'n-2}\,\log n\right)$ elements, but D contains $\exp\left(\frac{15}{512} 2^{2p'n-1}\,\log n\right)$ elements. This means that there exists a subset $C \subset D$ of D with

$$|C| \geq \frac{\exp\left(\frac{15}{512} 2^{2p'n-1}\,\log n\right)}{\exp\left(\frac{15}{512} 2^{2p'n-2}\,\log n\right)} = \exp\left(\frac{15}{512} 2^{2p'n-2}\,\log n\right)$$

and $h(c,c') > 2^{-4}\log n$ for all $c, c' \in C$, $c \neq c'$. Now let $c = c_{m_1} + c_{m_1+1} + \cdots + c_{m_2}$, $c' = c'_{m_1} + c'_{m_1+1} + \cdots + c'_{m_2} \in C$ with $c \neq c'$. Since c_m and c'_m are elements of Q_m, we obtain with (4.6.11) that

$$\begin{aligned}\|c-c'\|_p^p &= \left\|\sum_{k=m_1}^{m_2}(c_k-c_k')\right\|_p^p = \sum_{k=m_1}^{m_2}\|c_k-c_k'\|_p^p \\ &> 2^{-4}\log(n)\,\frac{15}{4096}\,2^{-2pn} = \frac{15}{2^{16}}\,\log(n)\,2^{-2pn}.\end{aligned}$$

Hence, we see that C is a subset of $E_{m_1}+E_{m_1+1}+\cdots+E_{m_2}$ which consists of at least $\exp\left(\frac{15}{512}2^{2p'n-2}\,\log n\right)$ elements and it holds that $\|c-c'\|_p > \left(\frac{15}{2^{16}}\right)^{1/p}(\log n)^{1/p}\,2^{-2n}$ for all $c,c'\in C$, $c\neq c'$. But this means that

$$\begin{aligned}&\log N\left(E_{m_1}+E_{m_1+1}+\cdots+E_{m_2},\frac{1}{2}\left(\frac{15}{2^{16}}\right)^{1/p}(\log n)^{1/p}\,2^{-2n}\right)\\ &\geq \log N\left(C,\frac{1}{2}\left(\frac{15}{2^{16}}\right)^{1/p}(\log n)^{1/p}\,2^{-2n}\right)\\ &\geq \frac{1}{\ln 2}\frac{15}{512}2^{2p'n-2}\,\log n \geq \frac{15}{2048}\,2^{2p'n}\,\log n\end{aligned}$$

and finishes the proof. ■

Now we are well prepared to prove (1.3.16). For every ε satisfying $0<\varepsilon\leq \frac{1}{2}\left(\frac{15}{2^{12}}\right)^{1/p}2^{-2^{17}}$ there exists a natural number $n\geq 2^{16}$ with

$$\frac{1}{2}\left(\frac{15}{2^{16}}\right)^{1/p}(\log(n+1))^{1/p}\,2^{-2(n+1)} < \varepsilon \leq \frac{1}{2}\left(\frac{15}{2^{16}}\right)^{1/p}(\log n)^{1/p}\,2^{-2n}.$$

Using Lemma 4.6.3 we get

$$\begin{aligned}\log N\Big(\operatorname{conv}(A),\varepsilon\Big) &\geq \log N\Big(E_{m_1}+E_{m_1+1}+\cdots+E_{m_2},\varepsilon\Big)\\ &\geq \log N\left(E_{m_1}+E_{m_1+1}+\cdots+E_{m_2},\frac{1}{2}\left(\frac{15}{2^{16}}\right)^{1/p}(\log n)^{1/p}\,2^{-2n}\right)\\ &\geq \frac{15}{2048}\,2^{2p'n}\,\log n.\end{aligned}$$

Observe that $\varepsilon \geq 2^{-3}\left(\frac{15}{2^{16}}\right)^{1/p}(\log n)^{1/p}\,2^{-2n}$ and, therefore, we have that

$$2^{2p'n} \geq 2^{-3p'}\left(\frac{15}{2^{16}}\right)^{p'/p}\varepsilon^{-p'}(\log n)^{p'/p}. \tag{4.6.12}$$

Furthermore, from $\frac{1}{\varepsilon}\leq 2^{n^2}$ we conclude that

$$\log\log\frac{1}{\varepsilon} \leq 2\log n. \tag{4.6.13}$$

Combining (4.6.12) and (4.6.13), we finally obtain

$$\begin{aligned}\log N\Big(\operatorname{conv}(A),\varepsilon\Big) &\geq \frac{15}{2048}\,2^{2p'n}\,\log n \geq \frac{15}{2048}\,2^{-3p'}\left(\frac{15}{2^{16}}\right)^{p'/p}\varepsilon^{-p'}(\log n)^{p'/p+1}\\ &= \frac{15}{2048}\,2^{-3p'}\left(\frac{15}{2^{16}}\right)^{p'/p}\varepsilon^{-p'}(\log n)^{p'}\\ &\geq \frac{15}{2048}\,2^{-4p'}\left(\frac{15}{2^{16}}\right)^{p'/p}\varepsilon^{-p'}\left(\log\log\frac{1}{\varepsilon}\right)^{p'}\\ &= 15^{p'}\,2^{-16p'/p-4p'-11}\,\varepsilon^{-p'}\left(\log\log\frac{1}{\varepsilon}\right)^{p'}.\end{aligned}$$

This finishes the proof of Theorem 1.3.9. ■

4.7 Proof of Theorem 1.3.10

In order to prove Theorem 1.3.10 we slightly modify the construction used in the proof of Theorem 1.3.9. Fix $\beta > 1$. For $k = 1, 2, \ldots$, let the subsets A_k of l_p be defined as in the proof of Theorem 1.3.9. Now, let

$$A_k^\beta := k^{1-\beta} A_k, \quad k = 1, 2, \ldots,$$

and

$$A^\beta := A_1^\beta + A_2^\beta + \cdots + A_k^\beta + \cdots \subset l_p.$$

The first step is to show that A^β satisfies (1.3.17). To this end, let $0 < \varepsilon \leq 1/2$ be arbitrary. Then there exists a natural number $n \in \mathbb{N}$ with

$$(n+1)^{-\beta}\,2^{-n} < \varepsilon \leq n^{-\beta}\,2^{-(n-1)}. \tag{4.7.1}$$

Let $a = a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots \in A^\beta$ be an element of A^β with $a_k \in A_k^\beta$. For $s_n := a_1 + a_2 + \cdots + a_n$ we compute that

$$\|a - s_n\|_p \leq \sum_{k=n+1}^{\infty} \|a_k\|_p \leq \sum_{k=n+1}^{\infty} k^{-\beta}\,2^{-k} \leq (n+1)^{-\beta} \sum_{k=n+1}^{\infty} 2^{-k} = (n+1)^{-\beta}\,2^{-n} \leq \varepsilon.$$

Hence, we see that $S_n^\beta := A_1^\beta + A_2^\beta + \cdots + A_n^\beta$ is an ε-net for A^β. As in the proof of Theorem 1.3.9, we get that

$$|S_n^\beta| \leq \exp(2^{p'n+1}).$$

Consequently, we conclude that

$$\log N\Big(A^\beta, \varepsilon\Big) \leq \frac{2^{p'n+1}}{\ln 2} \leq 2^{p'n+2}.$$

Finally, in view of (4.7.1) we have that $2^n \leq 2\, n^{-\beta}\, \varepsilon^{-1}$ and $\frac{1}{\varepsilon} \leq (n+1)^\beta\, 2^n \leq 2^{n(\beta+1)}$. This implies $\log \frac{1}{\varepsilon} \leq n\,(\beta+1)$. Hence, we see that

$$2^{p'n+2} \leq 2^{2+p'}\, n^{-\beta p'}\, \varepsilon^{-p'} \leq 2^{2+p'}\, (\beta+1)^{\beta p'}\, \varepsilon^{-p'} \left(\log \frac{1}{\varepsilon}\right)^{-\beta p'}$$

and this gives the desired estimate

$$\log N(A^\beta, \varepsilon) \leq 2^{2+p'}\, (\beta+1)^{\beta p'}\, \varepsilon^{-p'} \left(\log \frac{1}{\varepsilon}\right)^{-\beta p'}.$$

Now let us deal with (1.3.18). For $m \in \mathbb{N}$, let E_m be defined as in the proof of Theorem 1.3.9. At first we show that for $m \in \mathbb{N}$, $m \geq 4$ and

$$\exp\left(-2^{p' 2^{m-1}-1}\right) \leq \delta \leq 2^{-2^{m+1}}$$

it holds that

$$\log N\Big(E_m, 2^{-10}\delta\Big) \geq 2^{-5}\, \delta^{-p'}. \tag{4.7.2}$$

According to Lemma 4.6.2 there exists a subset $Q_m \subset E_m$ with

$$|Q_m| \geq \exp\left(\frac{15}{512}\, \delta^{-p'}\right) \quad \text{and} \quad \|q - q'\|_p > \left(\frac{15}{4096}\right)^{1/p} \delta$$

for all $q, q' \in Q_m$, $q \neq q'$. We conclude that

$$\log N\left(E_m, \frac{1}{2}\left(\frac{15}{4096}\right)^{1/p} \delta\right) \geq \frac{1}{\ln 2}\frac{15}{512}\, \delta^{-p'} \geq 2^{-5}\, \delta^{-p'}.$$

Furthermore, it holds that $\frac{1}{2}\left(\frac{15}{4096}\right)^{1/p} \geq \frac{15}{8192} \geq 2^{-10}$ and this yields the assertion. Now for $m \in \mathbb{N}$ we define

$$E_m^\beta := \operatorname{conv}\left(A^\beta_{2^{m-1}+1}\right) + \operatorname{conv}\left(A^\beta_{2^{m-1}+2}\right) + \cdots + \operatorname{conv}\left(A^\beta_{2^m}\right)$$

and prove that $2^{m(1-\beta)}\, E_m \subset E_m^\beta$. To this end, fix a natural number m and let $2^{m-1}+1 \leq k \leq 2^m$. First, observe that for every k in the range described above it holds that $2^{m(1-\beta)}\, \operatorname{conv}(A_k) \subset k^{1-\beta}\, \operatorname{conv}(A_k)$. Hence, we get

$$\begin{aligned} 2^{m(1-\beta)}\, E_m &= \sum_{k=2^{m-1}+1}^{2^m} 2^{m(1-\beta)}\, \operatorname{conv}(A_k) \subset \sum_{k=2^{m-1}+1}^{2^m} k^{1-\beta}\, \operatorname{conv}(A_k) \\ &= \sum_{k=2^{m-1}+1}^{2^m} \operatorname{conv}\left(A_k^\beta\right) = E_m^\beta. \end{aligned}$$

Since $E_m^\beta \subset \operatorname{conv}\left(A^\beta\right)$, we conclude that

$$\begin{aligned}\log N\Big(\operatorname{conv}(A^\beta),\gamma\Big) &\geq \log N\Big(E_m^\beta,\gamma\Big) \geq \log N\Big(2^{m(1-\beta)}\,E_m,\gamma\Big)\\ &= \log N\Big(E_m, 2^{m(\beta-1)}\,\gamma\Big) \end{aligned} \tag{4.7.3}$$

for $\gamma > 0$. The next step is to apply (4.7.2) with $\delta = 2^{10}\,2^{m(\beta-1)}\,\gamma$. However, the crucial step is to find the right choice of m and γ. First, observe that there exists a number $n_\beta \geq 2^{16}$, which depends on β, such that

$$2^{10}\,(\log n)^{\beta-1}\,2^{-2n} \leq n^{-2} \quad \text{for all } n \geq n_\beta. \tag{4.7.4}$$

Now let $0 < \varepsilon \leq 2^{-2n_\beta} = c(\beta)$ be arbitrary. Then there exists a natural number $n \geq n_\beta$ with $2^{-2(n+1)} < \varepsilon \leq 2^{-2n}$. Let $m = \lfloor \log\log n \rfloor \geq 4$ and $\gamma = 2^{-2n}$. In view of (4.7.4) we compute that

$$\delta = 2^{10}\,2^{m(\beta-1)}\,\gamma \leq 2^{10}\,(\log n)^{\beta-1}\,2^{-2n} \leq n^{-2} \leq 2^{-2^{\lfloor \log\log n\rfloor+1}} = 2^{-2^{m+1}}.$$

Hence, applying estimate (4.7.2) with $\delta = 2^{10}\,2^{m(\beta-1)}\,2^{-2n}$ gives

$$\log N\Big(E_m, 2^{m(\beta-1)}\,2^{-2n}\Big) \geq 2^{-10p'-5}\,2^{mp'(1-\beta)}\,2^{2p'n}.$$

Next, from $\varepsilon \leq 2^{-2n}$ and (4.7.3) with $\gamma = 2^{-2n}$ we obtain that

$$\log N\Big(\operatorname{conv}(A^\beta),\varepsilon\Big) \geq \log N\Big(\operatorname{conv}(A^\beta),2^{-2n}\Big) \geq 2^{-10p'-5}\,2^{mp'(1-\beta)}\,2^{2p'n}.$$

Finally, from $2^{2n} \geq 2^{-2}\,\varepsilon^{-1}$ and $2^m \leq \log n \leq \log\log\frac{1}{\varepsilon}$ we conclude that

$$2^{mp'(1-\beta)}\,2^{2p'n} \geq 2^{-2p'}\,\varepsilon^{-p'}\,\left(\log\log\frac{1}{\varepsilon}\right)^{p'(1-\beta)}.$$

This yields the desired estimate

$$\log N\Big(\operatorname{conv}(A^\beta),\varepsilon\Big) \geq 2^{-12p'-5}\,\varepsilon^{-p'}\,\left(\log\log\frac{1}{\varepsilon}\right)^{p'(1-\beta)}$$

and finishes the proof of Theorem 1.3.10. ■

4.8 Proof of Theorem 1.4.1

We may assume that A consists of more than one element. Let $0 < t < \infty$ be arbitrary. Applying [CKP99, Prop. 4.4] with $b_n = n^{1/t}$ gives

$$\sup_{1\leq k\leq n} k^{1/t}\,e_k(\operatorname{aco}(A)) \leq c_1(t)\left(\|A\| + 2^{1/t}\sup_{1\leq k\leq n} k^{1/t}\,\varepsilon_k(A)\right)$$

for all $n \in \mathbb{N}$. Next, we observe that

$$\|A\| = \frac{\|A\|}{\varepsilon_1(A)}\,\varepsilon_1(A) \le \frac{\|A\|}{\varepsilon_1(A)} \sup_{1\le k\le n} k^{1/t}\,\varepsilon_k(A)$$

and $1 \le \frac{\|A\|}{\varepsilon_1(A)}$. Hence, we conclude that

$$\|A\| + 2^{1/t} \sup_{1\le k\le n} k^{1/t}\,\varepsilon_k(A) \le (1+2^{1/t})\,\frac{\|A\|}{\varepsilon_1(A)} \sup_{1\le k\le n} k^{1/t}\,\varepsilon_k(A).$$

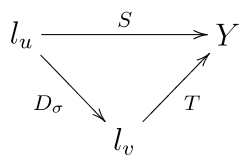
Consequently, for all natural numbers $n \in \mathbb{N}$ we obtain the estimate

$$\sup_{1\le k\le n} k^{1/t}\,e_k(\mathrm{aco}(A)) \le c_2(t)\,\frac{\|A\|}{\varepsilon_1(A)} \sup_{1\le k\le n} k^{1/t}\,\varepsilon_k(A),$$

where $c_2(t) = (1+2^{1/t})\,c_1(t)$ is a constant only depending on t. Now adapting the proof of Theorem 1.3.5 or Proposition 2.5.2 yields the assertion. ■

5 Operators factoring through general diagonal operators

This chapter is devoted to the study of entropy numbers of operators S which admit a factorization



,

where $D_\sigma : l_u \to l_v$ is a diagonal operator generated by a sequence $(\sigma_n)_n$ and $T : l_v \to Y$ is an arbitrary operator. A special case of this setting was considered by Carl in [C82]. We are interested in the case that the sequence $(\sigma_n)_n$ belongs to some generalized Lorentz sequence space $l_{p,q,\varphi}$. Assuming a *local* entropy estimate associated with the operator T, we want to determine the entropy behavior of the composition operator $TD_\sigma : l_u \to Y$. We start with formulating the main result of this chapter.

Theorem 5.0.1. *Let $1 \le u, v \le \infty$ and let φ be a slowly varying function. Suppose that $D_\sigma : l_u \to l_v$ is a diagonal operator generated by a sequence $(\sigma_n)_n \in l_{r,t,\varphi}$, where $0 < r < \infty$, $1/r > (1/v - 1/u)_+$ and $0 < t \le \infty$. Furthermore, suppose that $T : l_v \to Y$ is an arbitrary operator with image in a Banach space Y. If there are constants $\tau, \alpha > 0$ and $\beta \in \mathbb{R}$ such that for all natural numbers $n \in \mathbb{N}$ and all operators $R : l_v^n \to l_v$ the entropy estimate*

$$e_k(TR : l_v^n \to Y) \le \tau\, k^{-\alpha} \left(\log\left(\frac{n}{k} + 1\right)\right)^{\beta} \|T\|\, \|R\|\,, \quad 1 \le k \le n, \tag{5.0.1}$$

is satisfied, then

$$\Big(e_n(TD_\sigma : l_u \to Y)\Big)_n \in l_{s,t,\varphi} \quad \textit{for} \quad 1/s = 1/r + 1/u - 1/v + \alpha.$$

For the proof of Theorem 5.0.1 we make use of the so-called *quasi-norm-technique*; we do not need interpolation theory. For $s > 0$ and $T \in \mathcal{L}(X,Y)$ we define

$$L_{s,\infty}^{(e)}(T) := \sup_{k \in \mathbb{N}} k^{1/s}\, e_k(T)$$

and

$$\mathcal{L}_{s,\infty}^{(e)}(X,Y) := \left\{T \in \mathcal{L}(X,Y) : L_{s,\infty}^{(e)}(T) < \infty\right\}.$$

The properties of entropy numbers imply that $L_{s,\infty}^{(e)}$ defines a quasi-norm on the vector space $\mathcal{L}_{s,\infty}^{(e)}(X,Y)$ with quasi-norm-constant $Q = 2^{1/s}$. Due to Aoki and Rolewicz (cf. [Ao42], [Ro57]), every quasi-norm $\|\cdot\|$ with quasi-norm-constant Q is equivalent to the p-norm

$$|||x||| := \inf\left\{\left(\sum_{k=1}^{n}\|x_k\|^p\right)^{1/p} : x = \sum_{k=1}^{n} x_k,\ n = 1, 2, \ldots\right\},$$

where p is given by the equation $Q = 2^{1/p-1}$. Consequently, there exists a constant $c > 0$ such that for all natural numbers $m \in \mathbb{N}$ the estimate

$$L_{s,\infty}^{(e)}\left(\sum_{k=1}^{m} T_k\right) \le c\left(\sum_{k=1}^{m}\left(L_{s,\infty}^{(e)}(T_k)\right)^p\right)^{1/p}$$

is satisfied, where p is given by the equation $2^{1/s} = 2^{1/p-1}$, i.e. $1/p = 1 + 1/s$.

In preparation for the quasi-norm-technique, let us introduce some lemmata first. For the proofs we use ideas and techniques from [C81a], [C81b] and [C82] (see in particular [C81a, Lemma 1], [C81b, Th. 1], [C82, Lemma 2]).

Lemma 5.0.2. *Let X_n be a (real or complex) normed space with $\dim(X_n) = n < \infty$. Then for all $s > 0$ there exists a constant $c = c(s) > 0$ such that*

$$L_{s,\infty}^{(e)}(I_{X_n}) \le c \cdot n^{1/s}.$$

Proof. Taking Lemma 2.4.4 (*ii*) into account, we get

$$\sup_{k\in\mathbb{N}} k^{1/s}\, e_k(I_{X_n}) \le 3 \sup_{k\in\mathbb{N}} k^{1/s}\, 2^{-\frac{k-1}{2n}} \le 3 \cdot 2^{\frac{1}{2n}} \sup_{k\in\mathbb{N}} k^{1/s}\, 2^{-\frac{k}{2n}}.$$

Observe that the function $f(x) = x\, 2^{-\frac{x\,s}{2n}}$, $x > 0$, attains a maximum value of $\frac{2n}{s\cdot e\cdot\ln(2)}$. Hence, we conclude that

$$\sup_{k\in\mathbb{N}} k^{1/s}\, e_k(I_{X_n}) \le 3 \cdot 2^{\frac{1}{2n}}\left(\frac{2n}{s\cdot e\cdot\ln(2)}\right)^{1/s} \le 3\sqrt{2}\left(\frac{2}{s\cdot e\cdot\ln(2)}\right)^{1/s} n^{1/s},$$

which finishes the proof. ∎

In order to prove the next lemma, we need the following fact: Given $\alpha > 0$ and $\beta \in \mathbb{R}$, there exists a constant $c = c(\alpha, \beta) > 0$ such that for all natural numbers $n \in \mathbb{N}$ it holds that

$$\sup_{1\le k\le n} k^{\alpha}\left(\log\left(\frac{n}{k}+1\right)\right)^{\beta} \le c\cdot n^{\alpha}. \tag{5.0.2}$$

The proof is straightforward, and so is omitted.

Lemma 5.0.3. *Let X_n be a normed space with* $\dim(X_n) = n < \infty$ *and let Y be a Banach space. Suppose that $T \in \mathcal{L}(X_n, Y)$ is an operator satisfying the entropy estimate*

$$e_k(T : X_n \to Y) \le \tau\, k^{-\alpha} \left(\log\left(\frac{n}{k}+1\right)\right)^{\beta} \|T\|$$

for all natural numbers k with $1 \le k \le n$ and some constants $\tau, \alpha > 0$ and $\beta \in \mathbb{R}$. Then for all s with $\frac{1}{s} > \alpha$ there exists a constant $c = c(\alpha, \beta, s) > 0$ such that

$$L^{(e)}_{s,\infty}(T : X_n \to Y) \le c \cdot \tau\, n^{1/s-\alpha}\, \|T\|\,.$$

Proof. Since $1/s - \alpha > 0$, applying (5.0.2) gives

$$\begin{aligned}\sup_{1\le k\le n} k^{1/s}\, e_k(T : X_n \to Y) &\le \tau\, \|T\| \sup_{1\le k\le n} k^{1/s-\alpha} \left(\log\left(\frac{n}{k}+1\right)\right)^{\beta} \\ &\le c_1(\alpha,\beta,s)\, \tau\, n^{1/s-\alpha}\, \|T\|\,.\end{aligned}$$

The next step is to estimate $\sup\limits_{k>n} k^{1/s}\, e_k(T)$. To this end, we first observe that

$$\begin{aligned}\sup_{k>n} k^{1/s}\, e_k(T : X_n \to Y) &= \sup_{k\in\mathbb{N}} (n+k)^{1/s}\, e_{n+k}(T : X_n \to Y) \\ &\le e_n(T : X_n \to Y) \sup_{k\in\mathbb{N}} (n+k)^{1/s}\, e_k(I_{X_n}) \\ &\le 2^{1/s}\, e_n(T : X_n \to Y) \sup_{k\in\mathbb{N}} \left[n^{1/s} e_k(I_{X_n}) + k^{1/s} e_k(I_{X_n})\right] \\ &\le 2^{1/s}\, e_n(T : X_n \to Y) \left[n^{1/s} + L^{(e)}_{s,\infty}(I_{X_n})\right].\end{aligned}$$

Applying Lemma 5.0.2 yields

$$\sup_{k>n} k^{1/s}\, e_k(T : X_n \to Y) \le c_2(s)\, e_n(T : X_n \to Y)\, n^{1/s} \le c_2(s)\, \tau\, n^{1/s-\alpha}\, \|T\|\,.$$

Hence, for all natural numbers $k \in \mathbb{N}$ we have that

$$k^{1/s}\, e_k(T : X_n \to Y) \le \max\left\{c_1(\alpha,\beta,s), c_2(s)\right\}\, \tau\, n^{1/s-\alpha}\, \|T\|\,.$$

This finishes the proof. ■

Lemma 5.0.4. *Let $1 \le u, v \le \infty$. Then for all s with $1/s > (1/u - 1/v)_+$ there exists a constant $c = c(u, v, s) > 0$ such that*

$$L^{(e)}_{s,\infty}(id : l^n_u \to l^n_v) \le c \cdot n^{1/s+1/v-1/u}.$$

Proof. Let us start with the case $1 \le u < v \le \infty$. According to Schütt (cf. [Sch84]) the entropy estimate

$$e_k(id : l_u^n \to l_v^n) \le c_1(u,v)\, k^{-(1/u-1/v)} \left(\log\left(\frac{n}{k}+1\right)\right)^{1/u-1/v}$$

holds true for all natural numbers k with $1 \le k \le n$. Hence, applying Lemma 5.0.3 with $X_n = l_u^n$, $Y = l_v^n$, $T = id : l_u^n \to l_v^n$ and $\alpha = \beta = 1/u - 1/v$ gives

$$L_{s,\infty}^{(e)}(id : l_u^n \to l_v^n) \le c_2(u,v,s)\, n^{1/s+1/v-1/u}$$

for all s with $1/s > 1/u - 1/v$.

Now let us deal with the case $1 \le v \le u \le \infty$. From $\|id : l_u^n \to l_v^n\| = n^{1/v-1/u}$ and the multiplicativity of entropy numbers we obtain

$$\sup_{k\in\mathbb{N}} k^{1/s}\, e_k(id : l_u^n \to l_v^n) \le n^{1/v-1/u} \sup_{k\in\mathbb{N}} k^{1/s}\, e_k(I_{l_u^n}) = n^{1/v-1/u}\, L_{s,\infty}^{(e)}(I_{l_u^n}).$$

Taking Lemma 5.0.2 into account yields the desired estimate

$$L_{s,\infty}^{(e)}(id : l_u^n \to l_v^n) \le c_3(s)\, n^{1/s+1/v-1/u}$$

in the case $1 \le v \le u \le \infty$, where $s > 0$ is arbitrary. This finishes the proof. ∎

Now we are well prepared to prove Theorem 5.0.1.

Proof of Theorem 5.0.1. In what follows, $c_1, c_2, \ldots$ denote positive constants which may depend on u, v, φ, r, t, α and β. Without loss of generality we may assume that $|\sigma_1| \ge |\sigma_2| \ge \ldots \ge 0$. First, for $k = 0, 1, 2, \ldots$ we define canonical operators $Q_k \in \mathcal{L}\Big(l_u, l_u^{2^k}\Big)$, $D_k \in \mathcal{L}\Big(l_u^{2^k}, l_u^{2^k}\Big)$ and $J_k \in \mathcal{L}\Big(l_v^{2^k}, l_v\Big)$ by

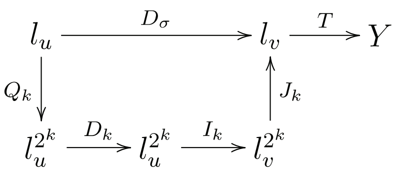
$$Q_k(\xi_1, \xi_2, \ldots) := (\xi_{2^k}, \xi_{2^k+1}, \ldots, \xi_{2^{k+1}-1}),$$

$$D_k(\xi_1, \ldots, \xi_{2^k}) := (\sigma_{2^k}\xi_1, \ldots, \sigma_{2^{k+1}-1}\xi_{2^k})$$

and

$$J_k(\xi_1, \ldots, \xi_{2^k}) := (\underbrace{0, \ldots, 0}_{2^k-1}, \xi_1, \ldots, \xi_{2^k}, 0, \ldots).$$

Clearly, it holds that $\|Q_k\| = \|J_k\| = 1$ and $\|D_k\| \le |\sigma_{2^k}|$. Furthermore, the identity operator from $l_u^{2^k}$ into $l_v^{2^k}$ is denoted by I_k. For the sake of clearness, we present the following diagram:



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We have that $D_\sigma = \sum_{k=0}^\infty J_k I_k D_k Q_k$ and, therefore,

$$TD_\sigma = \sum_{k=0}^\infty TJ_k I_k D_k Q_k.$$

Define $A_k := TJ_k I_k D_k Q_k$. In order to estimate the entropy of the operator TD_σ it seems convenient to estimate the entropy of A_k first. To this end, choose s_0 with $1/s_0 > \alpha$ and s_1 with $1/s_1 > (1/u - 1/v)_+$ and let $1/s = 1/s_0 + 1/s_1 > \alpha + (1/u - 1/v)_+$. Then the multiplicativity of entropy numbers yields

$$\begin{aligned} L^{(e)}_{s,\infty}(A_k) &\le 2^{1/s}\, L^{(e)}_{s_0,\infty}\Big(TJ_k : l_v^{2^k} \to Y\Big)\, L^{(e)}_{s_1,\infty}\Big(I_k : l_u^{2^k} \to l_v^{2^k}\Big)\, \|D_k\|\, \|Q_k\| \\ &\le 2^{1/s}\, L^{(e)}_{s_0,\infty}\Big(TJ_k : l_v^{2^k} \to Y\Big)\, L^{(e)}_{s_1,\infty}\Big(I_k : l_u^{2^k} \to l_v^{2^k}\Big)\, |\sigma_{2^k}|. \end{aligned}$$

Taking Lemma 5.0.3 into account gives

$$L^{(e)}_{s_0,\infty}\Big(TJ_k : l_v^{2^k} \to Y\Big) \le c_1\, \tau\, \left(2^k\right)^{1/s_0 - \alpha}\, \|T\|\, \|J_k\|$$

and Lemma 5.0.4 shows that

$$L^{(e)}_{s_1,\infty}\Big(I_k : l_u^{2^k} \to l_v^{2^k}\Big) \le c_2\, \left(2^k\right)^{1/s_1 + 1/v - 1/u}.$$

Hence, we obtain the estimate

$$L^{(e)}_{s,\infty}(A_k) \le c_3\, \tau\, 2^{k(1/s + 1/v - 1/u - \alpha)}\, |\sigma_{2^k}|\, \|T\| \tag{5.0.3}$$

for $1/s > \alpha + (1/u - 1/v)_+$ and $k \in \mathbb{N}$. Next we split the operator TD_σ into S_m and R_m, where

$$S_m := \sum_{k=0}^{m-1} A_k \quad \text{and} \quad R_m := \sum_{k=m}^{\infty} A_k.$$

First let us deal with the case $0 < t < \infty$. For this purpose, let $0 < q < \infty$ be a positive constant defined later. Choose s_0 such that $1/s_0 > \alpha + (1/u - 1/v)_+ + 1/q$. Since $L^{(e)}_{s_0,\infty}$ is equivalent to an $p = p(s_0)$ norm, it follows with (5.0.3) that

$$\begin{aligned} L^{(e)}_{s_0,\infty}(S_m) &\le c_4 \left(\sum_{k=0}^{m-1} \left(L^{(e)}_{s_0,\infty}(A_k) \right)^p \right)^{1/p} \\ &\le c_5\, \tau\, \|T\| \left(\sum_{k=0}^{m-1} \left(2^{k(1/s_0 + 1/v - 1/u - \alpha)}\, |\sigma_{2^k}| \right)^p \right)^{1/p}. \end{aligned} \tag{5.0.4}$$

Next we estimate the sum on the right hand side. Obviously, for all $0 \le k \le m$ it holds that

$$2^{k/q}\, |\sigma_{2^k}| \le \left(\sum_{j=0}^{k} 2^{jt/q}\, |\sigma_{2^j}|^t \right)^{1/t} \le \left(\sum_{j=0}^{m} 2^{jt/q}\, |\sigma_{2^j}|^t \right)^{1/t}.$$

This yields

$$\begin{aligned}&\left(\sum_{k=0}^{m-1}\left(2^{k(1/s_0+1/v-1/u-\alpha)}\,|\sigma_{2^k}|\right)^p\right)^{1/p}\\&=\left(\sum_{k=0}^{m-1}\left(2^{k(1/s_0+1/v-1/u-\alpha-1/q)}\,2^{k/q}\,|\sigma_{2^k}|\right)^p\right)^{1/p}\\&\le\left(\sum_{j=0}^{m}2^{jt/q}\,|\sigma_{2^j}|^t\right)^{1/t}\left(\sum_{k=0}^{m-1}\left(2^{k(1/s_0+1/v-1/u-\alpha-1/q)}\right)^p\right)^{1/p}.\end{aligned}$$

Since $1/s_0+1/v-1/u-\alpha-1/q>0$, we obtain

$$\left(\sum_{k=0}^{m-1}\left(2^{k(1/s_0+1/v-1/u-\alpha-1/q)}\right)^p\right)^{1/p}\le c_6\,2^{m(1/s_0+1/v-1/u-\alpha-1/q)}.$$

Consequently, we arrive at

$$L^{(e)}_{s_0,\infty}(S_m)\le c_7\,\tau\;\|T\|\;2^{m(1/s_0+1/v-1/u-\alpha-1/q)}\left(\sum_{j=0}^{m}2^{jt/q}\,|\sigma_{2^j}|^t\right)^{1/t}.$$

Finally, observe that

$$2^{(m-1)/s_0}\,e_{2^{m-1}}(S_m)\le L^{(e)}_{s_0,\infty}(S_m)$$

and therefore

$$e_{2^{m-1}}(S_m)\le c_8\,\tau\;\|T\|\;2^{m(1/v-1/u-\alpha-1/q)}\left(\sum_{j=0}^{m}2^{jt/q}\,|\sigma_{2^j}|^t\right)^{1/t}. \tag{5.0.5}$$

Now let us estimate the entropy of the remainder R_m. First, choose r_0 such that $1/r>1/r_0>(1/v-1/u)_+$. In addition, choose $s_1>0$ with $\alpha+(1/u-1/v)_+<1/s_1<\alpha+1/r_0+1/u-1/v$. Using the $p=p(s_1)$ norm property of $L^{(e)}_{s_1,\infty}$ and (5.0.3) we obtain

$$\begin{aligned}L^{(e)}_{s_1,\infty}(R_m)&\le c_9\left(\sum_{k=m}^{\infty}\left(L^{(e)}_{s_1,\infty}(A_k)\right)^p\right)^{1/p}\\&\le c_{10}\,\tau\;\|T\|\left(\sum_{k=m}^{\infty}\left(2^{k(1/s_1+1/v-1/u-\alpha)}\,|\sigma_{2^k}|\right)^p\right)^{1/p}.\end{aligned} \tag{5.0.6}$$

Obviously, for all $k\ge m$ we have that

$$2^{k/r_0}\,|\sigma_{2^k}|\le\left(\sum_{j=m}^{\infty}2^{jt/r_0}\,|\sigma_{2^j}|^t\right)^{1/t}$$

and, therefore,

$$\begin{aligned}
&\left(\sum_{k=m}^{\infty}\left(2^{k(1/s_1+1/v-1/u-\alpha)}\,|\sigma_{2^k}|\right)^p\right)^{1/p}\\
&=\left(\sum_{k=m}^{\infty}\left(2^{k(1/s_1+1/v-1/u-\alpha-1/r_0)}\,2^{k/r_0}\,|\sigma_{2^k}|\right)^p\right)^{1/p}\\
&\le\left(\sum_{j=m}^{\infty}2^{jt/r_0}\,|\sigma_{2^j}|^t\right)^{1/t}\left(\sum_{k=m}^{\infty}\left(2^{k(1/s_1+1/v-1/u-\alpha-1/r_0)}\right)^p\right)^{1/p}.
\end{aligned}$$

Since $1/s_1+1/v-1/u-\alpha-1/r_0<0$, the summation formula for the geometric series gives

$$\left(\sum_{k=m}^{\infty}\left(2^{k(1/s_1+1/v-1/u-\alpha-1/r_0)}\right)^p\right)^{1/p}\le c_{11}\,2^{m(1/s_1+1/v-1/u-\alpha-1/r_0)}.$$

Hence, we obtain that

$$L^{(e)}_{s_1,\infty}(R_m)\le c_{12}\,\tau\,\|T\|\;2^{m(1/s_1+1/v-1/u-\alpha-1/r_0)}\left(\sum_{j=m}^{\infty}2^{jt/r_0}\,|\sigma_{2^j}|^t\right)^{1/t}$$

and consequently

$$e_{2^{m-1}}(R_m)\le c_{13}\,\tau\,\|T\|\;2^{m(1/v-1/u-\alpha-1/r_0)}\left(\sum_{j=m}^{\infty}2^{jt/r_0}\,|\sigma_{2^j}|^t\right)^{1/t}. \tag{5.0.7}$$

The additivity of entropy numbers and the estimates given in (5.0.5) and (5.0.7) lead to

$$\begin{aligned}
e_{2^m}(TD_\sigma)&\le e_{2^{m-1}}(S_m)+e_{2^{m-1}}(R_m)\\
&\qquad\le c_{14}\,\tau\,\|T\|\left[2^{m(1/v-1/u-\alpha-1/q)}\left(\sum_{j=0}^{m}2^{jt/q}\,|\sigma_{2^j}|^t\right)^{1/t}\right.\\
&\qquad\qquad\left.+\,2^{m(1/v-1/u-\alpha-1/r_0)}\left(\sum_{j=m}^{\infty}2^{jt/r_0}\,|\sigma_{2^j}|^t\right)^{1/t}\right].
\end{aligned}$$

According to Lemma 2.3.3 it is enough to show that $\left(\varphi(2^m)\,2^{m/s}\,e_{2^m}(TD_\sigma)\right)_m$ belongs to l_t, where s is given by the equality $1/s=1/r+1/u-1/v+\alpha$. To this end, we first observe that

$$\begin{aligned}\left(\sum_{m=0}^{\infty}\Big(\varphi(2^m)\,2^{m/s}\,e_{2^m}(TD_\sigma)\Big)^t\right)^{1/t}&\\ \le c_{15}\,\tau\,\|T\|\Bigg(\sum_{m=0}^{\infty}(\varphi(2^m))^t\,2^{mt(1/r-1/q)}\sum_{j=0}^{m}2^{jt/q}\,|\sigma_{2^j}|^t&\\ +\sum_{m=0}^{\infty}(\varphi(2^m))^t\,2^{mt(1/r-1/r_0)}\sum_{j=m}^{\infty}2^{jt/r_0}\,|\sigma_{2^j}|^t\Bigg)^{1/t}&.\end{aligned}$$

Changing the order of summation gives

$$\begin{aligned}\left(\sum_{m=0}^{\infty}\Big(\varphi(2^m)\,2^{m/s}\,e_{2^m}(TD_\sigma)\Big)^t\right)^{1/t}&\\ \le c_{15}\,\tau\,\|T\|\Bigg(\sum_{j=0}^{\infty}2^{jt/q}\,|\sigma_{2^j}|^t\sum_{m=j}^{\infty}(\varphi(2^m))^t\,2^{mt(1/r-1/q)}&\\ +\sum_{j=0}^{\infty}2^{jt/r_0}\,|\sigma_{2^j}|^t\sum_{m=0}^{j}(\varphi(2^m))^t\,2^{mt(1/r-1/r_0)}\Bigg)^{1/t}&.\end{aligned}$$

Finally, we choose q such that $0 < q < r$. Then Lemma 3.0.4 gives

$$\sum_{m=j}^{\infty}(\varphi(2^m))^t\,2^{mt(1/r-1/q)} \le c_{16}\,(\varphi(2^j))^t\,2^{jt(1/r-1/q)}.$$

Furthermore, recall that $1/r > 1/r_0$. Hence, taking Lemma 3.0.3 into account, we obtain

$$\sum_{m=0}^{j}(\varphi(2^m))^t\,2^{mt(1/r-1/r_0)} \le c_{17}\,(\varphi(2^j))^t\,2^{jt(1/r-1/r_0)}.$$

Consequently, we arrive at

$$\left(\sum_{m=0}^{\infty}\Big(\varphi(2^m)\,2^{m/s}\,e_{2^m}(TD_\sigma)\Big)^t\right)^{1/t} \le c_{18}\,\tau\,\|T\|\left(\sum_{j=0}^{\infty}\Big(\varphi(2^j)\,2^{j/r}\,|\sigma_{2^j}|\Big)^t\right)^{1/t},$$

which shows that

$$(\sigma_n)_n \in l_{r,t,\varphi} \quad \text{implies} \quad \Big(e_n(TD_\sigma)\Big)_n \in l_{s,t,\varphi}$$

and finishes the proof. For the sake of completeness, we sketch the proof for the case $t = \infty$. This time, choose s_0 such that $1/s_0 > \alpha + (1/u - 1/v)_+ + 1/r$. Then, using

(5.0.4), we obtain

$$\begin{aligned}L^{(e)}_{s_0,\infty}(S_m) &\le c_5\,\tau\;\|T\|\;\left(\sum_{k=0}^{m-1}\left(2^{k(1/s_0+1/v-1/u-\alpha)}\,|\sigma_{2^k}|\right)^p\right)^{1/p}\\ &\le c_5\,\tau\;\|T\|\sup_{k\in\mathbb{N}_0}\varphi(2^k)\,2^{k/r}\,|\sigma_{2^k}|\times\\ &\qquad\times\left(\sum_{k=0}^{m-1}\left((\varphi(2^k))^{-1}\,2^{k(1/s_0+1/v-1/u-\alpha-1/r)}\right)^p\right)^{1/p}.\end{aligned}$$

Applying Lemma 3.0.3, the sum on the right hand side can be estimated by

$$\left(\sum_{k=0}^{m-1}\left((\varphi(2^k))^{-1}\,2^{k(1/s_0+1/v-1/u-\alpha-1/r)}\right)^p\right)^{1/p}\le c_{19}\,(\varphi(2^m))^{-1}\,2^{m(1/s_0+1/v-1/u-\alpha-1/r)}.$$

Hence, we get

$$e_{2^{m-1}}(S_m)\le c_{20}\,\tau\;\|T\|\;(\varphi(2^m))^{-1}\,2^{m(1/v-1/u-\alpha-1/r)}\sup_{k\in\mathbb{N}_0}\varphi(2^k)\,2^{k/r}\,|\sigma_{2^k}|.$$

Now consider the remainder R_m. This time choose s_1 such that $\alpha+(1/u-1/v)_+<1/s_1<\alpha+1/r+1/u-1/v$. Then, using (5.0.6), we obtain

$$\begin{aligned}L^{(e)}_{s_1,\infty}(R_m) &\le c_{10}\,\tau\;\|T\|\left(\sum_{k=m}^{\infty}\left(2^{k(1/s_1+1/v-1/u-\alpha)}\,|\sigma_{2^k}|\right)^p\right)^{1/p}\\ &\le c_{10}\,\tau\;\|T\|\sup_{k\in\mathbb{N}_0}\varphi(2^k)\,2^{k/r}\,|\sigma_{2^k}|\times\\ &\qquad\times\left(\sum_{k=m}^{\infty}\left((\varphi(2^k))^{-1}\,2^{k(1/s_1+1/v-1/u-\alpha-1/r)}\right)^p\right)^{1/p}.\end{aligned}$$

From Lemma 3.0.4 we know that

$$\left(\sum_{k=m}^{\infty}\left((\varphi(2^k))^{-1}\,2^{k(1/s_1+1/v-1/u-\alpha-1/r)}\right)^p\right)^{1/p}\le c_{21}\,(\varphi(2^m))^{-1}\,2^{m(1/s_1+1/v-1/u-\alpha-1/r)}$$

and this yields

$$e_{2^{m-1}}(R_m)\le c_{22}\,\tau\;\|T\|\;(\varphi(2^m))^{-1}\,2^{m(1/v-1/u-\alpha-1/r)}\sup_{k\in\mathbb{N}_0}\varphi(2^k)\,2^{k/r}\,|\sigma_{2^k}|.$$

Finally, we arrive at

$$\begin{aligned}\varphi(2^m)\,2^{m/s}\,e_{2^m}(TD_\sigma) &\le \varphi(2^m)\,2^{m/s}\,e_{2^{m-1}}(S_m)+\varphi(2^m)\,2^{m/s}\,e_{2^{m-1}}(R_m)\\ &\le c_{23}\,\tau\;\|T\|\sup_{k\in\mathbb{N}_0}\varphi(2^k)\,2^{k/r}\,|\sigma_{2^k}|.\end{aligned}$$

This shows that

$$(\sigma_n)_n \in l_{r,\infty,\varphi} \quad \text{implies} \quad \Big(e_n(TD_\sigma)\Big)_n \in l_{s,\infty,\varphi}$$

with $1/s = 1/r + 1/u - 1/v + \alpha$ and finishes the proof of Theorem 5.0.1. ■

Remark 5. The entropy estimate (5.0.1) is essential for Theorem 5.0.1 to hold. However, we can always estimate the entropy of an operator $TR : l_v^n \to Y$ by

$$e_k(TR : l_v^n \to Y) \le \|T\| \, \|R\| , \quad 1 \le k \le n.$$

Hence, we have (5.0.1) with $\tau = 1$ and $\alpha = \beta = 0$. This can be seen as a setting of Theorem 5.0.1 where we have no additional entropy information. Carefully reading the proof of Theorem 5.0.1 shows that the assertion of Theorem 5.0.1 remains true in this case, i.e.

$$\Big(e_n(TD_\sigma : l_u \to Y)\Big)_n \in l_{s,t,\varphi} \quad \text{with} \quad 1/s = 1/r + 1/u - 1/v.$$

Now we consider the special case of Theorem 5.0.1 where $Y = l_v$ and $T = id : l_v \to l_v$. Then the composition operator TD_σ is nothing but the diagonal operator $D_\sigma : l_u \to l_v$. Hence, with the help of Theorem 5.0.1 we can get insights into the entropy behavior of diagonal operators. The following result extends an earlier result of Carl given in [C81b, Th. 2] (see also [O78]) by adding slowly varying functions.

Corollary 5.0.5. *Let $1 \le u, v \le \infty$ and let φ be a slowly varying function. Suppose that $0 < r < \infty$, $1/r > (1/v - 1/u)_+$, $0 < t \le \infty$ and $1/s = 1/r + 1/u - 1/v$. Then we have that*

$$(\sigma_n)_n \in l_{r,t,\varphi} \quad \text{if and only if} \quad \Big(e_n(D_\sigma : l_u \to l_v)\Big)_n \in l_{s,t,\varphi}.$$

Proof. To prove the only-if-part we apply Theorem 5.0.1 with $Y = l_v$ and $T = I_{l_v}$. Using the trivial estimate $e_k(I_{l_v}R : l_v^n \to l_v) \le \|R\|$, i.e. $\tau = 1$ and $\alpha = \beta = 0$ (cf. Remark 5), leads to the desired assertion. The proof of the if-part is analogous to the proof of Proposition 1 in [C81b]. Without loss of generality we assume that $|\sigma_1| \ge |\sigma_2| \ge \ldots \ge 0$. Obviously, if there exists an index k such that $\sigma_k = 0$, then the statement is true. So assume that $|\sigma_1| \ge |\sigma_2| \ge \ldots > 0$. For $n \in \mathbb{N}$, define canonical operators $J_n \in \mathcal{L}(l_u^n, l_u)$ and $Q_n \in \mathcal{L}(l_v, l_v^n)$ by

$$J_n(\xi_1, \ldots, \xi_n) := (\xi_1, \ldots, \xi_n, 0, 0, \ldots)$$

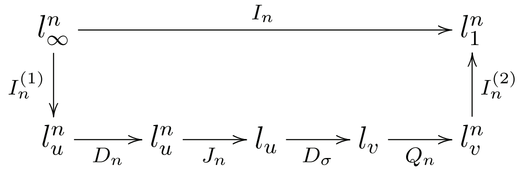
and

$$Q_n(\xi_1, \xi_2, \ldots) := (\xi_1, \ldots, \xi_n).$$

Clearly, we have that $\|J_n\| = \|Q_n\| = 1$. Furthermore, define $D_n \in \mathcal{L}(l_u^n, l_u^n)$ by

$$D_n(\xi_1, \ldots, \xi_n) := (\xi_1/\sigma_1, \ldots, \xi_n/\sigma_n)$$

and observe that $\|D_n\| \le |\sigma_n|^{-1}$. Then the identity operator $I_n : l_\infty^n \to l_1^n$ can be factorized as



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where $I_n^{(1)}$ and $I_n^{(2)}$ denote the respective identity operators. From [P78, 12.2.1.] we know that $e_n(I_n : l_\infty^n \to l_1^n) \ge \frac{n}{2e}$. Hence, the multiplicativity of entropy numbers yields

$$\begin{aligned}\frac{n}{2e} &\le e_n(I_n : l_\infty^n \to l_1^n)\\ &\le \left\|I_n^{(2)} : l_v^n \to l_1^n\right\| \, \|Q_n\| \; e_n(D_\sigma) \; \|J_n\| \; \|D_n\| \; \left\|I_n^{(1)} : l_\infty^n \to l_u^n\right\|\\ &\le n^{1-1/v} \, e_n(D_\sigma) \, |\sigma_n|^{-1} \, n^{1/u}.\end{aligned}$$

Consequently, we obtain the estimate

$$|\sigma_n| \le 2e \, n^{1/u-1/v} \, e_n(D_\sigma)$$

for all $n \in \mathbb{N}$ and, therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \left[\varphi(n) \, n^{1/r-1/t} \, |\sigma_n|\right]^t &\le (2e)^t \sum_{n=1}^{\infty} \left[\varphi(n) \, n^{1/r+1/u-1/v-1/t} \, e_n(D_\sigma)\right]^t\\ &= (2e)^t \sum_{n=1}^{\infty} \left[\varphi(n) \, n^{1/s-1/t} \, e_n(D_\sigma)\right]^t < \infty.\end{aligned}$$

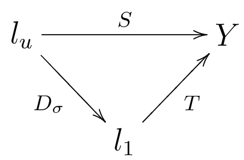
This implies $(\sigma_n)_n \in l_{r,t,\varphi}$. ■

We would like to highlight the special case of Corollary 5.0.5 where $t = \infty$. Here we get, for $1/r > (1/v - 1/u)_+$ and $1/s = 1/r + 1/u - 1/v$, that

$$\sigma_n \preccurlyeq n^{-1/r} \, \varphi(n) \quad \text{if and only if} \quad e_n(D_\sigma : l_u \to l_v) \preccurlyeq n^{-1/s} \, \varphi(n), \tag{5.0.8}$$

where φ is an arbitrary slowly varying function. Note that the only-if part of (5.0.8) can also be obtained from [Kü05, Th. 2.2] (see also [Kü01, Th. 2]).

Next, let us consider the special case



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where Y is a Banach space of type p. This setting was studied by Carl in [C82, Th. 2]. By adding slowly varying functions, we will extend Carl's results.

Corollary 5.0.6. *Let $1 \leq u \leq \infty$ and let φ be a slowly varying function. Suppose that $D_\sigma : l_u \to l_1$ is a diagonal operator generated by a sequence $(\sigma_n)_n \in l_{r,t,\varphi}$, where $0 < r < \infty$, $1/r > 1 - 1/u$ and $0 < t \leq \infty$. Furthermore, suppose that $T : l_1 \to Y$ is an arbitrary operator with image in a Banach space Y of type p, $1 < p \leq 2$. Then for the composition operator TD_σ it holds that*

$$\Big(e_n(TD_\sigma : l_u \to Y)\Big)_n \in l_{s,t,\varphi} \quad \textit{with} \quad 1/s = 1/r + 1/u - 1/p.$$

Proof. Applying Theorem 5.0.1 with $v = 1$ and $\alpha = \beta = 1 - 1/p$ (cf. Theorem 2.6.3) immediately yields the assertion. ■

Finally, we return to entropy estimates of absolutely convex hulls in the diagonal case and prove the statement given in (1.1.6).

Corollary 5.0.7. *Let $A = \{x_1, x_2, \ldots\} \subset X$ be a precompact subset of a Banach space X of type p, $1 < p \leq 2$. Suppose that $0 < r < \infty$, $0 < s \leq \infty$ and φ is a slowly varying function. Furthermore, suppose that $\|x_n\| \leq \sigma_n$ for all $n \in \mathbb{N}$, where $(\sigma_n)_n$ is a monotone decreasing null sequence with $(\sigma_n)_n \in l_{r,s,\varphi}$. Then we have that*

$$\Big(e_n(\mathrm{aco}(A))\Big)_n \in l_{q,s,\varphi} \quad \textit{for} \quad 1/q = 1/r + 1/p'.$$

Proof. Due to (1.1.2) we have that $e_n(\mathrm{aco}(A)) = e_n(SD_\sigma : l_1 \to X)$. Hence, applying Corollary 5.0.6 with $u = 1$ immediately yields the assertion. ■

6 Applications of entropy and Gelfand numbers of convex hulls

We start with studying $C(M)$-valued operators T_K defined by abstract kernels K. Using a general approach, we show how entropy and Kolmogorov numbers of such abstract kernel operators are connected to the entropy of the image of the abstract kernel. We consider the important case of a weakly singular integral operator $T_K : L_p[0,1] \to C[0,1]$ generated by a convolution kernel. Here, the Hilbert space case $p = 2$ is of particular interest. In addition, we also investigate the case when $T_K : L_2[0,1] \to L_q[0,1]$ for $1 \leq q < \infty$. Finally, we deal with entropy estimates of the classical Riemann-Liouville operator in different settings. In all these applications we need sharp estimates of entropy and Gelfand numbers of absolutely convex hulls.

6.1 Operators with values in $C(M)$

As already mentioned in the introduction, the entropy behavior of a compact operator is reflected by that of a $C(M)$-valued operator on a compact metric space (M,d). Thus, when studying entropy numbers of compact operators, $C(M)$-valued operators are universal. Following the monograph [CS90] we first recall some basic concepts related to $C(M)$-valued operators.

The *modulus of continuity* $\omega(f;\delta)$ of a bounded scalar-valued function f on a metric space (M,d) is defined by

$$\omega(f;\delta) := \sup\{|f(t) - f(s)| : s,t \in M, d(s,t) \leq \delta\}$$

for $0 \leq \delta < \infty$. If $T : X \to C(M)$ is an operator from a Banach space X in the space $C(M)$ of all continuous scalar-valued functions on a compact metric space (M,d), then the *modulus of continuity* $\omega(T;\delta)$ of the operator T is defined by

$$\omega(T;\delta) := \sup_{x \in B_X} \omega(Tx;\delta).$$

This canonical definition in fact makes sense for all $\delta \geq 0$ since

$$\omega(T;\delta) \leq 2\,\|T\|\,.$$

From the Arzelà-Ascoli theorem we know that an operator $T : X \to C(M)$ is compact if and only if the limit relation

$$\lim_{\delta \to 0+} \omega(T;\delta) = 0 \tag{6.1.1}$$

is fulfilled (cf. [CS90, Prop. 5.5.1]). In analogy to Hölder-continuous functions, a compact operator $T : X \to C(M)$ is called *Hölder-continuous of type* α, $0 < \alpha \leq 1$, if

$$|T|_\alpha := \sup_{\delta > 0} \frac{\omega(T;\delta)}{\delta^\alpha} < \infty.$$

Note that a Hölder-continuous operator T of type α actually maps X into the space $C^\alpha(M)$ of Hölder-continuous scalar-valued functions of type α on M. In the special case $\alpha = 1$, a Hölder-continuous operator of type 1 is said to be *Lipschitz-continuous*. The vector space $\mathcal{L}ip_\alpha\Big(X, C(M)\Big)$ of all operators from X into $C(M)$ which are Hölder-continuous of type α becomes a Banach space under the norm

$$\mathrm{Lip}_\alpha(T) := \max \left\{ \|T : X \to C(M)\|, |T|_\alpha \right\}.$$

For the space of all Lipschitz-continuous operators $T : X \to C(M)$ we simply write

$$\Big[\mathcal{L}ip\Big(X, C(M)\Big), \mathrm{Lip}\Big] := \Big[\mathcal{L}ip_1\Big(X, C(M)\Big), \mathrm{Lip}_1\Big].$$

By changing the metric d on M to d^α, $0 < \alpha \leq 1$, we reduce a Hölder-continuous operator of type α to a Lipschitz-continuous operator, i.e.

$$\mathcal{L}ip_\alpha\Big(X, C((M,d))\Big) = \mathcal{L}ip\Big(X, C((M,d^\alpha))\Big).$$

Now we represent compact and Hölder-continuous $C(M)$-valued operators by abstract kernels. To this end, let us introduce the vector space $C(M,Z)$ of all continuous Z-valued functions on a compact metric space (M,d), where Z is an arbitrary Banach space. It is clear that $C(M,Z)$ is a Banach space with respect to the supremum norm

$$\|K\|_\infty := \sup_{s \in M} \|K(s)\|_Z.$$

Just as for scalar-valued functions on M we define a modulus of continuity by

$$\omega_Z(K;\delta) := \sup \left\{ \|K(s) - K(t)\|_Z : s, t \in M, d(s,t) \leq \delta \right\}$$

for $0 \leq \delta < \infty$. Since $\omega_Z(K;\delta) \leq 2\,\|K\|_\infty$ holds true for all $\delta \geq 0$, this is well defined for arbitrary bounded Z-valued functions K on M. From the fact that a continuous function on a compact set is uniformly continuous, we see that a bounded Z-valued function is continuous if and only if

$$\lim_{\delta \to 0+} \omega_Z(K;\delta) = 0. \tag{6.1.2}$$

A stronger condition than the limit relation in (6.1.2) is that of *Hölder-continuity.* The continuous function $K \in C(M,Z)$ is said to be *Hölder-continuous of type α,* $0 < \alpha \le 1$, if

$$|K|_{Z,\alpha} := \sup_{\delta>0} \frac{\omega_Z(K;\delta)}{\delta^\alpha} < \infty.$$

As a direct consequence of this definition, we see that $\omega_Z(K;\delta) \le |K|_{Z,\alpha}\,\delta^\alpha$ holds for all $\delta \ge 0$. Consequently, for all $s,t \in M$ we have

$$\|K(s) - K(t)\|_Z \le |K|_{Z,\alpha} \Big(d(s,t)\Big)^\alpha$$

and this implies

$$\varepsilon_n(\mathrm{Im}(K)) \le |K|_{Z,\alpha} \Big(\varepsilon_n(M)\Big)^\alpha$$

for all $n \in \mathbb{N}$. The vector space $C^\alpha(M,Z)$ of all Hölder-continuous Z-valued functions of type α on M turns out to be a Banach space with respect to the norm

$$\|K\|_{Z,\alpha} := \max\left\{\|K\|_\infty, |K|_{Z,\alpha}\right\}.$$

If Z is a dual space, i.e. $Z = X'$ for some Banach space X, then an element $K \in C(M,X')$ gives rise to an operator $T_K : X \to C(M)$ according to the rule

$$(T_K x)(s) := \langle x, K(s)\rangle\,, \quad x \in X, s \in M, \tag{6.1.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X'. The function K is called the *abstract kernel* of the operator T_K. Obviously, T_K is a linear operator and an easy computation shows that it is also bounded:

$$\|T_K : X \to C(M)\| = \sup_{x\in B_X}\sup_{s\in M} |\langle x, K(s)\rangle| = \sup_{s\in M} \|K(s)\|_{X'} = \|K\|_\infty < \infty.$$

Moreover, we have that

$$\omega(T_K;\delta) = \omega_{X'}(K;\delta), \tag{6.1.4}$$

hence, by combining (6.1.1) and (6.1.2), we observe that the operator $T_K : X \to C(M)$ is compact. It does not surprise that stronger conditions on the kernel K are reflected in the properties of the generated operator T_K. Indeed, if the kernel K is even Hölder-continuous of type α, then $T_K : X \to C(M)$ is a Hölder-continuous operator of type α.

On the other hand, any compact or even Hölder-continuous operator T from X into $C(M)$ can be generated by an appropriate kernel in the sense of (6.1.3). More precisely, we define the abstract kernel $K : M \to X'$ by

$$K(s) := T'\delta_s, \tag{6.1.5}$$

where δ_s is the Dirac functional on $C(M)$ given by $\langle L, \delta_s \rangle := L(s)$ for $s \in M$, $L \in C(M)$. An easy computation shows that

$$\|K\|_\infty = \|T : X \to C(M)\| \quad \text{and} \quad \omega_{X'}(K;\delta) = \omega(T;\delta).$$

On the one hand, this implies $K \in C(M, X')$ for $T \in \mathcal{K}\Big(X, C(M)\Big)$, on the other hand we see by the very definition that $|K|_{X',\alpha} = |T|_\alpha$. Hence, if $T \in \mathcal{L}ip_\alpha\Big(X, C(M)\Big)$ is Hölder-continuous of type α, $0 < \alpha \le 1$, then for the kernel K given in (6.1.5) it holds that $K \in C^\alpha(M, X')$ and $\|K\|_{X',\alpha} = \mathrm{Lip}_\alpha(T)$. Moreover, for $x \in X$ and $s \in M$, we have

$$(T_K x)(s) = \langle x, K(s) \rangle = \langle x, T'\delta_s \rangle = \langle Tx, \delta_s \rangle = (Tx)(s),$$

which means that the original operator T coincides with the operator T_K generated by the kernel K as given in (6.1.3). Summarizing the above-mentioned facts we arrive at the following well-known statement (cf. [CS90, Prop. 5.13.1]).

Proposition 6.1.1. [CS90] *Let (M, d) be a compact metric space and let X be a Banach space. Then the map $\Phi : \mathcal{K}\Big(X, C(M)\Big) \to C(M, X')$ defined by*

$$\Phi(T)(s) = T'\delta_s, \quad s \in M,$$

is a metric isomorphism from $\mathcal{K}\Big(X, C(M)\Big)$ onto $C(M, X')$ as well as a metric isomorphism from the subclass $\mathcal{L}ip_\alpha\Big(X, C(M)\Big)$ of $\mathcal{K}\Big(X, C(M)\Big)$ onto the subclass $C^\alpha(M, X')$ of $C(M, X')$, for $0 < \alpha \le 1$.

Finally, in order to apply our previous results to $C(M)$-valued operators T_K generated by abstract kernels in the sense of (6.1.3), we have to find a link to absolutely convex hulls of precompact sets. To this end, let $S : l_1(M) \to X'$ be the operator defined on the canonical basis $(e_s)_{s\in M}$ of $l_1(M)$ by $Se_s = K(s)$. Furthermore, let J_∞ be the canonical embedding from $C(M)$ into $l_\infty(M)$ and let $\mathcal{K}_X$ be the canonical metric injection from X into the bidual X''. It follows directly from the definitions that

$$J_\infty T_K = S'\mathcal{K}_X \tag{6.1.6}$$

and

$$e_n(S) = e_n\Big(\operatorname{aco}(\operatorname{Im}(K))\Big) \quad \text{and} \quad c_n(S) = c_n\Big(\operatorname{aco}(\operatorname{Im}(K))\Big). \tag{6.1.7}$$

From (6.1.6) and the injectivity of the entropy numbers up to the factor two, we conclude that

$$e_n(T_K) \le 2\, e_n(J_\infty T_K) = 2\, e_n(S'\mathcal{K}_X) \le 2\, e_n(S').$$

The next step is to connect the entropy numbers of T_K with the entropy numbers of the absolutely convex hull of $\mathrm{Im}(K)$ by using duality relations. To this end, assume that X is a B-convex Banach space and that the abstract kernel $K : M \to X'$ is continuous. Since also X' is B-convex, the operator S maps into a B-convex Banach space. Furthermore, due to (6.1.7), the operator S is compact. Hence we can use Theorem 2.5.1 to relate the entropy numbers of S with the entropy numbers of S'. According to the previous considerations, this relates the entropy of the compact operator $T_K : X \to C(M)$ to the entropy numbers of the absolutely convex hull of $\mathrm{Im}(K)$ as a precompact subset of X'. In addition, the entropy numbers of T_K can also be related to the Gelfand numbers of $\mathrm{aco}(\mathrm{Im}(K))$ by using well known properties of the symmetrized approximation numbers. Indeed, by Theorem 5.3.2 in [CS90] and in view of (6.1.6) and (6.1.7) we have that

$$\begin{aligned} d_n(T_K) &= t_n(T_K) = t_n(J_\infty T_K) = t_n(S'\mathcal{K}_X) \le t_n(S') = t_n(S) \\ &\le c_n(S) = c_n\Big(\mathrm{aco}(\mathrm{Im}(K))\Big) \end{aligned} \tag{6.1.8}$$

and applying Theorem 2.1.1 to the operator T_K leads to the assertion. Note that we do not need the B-convexity of X in this case. Let us summarize these results in the following lemma.

Lemma 6.1.2. *Let X be a B-convex Banach space. Then for every $0 < \alpha < \infty$ there exists a constant $c = c(\alpha, X') > 0$, such that for the compact operator $T_K : X \to C(M)$ with kernel $K \in C(M, X')$ and all $n \in \mathbb{N}$ we have*

$$\sup_{1\le k\le n} k^\alpha e_k(T_K) \le c \sup_{1\le k\le n} k^\alpha e_k\Big(\mathrm{aco}(\mathrm{Im}(K))\Big).$$

In the case of an arbitrary Banach space X, this statement remains true if one replaces $e_k\Big(\mathrm{aco}(\mathrm{Im}(K))\Big)$ on the right hand side by $c_k\Big(\mathrm{aco}(\mathrm{Im}(K))\Big)$.

As a consequence of Lemma 6.1.2 and the Hardy-type inequalities given in Lemma 3.0.8 and 3.0.9 we obtain the following result. The proof is analogous to that of Theorem 1.3.5 or Proposition 2.5.2.

Corollary 6.1.3. *Let X be a B-convex Banach space and let $0 < r, s < \infty$. Suppose that φ is a slowly varying function. Then there exists a constant $c = c(r, s, \varphi, X') > 0$ such that for the compact operator $T_K : X \to C(M)$ with kernel $K \in C(M, X')$ and all $N \in \mathbb{N}$ it holds*

$$\sum_{n=1}^{N} (\varphi(n))^s\, n^{s/r-1} \Big(e_n(T_K)\Big)^s \le c \sum_{n=1}^{N} (\varphi(n))^s\, n^{s/r-1} \Big(e_n(\mathrm{aco}(\mathrm{Im}(K)))\Big)^s$$

and

$$\sup_{1\le n\le N} \varphi(n)\, n^{1/r}\, e_n(T_K) \le c \sup_{1\le n\le N} \varphi(n)\, n^{1/r}\, e_n\Big(\mathrm{aco}(\mathrm{Im}(K))\Big).$$

In particular, we see that

$$\Big(e_n(\mathrm{aco}(\mathrm{Im}(K)))\Big)_n \in l_{r,s,\varphi} \quad \textit{implies} \quad \Big(e_n(T_K)\Big)_n \in l_{r,s,\varphi}$$

for all $0 < r < \infty$, $0 < s \leq \infty$ *and any slowly varying function* φ. *In the case of an arbitrary Banach space* X, *the same holds true for the Gelfand numbers* $c_n\Big(\mathrm{aco}(\mathrm{Im}(K))\Big)$ *instead of* $e_n\Big(\mathrm{aco}(\mathrm{Im}(K))\Big)$.

Now we are well prepared to give entropy estimates of Lipschitz-continuous operators $T_K : X \to C(M)$ with kernel $K \in C^1(M, X')$ (cf. [CHK88], [CS90, Th. 5.10.1], [CKP99], [St99], [St00], [CE01], [CHP11]).

Theorem 6.1.4. *Let* M *be a compact metric space with the property that there are constants* $\tau, \sigma > 0$ *and* $\gamma \in \mathbb{R}$ *such that for all* $n \in \mathbb{N}$ *it holds*

(a) $\varepsilon_n(M) \leq \tau\, n^{-\sigma} (\log(n+1))^{-\gamma}$ *or*

(b) $e_n(M) \leq \tau\, n^{-\sigma} (\log(n+1))^{-\gamma}$.

Furthermore, let X *be a Banach space such that the dual Banach space* X' *is of type* p, $1 < p \leq 2$. *Moreover, let* $K \in C^1(M, X')$ *be a Lipschitz-continuous kernel and let* $T_K : X \to C(M)$ *be the corresponding induced operator given by (6.1.3). Then in the case (a) we have the entropy estimate*

$$e_n(T_K) \leq c\, n^{-\sigma - 1/p'} (\log(n+1))^{-\gamma}\, \|K\|_{X',1}$$

for all $n = 1, 2, 3, \ldots$ *and in the case (b) it holds*

$$e_n(T_K) \leq c \begin{cases} n^{-\sigma} (\log(n+1))^{-\gamma}\, \|K\|_{X',1}\,, & \sigma < 1/p', \gamma \in \mathbb{R}, \\ n^{-1/p'} (\log(n+1))^{1/p' - \sigma} (\log(\log(n+3)))^{-\gamma}\, \|K\|_{X',1}\,, & \sigma > 1/p', \gamma \in \mathbb{R}, \\ n^{-1/p'} (\log(n+1))^{1-\gamma}\, \|K\|_{X',1}\,, & \sigma = 1/p', \gamma < 1, \\ n^{-1/p'} \log\log(n+3)\, \|K\|_{X',1}\,, & \sigma = 1/p', \gamma = 1, \end{cases}$$

for all $n = 1, 2, 3, \ldots$, *where* c *is a constant which may depend on* τ, σ, γ, p *and the type constant of* X'. *Furthermore, if* $0 < r, s < \infty$, *then*

$$\Big(\varepsilon_n(M)\Big)_n \in l_{r,s} \quad \textit{implies} \quad \Big(e_n(T_K)\Big)_n \in l_{q,s}$$

with $1/q = 1/p' + 1/r$. *Moreover, we have that* $\Big(e_n(M)\Big)_n \in l_{r,s}$ *implies*

$$\Big(e_n(T_K)\Big)_n \in \begin{cases} l_{p',s,\varphi_1}, & 0 < r < p',\ 0 < s < \infty, \\ l_{p',\infty,\varphi_2}, & r = p',\ p \leq s < \infty, \\ l_{r,s}, & p' < r < \infty,\ 0 < s < \infty, \end{cases}$$

where $\varphi_1(n) = (\log(n+1))^{1/r - 1/p' - 1/s}$ *and* $\varphi_2(n) = (\log(n+1))^{1/s - 1}$.

Proof. First observe that for all $n \in \mathbb{N}$ it holds that $\varepsilon_n(\mathrm{Im}(K)) \le \|K\|_{X',1}\, \varepsilon_n(M)$. In a next step, we use the results from (1.3.19) - (1.3.29) to estimate the dyadic entropy numbers of $\mathrm{aco}(\mathrm{Im}(K))$. Finally, we carry over these estimates from $e_n\Big(\mathrm{aco}(\mathrm{Im}(K))\Big)$ to $e_n(T_K)$ by using Corollary 6.1.3. ■

Remark 6.

(i) Let (M,d) and X be as in Theorem 6.1.4, but assume that $K:(M,d)\to X'$ is a Hölder-continuous kernel of type α, $0<\alpha\le 1$. Then the estimates in the cases (a) and (b) of Theorem 6.1.4 remain true if we replace the exponents σ and γ by $\alpha\sigma$ and $\alpha\gamma$, respectively. For example, the estimate of the dyadic entropy numbers of T_K in the case (a) reads as

$$e_n(T_K) \le c\, n^{-\alpha\sigma-1/p'}\,(\log(n+1))^{-\alpha\gamma}\ \|K:(M,d)\to X'\|_{X',\alpha}$$

for all $n=1,2,3,\dots$. To see this, change the metric on M from d to d^α. Then $K:(M,d^\alpha)\to X'$ is a Lipschitz-continuous kernel,

$$\|K:(M,d^\alpha)\to X'\|_{X',1} = \|K:(M,d)\to X'\|_{X',\alpha}\,,$$

and applying Theorem 6.1.4 with (M,d^α) yields the assertion due to

$$\varepsilon_n(M,d^\alpha) = \Big(\varepsilon_n(M,d)\Big)^\alpha.$$

(ii) The assertion of Theorem 6.1.4 remains true for Lipschitz-continuous operators $T\in\mathcal{L}ip\Big(X,l_\infty(M)\Big)$ where (M,d) is a precompact metric space satisfying the entropy condition of Theorem 6.1.4. This is due to the fact that there exists a compact metric space $(\widehat{M},\widehat{d})$ and a Lipschitz-continuous operator $S\in\mathcal{L}ip\Big(X,C(\widehat{M})\Big)$ such that

$$\varepsilon_n\left(\widehat{M},\widehat{d}\right) \le \varepsilon_n(M,d) \le 2\,\varepsilon_n\left(\widehat{M},\widehat{d}\right)$$

and

$$\varepsilon_n(T:X\to l_\infty(M)) \le \varepsilon_n\left(S:X\to C(\widehat{M})\right) \le 2\,\varepsilon_n(T:X\to l_\infty(M))$$

for all $n\in\mathbb{N}$.

(iii) Consider an operator $T:l_1(M)\to X$, where (M,d) is a precompact metric space and X is a Banach space. Such an operator is said to be Lipschitz-continuous, if its dual T' belongs to $\mathcal{L}ip\Big(X',l_\infty(M)\Big)$ (cf. [CE01, Section 3]). Theorem 6.1.4 remains true for Lipschitz-continuous operators $T:l_1(M)\to X$, where X is a Banach space of type p, $1<p\le 2$, and (M,d) is a precompact metric space satisfying the entropy condition of Theorem 6.1.4.

(iv) For later use, we highlight the special case of Theorem 6.1.4 where $X = L_q$ for $1 < q < \infty$. Here we have in the case (a)

$$e_n(T_K) \leq c\, n^{-\sigma - \min\{1/q;1/2\}} \left(\log(n+1)\right)^{-\gamma} \|K\|_{X',1}$$

for all $n = 1, 2, 3, \ldots$ and in the case (b) we obtain for all $n = 1, 2, 3, \ldots$ the estimates

$$e_n(T_K) \leq c\, n^{-\sigma} \left(\log(n+1)\right)^{-\gamma} \|K\|_{X',1}$$

for $\sigma < \min\{1/q; 1/2\}$, $\gamma \in \mathbb{R}$,

$$e_n(T_K) \leq c\, n^{-\min\{1/q;1/2\}} (\log(n+1))^{\min\{1/q;1/2\}-\sigma} (\log(\log(n+3)))^{-\gamma} \|K\|_{X',1}$$

for $\sigma > \min\{1/q; 1/2\}$, $\gamma \in \mathbb{R}$,

$$e_n(T_K) \leq c\, n^{-\min\{1/q;1/2\}} \left(\log(n+1)\right)^{1-\gamma} \|K\|_{X',1}$$

for $\sigma = \min\{1/q; 1/2\}$, $\gamma < 1$ and

$$e_n(T_K) \leq c\, n^{-\min\{1/q;1/2\}} \log\log(n+3) \|K\|_{X',1}$$

for $\sigma = \min\{1/q; 1/2\}$, $\gamma = 1$.

This is due to the fact that $(L_q)' = L_{q'}$ is of optimal type $\min\{q'; 2\}$, where q' is given by the Hölder condition $1/q + 1/q' = 1$.

6.2 Abstract and integral operators

Let M be an arbitrary set and let $K : M \to X'$ be a bounded function from M into the dual X' of a Banach space X, i.e.

$$\|K\|_\infty = \sup_{s \in M} \|K(s)\|_{X'} < \infty.$$

Then K can be considered as an *abstract kernel* which gives rise to an operator $T_K : X \to l_\infty(M)$ from X into the space $l_\infty(M)$ of bounded scalar-valued functions on M by the rule

$$(T_K x)(s) := \langle x, K(s) \rangle, \quad x \in X, s \in M, \tag{6.2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X'. An easy computation shows that

$$\|T_K : X \to l_\infty(M)\| = \|K\|_\infty .$$

Let us define a pseudo-metric on M by

$$d(s,t) := \|K(s) - K(t)\|_{X'}, \quad s,t \in M.$$

Next, we introduce the cosets

$$[t] := \{s \in M : d(s,t) = 0\}, \quad t \in M,$$

and the family of cosets

$$\widehat{M} := \{[t] : t \in M\}.$$

Observe that if $s_1, s_2 \in [s]$ and $t_1, t_2 \in [t]$ then $d(s_1,t_1) = d(s_2,t_2)$. Hence, it makes sense to define a metric $\widehat{d}$ on $\widehat{M}$ by

$$\widehat{d}([s],[t]) := d(s,t).$$

It turns out that the entropy numbers of (M,d) and $(\widehat{M},\widehat{d})$ coincide,

$$\varepsilon_n(M,d) = \varepsilon_n(\widehat{M},\widehat{d}).$$

Usually, there is a natural metric $\tilde{d}$ given on M. If the entropy numbers of M with respect to the metric $\tilde{d}$ are known and if there is a relationship between $\tilde{d}$ and the pseudo-metric d, then it is generally easy to compute the entropy numbers of M with respect to the pseudo-metric d and, therefore, the entropy numbers of $\widehat{M}$ with respect to the metric $\widehat{d}$. As an example, consider the case $M = [0,1]$ with the natural distance $\tilde{d}(s,t) = |s-t|$ and entropy numbers

$$\varepsilon_n([0,1],\tilde{d}) = (2n)^{-1}.$$

If $d(s,t) \sim \Phi(|s-t|)$ for some continuous strictly increasing function $\Phi : [0,1] \to [0,\infty)$ with $\Phi(0) = 0$, then we obtain

$$\varepsilon_n([0,1],d) \sim \Phi\Big((2n)^{-1}\Big).$$

Our aim is to give entropy estimates of the operator $T_K : X \to l_\infty(M)$ mentioned above in (6.2.1). To this end, define the operator $S_K : X \to l_\infty(\widehat{M})$ by

$$(S_K x)([s]) := (T_K x)(s), \quad x \in X, [s] \in \widehat{M},$$

so that $e_n(T_K : X \to l_\infty(M)) = e_n(S_K : X \to l_\infty(\widehat{M}))$ for $n \in \mathbb{N}$. Then, for $x \in X$ and $[s],[t] \in \widehat{M}$, it holds

$$\begin{aligned} |(S_K x)([s]) - (S_K x)([t])| &= |(T_K x)(s) - (T_K x)(t)| = |\langle x, K(s) - K(t)\rangle| \\ &\leq \|x\|_X \, \|K(s) - K(t)\|_{X'} = \|x\|_X \, \widehat{d}([s],[t]), \end{aligned}$$

which means that $S_K x$ is a continuous function on the metric space $(\widehat{M},\widehat{d})$ and, moreover, that $S_K : X \to l_\infty(\widehat{M})$ is a Lipschitz-continuous operator. Now, given that the metric space $(\widehat{M},\widehat{d})$ is precompact, we can use Theorem 6.1.4 (cf. Remark 6 (ii)) to give entropy estimates of the operator S_K and, consequently, of the operator T_K.

In concrete cases, the Banach spaces X and X' are function spaces over some measure space (Ω,μ) such that the duality is given by integration with respect to μ,

$$\langle f,g\rangle = \int_\Omega f(x)g(x)\,\mathrm{d}\mu(x) \quad \text{for } f\in X, g\in X'.$$

The kernel K is given as a function $K : M\times\Omega \to \mathbb{R}$ such that $K(s)=K(s,\cdot)\in X'$, $s\in M$. Then the generated operator $T_K : X\to l_\infty(M)$ is given as a kernel integral operator by

$$(T_K f)(s) = \int_\Omega K(s,x)\,f(x)\,\mathrm{d}\mu(x).$$

To specialize even further, let us now assume that $X = L_p(\Omega,\mu)$ for some σ-finite measure space (Ω,μ) with $1<p<\infty$. Then $X' = L_{p'}(\Omega,\mu)$ has optimal type 2, if $1<p\le 2$ and optimal type p', if $2<p<\infty$. In this case, the crucial distance on M is given by

$$d(s,t) = \left(\int_\Omega |K(s,x)-K(t,x)|^{p'}\,\mathrm{d}\mu(x)\right)^{1/p'}.$$

6.3 Weakly singular integral operators from $L_p[0,1]$ to $C[0,1]$ generated by convolution kernels

In this section we give entropy estimates for weakly singular integral operators and weakly singular integral operators of Volterra-type generated by convolution kernels. We consider non-negative kernels K on $([0,1]\times[0,1])\setminus\{(0,0)\}$ so that T_K maps a function f on $[0,1]$ to the function

$$(T_K f)(t) = \int_0^1 K(t,x)f(x)\,\mathrm{d}x, \quad t\in[0,1].$$

In the remainder of this section we distinguish between the cases

$$(WS) \qquad K(t,x) := k(|t-x|),$$

$$(V) \qquad K(t,x) := \begin{cases} k(t-x), & \text{for } x<t,\\ 0, & \text{for } x\ge t,\end{cases}$$

of a weakly singular kernel (WS) and a Volterra-kernel (V), respectively. Here $k : (0,1] \to \mathbb{R}$ is a non-negative, continuous and strictly decreasing function with a singularity at 0, i.e.

$$\lim_{x\to 0+} k(x) = \infty.$$

Actually, it would be enough to assume that k is strictly decreasing only in a neighborhood of 0. Furthermore, we fix q with $1 < q < \infty$ and assume that $k \in L_q[0,1]$. These are standing assumptions in all of the results to follow.

The following lemma is the key to several examples.

Lemma 6.3.1. *Let $A \subset [0,1]$ be a subset of the interval $[0,1]$. Denote by $\varepsilon_n(A)$ the n-th entropy number of A with respect to the absolute value $|\cdot|$. Under the stated assumptions (cf. p.126)*

$$d(s,t) = \left(\int_0^1 |K(s,x) - K(t,x)|^q \, \mathrm{d}x\right)^{1/q}$$

defines a pseudo-metric on $[0,1]$. In the case (WS) we have

$$d(s,t) \le 4^{1/q} \left(\int_0^{|s-t|} (k(u))^q \, \mathrm{d}u\right)^{1/q} \quad \text{for } 0 \le s,t \le 1,$$

and

$$\varepsilon_n(A,d) \le 4^{1/q} \left(\int_0^{\varepsilon_n(A)} (k(u))^q \, \mathrm{d}u\right)^{1/q} \quad \text{for } n = 1,2,3,\ldots.$$

and in the case (V) it holds

$$\left(\int_0^{|s-t|} (k(u))^q \, \mathrm{d}u\right)^{1/q} \le d(s,t) \le 2^{1/q} \left(\int_0^{|s-t|} (k(u))^q \, \mathrm{d}u\right)^{1/q} \quad \text{for } 0 \le s,t \le 1,$$

and

$$\left(\int_0^{\varepsilon_n(A)} (k(u))^q \, \mathrm{d}u\right)^{1/q} \le \varepsilon_n(A,d) \le 2^{1/q} \left(\int_0^{\varepsilon_n(A)} (k(u))^q \, \mathrm{d}u\right)^{1/q}$$

for $n = 1,2,3,\ldots$.

Proof. In order to estimate the pseudo-metric d let $s > t$. From the inequality

$$|a-b|^q \le |a^q - b^q| \quad \text{for } a,b \ge 0 \text{ and } q \ge 1,$$

we obtain the estimate

$$\begin{aligned}(d(s,t))^q &= \int_0^1 |K(s,x) - K(t,x)|^q \, \mathrm{d}x \le \int_0^1 |(K(s,x))^q - (K(t,x))^q| \, \mathrm{d}x \\ &= \int_0^1 |(k(|s-x|))^q - (k(|t-x|))^q| \, \mathrm{d}x.\end{aligned}$$

A natural way to proceed is to decompose the integration region. To this end, put

$$I := \int_0^1 |(k(|s-x|))^q - (k(|t-x|))^q| \,\mathrm{d}x = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_0^t |(k(|s-x|))^q - (k(|t-x|))^q| \,\mathrm{d}x = \int_0^t \left[(k(t-x))^q - (k(s-x))^q\right] \,\mathrm{d}x \\ &= \int_0^t (k(u))^q \,\mathrm{d}u - \int_{s-t}^s (k(u))^q \,\mathrm{d}u = \int_0^{s-t} (k(u))^q \,\mathrm{d}u - \int_t^s (k(u))^q \,\mathrm{d}u, \end{aligned}$$

$$\begin{aligned} I_2 &= \int_t^s |(k(|s-x|))^q - (k(|t-x|))^q| \,\mathrm{d}x \le \int_t^s \left[(k(|s-x|))^q + (k(|t-x|))^q\right] \,\mathrm{d}x \\ &= \int_t^s (k(s-x))^q \,\mathrm{d}x + \int_t^s (k(x-t))^q \,\mathrm{d}x = 2\int_0^{s-t} (k(u))^q \,\mathrm{d}u \end{aligned}$$

and

$$\begin{aligned} I_3 &= \int_s^1 |(k(|s-x|))^q - (k(|t-x|))^q| \,\mathrm{d}x = \int_s^1 \left[(k(x-s))^q - (k(x-t))^q\right] \,\mathrm{d}x \\ &= \int_0^{1-s} (k(u))^q \,\mathrm{d}u - \int_{s-t}^{1-t} (k(u))^q \,\mathrm{d}u = \int_0^{s-t} (k(u))^q \,\mathrm{d}u - \int_{1-s}^{1-t} (k(u))^q \,\mathrm{d}u. \end{aligned}$$

We obtain

$$\begin{aligned} (d(s,t))^q &\le I = I_1 + I_2 + I_3 \\ &\le 4\int_0^{s-t} (k(u))^q \,\mathrm{d}u - \left[\int_t^s (k(u))^q \,\mathrm{d}u + \int_{1-s}^{1-t} (k(u))^q \,\mathrm{d}u\right] \\ &\le 4\int_0^{s-t} (k(u))^q \,\mathrm{d}u \end{aligned}$$

for all $0 \le t \le s \le 1$. This implies the desired estimates in the case (WS) of a weakly singular kernel. In the case (V) of a Volterra-kernel

$$K(t,x) = \begin{cases} k(t-x), & \text{for } x < t, \\ 0, & \text{for } x \ge t, \end{cases}$$

we get, for $s > t$, that

$$\begin{aligned} (d(s,t))^q &= \int_0^1 |K(s,x) - K(t,x)|^q \,\mathrm{d}x \\ &= \int_0^t |k(s-x) - k(t-x)|^q \,\mathrm{d}x + \int_t^s (k(s-x))^q \,\mathrm{d}x. \end{aligned}$$

Thus, similarly as before, we obtain the estimates

$$\begin{aligned}\int_t^s (k(s-x))^q \,\mathrm{d}x \le (d(s,t))^q \\ &\le \int_0^t \left[(k(t-x))^q - (k(s-x))^q\right] \mathrm{d}x + \int_t^s (k(s-x))^q \,\mathrm{d}x \\ &= 2\int_0^{s-t} (k(u))^q \,\mathrm{d}u - \int_t^s (k(u))^q \,\mathrm{d}u\end{aligned}$$

and therefore

$$\int_0^{s-t} (k(u))^q \,\mathrm{d}u \le (d(s,t))^q \le 2\int_0^{s-t} (k(u))^q \,\mathrm{d}u$$

for all $0 \le t \le s \le 1$. This yields the statement and finishes the proof. ■

Now we treat several important examples of weakly singular kernels.

Lemma 6.3.2. *Let $A \subset [0,1]$ be a subset of the interval $[0,1]$. Denote by $\varepsilon_n(A)$ the n-th entropy number of A with respect to the absolute value $|\cdot|$. Under the stated assumptions (cf. p.126) the following statements hold:*

(i) If the function $k : (0,1] \to \mathbb{R}$ is defined by

$$k(x) = x^{-\tau}, \quad 0 < \tau < \frac{1}{q},$$

then we have in the case (WS)

$$d(s,t) \preccurlyeq |s-t|^{1/q-\tau} \quad \textit{and} \quad \varepsilon_n(A,d) \preccurlyeq \Big(\varepsilon_n(A)\Big)^{1/q-\tau},$$

and in the case (V) it holds

$$d(s,t) \sim |s-t|^{1/q-\tau} \quad \textit{and} \quad \varepsilon_n(A,d) \sim \Big(\varepsilon_n(A)\Big)^{1/q-\tau}.$$

(ii) If the function $k : (0,1] \to \mathbb{R}$ is defined by

$$k(x) = x^{-1/q}(c_0 - \ln x)^{-\beta}, \quad \frac{1}{q} < \beta,$$

where c_0 is a positive constant, then we have in the case (WS)

$$d(s,t) \preccurlyeq (c_0 - \ln|s-t|)^{1/q-\beta} \quad \textit{and} \quad \varepsilon_n(A,d) \preccurlyeq \Big(c_0 - \ln \varepsilon_n(A)\Big)^{1/q-\beta},$$

and in the case (V) it holds

$$d(s,t) \sim (c_0 - \ln|s-t|)^{1/q-\beta} \quad \textit{and} \quad \varepsilon_n(A,d) \sim \Big(c_0 - \ln \varepsilon_n(A)\Big)^{1/q-\beta}.$$

Proof. Due to Lemma 6.3.1 it is enough to compute the expression

$$\left(\int_0^{|s-t|} (k(u))^q \,\mathrm{d}u\right)^{1/q} \quad \text{for } 0 \le s,t \le 1.$$

In the case (i), we have that

$$\left(\int_0^{|s-t|} (k(u))^q \,\mathrm{d}u\right)^{1/q} = (1-q\tau)^{-1/q}\,|s-t|^{1/q-\tau} \quad \text{for } 0 \le s,t \le 1.$$

Now let us turn to the case (ii). Observe that for any $c_0 > 0$ the function k is an element of $L_q[0,1]$ and is strictly decreasing in a neighborhood of 0. The technical assumption that k is strictly decreasing on the whole interval $(0,1]$ is ensured if we choose c_0 large enough, e.g. $c_0 > \beta q$. However, keep in mind that the upcoming results are true for any $c_0 > 0$. For the corresponding integral we get

$$\left(\int_0^{|s-t|} (k(u))^q \,\mathrm{d}u\right)^{1/q} = (\beta q-1)^{-1/q}\,(c_0 - \ln|s-t|)^{1/q-\beta} \quad \text{for } 0 \le s,t \le 1.$$

This finishes the proof. ■

Now we are well prepared to prove the following theorem. Since convolution operators from $L_p[0,1]$ into $C[0,1]$ are, in a sense, closely related to certain diagonal operators from l_p into l_∞, we may expect sharp estimates of entropy numbers of convolution operators only in the case where $2 \le p < \infty$. This is the reason why we restrict our applications to this case.

Theorem 6.3.3. *Under the stated assumptions (cf. p.126) the following statements hold:*

(i) If the function $k : (0,1] \to \mathbb{R}$ is defined by

$$k(x) = x^{-\tau}, \quad 2 \le p < \infty,\ 0 < \tau < \frac{1}{p'},$$

then T_K maps $L_p[0,1]$ into $C[0,1]$ and in the cases (WS) and (V) the entropy estimate

$$e_n(T_K : L_p[0,1] \to C[0,1]) \preccurlyeq n^{\tau-1}$$

holds.

(ii) If the function $k : (0,1] \to \mathbb{R}$ is defined by

$$k(x) = x^{-1/p'}(c_0 - \ln x)^{-\beta}, \quad 2 \le p < \infty,\ \frac{1}{p'} < \beta,\ c_0 > 0,$$

then T_K maps $L_p[0,1]$ into $C[0,1]$ and in the cases (WS) and (V) the following entropy estimates hold:

$$e_n(T_K : L_p[0,1] \to C[0,1]) \preccurlyeq \begin{cases} n^{1/p'-\beta}, & 1/p' < \beta < 1, \\ n^{-1/p}\,(\log(n+1))^{1-\beta}, & 1 < \beta < \infty, \\ n^{-1/p}\,\log(n+1), & \beta = 1. \end{cases}$$

Proof. The results follow from Lemma 6.3.2 with $A = [0,1]$, $q = p'$ and Remark 6 (iv) after Theorem 6.1.4 with $q = p$. For the proof of (i) we apply Theorem 6.1.4 (a) with $X = L_p[0,1]$, $\sigma = 1/p' - \tau$ and $\gamma = 0$. The proof of (ii) follows from Theorem 6.1.4 (b) with $X = L_p[0,1]$, $\sigma = \beta - 1/p'$ and $\gamma = 0$. ■

We can even go a step further and consider more general kernels given by kernel functions

$$k(x) = x^{-\tau}\,\varphi(1/x), \quad 0 < \tau < \frac{1}{q},\ 0 < x \le 1,$$

where φ is a slowly varying function defined on $[1,\infty)$. Note that such kernel functions always have a singularity at 0 (cf. Lemma 2.2.4). Furthermore, Lemma 2.2.5 tells us that k is up to multiplicative constants equivalent to a decreasing function. This enables us to apply analogous reasoning as before.

First we present an analogon of Lemma 6.3.2.

Lemma 6.3.4. *Under the stated assumptions (cf. p.126) the following statements hold. If the function $k : (0,1] \to \mathbb{R}$ is defined by*

$$k(x) = x^{-\tau}\,\varphi(1/x), \quad 0 < \tau < 1/q,$$

then we have in the case (WS) the estimates

$$d(s,t) \preccurlyeq |s-t|^{1/q-\tau}\,\varphi(|s-t|^{-1}) \quad \textit{and} \quad \varepsilon_n([0,1],d) \preccurlyeq n^{\tau-1/q}\,\varphi(2n)$$

and in the case (V) it holds

$$d(s,t) \sim |s-t|^{1/q-\tau}\,\varphi(|s-t|^{-1}) \quad \textit{and} \quad \varepsilon_n([0,1],d) \sim n^{\tau-1/q}\,\varphi(2n).$$

Proof. Again, it is enough to compute

$$\left(\int_0^r (k(u))^q\,\mathrm{d}u\right)^{1/q} \quad \text{for } 0 < r \le 1.$$

According to the definition, we have

$$\int_0^r (k(u))^q\,\mathrm{d}u = \int_0^r u^{-\tau q}\,(\varphi(1/u))^q\,\mathrm{d}u = \int_{1/r}^\infty z^{\tau q-2}\,(\varphi(z))^q\,\mathrm{d}z.$$

Now it follows from Lemma 2.2.6 that

$$\int_{1/r}^{\infty} z^{\tau q-2} \, (\varphi(z))^q \, \mathrm{d}z \sim (1/r)^{\tau q-1} \, (\varphi(1/r))^q.$$

Note that the arising constants depend on τ, q and the function φ. Hence, we obtain

$$\left(\int_0^r (k(u))^q \, \mathrm{d}u\right)^{1/q} \sim r^{1/q-\tau} \, \varphi(1/r),$$

which yields the assertion. ■

The resulting version of Theorem 6.3.3 then reads as follows. The proof is based on Theorem 1.3.1 *(i)* with $X = L_{p'}[0,1]$ and Corollary 6.1.3.

Theorem 6.3.5. *Under the stated assumptions (cf. p.126) the following statements hold. If $2 \leq p < \infty$ and the function $k : (0,1] \to \mathbb{R}$ is defined by*

$$k(x) = x^{-\tau} \, \varphi(1/x), \quad 0 < \tau < \frac{1}{p'},$$

then in the cases (WS) and (V) the following entropy estimate holds: For all $\beta \in \mathbb{R}$ there exists a constant $c = c(p,\tau,\varphi,\beta) > 0$ such that for all $n \in \mathbb{N}$ we have

$$n^{1-\tau} \, (\log(n+1))^\beta \, e_n(T_K : L_p[0,1] \to C[0,1]) \preccurlyeq c \sup_{1 \leq k \leq a_n} (\log(k+1))^\beta \, \varphi(2k),$$

where $a_n = n^{1+\frac{p'-1}{1-p'\tau}}$.

Finally, let us treat an important example by considering a double-logarithmic term.

Example 2. Under the stated assumptions (cf. p.126) the following statements hold. If the function $k : (0,1] \to \mathbb{R}$ is defined by

$$k(x) = x^{-\tau} \, (c_0 - \ln x)^{-\beta} \left(c_0 + \ln(c_0 - \ln x)\right)^{-\gamma}, \quad 0 < \tau \leq \frac{1}{q}, \beta, \gamma \in \mathbb{R},$$

where c_0 is a positive constant, then we have in the case (WS) the estimates

$$d(s,t) \preccurlyeq f(s,t,\tau,\beta,\gamma,c_0,q) \quad \text{and} \quad \varepsilon_n([0,1],d) \preccurlyeq g(n,\tau,\beta,\gamma,q)$$

and in the case (V) we obtain the asymptotic behavior

$$d(s,t) \sim f(s,t,\tau,\beta,\gamma,c_0,q) \quad \text{and} \quad \varepsilon_n([0,1],d) \sim g(n,\tau,\beta,\gamma,q),$$

where $f(s,t,\tau,\beta,\gamma,c_0,q) =$

$$\begin{cases} |s-t|^{1/q-\tau}\,(c_0 - \ln|s-t|)^{-\beta}\,(c_0 + \ln(c_0 - \ln|s-t|))^{-\gamma}, & \tau < 1/q, \beta \in \mathbb{R}, \gamma \in \mathbb{R}, \\ (c_0 - \ln|s-t|)^{1/q-\beta}\,(c_0 + \ln(c_0 - \ln|s-t|))^{-\gamma}, & \tau = 1/q, \beta > 1/q, \gamma \in \mathbb{R}, \\ (c_0 + \ln(c_0 - \ln|s-t|))^{1/q-\gamma}, & \tau = \beta = 1/q, \gamma > 1/q, \end{cases}$$

and $g(n,\tau,\beta,\gamma,q) =$

$$\begin{cases} n^{\tau-1/q}\,(\log(n+1))^{-\beta}\,(\log\log(n+3))^{-\gamma}, & \tau < 1/q,\ \beta \in \mathbb{R},\ \gamma \in \mathbb{R}, \\ (\log(n+1))^{1/q-\beta}\,(\log\log(n+3))^{-\gamma}, & \tau = 1/q,\ \beta > 1/q,\ \gamma \in \mathbb{R}, \\ (\log\log(n+3))^{1/q-\gamma}, & \tau = \beta = 1/q,\ \gamma > 1/q. \end{cases}$$

For this example, the resulting version of Theorem 6.3.3 reads as follows.

Proposition 6.3.6. *Under the stated assumptions (cf. p.126) the following statements hold. If* $2 \le p < \infty$ *and the function* $k : (0,1] \to \mathbb{R}$ *is defined by*

$$k(x) = x^{-\tau}\,(c_0 - \ln x)^{-\beta}\,\Big(c_0 + \ln(c_0 - \ln x)\Big)^{-\gamma}, \quad 0 < \tau \le \frac{1}{p'},\ \beta, \gamma \in \mathbb{R},\ c_0 > 0,$$

then in the cases (WS) *and* (V) *the following entropy estimates hold:*

$$e_n(T_K : L_p[0,1] \to C[0,1]) \preccurlyeq f(n,\tau,\beta,\gamma,p),$$

where $f(n,\tau,\beta,\gamma,p) =$

$$\begin{cases} n^{\tau-1}\,(\log(n+1))^{-\beta}\,(\log\log(n+3))^{-\gamma}, & 0 < \tau < 1/p',\ \beta \in \mathbb{R},\ \gamma \in \mathbb{R}, & (6.3.1) \\ n^{1/p'-\beta}\,(\log(n+1))^{-\gamma}, & \tau = 1/p',\ 1/p' < \beta < 1,\ \gamma \in \mathbb{R}, & (6.3.2) \\ n^{-1/p}\,(\log(n+1))^{1-\beta}\,(\log\log(n+3))^{-\gamma}, & \tau = 1/p',\ \beta > 1,\ \gamma \in \mathbb{R}, & (6.3.3) \\ n^{-1/p}\,(\log(n+1))^{1-\gamma}, & \tau = 1/p',\ \beta = 1,\ \gamma < 1, & (6.3.4) \\ n^{-1/p}\,\log\log(n+3), & \tau = 1/p',\ \beta = 1,\ \gamma = 1, & (6.3.5) \\ n^{-1/p}\,(\log(n+1))^{\delta}, & \tau = 1/p',\ \beta = 1,\ \gamma > 1, & (6.3.6) \\ (\log(n+1))^{1/p'-\gamma}, & \tau = \beta = 1/p',\ \gamma > 1/p', & (6.3.7) \end{cases}$$

and δ *in* (6.3.6) *is an arbitrary positive number.*

Remark 7. The estimate (6.3.6) of Proposition 6.3.6 is a consequence of (1.3.11). We use this weak estimate since we do not have precise entropy estimates of absolutely convex hulls in this critical case.

Finally, let us deal with the optimality of the results in the Volterra-kernel case (V). In order to prove lower bounds of the entropy of $T_K : L_p[0,1] \to C[0,1]$ we construct suitable distance nets in $T_K(B_{L_p[0,1]})$. The proofs are inspired by unpublished works of Linde and Lacey in the Hilbert space case $p = 2$ (cf. [Lif10, p. 1807], [Kl12a, Prop. 38]). However, it turns out that their techniques also work in a more general framework.

Let us start with (6.3.1), i.e. $0 < \tau < 1/p'$. Consider the 2^n functions

$$f_\varepsilon(x) := \sum_{i=1}^n \varepsilon_i \, \mathbb{1}_{[(i-1)/n, i/n]}(x), \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1,1\}^n .$$

Then $f_\varepsilon \in B_{L_p[0,1]}$ for every choice of $\varepsilon \in \{-1,1\}^n$. Now we estimate the mutual distance of the images of f_ε under T_K in the Volterra-kernel case (V). To this end, let $\varepsilon, \tilde{\varepsilon} \in \{-1,1\}^n$ with $\varepsilon \neq \tilde{\varepsilon}$. Let j be the least index such that $\varepsilon_j \neq \tilde{\varepsilon}_j$. Then

$$\|T_K f_\varepsilon - T_K f_{\tilde{\varepsilon}}\|_\infty \geq |(T_K f_\varepsilon)(j/n) - (T_K f_{\tilde{\varepsilon}})(j/n)| = 2 \int_0^{1/n} k(z) \, \mathrm{d}z.$$

Hence, we have found a distance net consisting of 2^n elements of $T_K(B_{L_p[0,1]})$ and therefore

$$e_n(T_K : L_p[0,1] \to C[0,1]) \geq \varepsilon_{2^n - 1}(T_K : L_p[0,1] \to C[0,1]) \geq \int_0^{1/n} k(x) \, \mathrm{d}x.$$

Using Lemma 2.2.6 we compute that

$$\begin{aligned} \int_0^{1/n} k(x) \, \mathrm{d}x &= \int_0^{1/n} x^{-\tau} (c_0 - \ln x)^{-\beta} \Big(c_0 + \ln(c_0 - \ln x)\Big)^{-\gamma} \mathrm{d}x \\ &= \int_n^\infty z^{\tau-2} (c_0 + \ln z)^{-\beta} \Big(c_0 + \ln(c_0 + \ln z)\Big)^{-\gamma} \mathrm{d}z \\ &\succcurlyeq n^{\tau-1} (c_0 + \ln n)^{-\beta} \Big(c_0 + \ln(c_0 + \ln n)\Big)^{-\gamma} \end{aligned}$$

and conclude

$$e_n(T_K : L_p[0,1] \to C[0,1]) \succcurlyeq n^{\tau-1} (\log(n+1))^{-\beta} (\log\log(n+3))^{-\gamma}.$$

This shows that estimate (6.3.1) is best possible.

Now let us deal with optimality in the case $\tau = 1/p'$. The idea is to construct a suitable distance net by using the kernel function $k \in L_{p'}[0,1]$. To this end, define functions

$$f_j(x) := \frac{1}{\alpha_m} \left(k(j/m - x)\right)^{p'/p} \mathbb{1}_{[(j-1)/m, j/m)}(x), \quad j = 1, 2, \dots, m, \tag{6.3.8}$$

where

$$\alpha_m = \left(\int_0^{1/m} (k(x))^{p'} \,\mathrm{d}x \right)^{1/p}.$$

Then $f_j \in B_{L_p[0,1]}$ for every $j = 1, 2, \dots, m$. Furthermore, for $1 \le i < j \le m$, we obtain

$$\|T_K f_i - T_K f_j\|_\infty \ge |(T_K f_i)(i/m) - (T_K f_j)(i/m)| = \frac{1}{\alpha_m} \int_0^{1/m} (k(z))^{p'} \,\mathrm{d}z = \alpha_m^{p-1}.$$

Hence, we have that

$$\varepsilon_{m-1}(T_K : L_p[0,1] \to C[0,1]) \ge \frac{1}{2}\, \alpha_m^{p-1}.$$

Now let $\tau = 1/p'$, $\beta > 1/p'$ and $\gamma \in \mathbb{R}$. Using Lemma 2.2.6 we compute that

$$\begin{aligned}
\alpha_m^{p-1} &= \left(\int_0^{1/m} (k(x))^{p'} \,\mathrm{d}x \right)^{1/p'} \\
&= \left(\int_0^{1/m} x^{-1} \,(c_0 - \ln x)^{-\beta p'} \left(c_0 + \ln(c_0 - \ln x) \right)^{-\gamma p'} \,\mathrm{d}x \right)^{1/p'} \\
&= \left(\int_{c_0 + \ln m}^{\infty} z^{-\beta p'} (c_0 + \ln z)^{-\gamma p'} \,\mathrm{d}z \right)^{1/p'} \\
&\succcurlyeq (c_0 + \ln m)^{1/p' - \beta} \left(c_0 + \ln(c_0 + \ln m) \right)^{-\gamma}.
\end{aligned}$$

Consequently, putting $m = 2^{n-1} + 1$, we get

$$e_n(T_K : L_p[0,1] \to C[0,1]) \succcurlyeq n^{1/p' - \beta} \,(\log(n+1))^{-\gamma}.$$

This shows that estimate (6.3.2) is best possible. Moreover, we see that in the critical case $\tau = 1/p'$ and $\beta = 1$ the estimate

$$e_n(T_K : L_p[0,1] \to C[0,1]) \succcurlyeq n^{-1/p} (\log(n+1))^{-\gamma}$$

holds for $\gamma \in \mathbb{R}$, cf. (6.3.4).

Now let us deal with the case $\tau = \beta = 1/p'$ and $\gamma > 1/p'$. Here we have

$$\alpha_m^{p-1} = (\gamma p' - 1)^{-1/p'} \left(c_0 + \ln(c_0 + \ln m) \right)^{1/p' - \gamma}$$

and therefore

$$e_n(T_K : L_p[0,1] \to C[0,1]) \succcurlyeq (\log(n+1))^{1/p' - \gamma}.$$

This shows that estimate (6.3.7) is best possible.

Finally, we show that also estimate (6.3.3) is best possible. To see this, we consider suitable means of the functions f_j defined in (6.3.8). For $J \subset \{1, 2, \ldots, m\}$ define

$$f_J(x) := |J|^{-1/p} \sum_{j \in J} f_j(x).$$

Then $f_J \in B_{L_p[0,1]}$ for every choice of $J \subset \{1, 2, \ldots, m\}$. Let $m > 1$ be a square number and define

$$\Phi_m := \left\{ f_J : J \subset \{1, 2, \ldots, m\} \text{ with } |J| = \sqrt{m} \right\}.$$

Then it holds that

$$\log |\Phi_m| = \log \binom{m}{\sqrt{m}} \geq \log \left(\left(\frac{m}{\sqrt{m}} \right)^{\sqrt{m}} \right) = \frac{1}{2} \sqrt{m} \log(m).$$

Let $f_J, f_L \in \Phi_m$ with $J \neq L$ and let i be the least element in the symmetric difference $(J \cup L) \setminus (J \cap L)$. Then, for $\tau = 1/p'$, $\beta > 1/p'$ and $\gamma \in \mathbb{R}$, we have

$$\begin{aligned}
\|T_K f_J - T_K f_L\|_\infty &\geq |(T_K f_J)(i/m) - (T_K f_L)(i/m)| \\
&= (\sqrt{m})^{-1/p} \, \alpha_m^{-1} \int_0^{1/m} (k(z))^{p'} \, \mathrm{d}z \\
&= (\sqrt{m})^{-1/p} \, \alpha_m^{p-1} \\
&\succcurlyeq (\sqrt{m})^{-1/p} \, (c_0 + \ln m)^{1/p' - \beta} \left(c_0 + \ln(c_0 + \ln m) \right)^{-\gamma} \\
&\succcurlyeq \left(\sqrt{m} \log(m) \right)^{-1/p} \left(\log \left(\sqrt{m} \log(m) + 1 \right) \right)^{1-\beta} \times \\
&\quad \times \left(\log \log \left(\sqrt{m} \log(m) + 3 \right) \right)^{-\gamma}
\end{aligned}$$

Hence, we have found at least $2^{\frac{1}{2}\sqrt{m} \log(m)}$ elements in $T_K(B_{L_p[0,1]})$ with mutual distance (up to some constant) at least

$$\left(\sqrt{m} \log(m) \right)^{-1/p} \left(\log \left(\sqrt{m} \log(m) + 1 \right) \right)^{1-\beta} \left(\log \log \left(\sqrt{m} \log(m) + 3 \right) \right)^{-\gamma},$$

where $m > 1$ is an arbitrary square number. Therefore,

$$e_n(T_K : L_p[0,1] \to C[0,1]) \succcurlyeq n^{-1/p} \, (\log(n+1))^{1-\beta} \, (\log \log(n+3))^{-\gamma}$$

and this shows that estimate (6.3.3) is the best possible.

6.4 Weakly singular integral operators from $L_2[0,1]$ to $C[0,1]$ and $L_q[0,1]$ generated by convolution kernels

Given a precompact subset A of a Banach space X of type p, we do not have exact entropy estimates of $\mathrm{aco}(A)$ in the critical case that

$$e_n(A) \preccurlyeq n^{-1/p'} (\log(n+1))^{\beta} \quad \text{with } \beta > 1.$$

In contrast to that, in the Hilbert space case we have such estimates (cf. (1.2.14)) and very recent results of Gao [G12] show that they are asymptotically optimal. Consequently, some of our estimates of $e_n(T_K)$ from the previous section can be refined in the Hilbert space case. This fact and the general importance of the Hilbert space case motivates this section.

In (6.1.8) we related the Kolmogorov numbers $d_n(T_K)$ of the operator T_K to the Gelfand numbers $c_n\Big(\mathrm{aco}(\mathrm{Im}(K))\Big)$ of the absolutely convex hull of $\mathrm{Im}(K)$. This relationship leads to fruitful results in the Hilbert space case. Indeed, we can use (1.2.11)-(1.2.15) to relate the entropy numbers of $\mathrm{Im}(K)$ to both $e_n\Big(\mathrm{aco}(\mathrm{Im}(K))\Big)$ and $c_n\Big(\mathrm{aco}(\mathrm{Im}(K))\Big)$. Hence, in the Hilbert space case we can estimate not only the dyadic entropy numbers $e_n(T_K)$ but also the Kolmogorov numbers $d_n(T_K)$ of the operator T_K. In the Hilbert space setting, the resulting version of Proposition 6.3.6 reads as follows.

Theorem 6.4.1. *Let (s_n) stand for the Kolmogorov numbers (d_n) or for the dyadic entropy numbers (e_n). Under the stated assumptions (cf. p.126) the following statements hold. If the function $k : (0,1] \to \mathbb{R}$ is defined by*

$$k(x) = x^{-\tau} (c_0 - \ln x)^{-\beta} \Big(c_0 + \ln(c_0 - \ln x)\Big)^{-\gamma}, \quad 0 < \tau \leq \frac{1}{2}, \ \beta, \gamma \in \mathbb{R}, \ c_0 > 0,$$

then in the cases (WS) and (V) the following estimates hold:

$$s_n(T_K : L_2[0,1] \to C[0,1]) \preccurlyeq f(n, \tau, \beta, \gamma),$$

where $f(n,\tau,\beta,\gamma) =$

$$\begin{cases} n^{\tau-1}\,(\log(n+1))^{-\beta}\,(\log\log(n+3))^{-\gamma}, & 0<\tau<1/2,\ \beta\in\mathbb{R},\ \gamma\in\mathbb{R}, & (6.4.1)\\ n^{1/2-\beta}\,(\log(n+1))^{-\gamma}, & \tau=1/2,\ 1/2<\beta<1,\ \gamma\in\mathbb{R}, & (6.4.2)\\ n^{-1/2}\,(\log(n+1))^{1-\beta}\,(\log\log(n+3))^{-\gamma}, & \tau=1/2,\ \beta>1,\ \gamma\in\mathbb{R}, & (6.4.3)\\ n^{-1/2}\,(\log(n+1))^{1-\gamma}, & \tau=1/2,\ \beta=1,\ \gamma<1, & (6.4.4)\\ n^{-1/2}\ \log\log(n+3), & \tau=1/2,\ \beta=1,\ \gamma=1, & (6.4.5)\\ n^{-1/2}\,(\log\log(n+3))^{1-\gamma}, & \tau=1/2,\ \beta=1,\ \gamma>1, & (6.4.6)\\ (\log(n+1))^{1/2-\gamma}, & \tau=\beta=1/2,\ \gamma>1/2. & (6.4.7)\end{cases}$$

According to Theorem 6.4.1, the behavior of entropy numbers as well as Kolmogorov numbers of the operator $T_K : L_2[0,1] \to C[0,1]$ differs significantly between the cases $0 < \tau < 1/2$, $\tau = 1/2$, $\beta > 1/2$ and $\tau = \beta = 1/2$. Furthermore, we see that for fixed $\tau = 1/2$ a sudden jump occurs if the parameter β crosses the point $\beta = 1$. In addition, for fixed $\tau = 1/2$ and $\beta = 1$, we recognize a sudden jump if the parameter γ crosses the point $\gamma = 1$.

We already know from the previous section that the entropy estimates given in (6.4.1), (6.4.2), (6.4.3) and (6.4.7) are the best possible. In the critical case (6.4.4), Lifshits [Lif10, Th. 3.2] proved that, for $\gamma = 0$,

$$e_n(T_K : L_2[0,1] \to C[0,1]) \preccurlyeq n^{-1/2}.$$

Hence, in the critical case, our general approach using absolutely convex hulls does not lead to a sharp upper estimate. We do not know whether the upper estimates given in (6.4.5) and (6.4.6) are the best possible. Furthermore, we would like to point out that Linde [Lin08] proved the lower estimate

$$e_n(T_K : L_2[0,1] \to C[0,1]) \succcurlyeq n^{-1/2}\,(\log(n+1))^{1/2-\beta}$$

in the case where $\tau = 1/2$, $\beta > 1/2$ and $\gamma = 0$.

The estimates (6.4.1), (6.4.2), (6.4.3) and (6.4.7) of the Kolmogorov numbers of T_K are also optimal. This can be derived from the optimality of the entropy estimates in this cases by using Theorem 2.1.1 in combination with a trick given in [C85, p. 106]. The following lemma deals with this subject.

Lemma 6.4.2. *Let $T \in \mathcal{L}(X,Y)$ be an arbitrary operator and $s \in \{a,c,d,t\}$. Suppose that there exist constants $c_1, c_2 > 0$ such that*

$$e_n(T) \ge c_1\, n^{-\alpha}\varphi(n) \quad \textit{and} \quad s_n(T) \le c_2\, n^{-\alpha}\varphi(n)$$

for all $n \in \mathbb{N}$, where $\alpha \geq 0$ and φ is a slowly varying function. Then there exists a positive constant $C = C(\alpha, \varphi, c_1, c_2)$ such that

$$s_n(T) \geq C\, n^{-\alpha} \varphi(n) \quad \textit{for all } n \in \mathbb{N}.$$

Proof. We restrict ourselves to prove the assertion for $\alpha > 0$. The proof in the case $\alpha = 0$ is analogous. According to Lemma 2.2.5 the function $x^\alpha \varphi(x)$ is up to multiplicative constants equivalent to an increasing function Φ, i.e.

$$C_1(\alpha, \varphi)\, \Phi(k) \leq k^\alpha \varphi(k) \leq C_2(\alpha, \varphi)\, \Phi(k)$$

for all $k \in \mathbb{N}$. Consequently, we get

$$\sup_{1 \leq k \leq mn} k^{2\alpha}\, e_k(T) \geq c_1 \sup_{1 \leq k \leq mn} k^\alpha\, \varphi(k) \geq c_1\, C_1\, \Phi(mn) \geq \frac{c_1\, C_1}{C_2}\, (mn)^\alpha\, \varphi(mn),$$

where m is a natural number which will be defined later. Taking Theorem 2.1.1 into account we obtain

$$\begin{aligned} \frac{c_1\, C_1}{C_2}\, (mn)^\alpha\, \varphi(mn) &\leq \sup_{1 \leq k \leq mn} k^{2\alpha}\, e_k(T) \leq C_3(\alpha) \sup_{1 \leq k \leq mn} k^{2\alpha}\, s_k(T) \\ &\leq C_3 \sup_{1 \leq k \leq n} k^{2\alpha}\, s_k(T) + C_3 \sup_{n \leq k \leq mn} k^{2\alpha}\, s_k(T). \end{aligned}$$

According to the assumption, we have that

$$\sup_{1 \leq k \leq n} k^{2\alpha}\, s_k(T) \leq c_2 \sup_{1 \leq k \leq n} k^\alpha\, \varphi(k) \leq c_2\, C_2\, \Phi(n) \leq \frac{c_2\, C_2}{C_1}\, n^\alpha\, \varphi(n).$$

Furthermore, the monotonicity of $(s_n(T))_n$ gives

$$\sup_{n \leq k \leq mn} k^{2\alpha}\, s_k(T) \leq (mn)^{2\alpha}\, s_n(T).$$

Hence,we conclude that

$$\frac{c_1\, C_1}{C_2}\, (mn)^\alpha\, \varphi(mn) \leq \frac{c_2\, C_2\, C_3}{C_1}\, n^\alpha\, \varphi(n) + C_3\, (mn)^{2\alpha}\, s_n(T).$$

For notational simplicity we define

$$\gamma_1 := \frac{c_1\, C_1}{C_2} \quad \text{and} \quad \gamma_2 := \frac{c_2\, C_2\, C_3}{C_1}$$

and obtain

$$s_n(T) \geq \frac{\gamma_1\, (mn)^\alpha\, \varphi(mn) - \gamma_2\, n^\alpha\, \varphi(n)}{C_3\, (mn)^{2\alpha}} = n^{-\alpha}\, \varphi(n)\, \frac{\gamma_1\, m^\alpha\, \frac{\varphi(mn)}{\varphi(n)} - \gamma_2}{C_3\, m^{2\alpha}}.$$

Choosing

$$m = \left\lfloor \left(\frac{\gamma_2 + 1}{\gamma_1} \right)^{1/\alpha} \right\rfloor + 1$$

we check that

$$\gamma_1 \, m^\alpha \, \frac{\varphi(mn)}{\varphi(n)} - \gamma_2 \geq (\gamma_2 + 1) \, \frac{\varphi(mn)}{\varphi(n)} - \gamma_2.$$

Since φ varies slowly, we have that

$$\lim_{n \to \infty} (\gamma_2 + 1) \, \frac{\varphi(mn)}{\varphi(n)} - \gamma_2 = 1.$$

The assertion follows. ■

In contrast to Theorem 6.4.1 we now study entropy and Kolmogorov numbers of convolution operators from $L_2[0,1]$ into $L_q[0,1]$ for $1 \leq q < \infty$. It turns out that the asymptotic behavior of those numbers significantly changes in the critical cases. This demonstrates the difficulties of estimating entropy and Kolmogorov numbers of convolution operators.

Let us start with recalling the l-norm of an operator $T : X \to Y$ (or absolutely γ-summing norm in [LP74]). Let l_2^n be the n-dimensional Euclidean space and $S : l_2^n \to Y$ an operator, then the *l-norm of S* is defined by

$$l(S) := \left(\int_{R^n} \|Sx\|^2 \, \mathrm{d}\gamma_n(x) \right)^{1/2},$$

where γ_n is the canonical Gaussian probability measure of $\mathbb{R}^n$. For an operator $T : X \to Y$ we define

$$l(T) := \sup \left\{ l(TA) : \|A : l_2^n \to X\| \leq 1, n \in \mathbb{N} \right\}.$$

If $A : X_0 \to X$ and $B : Y \to Y_0$ are operators acting between Banach spaces, then l has the ideal property (cf. [LP74])

$$l(BTA) \leq \|B\| \; l(T) \; \|A\| \, .$$

Furthermore, we need a refined version of a Sudakov-type inequality. The following theorem is due to Pajor and Tomczak-Jaegermann.

Theorem 6.4.3. [PT86] *There is a constant $c \geq 1$ such that for all operators $T : X \to H$ from a Banach space X into a Hilbert space H and all $n \in \mathbb{N}$,*

$$n^{1/2} \, c_n(T) \leq c \cdot l(T'). \tag{6.4.8}$$

By Gordon [Go88] we know that $c \le \sqrt{2}$.

For our purposes, we need an additional version of Pajor and Tomczak-Jaegermann's inequality (see also [CE03, Lemma A]). In order to formulate it we introduce the approximation numbers with respect to the l-norm. For an operator $T : X \to Y$ acting between Banach spaces X and Y the *approximation numbers with respect to the l-norm* are defined by

$$a_n(T;l) := \inf\{l(T-A) : A \in \mathcal{L}(X,Y) \text{ with } \operatorname{rank} A < n\}, \quad n = 1,2,\ldots.$$

Analogously, we define $a_n(T;\Pi_q)$ as the *approximation numbers with respect to the absolutely q-summing norm* (cf. [P87]). We remark that the approximation numbers with respect to the l-norm were used for some time in functional analysis with different notations (cf. e.g. [Pi89, Th. 9.1]). They also play a role in probability theory since they describe the approximability of Gaussian processes by finite sums (cf. e.g. [LL99]).

Lemma 6.4.4. *(i) For an operator $T : H \to X$ from a Hilbert space H into a Banach space X with $l(T) < \infty$ we have the inequality*

$$k^{1/2} d_{k+n-1}(T) \le \sqrt{2}\, a_n(T;l) \quad \textit{for } k,n \in \mathbb{N}.$$

(ii) Let $1 \le q < \infty$, then for all absolutely q-summing operators $T : H \to X$ from a Hilbert space H into a Banach space X we have the inequality

$$k^{1/2} d_{k+n-1}(T) \le \sqrt{2q}\, a_n(T;\Pi_q) \quad \textit{for } k,n \in \mathbb{N}.$$

Proof. Since $l(T) < \infty$, the operator $T : H \to X$ is compact (cf. [Pi89, Th. 5.5]) and, therefore, we have that $d_n(T) = c_n(T')$ (cf. [P78, 11.7.7], [CS90, Prop. 2.5.6]). Furthermore, it holds that $T'' = \mathcal{K}_X T \mathcal{K}_H^{-1}$, where $\mathcal{K}_Z$ is the canonical metric injection from a Banach space Z into its bidual Z''. We conclude that $l(T'') \le l(T)$ and taking (6.4.8) into account gives

$$n^{1/2} d_n(T) = n^{1/2} c_n(T') \le \sqrt{2}\, l(T'') \le \sqrt{2}\, l(T), \quad n \in \mathbb{N}.$$

Now let $A : H \to X$ be an operator with $\operatorname{rank}(A) < n$. Due to the additivity and rank property of the Kolmogorov numbers, we get

$$d_{k+n-1}(T) \le d_k(T-A) + d_n(A) = d_k(T-A)$$

for all $k,n \in \mathbb{N}$. Hence, we obtain

$$k^{1/2} d_{k+n-1}(T) \le k^{1/2} d_k(T-A) \le \sqrt{2}\, l(T-A)$$

for all $k, n \in \mathbb{N}$ and all operators $A : H \to X$ with $\operatorname{rank}(A) < n$. This yields the assertion

$$k^{1/2} d_{k+n-1}(T) \le \sqrt{2}\, a_n(T; l) \quad \text{for } k, n \in \mathbb{N}.$$

Now let us deal with the proof of (ii). By Linde and Pietsch [LP74] we have for an absolutely q-summing operator $T : H \to X$ the estimate

$$l(T) \le b_q\, \Pi_q(T),$$

where

$$b_q = \max\left\{1; 2^{1/2} \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{1}{2})}\right\} \le \sqrt{q}, \quad 1 \le q < \infty.$$

Combining this estimate with (i), we get the desired assertion. ∎

Now we are well prepared to prove the following theorem.

Theorem 6.4.5. *Let $k : (0,1] \to \mathbb{R}$ be a kernel function as stated on page 126 with $k \in L_2[0,1]$. Then for the weakly singular integral operator $T_K : L_2[0,1] \to L_q[0,1]$, $1 \le q < \infty$, given by*

$$(T_K f)(t) = \int_0^1 k(|t-x|)\, f(x)\, \mathrm{d}x$$

the inequality

$$n^{1/2} d_n(T_K : L_2[0,1] \to L_q[0,1]) \le c \sqrt{q} \left(\int_0^{1/n} (k(u))^2 \,\mathrm{d}u \right)^{1/2}, \quad n \in \mathbb{N},$$

holds true, where $c \ge 1$ is an absolute constant.

Proof. By Lemma 6.3.1 we have with

$$d(s,t) = \left(\int_0^1 |K(s,x) - K(t,x)|^2 \,\mathrm{d}x \right)^{1/2}$$

the estimate

$$\varepsilon_n([0,1], d) \le 2 \left(\int_0^{1/2n} (k(u))^2 \,\mathrm{d}u \right)^{1/2} \quad \text{for } n = 1, 2, 3, \ldots.$$

Using [CS90, Th. 5.6.1] (see also [RS96]) we get

$$a_{n+1}(T_K : L_2[0,1] \to C[0,1]) \le \varepsilon_n([0,1], d) \le 2 \left(\int_0^{1/2n} (k(u))^2 \,\mathrm{d}u \right)^{1/2}.$$

Moreover, for the identity operator $id : C[0,1] \to L_q[0,1]$ we have that $\Pi_q(id) = 1$ (cf. [P87, 1.3.8]). Using the inequality

$$a_{n+1}(T_K : L_2[0,1] \to L_q[0,1]; \Pi_q) \le \Pi_q(id : C[0,1] \to L_q[0,1]) \times \\ \times\, a_{n+1}(T_K : L_2[0,1] \to C[0,1])$$

we arrive at

$$a_{n+1}(T_K : L_2[0,1] \to L_q[0,1]; \Pi_q) \le 2 \left(\int_0^{1/2n} (k(u))^2 \right)^{1/2}.$$

Combining this estimate with Lemma 6.4.4 (ii) we finally obtain

$$k^{1/2} d_{k+n}(T_K : L_2[0,1] \to L_q[0,1]) \le 2\sqrt{2q} \left(\int_0^{1/2n} (k(u))^2 \right)^{1/2}$$

for $k, n \in \mathbb{N}$. Putting $k = n$ and $k = n-1$, respectively, we get with a new absolute constant $c \ge 1$ the desired estimate

$$n^{1/2} d_n(T_K : L_2[0,1] \to L_q[0,1]) \le c\,\sqrt{q} \left(\int_0^{1/n} (k(u))^2 \right)^{1/2}$$

for $n = 1, 2, \ldots$. ■

Now we give the corresponding result to Theorem 6.4.1 for weakly singular convolution operators from $L_2[0,1]$ into $L_q[0,1]$ for $1 \le p < \infty$.

Theorem 6.4.6. *Let (s_n) stand for the Kolmogorov numbers (d_n) or for the dyadic entropy numbers (e_n). Under the stated assumptions (cf. p.126) the following statements hold. If the function $k : (0,1] \to \mathbb{R}$ is defined by*

$$k(x) = x^{-\tau} (c_0 - \ln x)^{-\beta} \Big(c_0 + \ln(c_0 - \ln x) \Big)^{-\gamma}, \quad 0 < \tau \le \frac{1}{2},\ \beta, \gamma \in \mathbb{R},\ c_0 > 0,$$

then in the cases (WS) and (V) the following estimates hold for all $1 \le q < \infty$:

$$s_n(T_K : L_2[0,1] \to L_q[0,1]) \le c(\beta,\gamma)\,\sqrt{q}\, f(n,\tau,\beta,\gamma),$$

where $f(n,\tau,\beta,\gamma) =$

$$\begin{cases} n^{\tau-1} (\log(n+1))^{-\beta} (\log\log(n+3))^{-\gamma}, & 0 < \tau < 1/2,\ \beta \in \mathbb{R},\ \gamma \in \mathbb{R}, \quad (6.4.9) \\ n^{-1/2} (\log(n+1))^{1/2-\beta} (\log\log(n+3))^{-\gamma}, & \tau = 1/2,\ \beta > 1/2,\ \gamma \in \mathbb{R}, \quad (6.4.10) \\ n^{-1/2} (\log\log(n+3))^{1/2-\gamma}, & \tau = \beta = 1/2,\ \gamma > 1/2. \quad (6.4.11) \end{cases}$$

Proof. Using Theorem 6.4.5 we obtain the desired estimates for the Kolmogorov numbers of T_K. By applying Theorem 2.1.1 we get the same asymptotic estimates also for the dyadic entropy numbers of T_K. ■

If we compare the estimates of entropy and Kolmogorov numbers of weakly singular convolution operators from $L_2[0,1]$ into $C[0,1]$ given in Theorem 6.4.1 with those of Theorem 6.4.6 for the same convolution operator considered from $L_2[0,1]$ into $L_q[0,1]$ we observe a significant difference in the critical case $\tau = 1/2$, $\beta > 1/2$, $\gamma \in \mathbb{R}$ and in the super-critical case $\tau = \beta = 1/2$, $\gamma > 1/2$. In the critical case the difference is on the logarithmic scale, in the super-critical case it is even on the polynomial scale. In particular, we see that the estimates given in Theorem 6.4.1 are not the limiting case $q \to \infty$ of the estimates in Theorem 6.4.6.

6.5 Riemann-Liouville operators

This section deals with entropy and Kolmogorov numbers of the famous *Riemann-Liouville operator*

$$(R_\alpha f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)\,\mathrm{d}x$$

for $0 \le t \le 1$ and $\alpha > 0$. Singular numbers (= approximation numbers) of these operators between Hilbert spaces have been extensively studied by many authors in the literature (cf. e.g. [Bu07, DM97, Do93, Do95, FM86, M01]). Our results extend and complement results in the literature, especially of Lomakina and Stepanov [LS06], Li and Linde [LL99] and Linde [Lin04]. We start with recalling the fact that the classical Riemann-Liouville operator satisfies the semigroup property

$$R_\alpha(R_\beta f) = R_{\alpha+\beta} f, \quad \alpha, \beta > 0.$$

Furthermore, we need the following result for a general Volterra integration operator.

Lemma 6.5.1. *Let $k : (0,1] \to \mathbb{R}$ be a kernel function defined as on page 126 with $k \in L_1[0,1]$. Then for the Volterra-operator*

$$(T_K f)(t) = \int_0^t k(t-x) f(x)\,\mathrm{d}x$$

we have that

$$T_K : L_p[0,1] \to L_p[0,1], \quad 1 \le p \le \infty,$$

and for all $0 \le \delta \le 1$ the estimate

$$\|(T_K f)(\cdot + \delta) - (T_K f)(\cdot) | L_p[0,1]\| \le 2\, \|f | L_p[0,1]\| \int_0^\delta k(s)\,\mathrm{d}s$$

holds true.

Proof. Using the triangle inequality in L_p we see that

$$
\begin{aligned}
&\|(T_K f)(\cdot+\delta)-(T_K f)(\cdot)\|_p \\
&= \left(\int_0^{1-\delta} |(T_K f)(t+\delta)-(T_K f)(t)|^p \,\mathrm{d}t\right)^{1/p} \\
&= \left(\int_0^{1-\delta} \left|\int_0^{t+\delta} k(t+\delta-x) f(x)\,\mathrm{d}x - \int_0^{t} k(t-x) f(x)\,\mathrm{d}x\right|^p \mathrm{d}t\right)^{1/p} \\
&= \left(\int_0^{1-\delta} \left|\int_0^{t} \left[k(t+\delta-x)-k(t-x)\right] f(x)\,\mathrm{d}x + \int_t^{t+\delta} k(t+\delta-x) f(x)\,\mathrm{d}x\right|^p \mathrm{d}t\right)^{1/p} \\
&\le \left(\int_0^{1-\delta} \left|\int_0^{t} \left[k(t+\delta-x)-k(t-x)\right] f(x)\,\mathrm{d}x\right|^p \mathrm{d}t\right)^{1/p} \\
&\qquad + \left(\int_0^{1-\delta} \left|\int_t^{t+\delta} k(t+\delta-x) f(x)\,\mathrm{d}x\right|^p \mathrm{d}t\right)^{1/p} =: I_1 + I_2.
\end{aligned}
$$

Now we estimate the first integral I_1. Applying Minkowski's integral inequality (cf. [HLP88, Th. 202]) we obtain

$$
\begin{aligned}
I_1 &= \left(\int_0^{1-\delta} \left|\int_0^{t} \left[k(s+\delta)-k(s)\right] f(t-s)\,\mathrm{d}s\right|^p \mathrm{d}t\right)^{1/p} \\
&\le \left(\int_0^{1-\delta} \left(\int_0^{1-\delta} |k(s+\delta)-k(s)|\,|f(t-s)|\,\mathbb{1}_{[0,t]}(s)\,\mathrm{d}s\right)^p \mathrm{d}t\right)^{1/p} \\
&\le \int_0^{1-\delta} \left(\int_s^{1-\delta} |k(s+\delta)-k(s)|^p\,|f(t-s)|^p\,\mathrm{d}t\right)^{1/p} \mathrm{d}s \\
&= \int_0^{1-\delta} |k(s+\delta)-k(s)| \left(\int_s^{1-\delta} |f(t-s)|^p\,\mathrm{d}t\right)^{1/p} \mathrm{d}s \\
&\le \|f\|_p \int_0^{1-\delta} |k(s+\delta)-k(s)|\,\mathrm{d}s.
\end{aligned}
$$

Since k is non-negative and decreasing we can continue with

$$\begin{aligned}\int_0^{1-\delta} |k(s+\delta)-k(s)| \,\mathrm{d}s &= \int_0^{1-\delta} k(s)\,\mathrm{d}s - \int_0^{1-\delta} k(s+\delta)\,\mathrm{d}s \\ &= \int_0^{\delta} k(s)\,\mathrm{d}s - \int_0^{\delta} k(1-s)\,\mathrm{d}s \le \int_0^{\delta} k(s)\,\mathrm{d}s.\end{aligned}$$

Hence, we have that

$$I_1 \le \|f\|_p \int_0^{\delta} k(s)\,\mathrm{d}s.$$

Finally we deal with the second integral I_2. Using again Minkowski's integral inequality we see that

$$\begin{aligned} I_2 &\le \left(\int\limits_0^{1-\delta} \left(\int\limits_0^{\delta} k(\delta-s)\,|f(s+t)|\,\mathrm{d}s \right)^p \mathrm{d}t \right)^{1/p} \\ &\le \int_0^{\delta} \left(\int_0^{1-\delta} (k(\delta-s))^p\,|f(s+t)|^p\,\mathrm{d}t \right)^{1/p} \mathrm{d}s \\ &\le \|f\|_p \int_0^{\delta} k(\delta-s)\,\mathrm{d}s = \|f\|_p \int_0^{\delta} k(s)\,\mathrm{d}s. \end{aligned}$$

The assertion follows. ∎

Analogously to the above, also differences of higher order can be estimated. Now we are able to give upper entropy estimates of the classical Riemann-Liouville operator.

Proposition 6.5.2. *Let* $1 \le p,q \le \infty$ *and* $(1/p-1/q)_+ < \alpha < 1$. *Then*

$$e_n(R_\alpha : L_p[0,1] \to L_q[0,1]) \preccurlyeq n^{-\alpha}.$$

Proof. Applying Lemma 6.5.1 with the kernel function $k(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1}$ gives

$$\|(R_\alpha f)(\cdot+\delta) - (R_\alpha f)(\cdot)|L_p[0,1]\| \le \frac{2}{\alpha\Gamma(\alpha)}\,\delta^\alpha\, \|f|L_p[0,1]\|\,.$$

for all $\alpha > 0$. Hence, if $0 < \alpha < 1$, then the image $R_\alpha(L_p[0,1])$ belongs to the Besov space $B^\alpha_{p,\infty}[0,1]$ (cf. [Di03, Th. 3.18]). For $\alpha > (1/p-1/q)_+$ let $J_1 : B^\alpha_{p,\infty}[0,1] \to B^0_{q,1}[0,1]$ be the natural embedding. Then [C81c, Th. 2] (see also [ETr96], [Kö86]) gives

$$e_n(J_1 : B^\alpha_{p,\infty}[0,1] \to B^0_{q,1}[0,1]) \sim n^{-\alpha}.$$

Furthermore, the Besov space $B^0_{q,1}[0,1]$ is continuously embedded in $L_q[0,1]$ (cf. [Tr10, Prop. 2.5.7]). Denote this embedding by J_2 und let $I_{p,q} := J_2 J_1 : B^\alpha_{p,\infty}[0,1] \to L_q[0,1]$. Then for all $\alpha > (1/p-1/q)_+$ we get

$$e_n(I_{p,q} : B^\alpha_{p,\infty}[0,1] \to L_q[0,1]) \preccurlyeq n^{-\alpha}.$$

Now, for $(1/p - 1/q)_+ < \alpha < 1$ we factorize the Riemann-Liouville operator $R_\alpha : L_p[0,1] \to L_q[0,1]$ as $R_\alpha = I_{p,q}\, R_\alpha^B$, where $R_\alpha^B : L_p[0,1] \to B_{p,\infty}^\alpha[0,1]$ is the operator R_α with values in the Besov space. This yields the assertion due to the boundedness of the operator R_α^B. ■

In a next step, using Proposition 6.5.2 we give a complete overview about the entropy behavior of the classical Riemann-Liouville operator. Our results extend those of [LS06, Th. 2.2] from integers to positive real numbers.

Theorem 6.5.3. *Let* $1 \le p, q \le \infty$ *and* $(1/p - 1/q)_+ < \alpha < \infty$. *Then*

$$e_n(R_\alpha : L_p[0,1] \to L_q[0,1]) \sim n^{-\alpha}.$$

Proof. The upper estimate is a consequence of Proposition 6.5.2 and [LS06, Th. 2.2] in combination with the semigroup property of the Riemann-Liouville operator. First, let $1 < p, q < \infty$. In view of Proposition 6.5.2 and [LS06, Th. 2.2] it is enough to consider the case $\alpha > 1$ and $\alpha \notin \mathbb{N}$. We write $\alpha = \lfloor \alpha \rfloor + a$, $0 < a < 1$. Due to [LS06, Th. 2.2] we have

$$e_n(R_{\lfloor \alpha \rfloor} : L_p[0,1] \to L_q[0,1]) \preccurlyeq n^{-\lfloor \alpha \rfloor}$$

for $1 < p, q < \infty$ and Proposition 6.5.2 gives

$$e_n(R_a : L_p[0,1] \to L_p[0,1]) \preccurlyeq n^{-a}.$$

From the multiplicativity of entropy numbers we conclude that

$$e_{2n-1}(R_\alpha) = e_{2n-1}(R_{\lfloor \alpha \rfloor} R_a) \le e_n(R_{\lfloor \alpha \rfloor})\, e_n(R_a) \preccurlyeq n^{-\lfloor \alpha \rfloor}\, n^{-a} = n^{-\alpha}$$

for all $n = 1, 2, \ldots$. This yields the desired upper estimate for $1 < p, q < \infty$. It remains to deal with the cases $p \in \{1, \infty\}$, $1 \le q \le \infty$ and $1 \le p \le \infty$, $q \in \{1, \infty\}$. Let, for example, $p = 1$ and $1 < q < \infty$. For $\alpha > 1$, we write $\alpha = a + b$ with $b > 1$ and $0 < a < 1$. Choose $r > 1$ such that $1 - 1/r < a$ and consider $R_a : L_1[0,1] \to L_r[0,1]$ and $R_b : L_r[0,1] \to L_q[0,1]$. Due to Proposition 6.5.2 we have that $e_n(R_a : L_1[0,1] \to L_r[0,1]) \preccurlyeq n^{-a}$ and we have already proved that $e_n(R_b : L_r[0,1] \to L_q[0,1]) \preccurlyeq n^{-b}$. Hence,

$$e_{2n-1}(R_\alpha : L_1[0,1] \to L_q[0,1]) \preccurlyeq n^{-a}\, n^{-b} = n^{-\alpha}.$$

The other cases can be treated similarly.

In order to prove the lower estimate we use [LS06, Th. 2.2] and the upper estimate of $e_n(R_\alpha)$ we have already proved. First, let $1 < p, q < \infty$. For $\alpha > (1/p - 1/q)_+$ and $\alpha \notin \mathbb{N}$ we choose $0 < a < 1$ such that $\lfloor \alpha \rfloor + 1 = \alpha + a$. Then [LS06, Th. 2.2] gives

$$e_n(R_{\lfloor \alpha \rfloor + 1} : L_p[0,1] \to L_q[0,1]) \succcurlyeq n^{-\lfloor \alpha \rfloor - 1}.$$

Consequently, applying Proposition 6.5.2 we obtain

$$\begin{aligned} n^{-\lfloor\alpha\rfloor-1} &\preccurlyeq e_{2n-1}(R_{\lfloor\alpha\rfloor+1} : L_p[0,1] \to L_q[0,1]) \\ &\le e_n(R_a : L_p[0,1] \to L_p[0,1])\, e_n(R_\alpha : L_p[0,1] \to L_q[0,1]) \\ &\preccurlyeq n^{-a}\, e_n(R_\alpha : L_p[0,1] \to L_q[0,1]) \end{aligned}$$

and conclude that

$$e_n(R_\alpha : L_p[0,1] \to L_q[0,1]) \succcurlyeq n^{-\lfloor\alpha\rfloor-1+a} = n^{-\alpha}.$$

The remaining cases $p \in \{1,\infty\}$, $1 \le q \le \infty$ and $1 \le p \le \infty$, $q \in \{1,\infty\}$ can be treated analogously. ∎

Remark 8. Similar statements as in Theorem 6.5.3 can be obtained for a multiplicative s-number sequence $(s_n)_n$ if the s-numbers of the embedding

$$I_{p,q} : B^\alpha_{p,\infty}[0,1] \to B^0_{q,1}[0,1], \quad 1 \le p,q \le \infty, \quad \alpha > (1/p - 1/q)_+,$$

are known.

Finally, we deal with entropy and Kolmogorov numbers of weakly singular integral operators $T_K : L_p[0,1] \to C(A)$ for $2 \le p < \infty$, where $A \subset [0,1]$ is a compact subset of the interval $[0,1]$. The inequalities of the following theorem are modifications of the inequalities given in Theorem 1.2.2 and 1.3.1.

Theorem 6.5.4. *Let $2 \le p < \infty$ and $A \subset [0,1]$ be a compact subset of the interval $[0,1]$. Suppose that $k : (0,1] \to \mathbb{R}$ is a kernel function as stated on page 126 with $k \in L_{p'}[0,1]$. Then for the weakly singular integral operator $T_K : L_p[0,1] \to C(A)$,*

$$(T_K f)(t) = \int_0^1 k(|t-x|)\, f(x)\, \mathrm{d}x, \quad t \in A,$$

the following inequalities hold true:

(i)

$$\begin{aligned} &\sup_{1\le k\le n} k^{1/r+1/p}\, (\log(k+1))^\beta\, e_k\Big(T_K : L_p[0,1] \to C(A)\Big) \\ &\qquad\qquad \le c(p,r,\beta,K) \sup_{1\le k\le n^{1+\frac{r}{p}}} k^{1/r}\, (\log(k+1))^\beta\, \varepsilon_k(A,d) \end{aligned}$$

for $r > 0$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$, where $\varepsilon_n(A,d) \le 4^{1/p'} \left(\int_0^{\varepsilon_n(A)} (k(u))^{p'}\, \mathrm{d}u\right)^{1/p'}$. In the case $p = 2$ the inequality holds also for the Kolmogorov numbers $d_n\Big(T_K : L_2[0,1] \to C(A)\Big)$ of the operator T_K.

(ii)

$$n^{1/2} d_n\Big(T_K : L_2[0,1] \to C(A)\Big) \le c(K) \left(1 + \sum_{k=1}^{n} k^{-1/2}\, e_k(A,d)\right)$$

and

$$\begin{aligned} k^{1/2} d_{k+n}&\Big(T_K : L_2[0,1] \to C(A)\Big) \\ &\le c \left((\log(n+1))^{1/2}\, \varepsilon_n(A,d) + \sum_{j=n+1}^{\infty} \frac{\varepsilon_j(A,d)}{j\,(\log(j+1))^{1/2}} \right) \end{aligned}$$

for natural numbers $k, n \in \mathbb{N}$*, where* $\varepsilon_n(A,d) \le 2\left(\int_0^{\varepsilon_n(A)} (k(u))^2\, \mathrm{d}u\right)^{1/2}$ *and* $e_n(A,d) = \varepsilon_{2^{n-1}}(A,d)$*. In particular, we have that*

$$2^{n/2}\, d_{2^n}\Big(T_K : L_2[0,1] \to C(A)\Big) \le c \left(n^{1/2}\, e_n(A,d) + \sum_{j=n}^{\infty} j^{-1/2}\, e_j(A,d) \right)$$

for $n \in \mathbb{N}$*.*

Proof. From Theorem 1.3.1 and Corollary 6.1.3 we get the desired estimate (i) (see also Lemma 6.3.1). Furthermore, (6.1.8) gives $d_n(T_K) \le c_n\Big(\operatorname{aco}(\operatorname{Im}(K))\Big)$, where the kernel $K : A \to L_{p'}[0,1]$ maps A into $L_{p'}[0,1]$. In the Hilbert space case $p = 2$, we use the inequalities of Theorem 1.2.2 to obtain the desired result (ii). ■

Theorem 6.5.4 is the key to several estimates of entropy and Kolmogorov numbers of weakly singular integral operators. In the following, we complement results for the classical Riemann-Liouville operator given in [Lin04].

Proposition 6.5.5. *Let* $A \subset [0,1]$ *be a compact subset of the interval* $[0,1]$ *with* $\varepsilon_n(A) \preccurlyeq n^{-\delta}\,(\log(n+1))^{\theta}$ *for* $\delta \ge 1$ *and* $\theta \in \mathbb{R}$ *(note that since* $A \subset [0,1]$ *we necessarily have* $\theta \le 0$ *for* $\delta = 1$*). Then for the classical Riemann-Liouville operator* $R_\alpha : L_p[0,1] \to C(A)$*,* $2 \le p < \infty$*,* $\alpha > 1/p$*, we have the estimate*

$$e_n\Big(R_\alpha : L_p[0,1] \to C(A)\Big) \preccurlyeq n^{-1/p-\delta(\alpha-1/p)}\,(\log(n+1))^{\theta(\alpha-1/p)}$$

and in the case $p = 2$*,* $\alpha > 1/2$*, we also obtain*

$$d_n\Big(R_\alpha : L_2[0,1] \to C(A)\Big) \preccurlyeq n^{-1/2-\delta(\alpha-1/2)}\,(\log(n+1))^{\theta(\alpha-1/2)}.$$

Proof. For the kernel function $k(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1}$, $\alpha > 1/p$, we get

$$\begin{aligned} \varepsilon_n(A,d) &\le 4^{1/p'} \left(\int_0^{\varepsilon_n(A)} (k(u))^{p'}\, \mathrm{d}u\right)^{1/p'} = \frac{1}{\Gamma(\alpha)} \left(\frac{4}{p'(\alpha-1)+1}\right)^{1/p'} \Big(\varepsilon_n(A)\Big)^{\alpha-1/p} \\ &\preccurlyeq n^{-\delta(\alpha-1/p)}\,(\log(n+1))^{\theta(\alpha-1/p)}. \end{aligned}$$

Now we apply Theorem 6.5.4 (i) with $1/r = \delta(\alpha - 1/p)$ and $\beta = -\theta(\alpha - 1/p)$ to obtain the desired estimate

$$e_n\Big(R_\alpha : L_p[0,1] \to C(A)\Big) \preccurlyeq n^{-1/p-\delta(\alpha-1/p)}\,(\log(n+1))^{\theta(\alpha-1/p)}.$$

In the case $p = 2$ we get the corresponding estimate also for the Kolmogorov numbers of $R_\alpha : L_2[0,1] \to C(A)$. ∎

In order to illustrate the generality of the inequalities given in Theorem 6.5.4 we prove an analogon of Theorem 6.4.1 for the limiting case $\tau = 1/2$ and $1/2 < \beta < \infty$.

Proposition 6.5.6. *Let $A \subset [0,1]$ be a compact subset and $T_K : L_2[0,1] \to C(A)$ the weakly singular integral operator generated by the kernel function*

$$k(x) = x^{-1/2}\,(c_0 - \ln x)^{-\beta}, \quad 1/2 < \beta < \infty,\ c_0 > 0.$$

Furthermore, let (s_n) stand for the Kolmogorov numbers (d_n) or for the dyadic entropy numbers (e_n). Then the following statements hold true:

(i) If $\varepsilon_n(A) \sim n^{-\delta}$ for $\delta \geq 1$ then

$$s_n\Big(T_K : L_2[0,1] \to C(A)\Big) \preccurlyeq \begin{cases} n^{1/2-\beta}, & 1/2 < \beta < 1, \\ n^{-1/2}\,\log(n+1), & \beta = 1, \\ n^{-1/2}\,(\log(n+1))^{1-\beta}, & 1 < \beta < \infty. \end{cases}$$

(ii) If $\varepsilon_n(A) \sim e^{-n^\delta}$ for $\delta > 0$ then

$$s_n\Big(T_K : L_2[0,1] \to C(A)\Big) \preccurlyeq n^{-1/2-\delta(\beta-1/2)}.$$

Proof. Both results follow from Theorem 6.5.4. At first, we compute that

$$\varepsilon_n(A,d) \leq 2\left(\int_0^{\varepsilon_n(A)} (k(u))^2\,\mathrm{d}u\right)^{1/2} \preccurlyeq \Big(c_0 - \ln\varepsilon_n(A)\Big)^{1/2-\beta}.$$

Consequently, in the case (i) we obtain $\varepsilon_n(A,d) \preccurlyeq (\log(n+1))^{1/2-\beta}$ and $e_n(A,d) \preccurlyeq n^{1/2-\beta}$. Applying the inequalities of Theorem 6.5.4 (ii) leads to the desired estimates for the Kolmogorov numbers of T_K. For example, if $1/2 < \beta < 1$ then

$$\sum_{k=1}^n k^{-1/2}\,e_k(A,d) \preccurlyeq \sum_{k=1}^n k^{-\beta} \preccurlyeq n^{1-\beta}$$

and this yields

$$n^{1/2}\,d_n\Big(T_K : L_2[0,1] \to C(A)\Big) \preccurlyeq n^{1-\beta}.$$

The other cases can be treated similarly. Using Theorem 2.1.1 we obtain the same asymptotic estimates also for the dyadic entropy numbers of T_K. In the case (ii) we get $\varepsilon_n(A,d) \preccurlyeq n^{\delta(1/2-\beta)}$. Consequently, applying Theorem 6.5.4 (i) with $1/r = -\delta(1/2-\beta)$ and $\beta = 0$ yields

$$s_n\Big(T_K : L_2[0,1] \to C(A)\Big) \preccurlyeq n^{-1/2+\delta(1/2-\beta)}$$

and finishes the proof. ■

Remark 9. The upper estimates given in Proposition 6.5.6 (i) are the same as for the whole interval $A = [0,1]$ (cf. Theorem 6.4.1). The parameter δ only affects the constant.

We can also prove an analogon of Theorem 6.4.5.

Theorem 6.5.7. *Let $k : (0,1] \to \mathbb{R}$ be a kernel function as stated on page 126 with $k \in L_2[0,1]$. Furthermore, let $A \subset [0,1]$ be a compact subset of the interval $[0,1]$ and μ a Hausdorff measure on A. Then for the weakly singular integral operator $T_K : L_2[0,1] \to L_q(A,\mu)$, $1 \le q < \infty$, given by*

$$(T_K f)(t) = \int_0^1 k(|t-x|)\, f(x)\,\mathrm{d}x$$

the estimate

$$n^{1/2} d_{2n}\Big(T_K : L_2[0,1] \to L_q(A,\mu)\Big) \le c\,(\mu(A))^{1/q}\,\sqrt{q}\,\left(\int_0^{\varepsilon_n(A)} (k(u))^2\,\mathrm{d}u\right)^{1/2}, \quad n \in \mathbb{N},$$

holds true, where $c \ge 1$ is an absolute constant. In particular, for $k(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1}$, $\alpha > 1/2$, we obtain that

$$d_{2n}\Big(T_K : L_2[0,1] \to L_q(A,\mu)\Big) \preccurlyeq n^{-1/2}\,(\varepsilon_n(A))^{\alpha-1/2}.$$

Proof. The proof is analogous to that of Theorem 6.4.5. This time we use that

$$a_{n+1}\Big(L_2[0,1] \to C(A)\Big) \le \varepsilon_n(A,d) \le 2\left(\int_0^{\varepsilon_n(A)} (k(u))^2\,\mathrm{d}u\right)^{1/2}$$

and

$$\Pi_q\Big(I : C(A) \to L_q(A,\mu)\Big) \le (\mu(A))^{1/q}, \quad 1 \le q < \infty.$$

If the kernel function is given by $k(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1}$ with $\alpha > 1/2$, then we compute that

$$\left(\int_0^{\varepsilon_n(A)} (k(u))^2\,\mathrm{d}u\right)^{1/2} = \frac{1}{\Gamma(\alpha)}\left(\frac{(\varepsilon_n(A))^{2(\alpha-1)+1}}{2(\alpha-1)+1}\right)^{1/2}.$$

The assertion follows. ■

Bibliography

[Ao42] T. Aoki, Locally bounded linear topological spaces. Proc. Imp. Acad. Tokyo **18** (1942), 588–594

[AMS03] S. Artstein, V. Milman, S. J. Szarek, Duality of metric entropy in Euclidean space. C. R. Acad. Sci. Paris **337** (2003), no. 11, 711–714

[AMS04] S. Artstein, V. Milman, S. J. Szarek, Duality of metric entropy. Ann. of Math. (2) **159** (2004), no. 3, 1313–1328

[AMST04] S. Artstein, V. Milman, S. J. Szarek, N. Tomczak-Jaegermann, On convexified packing and entropy duality. Geom. Funct. Anal. **14** (2004), no. 5, 1134–1141

[BP90] K. Ball, A. Pajor, The entropy of convex bodies with few extreme points. In: Geometry of Banach spaces (Strobl, 1989), 25–32, London Math. Soc. Lecture Note Ser. **158**, Cambridge Univ. Press, Cambridge, 1990

[BGT87] N. H. Bingham, C. M. Goldie, J. L. Teugels, Regular Variation. Cambridge Univ. Press, Cambridge, 1987

[BS67] M. Š. Birman, M. Z. Solomjak, Piecewise polynomial approximations of functions of classes W_p^α. (Russian) Mat. Sb. (N.S.) **73 (115)** (1967), 331–355

[BPST89] J. Bourgain, A. Pajor, S. J. Szarek, N. Tomczak-Jaegermann, On the duality problem for entropy numbers of operators. Geometric aspects of functional analysis (1987-88), 50–63, Lecture Notes in Math. **1376**, Springer-Verlag, Berlin, 1989

[Bu07] P. Burman, Sharp bounds for singular values of fractional integral operators. (English summary) J. Math. Anal. Appl. **327** (2007), no. 1, 251–256

[C81a] B. Carl, Entropy numbers, s-numbers, and eigenvalue problems. J. Funct. Anal. **41** (1981), no. 3, 290–306

[C81b] B. Carl, Entropy numbers of diagonal operators with an application to eigenvalue problems. J. Approx. Theory **32** (1981), no. 2, 135–150

[C81c] B. Carl, Entropy numbers of embedding maps between Besov spaces with an application to eigenvalue problems. Proc. Roy. Soc. Edinburgh Sect. A **90** (1981), no. 1–2, 63–70

[C82] B. Carl, On a characterization of operators from l_q into a Banach space of type p with some applications to eigenvalue problems. J. Funct. Anal. **48** (1982), no. 3, 394–407

[C85] B. Carl, Inequalities of Bernstein-Jackson type and the degree of compactness of operators in Banach spaces. Ann. Inst. Fourier (Grenoble) **35** (1985), no. 3, 79–118

[C97] B. Carl, Metric entropy of convex hulls in Hilbert spaces. Bull. London Math. Soc. **29** (1997), no. 4, 452–458

[CE01] B. Carl, D. E. Edmunds, Entropy of $C(X)$-valued operators and diverse applications. J. Inequal. Appl. **6** (2001), no. 2, 119–147

[CE03] B. Carl, D. E. Edmunds, Gelfand numbers and metric entropy of convex hulls in Hilbert spaces. Studia Math. **159** (2003), no. 3, 391–402

[CHK88] B. Carl, S. Heinrich, T. Kühn, s-numbers of integral operators with Hölder continuous kernels over metric compacta. J. Funct. Anal. **81** (1988), no. 1, 54–73

[CHP11] B. Carl, A. Hinrichs, A. Pajor, Gelfand numbers and metric entropy of convex hulls in Hilbert spaces. Positivity **17** (2013), no. 1, 171–203

[CHR12] B. Carl, A. Hinrichs, P. Rudolph, Entropy numbers of convex hulls in Banach spaces and applications. Submitted to J. Funct. Anal.

[CKP99] B. Carl, I. Kyrezi, A. Pajor, Metric entropy of convex hulls in Banach spaces. J. London Math. Soc. (2) **60** (1999), no. 3, 871–896

[CPa88] B. Carl, A. Pajor, Gelfand numbers of operators with values in Hilbert spaces. Invent. Math. **94** (1988), no. 3, 459–504

[CP76] B. Carl, A. Pietsch, Entropy numbers of operators in Banach spaces. Proc. Fourth Prague Topological Sympos., Prague 1976, Part A, pp. 21–33, Lecture Notes in Math. **609**, Springer-Verlag, Berlin, 1977

[CR13] B. Carl, P. Rudolph, Operators factoring through general diagonal operators. Submitted to Rev. Mat. Complut.

[CS90] B. Carl, I. Stephani, Entropy, Compactness and Approximation of Operators. Cambridge Univ. Press, Cambridge, 1990

[CTr80] B. Carl, H. Triebel, Inequalities between eigenvalues, entropy numbers, and related quantities of compact operators in Banach spaces. Math. Ann. **251** (1980), no. 2, 129–133

[CrSt02] J. Creutzig, I. Steinwart, Metric entropy of convex hulls in type p spaces – the critical case. Proc. Amer. Math. Soc. **130** (2002), no. 3, 733–743

[DJT95] J. Diestel, H. Jarchow, A. Tonge, Absolutely Summing Operators. Cambridge Univ. Press, Cambridge, 1995

[Di03] S. Dispa, Intrinsic characterizations of Besov spaces on Lipschitz domains. Math. Nachr. **260** (2003), 21–33

[Do93] M. R. Dostanic, Asymptotic behavior of the singular values of fractional integral operators. J. Math. Anal. Appl. **175** (1993), no. 2, 380–391

[Do95] M. R. Dostanic, An estimation of singular values of convolution operators. Proc. Amer. Math. Soc. **123** (1995), no. 5, 1399–1409

[DM97] M. R. Dostanic, D. Z. Milinkovic, Asymptotic behavior of singular values of certain integral operators. Publ. Inst. Math. (Beograd) (N.S.) **62 (76)** (1997), 83–98

[DPR72] E. Dubinsky, A. Pełczyński, H. P. Rosenthal, On Banach spaces X for which $\Pi_2(\mathcal{L}_\infty, X) = B(\mathcal{L}_\infty, X)$. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, VI. Studia Math. **44** (1972), 617–648

[D87] R. M. Dudley, Universal Donsker classes and metric entropy, Ann. Probab. **15** (1987), no. 4, 1306–1326

[EE86] D. E. Edmunds, R. M. Edmunds, Entropy numbers of compact operators. Bull. London Math. Soc. **18** (1986), no. 4, 392–394

[ETr96] D. E. Edmunds, H. Triebel, Function Spaces, Entropy Numbers and Differential Operators. Cambridge Univ. Press, Cambridge, 1996

[FM86] V. Faber, G. M. Wing, Singular values of fractional integral operators: a unification of theorems of Hille, Tamarkin, and Chang. J. Math. Anal. Appl. **120** (1986), no. 2, 745–760

[G01] F. Gao, Metric entropy of convex hulls. Israel J. Math. **123** (2001), 359–364

[G04] F. Gao, Entropy of absolute convex hulls in Hilbert spaces. Bull. London Math. Soc. **36** (2004), no. 4, 460–468

[G12] F. Gao, Optimality of CKP-inequality in the critical case, to appear in Proc. Amer. Math. Soc.

[Go88] Y. Gordon, On Milman's inequality and random subspaces which escape through a mesh in R^n. Geometric aspects of functional analysis (1986/87), 84–106, Lecture Notes in Math. **1317** (1988), Springer-Verlag, Berlin, 1988

[Ha82] U. Haagerup, The best constants in the Khintchine inequality. Studia Math. **70** (1982), no. 3, 231–283

[HLP88] G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities. Reprint of the 1952 edition. Cambridge Mathematical Library. Cambridge Univ. Press, Cambridge, 1988

[HK85] S. Heinrich, T. Kühn, Embedding maps between Hölder spaces over metric compacta and eigenvalues of integral operators, Nederl. Akad. Wetensch. Indag. Math. **47** (1985), no. 1, 47–62

[Hi03] J. Hildebrandt, Entropie p-konvexer Hüllen. (German) Dissertation (2003), Friedrich-Schiller-Universität Jena

[Ho74] J. Hoffmann-Jørgensen, Sums of independent Banach space valued random variables. Studia Math. **52** (1974), 159–186

[Hö80] K. Höllig, Diameters of classes of smooth functions. Quantitative Approximation (Proc. Internat. Sympos., Bonn, 1979), pp. 163–175, Academic Press, New York-London, 1980

[Ka68] J. P. Kahane, Some Random Series of Functions. Heath. Math. Monographs 1968, Second ed., Cambridge Univ. Press, Cambridge, 1985

[Ka30] J. Karamata, Sur un mode de croissance régulière des fonctions, Mathematica (Cluj) **4** (1930), 38–53

[Kl12a] O. Kley, Entropy of Convex Hulls and Kuelbs-Li Inequalities. Dissertation (2012), Friedrich-Schiller-Universität Jena

[Kl12b] O. Kley, Kuelbs-Li inequalities and metric entropy of convex hulls. J. Theoret. Probab. (2012), DOI: 10.1007/s10959-012-0408-5

[K56] A. N. Kolmogorov, On certain asymptotic characteristics of completely bounded metric spaces. (Russian) Dokl. Akad. Nauk SSSR **108** (1956), 585–589

[KTi61] A. N. Kolmogorov, V. M. Tihomirov, ε-entropy and ε-capacity of sets in function spaces. (Russian) Uspehi Mat. Nauk **14** (1959), no. 2 (86), 3-86; English transl.: Amer. Math. Soc. Transi. **17** (1961), no. 2, 277–364

[Kö86] H. König, Eigenvalue Distribution of Compact Operators. Operator Theory: Advances and Applications **16**, Birkhäuser, Basel, 1986

[KöM86] H. König, V. D. Milman, On the Covering Numbers of Convex Bodies. Geometrical aspects of functional analysis (1985/86), 82–95, Lecture Notes in Math. **1267**, Springer-Verlag, Berlin, 1987

[KöMT86] H. König, V. D. Milman, N. Tomczak-Jaegermann, Entropy numbers and duality for operators with values in a Hilbert space. Probability in Banach spaces **6** (Sandbjerg, 1986), 219–233, Progr. Probab. **20**, Birkhäuser Boston, Boston, 1990

[Kü01] T. Kühn, Entropy numbers of diagonal operators of logarithmic type. Georgian Math. J. **8** (2001), no. 2, 307–318

[Kü05] T. Kühn, Entropy numbers of general diagonal operators. Rev. Mat. Complut. **18** (2005), no. 2, 479–491

[Ky00] I. Kyrezi, On the entropy of the convex hull of finite sets. Proc. Amer. Math. Soc. **128** (2000), no. 8, 2393–2403

[LO94] R. Latala, K. Oleszkiewicz, On the best constant in the Khinchin-Kahane inequality. Studia Math. **109** (1994), no. 1, 101–104

[LL99] W. V. Li, W. Linde, Approximation, metric entropy and small ball estimates for Gaussian measures. Ann. Probab. **27** (1999), no. 3, 1556–1578

[LL00] W. V. Li, W. Linde, Metric entropy of convex hulls in Hilbert spaces. Studia Math. **139** (2000), no. 1, 29–45

[Lif10] M. A. Lifshits, Bounds for entropy numbers of some critical operators. Trans. Amer. Math. Soc. **364** (2012), no. 4, 1797–1813

[Lin04] W. Linde, Kolmogorov numbers of Riemann-Liouville operators over small sets and applications to Gaussian processes. J. Approx. Theory **128** (2004), no. 2, 207–233

[Lin08] W. Linde, Nondeterminism of linear operators and lower entropy estimates, J. Fourier Anal. Appl. **14** (2008), no. 4, 568–587

[LP74] W. Linde, A. Pietsch, Mappings of Gaussian measures of cylindrical sets in Banach spaces. (Russian) Teor. Verojatnost. i Primenen. **19** (1974), 472–487

[LR69] J. Lindenstrauss, H. P. Rosenthal, The $\mathcal{L}_p$ spaces. Israel J. Math. **7** (1969), 325–349

[LS06] E. N. Lomakina, V. D. Stepanov, Asymptotic estimates for the approximation and entropy numbers of the one-weight Riemann-Liouville operator. (Russian) Mat. Tr. **9** (2006), no. 1, 52–100

[M74] M. B. Marcus, The ε-entropy of some compact subsets of l^p. J. Approximation Theory **10** (1974), 304–312

[MaPi76] B. Maurey, G. Pisier, Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach. (French) Studia Math. **58** (1976), no. 1, 45–90

[M01] A. Meskhi, Asymptotic behavior of singular and entropy numbers for some Riemann-Liouville type operators. Georgian Math. J. **8** (2001), no. 2, 323–332

[M07] E. Milman, A remark on two duality relations. Integral Equations Operator Theory **57** (2007), no. 2, 217–228

[MS86] V. D. Milman, G. Schechtman, Asymptotic Theory of Finite Dimensional Normed Spaces. With an Appendix by M. Gromov. Lecture Notes in Math. **1200**, Springer-Verlag, Berlin, 1986

[O78] R. Oloff, Entropieeigenschaften von Diagonaloperatoren. (German) Math. Nachr. **86** (1978), 157–165

[PT86] A. Pajor, N. Tomczak-Jaegermann, Subspaces of small codimension of finite-dimensional Banach spaces. Proc. Amer. Math. Soc. **97** (1986), no. 4, 637–642

[P74] A. Pietsch, s-numbers of operators in Banach spaces. Studia Math. **51** (1974), 201–223

[P78] A. Pietsch, Operator Ideals. Mathematische Monographien [Mathematical Monographs] **16**, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978

[P87] A. Pietsch, Eigenvalues and s-Numbers. Cambridge Studies in Advanced Mathematics **13**, Cambridge Univ. Press, Cambridge, 1987

[P07] A. Pietsch, History of Banach Spaces and Linear Operators. Birkhäuser Boston, Boston, 2007

[Pi73a] G. Pisier, "Type" des espaces normés. (French) École Polytech. Palaiseau, Séminaire Maurey-Schwartz 1973/74, Exposé 3

[Pi73b] G. Pisier, Sur les espaces qui ne contiennent pas de l_n^1 uniformément. (French) École Polytech. Palaiseau, Séminaire Maurey-Schwartz 1973/74, Exposé 7

[Pi81] G. Pisier, Remarques sur un résultat non publié de B. Maurey. (French) École Polytech. Palaiseau, Séminaire d'Analyse Fonctionnelle 1980/81, Exposé 5

[Pi89] G. Pisier, The Volume of Convex Bodies and Banach Space Geometry. Cambridge Univ. Press, Cambridge, 1989

[PS32] L. Pontrjagin, L. Schnirelmann, Sur une propriété métrique de la dimension. (French) Ann. of Math. (2) **33** (1932), no. 1, 156–162

[RB97] C. Richter, W. Börner, Entropy, capacity and arrangements on the cube. Beiträge Algebra Geom. **38** (1997), no. 2, 227–232

[RS96] C. Richter, I. Stephani, Entropy and the approximation of bounded functions and operators. Arch. Math. (Basel) **67** (1996), no. 6, 478–492

[Ro57] S. Rolewicz, On a certain class of linear metric spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 471–473

[Ru10] P. Rudolph, Entropy Numbers of Operators in Banach Spaces. Diplomarbeit (2010), Friedrich-Schiller-University Jena

[Sch30] J. Schauder, Über lineare, vollstetige Funktionaloperationen, Studia Math. **2** (1930), 183–196

[Sch84] C. Schütt, Entropy numbers of diagonal operators between symmetric Banach spaces. J. Approx. Theory **40** (1984), no. 2, 121–128

[St99] I. Steinwart, Entropy of $C(K)$-Valued Operators and Some Applications. Dissertation (1999), Friedrich-Schiller-Universität Jena

[St00] I. Steinwart, Entropy of $C(K)$-valued operators. J. Approx. Theory **103** (2000), no. 2, 302–328

[St04] I. Steinwart, Entropy of convex hulls - some Lorentz norm results. J. Approx. Theory **128** (2004), no. 1, 42–52

[Ta87] M. Talagrand, Regularity of Gaussian processes. Acta Math. **159** (1987), no. 1–2, 99–149

[Ta93] M. Talagrand, New Gaussian estimates for enlarged balls. Geom. Funct. Anal. **3** (1993), no. 5, 502–526

[To87] N. Tomczak-Jaegermann, Dualité des nombres d'entropie pour des opérateurs à valeurs dans un espace de Hilbert. (French) C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 7, 299–301

[Tr75] H. Triebel, Interpolation properties of ε-entropy and of diameters. Geometric characteristics of the imbedding of function spaces of Sobolev-Besov type. (Russian) Mat. Sb. (N.S.) **98 (140)** (1975), no. 1 (9), 27–41, 157

[Tr10] H. Triebel, Theory of Function Spaces. Modern Birkhäuser Classics, Birkhäuser, 2010

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