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Effects of dynamic and non-dynamic element changes in RLC networks

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Abstract

The paper deals with the redesign of passive electric networks by changes of single dynamic and non-dynamic elements which may retain, or affect the natural topology of the network. It also deals with the effect of such changes on the natural dynamics of the network, the natural frequencies. The impedance and admittance modeling for passive electrical networks is used which provides a structured, symmetric, integral-differential description, which in the special cases of RC and RL networks is reduced to matrix pencil descriptions. The transformations on the network are expressed as those preserving, or modifying the two natural topologies of the network, the impedance graph and the admittance graph topologies. For the special cases of RC and RL networks we consider the problem of the effect of changes of a single dynamic, or non-dynamic element on the natural frequencies. Using the Determinantal Assignment Framework, it is shown that the family of single parameter variation problems is reduced to equivalent Root Locus problems with the possibility of fixed modes. An explicit characterization of the fixed modes is given and a number of interesting properties of the spectrum are derived such as the interlacing property of poles and zeros for the entire family of Root Locus problems.

Keywords: Passive networks, network redesign, matrix pencils, root locus, interlacing property, robustness

1 Introduction

The problem of redesigning autonomous (no inputs or outputs) passive electric networks [9] aims to change the natural dynamics of the network (natural frequencies) by modification of the network. As such, this is a problem that differs from a standard control problem, since it involves changing the values of the elements and possibly the topology of the network to achieve the desirable natural frequencies. In fact, this problem involves the selection of alternative values for dynamic elements (inductances, capacitances) and non-dynamic elements (resistances) within

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a fixed interconnection topology and/or alteration of the interconnection topology and possible evolution of the network (increase of elements, branches). We use *impedance* and *admittance* modeling [23] for passive electrical networks. Integral part of the study is the investigation of transformations of *structured models*. In our study we identify two natural topologies expressing the structured transformations which are identified as the *impedance graph* and the *admittance graph* of the network. Within this structured framework we will consider the effect of changes of a single dynamic, or non-dynamic element on the natural frequencies. We consider issues related to the movement of the natural frequencies rather than problems of frequency assignment [9, 14]. It is shown that the single parameter variation problem (dynamic or non-dynamic) is equivalent to a Root Locus problem [2, 18].

The general case of RLC networks may be considered within the impedance and admittance modeling framework we are considering. A simplification of the problem is achieved by restricting our study to the case of RL (resistor-inductor) or RC (resistor-capacitor) networks where the corresponding impedance, or admittance models become matrix pencils. Our study requires the representation of transformations that preserve, or transform the network topology. This is achieved by defining appropriate matrices expressing such transformations. We use the Determinantal Assignment approach [7, 13] for the analysis of the spectrum and it is shown that the single parameter variation problem is equivalent to a standard single-input single-output Root Locus problem [18]. The polynomials defining the Root Locus problem are explicitly defined from the network description and the nature of the transformation and it is shown that the problem may have fixed modes [8]. Such Root Locus problems are based on pole and zero polynomials formed by fixing the transformation under study. The selection of the transformation fixes the Root Locus problem and in some cases results in the emergence of fixed points in the Root Locus. Such points are readily identified within the exterior algebra framework used and their computation is reduced to a Greatest Common Divisor calculation [10]. The properties of the resulting Root Locus problems stem from the symmetry of the admittance, or impedance operators together with the passivity of the system. A number of interesting properties of the spectrum are derived such as the interlacing property of poles and zeros for the entire family of Root Locus problems. It is also shown that there is a common direction of movement of the poles under single parameter variations.

The paper deals primarily with network formulations coming from the electrical domain. The results however are equally relevant for the redesign of mechanical networks, as well as redesign of structures in structural (civil) engineering problems.

The paper is organized as follows: In Section 2 we introduce the frameworks of impedance and admittance modeling of passive networks. We present the corresponding loop and node methods by means of an illustrative example and also mention the natural vertex and loop topologies. In Section 3 we state the problem of network redesign for the cases of RL and RC networks and show that the impedance and admittance models become matrix pencils in these cases. Then we introduce the types of transformations of the network that we consider in this article: single dynamic, or non-dynamic element changes. We close Section 3 by showing that this single parameter variation problem is equivalent to a standard Root Locus problem. Section 4 starts with some remarks on the identification of fixed modes using the exterior algebra framework. After introducing some crucial results about symmetric matrix pencils we then prove that the Root Locus problem has an interlacing property. The remainder of Section 4 is then dedicated to the proof of another essential property of the Root Locus: The existence of a zero which is

larger than every pole and, as a consequence, the common direction of movement of the poles under single parameter variations. Some conclusions are derived in Section 5.

We close the introduction with the nomenclature used in this paper.

$\mathbb{N}, \mathbb{R}, \mathbb{C}$	the set of natural, real and complex numbers, resp.
$\mathbb{K}^{m \times n}$	the set of $m \times n$ matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$
A^*	$= \overline{A}^\top$, the conjugate transpose of $A \in \mathbb{C}^{m \times n}$
$\ x\ $	$= \sqrt{x^*x}$, the Euclidean norm of $x \in \mathbb{C}^n$
$\ A\ $	$= \max \{\ Ax\ \mid x \in \mathbb{C}^n, \ x\ = 1\}$, induced matrix norm of $A \in \mathbb{C}^{m \times n}$
$\ker A$	the kernel of the matrix $A \in \mathbb{C}^{m \times n}$
$\text{im } A$	the image of the matrix $A \in \mathbb{C}^{m \times n}$
$\text{rk } A$	the rank of the matrix $A \in \mathbb{C}^{m \times n}$
$\text{adj } A$	the adjugate of the matrix $A \in \mathbb{C}^{m \times n}$
$A > (\geq) 0 \Leftrightarrow$	$A \in \mathbb{C}^{m \times n}$ is positive (semi-)definite; $A < (\leq) 0 \Leftrightarrow -A > (\geq) 0$
$\mathcal{R}[s]$	the ring of polynomials with coefficients in a ring \mathcal{R}
$\deg p(s)$	the degree of the polynomial $p(s) \in \mathcal{R}[s]$

2 Impedance and admittance modeling of linear passive networks

2.1 Background

The vertex and path laws (Kirchhoff current and voltage laws for electrical circuits) together with the elemental equations (Ohm's law and impedance relations for electrical circuits) allow the system equations to be formulated. We consider networks with b branches, n vertices and s branches which are sources. There are two basic problems in deriving a mathematical model for the network. The first deals with the question of defining the required number of equations and the second is how to solve the obtained set in a systematic way. Such considerations are important especially when we deal with rather large systems. Regarding the first problem the results from network theory state the following [21].

Lemma 2.1. *A sufficient set of equations for determining the systems equations for any output of any system (linear or nonlinear) is obtained by using a set of $(n - 1)$ linearly independent vertex equations, a set of $(b - n + 1)$ linearly independent path equations and $(b - s)$ elemental equations.*

For a linear system the above set is also necessary to obtain a solution. The above clearly implies that for a graph that has b branches, where s are the sources, $(2b - s)$ independent equations can be stated. This set of equations in $(2b - s)$ unknowns (one through- and across-variable for each branch; for a source there is either a through- or an across-variable) may be solved for

any selected unknowns and this leads to the development of system equations for the selected outputs. There are basically two methods leading to a systematic derivation of a reduced set equations describing the network. These methods are more easily presented in terms of the variables which are selected as primary unknowns and are referred to as the *node* and *loop methods* [21, 23].

The node method: In this method, the across-variables from each vertex to some reference vertex are chosen as the unknowns in terms of which the final set of equations is formulated. Such variables are called *node variables*. These variables automatically satisfy the path laws, since the across-variables between nodes are expressed simply as the difference between the appropriate variables. The vertex equation is then written at each node, and the through-variables are expressed directly in terms of the node variables as related by the elemental equations. The process eliminates all variables except the node variables and ends up with a number of equations which is in general $(n - 1)$.

The loop method: In the loop method, the variable are selected such that the vertex law is automatically satisfied. Here we consider only planar graphs. We then consider the variables associated with each of the meshes and define these as the *loop variables*. This approach leads to that each branch through-variable will be the difference between the loop variables on each side of the branch. This way the vertex equations are automatically satisfied. The path law is then written for each mesh and substitutions are made for the across-variables in terms of the loop variables using the elemental equations. This way the overall system is reduced to a number of equations which is $b - n + 1$.

The two above methods lead to mathematical descriptions referred to as *node* or *admittance description* and *loop* or *impedance description* respectively, and their form is described in the following subsections.

2.2 Loop method formulation

The process of working out the equations involves the selection of internal independent loops, the definition of loop currents and the transformation of current sources to equivalent voltage sources (Thévenin's theorem). If we denote by (f_1, f_2, \dots, f_q) the set of the Laplace transforms of the loop currents and by $(v_{s_1}, \dots, v_{s_q})$ the set of Laplace transforms of equivalent voltage sources, then the *loop* or *impedance model* is defined by

$$\begin{bmatrix} z_{11} & -z_{12} & -z_{13} & \cdots & -z_{1q} \\ -z_{12} & z_{22} & -z_{23} & \cdots & -z_{2q} \\ -z_{13} & -z_{22} & z_{33} & \cdots & -z_{3q} \\ \vdots & \vdots & \vdots & & \vdots \\ -z_{1q} & -z_{2q} & -z_{3q} & \cdots & z_{qq} \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_q \end{pmatrix} = \begin{pmatrix} v_{s_1} \\ v_{s_2} \\ v_{s_3} \\ \vdots \\ v_{s_q} \end{pmatrix}, \quad (2.1)$$

where in (2.1):

- (i) $z_{ii}(s)$ is the sum of impedances in loop i ,

(ii) $z_{ij}(s)$ is the sum of the impedances common between loops i and j .

(2.1) can be written in short as

$$Z(s)f(s) = v_s(s).$$

This is referred to as the *loop* or *impedance model* and the symmetric matrix $Z(s)$ is referred to as the *network impedance matrix*.

2.3 Node method formulation

The node method is dual to the loop method and the basic steps involve the selection of internal nodes, definition of the corresponding node voltages and transformation of the voltage sources to equivalent current sources (Norton's theorem). If we denote by (v_1, v_2, \dots, v_n) the Laplace transforms of the reduced node voltages and by $(i_{s_1}, \dots, i_{s_n})$ the set of Laplace transforms of equivalent current sources, then the *node* or *admittance model* is defined by

$$\begin{bmatrix} y_{11} & -y_{12} & -y_{13} & \cdots & -y_{1n} \\ -y_{12} & y_{22} & -y_{23} & \cdots & -y_{2n} \\ -y_{13} & -y_{22} & y_{33} & \cdots & -y_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ -y_{1n} & -y_{2n} & -y_{3n} & \cdots & y_{nn} \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} i_{s_1} \\ i_{s_2} \\ i_{s_3} \\ \vdots \\ i_{s_n} \end{pmatrix}, \quad (2.2)$$

where in (2.2):

(i) $y_{ii}(s)$ is the sum of admittances in node i ,

(ii) $y_{ij}(s)$ is the sum of the admittances common between nodes i and j .

The mathematical model (2.1) is denoted in short as

$$Y(s)v(s) = i_s(s)$$

and is referred to as *node* or *admittance model*. The symmetric matrix $Y(s)$ is referred to as the *network admittance matrix*.

Remark 2.2. By the construction of the node and loop models the matrices $Z(s)$ and $Y(s)$ are symmetric. Furthermore, it should be remembered that the network is passive and this affects the properties of the characteristic roots. \diamond

2.4 Example

Consider the mechanical translational system depicted in Figure 1.

The associated linear graph is shown in Figure 2.

For the node method we transform the voltage source V into an equivalent current source. This eliminates the node voltage v_4 and the reduced node equations are

$$\begin{bmatrix} b_1 + b_2 + \frac{k_1}{s} & -b_2 & 0 \\ -b_2 & m_1s + b_2 + \frac{k_2}{s} & -\frac{k_2}{s} \\ 0 & -\frac{k_2}{s} & m_2s + \frac{k_2}{s} \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{k_1}{s}V \\ 0 \\ f \end{pmatrix}.$$

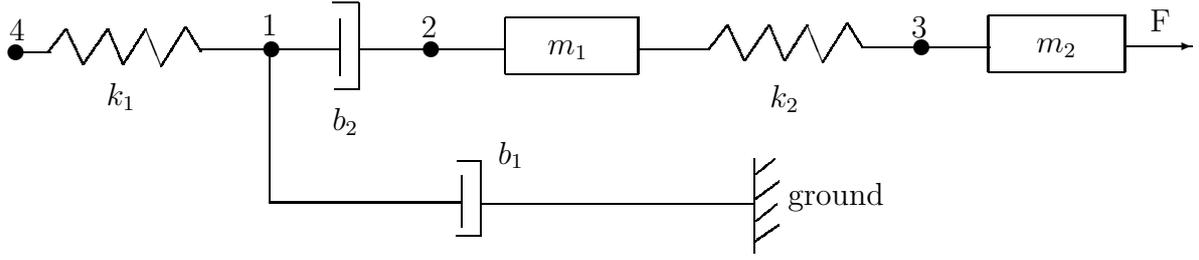


Figure 1: Mechanical translational system

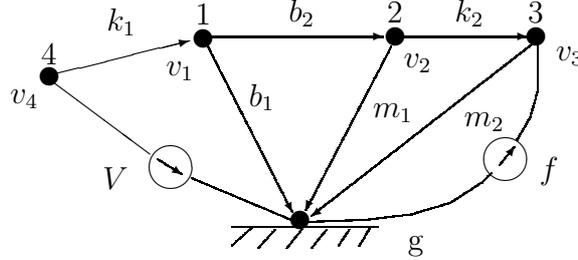


Figure 2: Graph associated with the mechanical system

For the loop method we transform the current source f into an equivalent voltage source and this leads to an elimination of the loop current f_4 . The reduced equations are

$$\begin{bmatrix} \frac{1}{b_1} + \frac{s}{k_1} & -\frac{1}{b_1} & 0 \\ -\frac{1}{b_1} & \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{m_1 s} & -\frac{1}{m_1 s} \\ 0 & -\frac{1}{m_1 s} & \frac{1}{m_1 s} + \frac{1}{m_2 s} + \frac{s}{k_2} \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} V \\ 0 \\ -\frac{f}{m_2 s} \end{pmatrix}.$$

2.5 The natural vertex and loop topologies

Network modeling uses the *system graph*, which is the basic topological structure that generates the system equations. Apart from the system graph we may introduce some additional topologies which are linked to the specifics of the node and loop analysis. These structures are introduced in the following.

The vertex topology: Every network can be presented in terms of a set of vertices, or nodes, and all branches between two vertices may be represented by an admittance function. Specification of the values of the across-variables of the vertices defines the values of all through-variables in the network. The vertex methodology implies the substitution of all across-variable sources by equivalent through-variable sources and defines the resulting topology.

For the example of Figure 2 the equivalent graph is depicted in Figure 3.

As it can be seen the number of independent vertices is reduced. The graph 3 without the sources completely defines the admittance matrix and it will be referred to as the *natural vertex graph* of the network.

Remark 2.3. The nature of sources in the network plays a key role in deriving the natural vertex graph from the system graph. The network graph acts as a progenitor of the natural vertex graph. \diamond

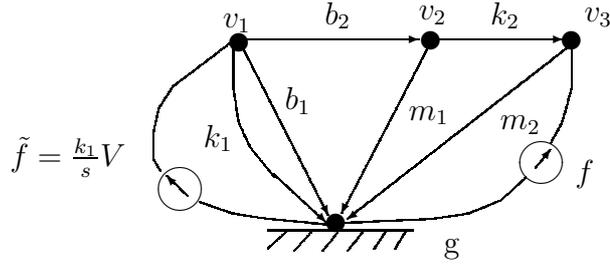


Figure 3: Vertex graph of the mechanical system

The nature of the elements in the branches of the natural vertex graph defines an element dependent topology which is characterized by adjacency type matrices. If we set the external sources to zero, the reduced graph will be referred to as the *kernel vertex graph*. The kernel vertex graph contains subgraphs defined by the nature of the elements associated with the branches (edges) and these are defined in the following.

Remark 2.4. All ideal lumped parameter elements may be classified on the basis of relationships between the through- and across-variables into the classes: A-type (mass, inertance, capacitance), T-type (spring, inductance), D-type (damper, resistance). \diamond

Definition 2.5. For a given kernel vertex graph we define the *A-vertex subgraph* by eliminating from the kernel graph all T- and D-type edges. Similarly, we define the *T-vertex subgraph* by eliminating all A- and D-type edges and the *D-vertex subgraph* by eliminating all A- and T-type edges. The subgraph of the natural vertex graph obtained by eliminating all T-, D- and A-type elements represents the location of the through-variable sources and will be called the *source-vertex subgraph*, or simply *S-vertex subgraph*. \diamond

Remark 2.6. The A-, T-, D- and S-vertex subgraphs are by construction simple graphs, that is they have no loops or parallel edges. The corresponding adjacency matrices are all symmetric boolean matrices. \diamond

The **loop topology** is a notion dual to that of the vertex topology and it is defined along similar lines.

3 Redesign of RC and RL networks for natural frequencies improvements

3.1 Problem statement

The general modeling for passive network provides a description of networks in terms of symmetric integral-differential operators, the impedance and admittance models which are described in a general way by

$$W(s) = sB + s^{-1}C + D, \quad (3.1)$$

where for the case of impedance we have that B is a matrix of T-type elements (spring, inductance), C is a matrix of A-type elements (mass, inertance, capacitance) and D is a matrix of D-type elements (damper, resistance). For the case of admittance modeling the reverse holds

true, i.e., B is a matrix of A-type elements, C is a matrix of T-type elements and D is again a matrix of D-type elements. The symmetric operator $W(s)$ is thus a common description of the $Z(s)$ (see (2.1)) and $Y(s)$ (see (2.2)) matrices. For the special cases where the network is characterized only by A- and D-type elements, or only by T- and D-type elements the operator $W(s)$ takes the following special forms:

$$\begin{aligned}\tilde{W}(s) &= sB + D, \\ \hat{W}(s) &= \hat{s}C + D, \quad \hat{s} = s^{-1},\end{aligned}$$

which are symmetric matrix pencils [3] derived from passive networks and thus inherit the passivity properties [4, 22].

The operator $W(s)$ describes the dynamics of the network and of special interest are the properties of its zeros. The specific problem we address here is the effect of changing individual elements of the network, i.e., changes of A-, T- or D-type elements, on these natural frequencies. We explicitly allow for the addition of such elements to the network within this framework - in particular, we may change the topology by a single addition and we can still treat it within the same framework. This problem is a special case of the more general network redesign problem [5, 9]. Our study will be focussed on the special cases where $W(s)$ is a symmetric matrix pencil.

3.2 Network transformation by change of an element

The study of the effect of changing an element in the network on the natural frequencies presupposes the representation of these transformations as operations on the operator $W(s)$ and this is the problem considered here. We will use an example to illustrate the nature of transformations and then generalize the observations.

Example 3.1. Consider the electrical network described in Figure 4.

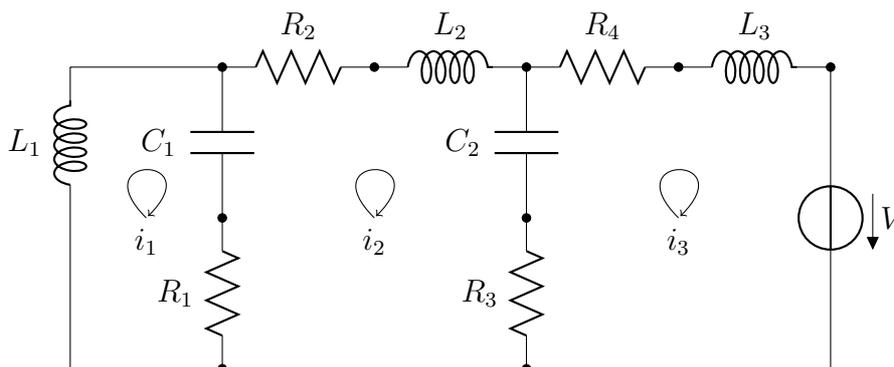


Figure 4: Electrical RLC network

For this example we are using loop analysis where the network variables are the loop currents i_1, i_2, i_3 . The impedance model expresses the structure of the impedances in the three loops and

has the form

$$\begin{aligned}
 Z(s) &= \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1} + \frac{1}{C_2} & -\frac{1}{C_2} \\ 0 & -\frac{1}{C_2} & \frac{1}{C_2} \end{bmatrix} s^{-1} + \begin{bmatrix} R_1 & -R_1 & 0 \\ -R_1 & R_1 + R_2 + R_3 & -R_3 \\ 0 & -R_3 & R_3 + R_4 \end{bmatrix} + \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix} s \\
 &= s^{-1}C + D + sB.
 \end{aligned} \tag{3.2}$$

◇

Example 3.1 clearly leads to the statement of the following general observations.

Remark 3.2. The presence of an element of A-, T- or D-type is expressed by an entry in the corresponding matrix C , B or D , respectively. In particular,

- (i) if an element is present in the i -th loop (node), then its value is added to the (i, i) position of the respective matrix.
- (ii) if an element is common to the i -th and j -th loop (node), then its value is added to the (i, i) and (j, j) positions, as well as subtracted from the (i, j) and (j, i) positions of the corresponding matrix. ◇

We demonstrate the above by means of Example 3.1.

Example 3.3. Consider the following network obtained from that of Figure 4 by the addition of elements L_4 , R_5 , C_3 as shown in Figure 5.

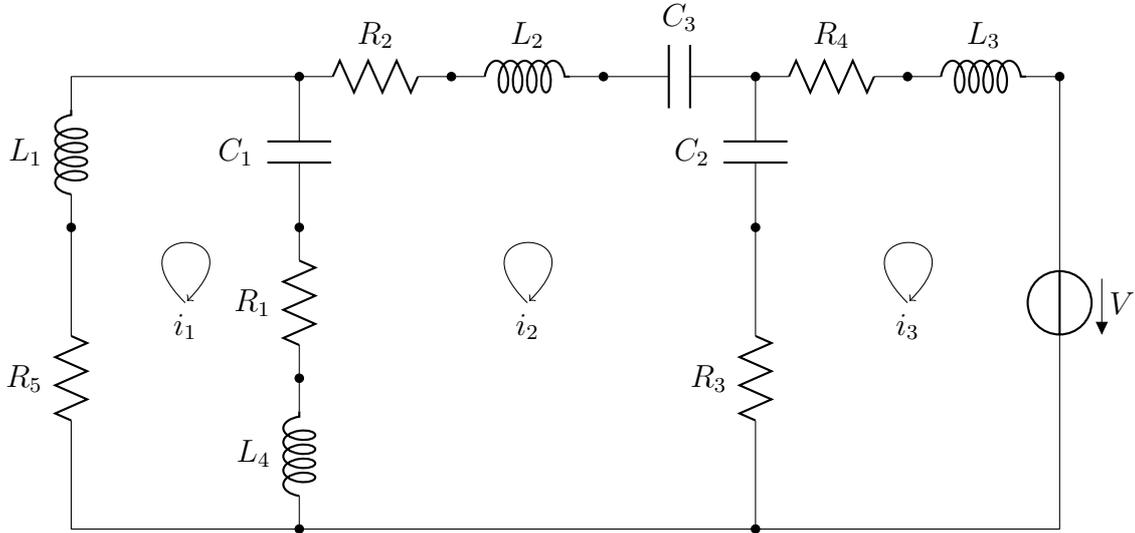


Figure 5: Electrical RLC network with additional elements

The resulting transformations on the corresponding matrices are then indicated below as

$$C = \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} & -\frac{1}{C_2} \\ 0 & -\frac{1}{C_2} & \frac{1}{C_2} \end{bmatrix}, \quad D = \begin{bmatrix} R_1 + R_5 & -R_1 & 0 \\ -R_1 & R_1 + R_2 + R_3 & -R_3 \\ 0 & -R_3 & R_3 + R_4 \end{bmatrix},$$

$$B = \begin{bmatrix} L_1 + L_4 & -L_4 & 0 \\ -L_4 & L_2 + L_4 & 0 \\ 0 & 0 & L_3 \end{bmatrix}.$$

◇

Remark 3.4. Removing elements without changing the corresponding topology can be achieved by assuming reduction of the values of these elements until they become zero and then modeling them following the previous rule established in Remark 3.2. ◇

The problem which we are now to investigate is the effect of single element changes on the corresponding natural frequencies of the network. Our current study will be restricted to the RC and RL cases where the operator $W(s)$ becomes a matrix pencil. The formulation of these problems is given below.

Single parameter variations: Given the symmetric matrix pencil $sF + G \in \mathbb{R}^{k \times k}[s]$, investigate the effect of simple perturbations on the pencil characteristic frequencies, where the variations are

$$F' = F + \bar{F}(x, b), \quad G' = G + \bar{G}(x, b)$$

and the matrices \bar{F} and \bar{G} , depending on the real parameter $x \in \mathbb{R}$ and the position vector $b \in \mathbb{R}^k$, have the form

$$(\bar{F}(x, b) =)\bar{G}(x, b) = xbb^\top \quad \text{for } b = e_i \text{ or } b = e_i - e_j, \quad i \neq j. \quad (3.3)$$

◇

The problem of single parameter variation is then equivalent to the investigation of the roots of

$$\tilde{f}(s; F, G, x, b) = \det(s(F + \bar{F}(x, b)) + G) \quad (3.4)$$

or

$$\bar{f}(s; F, G, x, b) = \det(sF + G + \bar{G}(x, b)) \quad (3.5)$$

as functions of the parameter x . In these problems the position vector b is fixed but arbitrary. Clearly, the above two problems are dual and thus we restrict ourselves to the study of (3.4) for the sake of convenience.

Preliminaries from exterior algebra: In the following we will use some results from exterior algebra (see e.g. [15]) and the representation of multi-linear maps in terms of compound matrices [16]. Some relevant notation is given below:

Let $Q_{k,n}$ denote the set of lexicographically ordered, strictly increasing sequences of k integers from $\{1, 2, \dots, n\}$. If $\{x_{i_1}, \dots, x_{i_k}\}$ is a set of vectors of a vector space V , $\omega = (i_1, \dots, i_k) \in Q_{k,n}$, then $x_{i_1} \wedge \dots \wedge x_{i_k} = x_\omega \wedge$ denotes the exterior product and by $\wedge^r V$ we denote the r -th exterior power of V . If $H \in \mathbb{K}^{m \times n}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and $r \leq \min\{m, n\}$, then by $C_r(H)$ we denote the r -th compound matrix of H . Let $M(s) \in \mathbb{R}^{p \times r}[s]$, $r \leq p$, be such that $\text{rk } M(s) = r$, and $m_i(s), i = 1, \dots, r$, are the columns of $M(s)$, then

$$C_r(M(s)) = m_1(s) \wedge \dots \wedge m_r(s) = m(s) \wedge \in \mathbb{R}^\sigma[s], \quad \sigma = \binom{p}{r},$$

Lemma 3.6. Consider a network with a second order variation that takes place in the i and j index variables, i.e., $b = e_i - e_j$, $i < j$. Then the vector $p(s; x, b)$ has the form

$$p(s; x, e_i - e_j)^\top = (1, 0, \dots, 0, \varepsilon_1 sx, 0, \dots, 0, \varepsilon_2 sx, 0, \dots, 0, \varepsilon_3 sx, 0, \dots, 0, \varepsilon_4 sx, 0, \dots, 0),$$

where there are four nonzero elements in the positions of the minors characterized by

$$\left. \begin{aligned} \omega_1 &= (1, 2, \dots, j-1, j+1, \dots, k, k+i), \\ \omega_2 &= (1, 2, \dots, j-1, j+1, \dots, k, k+j), \\ \omega_3 &= (1, 2, \dots, i-1, i+1, \dots, k, k+i), \\ \omega_4 &= (1, 2, \dots, i-1, i+1, \dots, k, k+j), \end{aligned} \right\} \in Q_{k, 2k} \quad (3.9)$$

and $\varepsilon_\mu = \pm 1$, $\mu = 1, \dots, 4$, which are determined by the sign of the corresponding determinants.

Proof: The proof is established along lines similar to the proof of Lemma 3.5. In fact the submatrix of $[I_k, s\bar{F}(x, e_i - e_j)]^\top$ that contributes to maximal order nonzero minors is defined by the nonzero rows, i.e.,

$$R_{ij} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \\ 0 & \cdots & sx & \cdots & -sx & \cdots & 0 \\ 0 & \cdots & -sx & \cdots & sx & \cdots & 0 \end{bmatrix} \begin{array}{l} \leftarrow 1 \\ \vdots \\ \leftarrow i \\ \vdots \\ \leftarrow j \\ \vdots \\ \leftarrow k \\ \leftarrow k+i \\ \leftarrow k+j \end{array} \in \mathbb{R}^{(k+2) \times k}. \quad (3.10)$$

We note that due to the structure of the $k \times k$ upper part as an identity matrix and the fact that the last two rows are linearly dependent, any maximal $k \times k$ minor that includes the last two rows is zero. The only nonzero minors of R_{ij} are:

- (i) The first that has value 1.
- (ii) The two minors of R_{ij} obtained by eliminating the j -th row and then considering the $(k+i)$ - and $(k+j)$ -row. These minors will be denoted by $\det \left(R_{ij}^{j, k+i} \right) = \varepsilon_1 sx$, $\det \left(R_{ij}^{j, k+j} \right) = \varepsilon_2 sx$, $\varepsilon_\mu = \pm 1$.
- (iii) Similarly we define two nonzero minors of R_{ij} by eliminating the i -th row and then considering first the $(k+i)$ - and then the $(k+j)$ -row. These minors are denoted by $\det \left(R_{ij}^{i, k+i} \right) = \varepsilon_3 sx$, $\det \left(R_{ij}^{i, k+j} \right) = \varepsilon_4 sx$, $\varepsilon_\mu = \pm 1$.

The above clearly yields the assertion. □

Using the determinantal formulation given by (3.6) and Lemmas 3.5 and 3.6 we are led to the following main result.

Theorem 3.7 (Root locus problem). *For RC or RL networks the characteristic frequencies problem under first or second order variations in a single parameter of the network is equivalent to a classical root locus problem and*

$$\tilde{f}(s; F, G, x, b) = p_{F,G}(s) + sxz_{F,G,b}(s), \quad (3.11)$$

where $p_{F,G}(s) = \det(sF + G)$ and $z_{F,G,b}(s)$ is a polynomial formed from the components of $g(s; F, G)^\top$ which depend on the network transformation, i.e., on the position vector b .

Proof: The result is a direct consequence of (3.6) and Lemmas 3.5 and 3.6. \square

The study of the root locus problem (3.11) requires determining the nature of the polynomial $z_{F,G,b}(s)$. This is defined next.

Corollary 3.8. *For RC or RL networks under first or second order single parameter variation, the polynomial $z_{F,G,b}(s)$ is defined as follows.*

(i) *If $b = e_\mu$, then*

$$z_{F,G,b}(s) = \varepsilon_\mu z_{F,G,\mu}(s),$$

where $z_{F,G,\mu}(s)$ is the component of $g(s; F, G)^\top$ defined by the minor characterized by $\omega_\mu = (1, 2, \dots, \mu - 1, \mu + 1, \dots, k, k + \mu) \in Q_{k,2k}$ and ε_μ is the sign of the permutation $\bar{\omega}_\mu = (1, 2, \dots, \mu - 1, \mu + 1, \dots, k, \mu)$.

(ii) *If $b = e_i - e_j$, $i < j$, then*

$$z_{F,G,b}(s) = \sum_{l=1}^4 \varepsilon_l z_{F,G,\omega_l}(s),$$

where $z_{F,G,\omega_l}(s)$ is the component of $g(s; F, G)^\top$ defined by the minor characterized by $\omega_l \in Q_{k,2k}$, $l = 1, \dots, 4$ in (3.9) and ε_l is their corresponding sign.

Proof: The derivation of the polynomials readily follows from the proofs of Lemmas 3.5 and 3.6. \square

The explicit derivation of the polynomials $p_{F,G}(s)$ and $z_{F,G,b}(s)$ allows for the study of the respective root locus problem which is considered in the next section.

4 Root locus problem and properties

The derivation of the formulation of the polynomials $p_{F,G}(s)$ and $z_{F,G,b}(s)$ defining the root locus (3.11) indicates that the problem may be characterized by fixed modes. The emergence of fixed modes is intimately linked to the structured nature of the transformation and it is similar to the emergence of fixed modes in the Decentralized Assignment Problem [8]. We first note:

Remark 4.1. The expression of the polynomial characterizing the natural frequencies as defined by (3.6) indicates that this study as a root locus problem may have fixed modes. In fact, the polynomial multi-vector $g(s; F, G)$ (see (3.7)) is always coprime, but the presence of fixed zeros in $p(s; x, b)$ (see (3.8)) for either types of transformations implies that a sub-vector of $g(s; F, G)$,

say $g^+(s; F, G)$, is the essential part that defines the resulting polynomial. The reduced vector $g^+(s; F, G)$ may not necessarily be coprime. If this vector is not coprime, then its greatest common divisor (GCD) will define the fixed zeros of the resulting root locus problem. The computation of the fixed zeros requires the computation of the GCD of many polynomials defined by $g^+(s; F, G)$. This may be readily achieved by using the well developed numerical methods such as the ERES methodology [1, 17], the matrix pencil methodology [10], or the optimization approach [6]. \diamond

Now we investigate the root locus problem defined by (3.11), i.e., the problem

$$1 + x \frac{s z_{F,G,b}(s)}{p_{F,G}(s)} = 0 \quad (4.1)$$

with parameter $x \in \mathbb{R}$, and its properties. For some background on the root locus theory see e.g. [2, 18].

Throughout this section we have to state the additional, but reasonable, assumption that

$$\text{the pencil } s(F + xbb^\top) + G \text{ is regular for all } x \in \mathbb{R}. \quad (4.2)$$

In the next subsection we first provide some results on symmetric matrix pencils.

4.1 Some preliminary results on symmetric matrix pencils

Consider a matrix pencil $sF + G \in \mathbb{R}^{k \times k}[s]$ corresponding to an RC or RL network. We first note that due to the modeling, see in particular Remark 3.2, the pencil has the following properties.

Corollary 4.2. *The matrix pencil $sF + G$ corresponding to an RC or RL network has the following properties:*

- (i) $sF + G$ is regular, i.e., $\det(sF + G) \not\equiv 0$,
- (ii) $F = F^\top \geq 0$, $G = G^\top \geq 0$.

The natural frequencies of the network now coincide with the finite eigenvalues of the pencil $sF + G$, i.e., with those numbers $\lambda \in \mathbb{C}$ where $\det(\lambda F + G) = 0$. Matrix pencils may also have infinite eigenvalues, see e.g. [3], in the case where $\det F = 0$.

Note that for regular matrix pencils, symmetry does not imply that all finite eigenvalues are real. This is illustrated by the following example [19, p. 304]: Consider

$$F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Then, for $x = (i, 1)^\top$ we have

$$-Gx = \begin{pmatrix} 1 \\ i \end{pmatrix} = i \begin{pmatrix} -i \\ 1 \end{pmatrix} = iFx,$$

which shows that the complex number i is a finite eigenvalue of $sF + G$.

However, as in the RC or RL network we additionally have that F and G are positive semi-definite, we may deduce that all eigenvalues (natural frequencies of the network) are real and non-positive.

Lemma 4.3 ([20]). *Let $sF + G \in \mathbb{R}^{k \times k}[s]$ be such that $F = F^\top \geq 0$, $G = G^\top \geq 0$ and $\ker F \cap \ker G = \{0\}$. Then there exist nonsingular $T \in \mathbb{R}^{k \times k}$ and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_{k_f})$ with $\lambda_i > 0$ such that*

$$T^\top (sF + G)T = \begin{bmatrix} sI_{k_f} + \Lambda & & \\ & sI_{k_0} & \\ & & I_{k_\infty} \end{bmatrix}.$$

Note, that regularity of a matrix pencil implies the condition $\ker F \cap \ker G = \{0\}$, however the opposite does in general not hold true: consider, for instance, $F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $G = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

We will also need another result which guarantees that all eigenvalues of the pencil are real. This requires only symmetry of F , however positive (or negative) semi-definiteness of G . Another fact which follows from these properties is that the index (see e.g. [11, Def. 2.9]) of the pencil $sF + G$ is at most one.

Lemma 4.4. *Let $sF + G \in \mathbb{R}^{k \times k}[s]$ be regular and such that $F = F^\top$ and either $G = G^\top \geq 0$ or $G = G^\top \leq 0$. Then all finite eigenvalues of $sF + G$ are real, i.e.,*

$$\forall \lambda \in \mathbb{C} : \det(\lambda F + G) = 0 \implies \lambda \in \mathbb{R}.$$

Furthermore, the pencil $sF + G$ has index at most one.

Proof: We show that all eigenvalues are real. Let $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^k \setminus \{0\}$ be such that $(\lambda F + G)x = 0$. Then

$$\lambda(x^* F x) = -x^* G x.$$

Suppose $x^* F x = 0$, then $x^* G x = 0$, and hence $Gx = 0$. If $\lambda = 0$, the assertion is shown, and if $\lambda \neq 0$, then $Fx = 0$ and as $(sF + G)x = 0$ for all $s \in \mathbb{C}$, this contradicts regularity of $sF + G$. Thus

$$\lambda = -\frac{x^* G x}{x^* F x} \in \mathbb{R} \setminus \{0\}.$$

In order to proof the index property suppose $sF + G$ has index greater than 1. Then there exist $x, y \in \mathbb{C}^k \setminus \{0\}$ such that

$$Gx = Fy, \quad Fx = 0,$$

and hence $x^* G x = x^* F y = y^* F x = 0$, which yields $Gx = 0$ and contradicts regularity of $sF + G$ as above. \square

It is important to note that Lemma 4.3 implies that $sF + G$ always has index ≤ 1 . For matrix pencils with this property the degree of the characteristic polynomial coincides with the rank of F .

Lemma 4.5 ([11]). *Consider a regular matrix pencil $sF + G \in \mathbb{R}^{k \times k}[s]$ of index ≤ 1 . Then*

$$\deg \det(sF + G) = \text{rk } F.$$

4.2 Interlacing property

In this subsection we show that the poles and zeros of the root locus problem (4.1) have an interlacing property. As usual for root locus problems [2, 18] we will refer to the roots of the polynomial $p_{F,G}(s)$ as the poles of the root locus problem (4.1) and to the roots of the polynomial $z_{F,G,b}(s)$ as the zeros of the root locus problem (4.1).

Interlacing property means that the poles and zeros are located on the real axis in an alternating pattern, so that (after cancelation of fixed points) there are no poles and zeros of multiplicity higher than one and no two poles or two zeros which are next to each other, resp. Interlacing properties for eigenvalue perturbation problems are well known, see e.g. [12], where an interlacing property for the dual problem (3.5) has been obtained in [12, Thm. 4.2] under the condition that F is invertible.

The next lemma establishes some important properties of the root locus.

Lemma 4.6. *Consider an RC or RL network with corresponding matrix pencil $sF+G \in \mathbb{R}^{k \times k}[s]$ with a first or second order parameter variation given by $b \in \mathbb{R}^k$. Suppose that (4.2) holds. The root locus problem (4.1) has the following properties:*

- (i) *For both $x > 0$ and $x < 0$ all branches of the root locus are restricted to the real axis. Furthermore, for $x > 0$ all branches are restricted to the negative real axis.*
- (ii) *If there exists a multiple pole, i.e., $p \leq 0$ and $\mu \in \mathbb{N}$, $\mu \geq 2$ such that $p_{F,G}(s) = \eta(s)(s-p)^\mu$, where p is not a root of $\eta(s)$, then p is a zero of multiplicity at least $\mu-1$, i.e., $sz_{F,G,b}(s) = \zeta(s)(s-p)^{\mu-1}$ (p might still be a root of $\zeta(s)$). In particular this means that p is a (multiple) fixed point of the root locus.*
- (iii) *Suppose all fixed points in (4.1) have been canceled out. Then, in the reduced problem, there cannot be two poles, i.e., roots of $p_{F,G}(s)$, next to each other. More precise, it is not possible that there are two points $p_1 < p_2 \in \mathbb{R}$, both roots of $p_{F,G}(s)$, such that there is no root of $sz_{F,G,b}(s)$ in the interval (p_1, p_2) .*

Proof: As $\tilde{f}(s; F, G, x, b) = \det(s(F + \bar{F}(x, b)) + G)$ and $G = G^\top \geq 0$, $(F + \bar{F}(x, b)) = (F + \bar{F}(x, b))^\top$, it follows from Lemma 4.4 that all roots s of $\tilde{f}(s; F, G, x, b)$ are real, for all $x \in \mathbb{R}$. Furthermore, for $x > 0$ we have $(F + \bar{F}(x, b)) \geq 0$ since $F \geq 0$ and $\bar{F}(x, b) \geq 0$, thus the last statement of (i) follows from Lemma 4.3.

If (ii) were not true, then p would be root of $\frac{d}{ds}(p_{F,G}(s) + sz_{F,G,b}(s))$ (as the equation (4.1) must hold true), thus being a breakaway point of the root locus in contradiction to (i).

In order to prove (iii) suppose that there is no root of $sz_{F,G,b}(s)$ in the interval (p_1, p_2) . Then, for either $x > 0$ or $x < 0$ there is a breakaway point in the root locus, which contradicts (i). \square

Theorem 4.7 (Interlacing property). *Consider an RC or RL network with corresponding matrix pencil $sF+G \in \mathbb{R}^{k \times k}[s]$ with a first or second order parameter variation given by $b \in \mathbb{R}^k$. Suppose that (4.2) holds. Then the root locus problem (4.1) has the interlacing property, that is, after canceling out all fixed points, the following holds true:*

- (i) *All poles and zeros are located on the real axis.*

(ii) *There are no poles and zeros of multiplicity higher than one, resp.*

(iii) *There cannot be two poles or two zeros next to each other, resp. That is, it is not possible that there are two points $p_1 < p_2 \in \mathbb{R}$ ($z_1 < z_2 \in \mathbb{R}$), both roots of $p_{F,G}(s)$ ($z_{F,G,b}(s)$), such that there is no root of $s z_{F,G,b}(s)$ ($p_{F,G}(s)$) in the interval (p_1, p_2) ((z_1, z_2)).*

Proof: The assertion follows immediately from applying Lemma 4.6 to the root locus problem (4.1) and to the inverse problem $1 + y \frac{p_{F,G}(s)}{s z_{F,G,b}(s)} = 0$ (obtained from (4.1) by considering $y = x^{-1}$). \square

Remark 4.8. The interlacing property implies that for positive (negative) parameter in the root locus problem (4.1) all the branches head in the same direction; we say that all the poles do either not change at all or move in the same direction, that is to the left or to right on the real axis. \diamond

4.3 Direction of the movement of the poles

In this subsection we show that there is a determinable common direction of movement of the poles under single parameter variations. Using the interlacing property established in the previous subsection, we prove that for positive parameter the poles always move to the right and for negative parameter the poles always move to the left. In order to obtain this result we consider the following reformulation of (3.4):

By Lemma 4.3 there exist nonsingular $T \in \mathbb{R}^{k \times k}$ and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_{k_f})$ with $\lambda_i > 0$ such that

$$T^\top (sF + G)T = \begin{bmatrix} sI_{k_f} + \Lambda & & \\ & sI_{k_0} & \\ & & I_{k_\infty} \end{bmatrix}.$$

Let $\tilde{b} = (b_1^\top, b_2^\top, b_3^\top)^\top := T^\top b$ and observe that, with $c := \det(TT^\top)$, we have

$$\begin{aligned} \det(s(F + xbb^\top) + G) &= c \det \left(\begin{bmatrix} sI_{k_f} + \Lambda & & \\ & sI_{k_0} & \\ & & I_{k_\infty} \end{bmatrix} + sx\tilde{b}\tilde{b}^\top \right) \\ &= cs^{k_0} \det(sI_{k_f} + \Lambda) \det \left(I + sx \begin{bmatrix} sI_{k_f} + \Lambda & & \\ & sI_{k_0} & \\ & & I_{k_\infty} \end{bmatrix}^{-1} \tilde{b}\tilde{b}^\top \right) \\ &= cs^{k_0} \det(sI_{k_f} + \Lambda) \left(1 + sx\tilde{b}^\top \begin{bmatrix} sI_{k_f} + \Lambda & & \\ & sI_{k_0} & \\ & & I_{k_\infty} \end{bmatrix}^{-1} \tilde{b} \right) \\ &= cs^{k_0} \det(sI_{k_f} + \Lambda) \left(1 + sx [b_1^\top (sI_{k_f} + \Lambda)^{-1} b_1 + s^{-1} b_2^\top b_2 + b_3^\top b_3] \right). \end{aligned}$$

It is therefore immediate that the pole and zero polynomial have the following representations:

$$p_{F,G}(s) = cs^{k_0} \det(sI_{k_f} + \Lambda), \quad (4.3)$$

$$\begin{aligned} z_{F,G,b}(s) &= cs^{k_0} \det(sI_{k_f} + \Lambda) [b_1^\top (sI_{k_f} + \Lambda)^{-1} b_1 + s^{-1} b_2^\top b_2 + b_3^\top b_3] \\ &= cs^{k_0} [b_1^\top \text{adj}(sI_{k_f} + \Lambda) b_1 + \|b_2\|^2 s^{-1} \det(sI_{k_f} + \Lambda) + \|b_3\|^2 \det(sI_{k_f} + \Lambda)]. \end{aligned} \quad (4.4)$$

Note that $s^{-1} \det(sI_{k_f} + \Lambda)$ is not a polynomial but a rational function. However, $s^{k_0} (s^{-1} \det(sI_{k_f} + \Lambda))$ (and therefore $z_{F,G,b}(s)$) is always a polynomial, as $k_0 = 0$ if, and only if, b_2 is absent. Note further, that b_1 and b_3 cannot vanish at the same time, because if $b_1 = 0$ and $b_3 = 0$, then, for $x = -\|b_2\|^{-2}$ ($b_2 \neq 0$ follows from $b \neq 0$), we have

$$T^\top (s(F + xbb^\top) + G)T \begin{pmatrix} 0 \\ b_2 \\ 0 \end{pmatrix} = \begin{bmatrix} sI_{k_f} + \Lambda & 0 & 0 \\ 0 & s(I_{k_0} - \frac{b_2 b_2^\top}{\|b_2\|^2}) & 0 \\ 0 & 0 & I_{k_\infty} \end{bmatrix} \begin{pmatrix} 0 \\ b_2 \\ 0 \end{pmatrix} = 0$$

for all $s \in \mathbb{C}$, thus $s(F + xbb^\top) + G$ is not a regular pencil in contradiction to (4.2).

The degrees of the pole polynomial $p_{F,G}(s)$ and the zero polynomial $z_{F,G,b}(s)$ are related by the type of the parameter variation. It is clear from formulas (4.3) and (4.4) that $\deg p_{F,G}(s) = \deg z_{F,G,b}(s)$ if, and only if, $b_3 \neq 0$ and that $\deg p_{F,G}(s) = \deg z_{F,G,b}(s) + 1$ if, and only if, $b_3 = 0$. In the following we relate this to the property of the parameter variation to increase the rank of F or not.

Lemma 4.9. *We have the following equivalences:*

$$\begin{aligned} b_3 \neq 0 &\iff \text{rk}(F + bb^\top) = \text{rk } F + 1, \\ b_3 = 0 &\iff \text{rk}(F + bb^\top) = \text{rk } F. \end{aligned}$$

Proof: Applying Schur's complement we find that, with $\hat{b} = (b_1^\top, b_2^\top)^\top$,

$$\begin{aligned} \text{rk}(F + bb^\top) &= \text{rk} \begin{bmatrix} I_{k_f+k_0} + \hat{b}\hat{b}^\top & \hat{b}b_3^\top \\ b_3\hat{b}^\top & b_3b_3^\top \end{bmatrix} \\ &= \text{rk} \begin{bmatrix} I & 0 \\ b_3\hat{b}^\top(I + \hat{b}\hat{b}^\top)^{-1} & I \end{bmatrix} \begin{bmatrix} I + \hat{b}\hat{b}^\top & 0 \\ 0 & b_3b_3^\top - b_3\hat{b}^\top(I + \hat{b}\hat{b}^\top)^{-1}\hat{b}b_3^\top \end{bmatrix} \begin{bmatrix} I & (I + \hat{b}\hat{b}^\top)^{-1}\hat{b}b_3^\top \\ 0 & I \end{bmatrix} \\ &= \text{rk}(I + \hat{b}\hat{b}^\top) + \text{rk} \left(b_3b_3^\top - b_3\hat{b}^\top(I + \hat{b}\hat{b}^\top)^{-1}\hat{b}b_3^\top \right), \end{aligned}$$

where it was used that clearly $I + \hat{b}\hat{b}^\top$ is nonsingular. Now $\text{rk}(I + \hat{b}\hat{b}^\top) = \text{rk } F$ and, if $b_3 \neq 0$,

$$\text{rk} \left(b_3b_3^\top - b_3\hat{b}^\top(I + \hat{b}\hat{b}^\top)^{-1}\hat{b}b_3^\top \right) = \begin{cases} 1, & \hat{b}^\top(I + \hat{b}\hat{b}^\top)^{-1}\hat{b} \neq 1 \\ 0, & \text{otherwise.} \end{cases}$$

However,

$$|\hat{b}^\top(I + \hat{b}\hat{b}^\top)^{-1}\hat{b}| \leq \|\hat{b}\|^2 \|(I + \hat{b}\hat{b}^\top)^{-1}\| \leq \|\hat{b}\|^2(1 + \|\hat{b}\|^2)^{-1} < 1,$$

which proves the assertion of the lemma. \square

The following is immediate from the above.

Corollary 4.10. *Consider an RC or RL network with corresponding matrix pencil $sF + G \in \mathbb{R}^{k \times k}[s]$ with a first or second order parameter variation given by $b \in \mathbb{R}^k$. Suppose that (4.2) holds. The following relations hold true:*

(i) If

$$\text{rk}(F + bb^\top) = \text{rk } F + 1, \quad (4.5)$$

then $\deg p_{F,G}(s) = \deg z_{F,G,b}(s)$.

(ii) If

$$\text{rk}(F + bb^\top) = \text{rk } F, \quad (4.6)$$

then $\deg p_{F,G}(s) = \deg z_{F,G,b}(s) + 1$.

Remark 4.11. Note that (4.5) and (4.6) fully characterize the relation between the ranks of F and $F + bb^\top$ - either the one or the other is true. That the rank of F cannot increase by more than one is clear as bb^\top is a rank-1 matrix, and that it cannot drop is clear from the structure of the problem. In the following we will also show that:

(i) If (4.5), then

$$\forall x \neq 0 : \text{rk}(F + \bar{F}(x, b)) = \text{rk } F + 1,$$

i.e., the rank of F increases for all values of the parameter.

(ii) If (4.6), then there exists one value $\tilde{x} \in \mathbb{R}$ of the parameter such that

$$\forall x \neq \tilde{x} : \text{rk}(F + \bar{F}(x, b)) = \text{rk } F, \quad \text{rk}(F + \bar{F}(\tilde{x}, b)) = \text{rk } F - 1,$$

i.e., the rank of F stays unchanged for all but one value of the parameter and decreases at this single point.

In terms of the matrix pencils $sF + G$ and $s(F + xbb^\top) + G$ this can be interpreted as follows: If (4.5), then, in the ‘‘perturbed’’ pencil $s(F + xbb^\top) + G$, one infinite eigenvalue of $sF + G$ becomes finite for all $x \neq 0$. If (4.6), then the ‘‘perturbed’’ pencil $s(F + xbb^\top) + G$ has the same number of finite and infinite eigenvalues as $sF + G$ for all $x \neq \tilde{x}$ and in $s(F + \tilde{x}bb^\top) + G$ one finite eigenvalue of $sF + G$ becomes infinite. \diamond

In order to obtain the direction of the movement of the poles the next theorem is crucial, as it shows that there always exists a zero which is larger than every pole.

Theorem 4.12 (Largest zero). *Consider an RC or RL network with corresponding matrix pencil $sF + G \in \mathbb{R}^{k \times k}[s]$ with a first or second order parameter variation given by $b \in \mathbb{R}^k$. Suppose that (4.2) holds. Then, in the root locus problem (4.1), after cancelation of all fixed points, we have that there exists a zero $z \leq 0$ (root of $sz_{F,G,b}(s)$) such that for all poles p (root of $p_{F,G}(s)$) it holds $p < z$.*

Proof: We consider the representations of $p_{F,G}(s)$ as in (4.3) and $z_{F,G,b}(s)$ as in (4.4) and distinguish two cases.

Case 1: $b_2 = 0$ or b_2 absent ($k_0 = 0$). In this case we have

$$sz_{F,G,b}(s) = cs^{k_0+1} [b_1^\top \text{adj}(sI_{k_f} + \Lambda)b_1 + \|b_3\|^2 \det(sI_{k_f} + \Lambda)]$$

with $k_0 \geq 0$. As an evaluation of $b_1^\top \text{adj}(sI_{k_f} + \Lambda)b_1 + \|b_3\|^2 \det(sI_{k_f} + \Lambda)$ at $s = 0$ yields the value

$$\sum_{i=1}^{k_f} \left(b_{1,i}^2 \prod_{j \neq i} \lambda_j \right) + \|b_3\|^2 \prod_{i=1}^{k_f} \lambda_i > 0,$$

we find that in the root locus problem (4.1) we have exactly k_0 fixed points at $s = 0$ and after cancelation of all these fixed points one zero of multiplicity one remains, so we may define $z := 0$. The remaining poles p are the values $-\lambda_i < 0$, $i = 1, \dots, k_f$, where possible higher multiplicities do not matter at the moment. It is clear that the assertion of the theorem follows in this case. *Case 2: $b_2 \neq 0$.* In this case we have $k_0 \geq 1$ and in the root locus problem (4.1) we have exactly k_0 fixed points at $s = 0$. After cancelation of these fixed points we find that in the reduced problem

$$\begin{aligned} p_{F,G}(s) &= c \det(sI_{k_f} + \Lambda), \\ s z_{F,G,b}(s) &= c [s b_1^\top \text{adj}(sI_{k_f} + \Lambda) b_1 + \|b_2\|^2 \det(sI_{k_f} + \Lambda) + \|b_3\|^2 s \det(sI_{k_f} + \Lambda)]. \end{aligned}$$

We may now observe that for $x = -\|b_2\|^{-2}$, $s = 0$ is a solution of the problem (4.1), by evaluating the above polynomials at $s = 0$. Therefore, the origin is part of the root locus branches corresponding to negative parameters x . Using that $b_1^\top \text{adj}(sI_{k_f} + \Lambda) b_1 = \sum_{i=1}^{k_f} (b_{1,i}^2 \prod_{j \neq i} (s + \lambda_j))$ and $\det(sI_{k_f} + \Lambda) = \prod_{i=1}^{k_f} (s + \lambda_i)$ we may rewrite (4.1) as a problem of finding all solutions of

$$F(s, x) = x + \left(\frac{s z_{F,G,b}(s)}{p_{F,G}(s)} \right)^{-1} = x + \left(\sum_{i=1}^{k_f} \frac{b_{1,i}^2 s}{s + \lambda_i} + \|b_2\|^2 + s \|b_3\|^2 \right)^{-1} = 0.$$

Clearly, F is continuously differentiable in a neighborhood of the solution $(0, -\|b_2\|^{-2})$ and the implicit function theorem yields existence of a local, continuously differentiable, solution $x = x(s)$, which in our case is part of the root locus branch through $s = 0$. Obviously, it is easy to explicitly obtain $x(s)$ from the above equation. We may now obtain the direction in which the branch goes through the origin by calculating the derivative $\dot{x}(0)$, which is

$$\dot{x}(0) = -\frac{1}{\|b_2\|^4} \left(\sum_{i=1}^{k_f} \frac{b_{1,i}^2}{\lambda_i} + \|b_3\|^2 \right).$$

Then $\dot{x}(0) < 0$ follows from the fact that b_1 and b_3 cannot vanish at the same time as the observations at the beginning of this subsection revealed. Therefore, $x(s)$ must be monotonically decreasing in a neighborhood of $s = 0$, that is the root locus branch through the origin moves to the left.

Now, observe that there must always be at least one finite zero: If $k_f \geq 1$ this is clear, as the number of finite zeros is greater or equal to the number of finite poles. If $k_f = 0$, that is there are no finite poles, then b_1 is absent and by the observations directly before Lemma 4.9 it follows $b_3 \neq 0$ and then Lemma 4.9 and Corollary 4.10 yield $\deg p_{F,G}(s) = \deg z_{F,G,b}(s)$, that is the root locus problem (4.1) is improper, so we have a finite zero.

By these observations and the fact that the whole positive real axis $(0, \infty)$ must be part of the root locus branch through the origin (as no poles can be located in the interval $(0, \infty)$), it follows that there must be a finite zero z to which this root locus branch tends to, located on the negative real axis $(-\infty, 0)$. This argumentation and the fact that all fixed points have been canceled out now gives that all the finite poles must be smaller than z . \square

Example 4.13. This example illustrates the fact that it is possible to have a complete pole-zero-cancellation at the origin and nevertheless one zero which is larger than every pole. To this end consider

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now we do a single parameter variation in F to get

$$F + xbb^\top \quad \text{with} \quad b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Calculating the polynomials $p_{F,G}(s)$ and $z_{F,G,b}(s)$ from this we obtain

$$p_{F,G}(s) = s^2 + s, \quad sz_{F,G,b}(s) = s(2s + 1).$$

Therefore, in the root locus problem (4.1) we have finite zeros at $z_1 = 0$ and $z_2 = -1/2$ and finite poles at $p_1 = 0$ and $p_2 = -1$, therefore having a fixed point at the origin, but neither a remaining zero nor a remaining pole at the origin. Nevertheless, the zero z_2 is greater than the pole p_2 . \diamond

In the following we exploit the root locus problem (4.1) to get some statements about the direction of the movement of the natural frequencies of the network under single parameter changes. We use the interlacing property proved in Section 4.2 and show in which direction the poles move for positive or negative parameters, resp. For doing so, the result of Theorem 4.12 is crucial. We have the following theorem for positive single parameter variations.

Theorem 4.14 (Movement of the poles). *Consider an RC or RL network with corresponding matrix pencil $sF + G \in \mathbb{R}^{k \times k}[s]$ with a first or second order parameter variation given by $b \in \mathbb{R}^k$ and the problem (3.4). Suppose that (4.2) holds. Then, in the root locus problem (4.1) for increasing $x > 0$, the natural frequencies of the network either shift to the right or do not change at all. More precise, if p is a pole (root of $p_{F,G}(s)$) of multiplicity μ , then either μ or $\mu - 1$ of these poles do not change and at most one pole moves to the right, however staying in the interval $(-\infty, 0]$. In particular, if (4.5) holds true, then an infinite pole becomes finite and moves to the right on the negative real axis as the parameter x increases.*

Proof: We consider the root locus problem (4.1). Suppose that all fixed points in the root locus have been canceled out and the reduced problem is considered, so that by Theorem 4.7 only distinct poles and zeros of multiplicity at most one are remaining. Let $\deg p_{F,G}(s) = q$ and $p_1 < \dots < p_q < 0$ be the roots of $p_{F,G}(s)$, with multiplicity exactly one, which are no roots of $sz_{F,G,b}(s)$. By Corollary 4.10 the degree of $z_{F,G,b}(s)$ equals either $\deg p_{F,G}(s)$ or $\deg p_{F,G}(s) - 1$. We consider these two cases separately.

Case 1: $\deg z_{F,G,b}(s) = \deg p_{F,G}(s) - 1$, i.e., $\text{rk}(F + bb^\top) = \text{rk} F$. Since $\deg sz_{F,G,b}(s) = \deg p_{F,G}(s)$, the root locus problem (4.1) is proper and we have q finite zeros and q finite poles. Furthermore, by Theorem 4.12, one of these zeros, say z_q , is larger than every pole. The interlacing property of Theorem 4.7 now gives that the poles and zeros are ordered in the pattern

$$p_1 < z_1 < p_2 < \dots < z_{q-1} < p_q < z_q \leq 0.$$

This yields, since the poles tend to the zeros, that they shift to the right as x increases.

Case 2: $\deg z_{F,G,b}(s) = \deg p_{F,G}(s)$, i.e., $\text{rk}(F + bb^\top) = \text{rk } F + 1$. In this case we have an improper root locus problem with $q + 1$ finite zeros, q finite poles and one infinite pole. Repeating the reasoning of Case 1 we may conclude that the relative positions of the finite poles and q finite zeros z_2, \dots, z_{q+1} are

$$p_1 < z_2 < p_2 < \dots < z_q < p_q < z_{q+1} \leq 0.$$

Now, there is one finite zero z_1 left of which the place is still unknown. However, since we know that the roots of $\tilde{f}(s; F, G, x, b) = \det(s(F + \bar{F}(x, b)) + G)$ always lie on the negative real axis for all $x > 0$ (by Lemma 4.3) it cannot be the case that the root locus branch corresponding to the infinite pole lies on the positive real axis, tending to z_{q+1} from the right, thus producing roots on the positive real axis. Hence it must start at $-\infty$ and tend to the right. This however, rises the need for a zero located between $-\infty$ and p_1 (repeating the argumentation of Case 1), thus we have

$$z_1 < p_1 < z_2 < p_2 < \dots < z_q < p_q < z_{q+1} \leq 0.$$

This gives, since the poles tend to the zeros, that they shift to the right as x increases and, in particular, the infinite pole becomes finite. \square

Remark 4.15. From the proof of Theorem 4.12 we see that in the case $b_2 = 0$ or b_2 absent ($k_0 = 0$) we have that the largest finite zero is $z = 0$ and it does not cancel out as a fixed point. Hence, there will always be a pole $p(0)$ of the original system (eigenvalue of $sF + G$) with the property $\lim_{x \rightarrow \infty} p(x) = 0$. \diamond

We illustrate Theorem 4.14 by an example.

Example 4.16. Consider an RC circuit with two nodes, where at each node we have an earthed resistor and at one node an earthed capacitor. Using the node method we may describe this network by a pencil $sF + G$, where

$$F = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1/R_1 & 0 \\ 0 & 1/R_2 \end{bmatrix}.$$

Now we add an earthed capacitor with capacitance x to the second node, that is we get the new matrix

$$F + xbb^\top \quad \text{with} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Calculating the polynomials $p_{F,G}(s)$ and $z_{F,G,b}(s)$ from this is simple and we obtain

$$p_{F,G}(s) = 1/R_2(sC_1 + 1/R_1), \quad z_{F,G,b}(s) = sC_1 + 1/R_1.$$

Therefore, in the root locus problem (4.1) we have finite zeros at $z_1 = -1/(R_1C_1)$ and $z_2 = 0$, a finite pole at $p_1 = -1/(R_1C_1)$ and an infinite pole. This means that $-1/(R_1C_1)$ is a fixed point of the root locus and the infinite pole will become finite and move from $-\infty$ to 0. We may also calculate this behaviour:

$$p_{F,G}(s) + xsz_{F,G,b}(s) = (sC_1 + 1/R_1)(sx + 1/R_2),$$

i.e., we have roots at $-1/(R_1C_1)$, the fixed point, and at $-1/(xR_2)$ which tends to $-\infty$ for $x \rightarrow 0$ and for $x > 0$ it is always negative and moves from $-\infty$ to the right, reaching 0 for $x \rightarrow \infty$. \diamond

In the following we investigate the behaviour of the natural frequencies for negative single parameter variations. While this may be not of circuit theoretic interest so much (however it may correspond to active elements), it corresponds to the behaviour if uncertainties in the elements of the network are present and thus provides a robustness result under perturbations of the elements. First we illustrate the difference to the case $x > 0$ by an example.

Example 4.17. Consider the RC circuit with matrices F and G of Example 4.16. Now we add a capacitor with capacitance x in parallel connection to the existing capacitor of the circuit, that is we get the new matrix

$$F + xbb^\top \quad \text{with} \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Calculating the polynomials $p_{F,G}(s)$ and $z_{F,G,b}(s)$ we obtain

$$p_{F,G}(s) = 1/R_2(sC_1 + 1/R_1), \quad z_{F,G,b}(s) = 1/R_2.$$

Therefore, in the root locus problem (4.1) we have a finite zero at $z_1 = 0$ and a finite pole at $p_1 = -1/(R_1C_1)$. So, for $x > 0$ this pole moves towards 0. In fact we may calculate that

$$p_{F,G}(s) + xsz_{F,G,b}(s) = 1/R_2(s(C_1 + x) + 1/R_1)$$

has exactly one root at $p(x) := -1/(R_1(C_1 + x))$ which is negative for all $x > 0$. However, if we consider $x < 0$ now, then $(-\infty, -1/(R_1C_1))$ and $(0, \infty)$ are branches of the root locus, the pole always moving towards the left. This can be explained by means of the calculated root as follows:

- (i) if $-C_1 < x < 0$, then $p(x)$ is always negative and moves towards $-\infty$ for $x \searrow -C_1$;
- (ii) if $x = -C_1$ then we have a pole at infinity, this can also be seen from the matrix $F + xbb^\top$ which has a rank drop for $x = -C_1$;
- (iii) if $x < -C_1$, then $p(x)$ is always positive and we have $p(x) \rightarrow \infty$ for $x \nearrow -C_1$ and $p(x) \searrow 0$ for $x \rightarrow -\infty$, thus the pole moves to the left from ∞ towards 0.

This shows that for $x < 0$ there may be single values of the gain which cause finite poles to become infinite. \diamond

In fact, Example 4.17 shows everything which can happen for $x < 0$ compared to $x > 0$ investigated in Theorem 4.14. Thus the proof of the following corollary is immediate from the proof of Theorem 4.14 and taking this example into account. Lemma 4.4 guarantees that also for $x < 0$ the pencil $s(F + xbb^\top) + G$ has only real finite eigenvalues and index ≤ 1 .

Corollary 4.18. *Consider an RC or RL network with corresponding matrix pencil $sF + G \in \mathbb{R}^{k \times k}[s]$ with a first or second order parameter variation given by $b \in \mathbb{R}^k$ and the problem (3.4). Suppose that (4.2) holds. Then, in the root locus problem (4.1) for decreasing $x < 0$, the natural frequencies of the network either shift to the left or do not change at all. More precise, if p is a pole (root of $p_{F,G}(s)$) of multiplicity μ , then either μ or $\mu - 1$ of these poles do not change and at most one pole moves to the left. Furthermore, if $z \leq 0$ is the largest zero of the root locus problem (4.1) given by Theorem 4.12, then we have the following:*

- (i) If $\text{rk}(F + bb^\top) = \text{rk } F + 1$, then an infinite pole becomes finite and moves to the left on the positive real axis as x decreases, reaching the largest zero z , which does not cancel out with a pole, for $x \rightarrow -\infty$; the corresponding root locus branch is (z, ∞) .
- (ii) If $\text{rk}(F + bb^\top) = \text{rk } F$, then the smallest pole $p(x)$ moves to the left towards $-\infty$ and there exists some constant $\kappa > 0$ such that $p(x)$ is negative for $-\kappa < x < 0$ and $p(x) \rightarrow -\infty$ for $x \searrow -\kappa$, $p(-\kappa)$ is an infinite pole of the perturbed pencil (i.e., a finite pole becoming infinite under the perturbation), and $p(x)$ is positive for $x < -\kappa$ and $p(x) \searrow z$ for $x \rightarrow -\infty$; the corresponding root locus branch is $(-\infty, p(0)) \cup (z, \infty)$.

Note that the fact, that for $\text{rk}(F + bb^\top) = \text{rk } F$ there is exactly one value of $x < 0$ for which a finite pole becomes infinite, follows from the fact that $\deg p_{F,G}(s) = \deg z_{F,G,b}(s) + 1$ in this case implies that the leading coefficient of $p_{F,G}(s) + x s z_{F,G,b}(s)$ (where $p_{F,G}(s)$ and $z_{F,G,b}(s)$ have positive leading coefficients) can only be zero for one single value of $x < 0$. That the drop of degree is exactly one in this case follows from the root locus, as only one finite pole can move to ∞ , the others moving to the finite zeros (except the largest one).

Remark 4.19. As by Corollary 4.18 it is possible that there exists a value $\tilde{x} < 0$ of the parameter so that $\text{rk}(F + \tilde{x}bb^\top) = \text{rk } F - 1$, that is a finite eigenvalue of $sF + G$ becomes infinite in the “perturbed” pencil $s(F + \tilde{x}bb^\top) + G$. In some cases the occurrence of this infinite pole corresponds to an element of the RC (RL) circuit being removed for the value \tilde{x} of the parameter. However, this is not a general rule as the following example shows: Consider

$$F + xbb^\top = \begin{bmatrix} C_1 + x & -C_1 & -x \\ -C_1 & C_1 + C_2 & -C_2 \\ -x & -C_2 & C_2 + x \end{bmatrix}.$$

We find that $\text{rk}(F + xbb^\top) = 1$ for $x = -\frac{C_1 C_2}{C_1 + C_2}$ and otherwise the rank is 2. However, the rank drop for $x = -\frac{C_1 C_2}{C_1 + C_2}$ does not correspond to removing any of the elements (in this case the capacitor C_1 or C_2), as none of the values of the elements is reduced to zero. \diamond

Remark 4.20. The jump of the root locus branch described in Corollary 4.18(ii), that is $(-\infty, p(0)) \cup (z, \infty)$, can be easily explained using the standard compactification of the real axis into the complex plane: Of course we may write, for any $s \in \mathbb{R}$, $s = \frac{\sin t}{\cos t}$ for some $t \in (-\pi, \pi)$. If we now identify s with $(\cos t + i \sin t) \in \mathbb{C}$ we get a one-to-one map of the real axis onto the unit circle in \mathbb{C} (without the point $-1 \in \mathbb{C}$). In particular, $(s = 0) \sim (\cos 0 + i \sin 0) = 1$ and for $t \rightarrow \pi$ (movement in the upper half of the circle) we have $s \rightarrow \infty$ so we may identify $(s = \infty) \sim \lim_{t \rightarrow \pi} (\cos t + i \sin t) = -1$. On the other hand side, for $t \rightarrow -\pi$ (movement in the lower half of the circle) we have $s \rightarrow -\infty$ so we may identify $(s = -\infty) \sim \lim_{t \rightarrow -\pi} (\cos t + i \sin t) = -1$. Now $s = \infty$ and $s = -\infty$ correspond to the same point on the unit circle and therefore the jump of the pole $p(x)$ in Corollary 4.18(ii) from $-\infty$ to ∞ is now a continuous movement on the circle. \diamond

An immediate consequence of Theorem 4.14 and Corollary 4.18 is the following property:

Any given RC or RL network has the property that for small uncertainties in the values of the elements of the circuit, all natural frequencies of the network do change only a little bit from

their original position, so RC and RL networks are robust with respect to perturbations of their elements! This property still holds true if a new element is added in linear dependence of the existing elements (of the same type), that is, if its value is sufficiently small, then the natural frequencies do only change little.

An observation following from the previous analysis is that in a passive RC or RL network it always causes the transfer function of the network to become worse in its behaviour as additional capacitors or inductors are placed into the circuit - no matter where. As the capacitances (inductances) of these additional capacitors (inductors) increase, the natural frequencies will move to the right. Hence it can be deduced that in an RC or RL network, as few A- and T-type elements as possible with as low values as possible should be placed in order to obtain the best possible behaviour of the transfer function. In contrast to this performance worsening effect of capacitors and inductors it can be inferred from a consideration of the dual problem (3.5) along similar lines that resistors do always have an improving effect - the natural frequencies will always move to the left for increasing resistances.

Remark 4.21. Note that the whole analysis of Section 4 is based only on the assumptions that F and G are symmetric and positive semi-definite, b is any vector, and $s(F + xbb^\top) + G$ is a regular pencil for all $x \in \mathbb{R}$. By the argumentation as in this subsection it is in particular clear that the root locus problem (4.1) can be derived under only these assumptions. The use of compound matrices as in Section 3 is only important for the calculation of the fixed characteristic frequencies.

Due to the above generality, the results obtained in Section 4 results may be valid for a much larger class than passive RC or RL circuits, for instance also for non-planar networks. \diamond

5 Conclusions

We have studied the redesign of electrical passive networks by the change of single elements, which preserve or change the natural network topology and their effects on the natural frequencies of the network. We have used the impedance and admittance network models, which in the case of RC and RL networks are reduced to symmetric matrix pencil descriptions. For the case of RC and RL networks, it has been shown that such a single parameter modification is equivalent to a classical root locus problem. The explicit form of the pole and zero polynomials is given and the presence of fixed modes is discussed. The properties of the root locus follow from the symmetry and passivity of the pencil description and the main feature is the interlacing property of the pole zero structure as well as the common direction of movement of the poles. Aspects of the perturbation properties of the natural frequencies are examined within the framework of the natural network topologies. The work has direct applications in the modification of mechanical networks and in structural engineering. So far we have examined the case of RC and RL networks. Extension of the work to the general case of passive RLC networks, described by integral-differential, symmetric and structured models is under investigation.

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