Covering numbers and image dimensions of some stochastic processes

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Einleitung

Die vorliegende Arbeit beschäftigt sich mit der Größenmessung von reellen, zufälligen Mengen, die als Bilder von stochastischen Prozessen entstehen. Zum Größenvergleich können viele Parameter herangezogen werden. Möglich wäre es, den Durchmesser, die Kardinalität oder das Lebesguemaß solcher Mengen zu untersuchen. In dieser Arbeit dienen stattdessen Dimensionen und Überdeckungszahlen als Vergleichsgrößen. Schon seit Jahrzehnten hat man sich in der Literatur mit der Hausdorff- und Packungsdimension von Bildern von stochastischen Prozessen beschäftigt, entsprechend zahlreich sind die heute verfügbaren Arbeiten auf diesen Gebieten (u.a. [5]-[10],[13]-[20],[27],[35],[37]-[39]). Die Vorteile dieser Dimensionen sind, dass einige gut anwendbare Resultate zur Dimensionsabschätzung zur Verfügung stehen (etwa das Frostman-Lemma, vgl. [11]) und dass die beiden Dimensionen für beliebige Teilmengen der reellen Zahlen erklärt sind. Außerdem haben beide Dimensionsbegriffe die Eigenschaft, dass die Dimension von Mengen mit höchstens abzählbar unendlicher Kardinalität verschwindet. Dies kann von Vorteil sein, wenn man die untersuchten Mengen aus technischen Gründen leicht (nur an abzählbar vielen Stellen) modifizieren muss, ohne die Dimension verändern zu wollen.

Andererseits kann man es natürlich auch als Nachteil dieser Dimensionsbegriffe ansehen, wenn zum Beispiel die Menge der rationalen Zahlen aus dem Intervall $[0, 1]$ und die Menge $\{1\}$ im Sinne der beiden Dimensionen nicht unterschiedlich sind. Der Begriff der Entropiedimension bewertet die beiden eben genannten Mengen dagegen anders: Die rationalen Zahlen haben hier die größtmögliche Dimension 1, die einelementige Menge $\{1\}$ dagegen die Dimension 0. Grundlage für diesen Dimensionsbegriff sind die sogenannten Überdeckungszahlen $N(T, |\cdot|, \varepsilon)$. Diese geben für eine beschränkte, reelle Menge T die kleinstmögliche Anzahl von Intervallen der Länge 2ε an, mit denen T überdeckt werden kann. Die genaue Berechnung der Überdeckungszahlen erweist sich in den meisten Fällen als unmöglich, daher begnügt man sich damit, das Verhalten der Größen für ε gegen 0 zu untersuchen (Überdeckungsrate). Die polynomielle Ordnung, mit der $N(T, |\cdot|, \varepsilon)$ für ε gegen 0 wächst, ist dann die Entropiedimension. Da eine beschränkte Menge immer durch eine Anzahl von Kugeln in der Größenordnung $\frac{\text{Intervalllänge}}{\varepsilon}$ überdeckt werden kann, ist die 1 stets eine obere Schranke der Entropiedimension. Die Rate ε^{-1} ist für ein Intervall sogar optimal.

Aus dieser Definition ergibt sich der erste Nachteil der Entropiedimension: Ein wie oben definierter Wert für die Dimension muss gar nicht existieren. Gesichert ist aber zumindest die Existenz einer oberen und einer unteren Dimension, die bei Gleichheit dann *die* Dimension ergeben. Im Unterschied zur Hausdorff- und Packungsdimension kann die Entropiedimension außerdem nur sinnvoll für beschränkte Mengen bestimmt werden. Ein Vorteil ist jedoch, dass zur Bestimmung der Dimension auch andere Größen als die Überdeckungszahlen zur Verfügung stehen (z.B. die Kardina-

lität eines maximalen ε-Netzes). Je nach vorliegender Menge kann man die in der jeweiligen Situation günstigste Lösung wählen.
In dieser Arbeit soll es vor allem um die Bestimmung der Ordnungen von Überdeckungszahlen gehen, dabei werden die Betrachtungen aus Gründen der Übersichtlichkeit auf einige wenige Prozesse beschränkt. Eine genaue Analyse der Beweise zeigt aber, dass für kein Einzelresultat sämtliche Eigenschaften eines Prozesses gebraucht werden, eine allgemeinere Formulierung der Ergebnisse sollte also möglich sein. Ebenso werden bereits bei diesen wenigen Prozessen die Grenzen der benutzten Beweistechniken sichtbar. Eine Diskussion möglicher Erweiterungen und Probleme findet im letzten Kapitel statt.
Die Situation ist klar, falls die Indexmenge T ein Intervall enthält und der Prozess X (fast sicher) stetig ist. Dann enthält das zufällige Bild ebenfalls (fast sicher) ein Intervall und das Bild lässt sich mit Rate ε^{-1} optimal überdecken. Es bleiben noch drei Fälle übrig:

1. X ist stetig, aber T enthält kein Intervall.
2. X ist nicht stetig und T enthält ein Intervall.
3. X ist nicht stetig und T enthält kein Intervall.

In die erste Kategorie fällt der Prozess der Fraktalen Brownschen Bewegung mit einer konvexen Folge als Indexmenge, aber auch selbstähnliche Indexmengen zählen dazu. Zur zweiten Kategorie gehören zum Beispiel die α-stabilen Prozesse auf dem Intervall $[0, 1]$. Und schließlich gehören zur dritten Kategorie, neben den stabilen Prozessen mit reellen Zahlenfolgen als Parametermengen, insbesondere die zufälligen Folgen vom Typ $X = (\alpha_n \xi_n)$, wobei (α_n) eine reelle Zahlenfolge und (ξ_n) eine Folge zufälliger Größen ist. Hierbei wird insbesondere auch die Möglichkeit betrachtet, dass ξ_n das Produkt zweier unabhängiger Größen U_n und ζ_n ist, wobei U_n nur die Werte 0 und 1 annehmen kann.
Die eigenen Resultate in dieser Arbeit gehören hauptsächlich zur ersten und dritten Kategorie. Gelegentlich machen manche Beweistechniken aus der dritten Kategorie aber nicht von einer speziellen Struktur der Indexmenge Gebrauch, so dass auch Ergebnisse aus der zweiten Kategorie gewonnen werden können.
Die Motivation für diese Arbeit stammt aus verschiedenen Quellen. Zuerst ist hier die vorhandene Literatur zu nennen: Es gibt für die betrachteten Prozesse bereits viele Packungs- und Hausdorffdimensionsresultate, insbesondere für die Hausdorffdimension bleibt oftmals keine Frage mehr offen. Schwieriger wird es bereits für die Packungsdimension, aber auch diese wurde in den letzten beiden Jahrzehnten ausführlich betrachtet. Überdeckungszahlen und Entropiedimension kommen meist vor, wenn es darum geht, obere Abschätzungen für die anderen beiden Dimensionen zu finden. Der zweite Grund liegt in der Natur der Entropiedimension. Im Gegensatz zur Hausdorff- und Packungsdimension können abzählbare Mengen hier höchst

unterschiedlich sein. Daher kann es in diesem Fall schon lohnenswert sein, sich auf die Betrachtung interessanter abzählbarer Mengen zu beschränken.
Eine dritte Motivationsquelle bilden die Arbeiten [25] und [26]. Dort wird die Überdeckungsrate von $X(K)$ berechnet, wobei X ein Subordinator und K eine kompakte reelle Menge ist. Die dabei genutzten Beweistechniken lassen sich aber nicht unbedingt auf allgemeinere Prozesse übertragen, da für die ursprünglichen Beweise die Monotonie der Subordinatoren eine wesentliche Eigenschaft ist.
Ferner wurde die Arbeit durch ein Resultat aus der Arbeit [2] angeregt. Dort taucht in der Proposition 17 die Ungleichung

$$\mathbb{P}\left\{N(Y([0,1]),|\cdot|,\varepsilon) < \delta\left\lceil\varepsilon^{-\frac{\alpha}{1+\alpha}}\right\rceil\right\} \leqslant \exp(-c\left\lceil\varepsilon^{-\frac{\alpha}{1+\alpha}}\right\rceil) \tag{1}$$

auf, die für jedes $\varepsilon > 0$ gilt, wobei Y ein selbstähnlicher α-stabiler Lévyprozess ist, c und δ Konstanten sind und $N(\cdot)$ für die Überdeckungszahlen steht. Anders ausgedrückt, $\varepsilon^{-\frac{\alpha}{1+\alpha}}$ ist auf einer Menge mit großer Wahrscheinlichkeit eine untere Schranke für $N(Y([0,1]),|\cdot|,\varepsilon)$. Ähnliche Ausdrücke tauchen auch in dieser Arbeit auf: Für einen Prozess X und eine geeignete Indexmenge K wird eine Schranke für $N(X(K),|\cdot|,\varepsilon)$ gesucht, die für ε gegen 0 auf einer Menge mit großer Wahrscheinlichkeit gültig ist. Gelegentlich sind auch „fast sichere“ Aussagen möglich, das heißt, die Aussagen gelten auf einer Menge mit Wahrscheinlichkeit 1.
Es bleibt noch, eine unveröffentlichte Arbeit von M.Lifshits zu erwähnen ([23]). In ihr wird die Vermutung widerlegt, dass die Überdeckungsrate des Bildes eines Gaußschen Prozesses durch die Überdeckungsrate der zu Grunde liegenden Indexmenge bezüglich der vom Prozess induzierten Metrik gegeben ist. Die Suche nach weiteren Gegenbeispielen führt zum Studium der konvexen Folgen, welche sich als gute Klasse von Indexmengen erweisen, da ihre Überdeckungsrate bekannt ist.
Um einige Ergebnisse klarer und übersichtlicher präsentieren zu können, wird in Anlehnung an die Konvergenz in Wahrscheinlichkeit eine Relation $\overset{\mathbb{P}}{\succcurlyeq}$ (bzw. $\overset{\mathbb{P}}{\preccurlyeq}$ und $\overset{\mathbb{P}}{\approx}$) eingeführt. Die Aussage von Gleichung (1) vereinfacht sich dann zu

$$N(Y([0,1]),|\cdot|,\varepsilon) \overset{\mathbb{P}}{\succcurlyeq} \varepsilon^{-\frac{\alpha}{1+\alpha}}.$$

Mit den Methoden dieser Arbeit kann diese Aussage sogar verbessert werden:

$$N(Y([0,1]),|\cdot|,\varepsilon) \overset{\mathbb{P}}{\succcurlyeq} \begin{cases} \varepsilon^{-1} : \alpha \in (1,2) \\ \varepsilon^{-\alpha} : \alpha \in (0,1). \end{cases}$$

Da zur Definition der gewöhnlichen Entropiedimension $\dim_E$ die polynomielle Komponente der Überdeckungsrate herangezogen wird, liegt es nahe, ähnlich vorzugehen, um aus der Relation $\overset{\mathbb{P}}{\succcurlyeq}$ einen Dimensionsbegriff $\dim_E^{\mathbb{P}}$ abzuleiten. Ein typisches Resultat dieser Arbeit kann dann folgendermaßen formuliert werden: Es sei X

die Fraktale Brownsche Bewegung mit Index H oder ein selbstähnlicher α-stabiler Lévyprozess mit $\alpha \in (1, 2)$ und es sei $T = (\alpha_n)$ eine streng konvexe Folge (die gegebenenfalls noch eine Regularitätsbedingung erfüllen muss) und $\dim_E T$ möge existieren. Dann gilt

$$\dim_E^{\mathbb{P}} X(T) = \frac{\dim_E T}{y + (1-y)\dim_E T},$$

wobei im Fall der Fraktalen Brownschen Bewegung $y = H$ und im Fall des stabilen Prozesses $y = \frac{1}{\alpha}$ gesetzt wird. Ist allerdings $\alpha \in (0, 1)$, dann gilt

$$\dim_E^{\mathbb{P}} X(T) = \alpha \cdot \dim_E T.$$

Etwas anders zu handhaben sind die Prozesse der Art $(\alpha_n \xi_n)$ und $(\alpha_n U_n \xi_n)$, denn hier besteht die Indexmenge stets aus den natürlichen Zahlen. Hierbei ist (α_n) eine reelle Zahlenfolge und (U_n) und (ξ_n) sind unabhängige zufällige Folgen, wobei die Unabhängigkeit auch für die Komponenten jeder Folge gelten soll. Außerdem sollen die (positiven) Größen ξ_n identisch verteilt sein und die U_n sollen nur die Werte 0 und 1 annehmen. Da die Bilder der beiden Prozessarten jeweils abzählbar sind, besteht keine Notwendigkeit, Arbeiten über die Packungs- oder Hausdorffdimension zu verfassen, denn diese sind beide 0. Der Frage nach der Überdeckungsrate wurde in der Literatur aber noch nicht nachgegangen. Teilweise können hier Techniken vom zeitstetigen Fall übernommen werden, oft müssen aber andere Ansätze verfolgt werden, um Resultate zu bekommen. Als geeignet erweist sich hier das Mittel, die Konvergenzrate der absteigenden Anordnung der betreffenden Folgen zu ermitteln. In einigen Fällen ist es sogar möglich, eine Funktion f zu bestimmen, so dass

$$\lim_{n\to\infty} \frac{(\alpha_\cdot \xi_\cdot)^*_n}{f(n)} = 1$$

fast sicher gilt. Hierbei bezeichnet $(\alpha_\cdot \xi_\cdot)^*$ die absteigende Anordnung der Folge $(\alpha_n \xi_n)$. Dieses Resultat kann ebenfalls benutzt werden, um beliebige Momente der zufälligen Größe ξ_1 zu approximieren (sofern diese existieren).
Sowohl bei den zeitstetigen als auch bei den zeitdiskreten Prozessen muss vor der Berechnung der Überdeckungsraten geklärt werden, ob diese Aufgabe überhaupt sinnvoll ist. Mit anderen Worten: Es ist die Frage zu beantworten, ob die Bilder der Prozesse (fast sicher) beschränkt sind. Für die in dieser Arbeit betrachteten zeitstetigen Prozesse lässt sich diese Frage leicht beantworten, da gewisse Stetigkeitseigenschaften genutzt werden können. Bei den zeitdiskreten Prozessen hängt die Beschränktheit vom Zusammenspiel der Folge (α_n) mit der Verteilung von ξ_1 ab. Für den Prozesstyp $(\alpha_n \xi_n)$ kann auf Vorarbeiten aus [1] zurückgegriffen werden, eine Modifikation der dort genutzten Beweistechnik beantwortet die Beschränktheitsfrage auch für $(\alpha_n U_n \xi_n)$.

Im Hinblick auf Anwendungen ist die Bestimmung von Überdeckungsraten eine gerechtfertigte Aufgabe, da Rückschlüsse über Größe und Kompaktheit der betrachteten Mengen gezogen werden können.
Nach der Einleitung wird die Arbeit in Kapitel 1 mit der Einführung der notwendigen Definitionen, Eigenschaften und Notationen fortgesetzt. Anschließend werden in Kapitel 2 die bekannten und eigenen Dimensions- und Überdeckungsresultate für die betrachteten Prozesstypen aufgelistet. Die Beweise der eigenen Ergebnisse folgen im Abschnitt 3. Die Arbeit endet mit einer kurzen Diskussion einiger offener Fragen und möglicher Verallgemeinerungen (Kapitel 4).

Introduction

Given a subset A of the real line, there are many possibilities to measure the size of A. For example, one could study the diameter $(\sup_{x,y\in A} |x-y|)$, the number of elements or the cardinality (finite, countable, not countable), the Lebesgue measure of A, compactness properties (covering properties) or its (Hausdorff, packing, entropy) dimension. Now, let $f : \mathbb{R} \to \mathbb{R}$ be some function and consider the set $f(A)$. Is it possible to express the size of $f(A)$ by the size of A and characteristics of f? The situation will be even more complicated if f is a random function.
In this thesis, we aim to establish results on covering numbers (which represent compactness properties) and dimensions of random sets, which are given as images of stochastic processes, i.e., the set A has the form $A = X(T)$, where X is some real-valued stochastic process given on a subset T of the real line. Computing a dimension for A means finding a real number between 0 and 1 to describe the size of A. There are different possibilities to do this. Some possibilities yield equivalent definitions of dimension, others are very different. An introduction to the most important dimension concepts can be found below (cf. section 1.2). Computing covering numbers means finding a minimal finite number of intervals with radius $\varepsilon > 0$ such that the union of these intervals covers A. For bounded sets it is always possible to do this. In fact, the covering rate function of a bounded, real set A is always bounded by a multiple of ε^{-1} because one can cover $[\inf\{a : a \in A\}, \sup\{a : a \in A\}]$ by $\left\lceil \frac{\sup\{a:a\in A\}-\inf\{a:a\in A\}}{2\varepsilon} \right\rceil$ intervals. In general, this number depends on ε and the computation is very complicated. Therefore, one changes the task and does not compute the exact value, but one tries to find its behavior for ε tending to zero (covering rate). The polynomial part of the covering rate function is the foundation for the so-called entropy or box counting dimension.
We are not interested in the case of T being any interval and X being (almost surely) continuous. Here, the situation is clear: The random image is an interval $[\inf_{t\in T} X_t, \sup_{t\in T} X_t]$ having a low degree of compactness and having dimension 1 (in all the dimension senses we introduce below). Because of the dimensions' monotonicity properties, the case "T is an interval" is equivalent to the case "T contains at least one interval". Excluding that case, there remain three situations:

- The process X is (almost surely) continuous and the index set T is fractal.
- X is not continuous and T is an interval.
- X is not continuous and T is fractal.

Here, fractal means that T does not contain any interval and continuity refers to the absolute value metric. In this thesis we mainly study the first (X being fractional Brownian motion, cf. section 2.1) and the third case (X being α- stable or a random sequence with independent components, cf. sections 2.2 and 2.3), where T is often

countable, but some results do not require T to have a special structure or do not require X to be a special process, so, second case results can appear, too.
Our own results concern the so-called entropy or box-counting dimension, but we give an overview about known Hausdorff and packing dimension results for some important stochastic processes. The task was motivated by the papers [25] and [26] where some processes with random index sets where studied and it was necessary to gain knowledge about the random sets' compactness properties. But in contrast to the mentioned papers, we cannot use any monotonicity property of our processes. Therefore, other tools have to be used.
A second source of motivation is given by the paper [2]: Proposition 17 in [2] asserts that for a given strictly α- stable Lévy process Y one can find constants δ and c such that

$$\mathbb{P}\left\{N(Y([0,1]),|\cdot|,\varepsilon) < \delta\left\lceil\varepsilon^{-\frac{\alpha}{1+\alpha}}\right\rceil\right\} \leqslant \exp(-c\left\lceil\varepsilon^{-\frac{\alpha}{1+\alpha}}\right\rceil) \tag{2}$$

is valid for all $\varepsilon > 0$, where $N(\cdot)$ denotes the covering numbers. In other words, if ε is small, $\varepsilon^{-\frac{\alpha}{1+\alpha}}$ will be a lower bound for $N(Y([0,1]),|\cdot|,\varepsilon)$, at least on a set with large probability. In this thesis appear similar inequalities, where we relate the rate of the covering numbers of the index set with some property of the underlying process (for stable Lévy processes the important property is the value of α). Contrary to (2), we do not want to find exponential inequalities, but we aim to find the rate of $N(Y(K),|\cdot|,\varepsilon)$, for ε to zero, being valid on sets which large probability (K is some index set). Sometimes "almost surely" results are possible, i.e., the rates are valid on sets with probability one.
And finally, the third source of motivation was an unpublished counterexample by M.Lifshits ([23]). This counterexample showed that in general the covering numbers of a centered Gaussian process given on an index set K are not determined by the covering numbers of K computed with respect to the metric induced by the process. The wish to find more counterexamples resulted in studying convex sequences, which turned out to be a valuable source of index sets.
To state some results, we introduce a convergence relation $\overset{\mathbb{P}}{\succcurlyeq}$ (and $\overset{\mathbb{P}}{\preccurlyeq}$ and $\overset{\mathbb{P}}{\approx}$) , which is similar to convergence in probability. In terms of that relation, inequality (2) results in

$$N(Y([0,1]),|\cdot|,\varepsilon) \overset{\mathbb{P}}{\succcurlyeq} \varepsilon^{-\frac{\alpha}{1+\alpha}},$$

being valid for $\alpha \in (0,2)$. Using our tools, we can also improve the exponent. In fact, we show that

$$N(Y([0,1]),|\cdot|,\varepsilon) \overset{\mathbb{P}}{\succcurlyeq} \begin{cases} \varepsilon^{-1} : \alpha \in (1,2) \\ \varepsilon^{-\alpha} : \alpha \in (0,1) \end{cases}$$

is true. The polynomial part of the (classical) covering number rate yields the entropy dimension, we use this approach to define a dimension $\dim_E^{\mathbb{P}}$ based on the covering numbers' behavior in $\overset{\mathbb{P}}{\succcurlyeq}$. A typical result of this thesis is the following: Let X be the fractional Brownian motion with index H or let X be strictly α- stable with $\alpha \in (1, 2)$ and let $T = (\alpha_n)$ be a strictly convex sequence (fulfilling some regularity condition) with existing entropy dimension $\dim_E T$. Then,

$$\dim_E^{\mathbb{P}} X(T) = \frac{\dim_E T}{y + (1 - y) \dim_E T},$$

where $y = H$ in the case of the fractional Brownian motion and $y = \frac{1}{\alpha}$ in the stable case. But if $\alpha \in (0, 1)$, then

$$\dim_E^{\mathbb{P}} X(T) = \alpha \cdot \dim_E T$$

is valid. For $\alpha = 1$ both equations are valid.

There are two main reasons why we are interested in covering numbers and entropy dimension. Firstly, there are already many papers dealing with Hausdorff and packing dimension, even in the multidimensional case. Covering number results are rare and when they appear in the literature, they are mainly used to get upper bounds for the other dimensions. Secondly, Hausdorff and packing dimension "neglect" countable sets, but, as it can be seen below, these sets are the foundation of some interesting results.

This thesis should be considered as an introduction to the topic. We study only a limited number of processes and index sets in order to demonstrate what kind of results can be expected, which tools can be used to derive these results and where generalizations could be possible. Of course, we also see the limits of our methods.

Somewhat different in comparison to the fractional Brownian motion and stable processes is the process type of random sequences because in this case the index set is always $\mathbb{N}$. Here, the process has the form $(\alpha_n \xi_n)$ or $(\alpha_n U_n \xi_n)$, where (α_n) is a sequence of real numbers, (ξ_n) is a sequence of independent and identically distributed random variables and (U_n) is a sequence of independent random variables with $\mathbb{P}\{U_n \in \{0, 1\}\} = 1$. Moreover, (ξ_n) is independent of (U_n). The random paths of these processes are countable, thus, it is not a challenging task to compute their Hausdorff and packing dimensions (they are zero). But it is challenging to find results concerning covering numbers. Because computing these numbers requires having bounded sets, we must find conditions for $(\alpha_n \xi_n)$ or $(\alpha_n U_n \xi_n)$ being (almost surely) bounded. In [1] this was done for the first process type, we adapt those ideas to give boundedness conditions for $(\alpha_n U_n \xi_n)$. In order to get results for the random sequences, we also compute the behavior of the decreasing rearrangements $(\alpha_n \xi_n)^*$ and $(\alpha_n U_n \xi_n)^*$. In some cases, we see how the components of the rearrangements must be scaled so that the resulting series of fractions converges to 1, i.e., we compute a function f in such a way that

$$\lim_{n\to\infty} \frac{(\alpha_. \xi_.)_n^*}{f(n)} = 1, \text{ a.s.}$$

Furthermore, we remark how our knowledge of decreasing rearrangements can be used to approximate arbitrary moments of (positive) random numbers. In view of applications, we think that computing covering numbers is a justified task because having information about covering numbers enables us to evaluate the compactness of sets.

The organisation of this thesis is as follows: In section 1, we introduce the most important mathematical objects. Further tools are introduced when they are needed. Section 2 contains known and new results concerning dimensions and covering numbers of images of stochastic processes. The proofs of the new results and examples can be found in section 3. We finish with a brief discussion on open problems and possible generalizations (section 4).

1 Preliminaries

In this section, we introduce the necessary mathematical objects. We start with some notation for asymptotics. For two functions f and g, $f(x) \preccurlyeq g(x)$, as $x \to 0$, means that there is a finite constant $c > 0$ such that $f(x) \leqslant cg(x)$ is valid for all x small enough. We also write $g(x) \succcurlyeq f(x)$ in this case. If $f(x) \preccurlyeq g(x)$ and $g(x) \preccurlyeq f(x)$, as $x \to 0$, we will write $f(x) \approx g(x)$. If $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = 1$, we will write $f(x) \sim g(x)$, as $x \to 0$. The notation is defined analogously for $x \to \infty$ and for sequences. It is useful to introduce the following class of functions:

$$\mathcal{D} := \{A : (0,1) \to \mathbb{R}^+ | A \text{ monotone}, \lim_{\varepsilon\to 0} A(\varepsilon) = \infty, A(\varepsilon) > 0 \ \forall \varepsilon \in (0,1)\}.$$

We make use of the class $\mathcal{D}$ when we study the limit behavior for $\varepsilon \to 0$. Consequently, it is sufficient that the functions being elements of $\mathcal{D}$ are defined on $(0,1)$. Tacitly, we assume that ε is always chosen smaller than one.

The symbol $\lceil x \rceil$ denotes the smallest integer which is greater than or equal to x and $\lfloor x \rfloor$ denotes the greatest integer which is smaller than or equal to x.

The expression log marks the natural logarithm and if a function f is invertible, f^{-1} will describe the inverse function. We use $\mathbb{E}$ for expectations and $\mathbb{1}$ for indicator functions. The decreasing rearrangement of a positive sequence (α_n) is symbolized by $(\alpha_n)^*$.

Furthermore, we denote constants, whose exact values are unimportant, by $c_1, c_2, \ldots$ or some other letters.

1.1 Measuring compactness

Let $\varepsilon > 0$ and $x \in \mathbb{R}$. We denote by $B_\varepsilon(x)$ the closed ball of radius ε centered at x with respect to the Euclidean metric. It is well-known that for a bounded set $A \subseteq \mathbb{R}$, we can find a finite number of ε-balls such that A can be covered by the union of these balls. The necessary number of balls is at most $\left\lceil \frac{\sup_{x,y\in A} |x-y|}{2\varepsilon} \right\rceil$. This justifies the following

Definition 1.1. Let $\varepsilon > 0$ and $A \subseteq \mathbb{R}$ be bounded. The value

$$N(A, |\cdot|, \varepsilon) := \inf\{n \in \mathbb{N} : \exists x_1, \ldots, x_n \in \mathbb{R} : A \subseteq \bigcup_{k=1}^{n} B_\varepsilon(x_k)\}$$

is called covering number of A.

There are some similar quantities. Let us denote by $\overline{N}(A, |\cdot|, \varepsilon)$ the minimal number of ε-balls needed to cover A, where the balls are centered at elements of A. The packing number $M(A, |\cdot|, \varepsilon)$ is defined as the maximum cardinality of an ε-separated subset of A, i.e.,

$$M(A, |\cdot|, \varepsilon) := \sup\{n \in \mathbb{N} : \exists x_1, \dots, x_n \in A : |x_i - x_j| > \varepsilon, \forall i \neq j\}.$$

The following lemma shows how these quantities are related. We omit the proof.

Lemma 1.2. *Let $\varepsilon > 0$ and let $A \subseteq \mathbb{R}$ be bounded. Then*

$$N(A, |\cdot|, \varepsilon) \leqslant \overline{N}(A, |\cdot|, \varepsilon) \leqslant 2N(A, |\cdot|, \varepsilon),$$
$$N(A, |\cdot|, \varepsilon) \leqslant M(A, |\cdot|, \varepsilon) \leqslant N(A, |\cdot|, \frac{\varepsilon}{2}).$$

The compactness of a bounded real set A is measured by the rate of $N(A, |\cdot|, \varepsilon)$ (or $\overline{N}$, or M, by Lemma 1.2) for ε tending to zero. The more balls are necessary to cover A, the lower is the degree of the compactness of A.

1.2 Dimensions

In this section, we introduce different possibilities to define dimensions for real sets. The introduction is restricted to the case of the real line equipped with the Euclidean metric, but the expansion to general metric spaces is possible. We follow the representation in [11].

1.2.1 Hausdorff dimension

Given $\emptyset \neq A \subseteq \mathbb{R}$, define $|A| := \sup\limits_{x,y \in A} |x - y|$ and set $|\emptyset| := 0$. Let $\alpha \geqslant 0$ and $\delta > 0$. Define a set function by

$$\mathcal{H}^\alpha_\delta(A) := \inf\{\sum_{i=1}^\infty |A_i|^\alpha : A \subseteq \bigcup_{i=1}^\infty A_i, |A_i| \leqslant \delta\}.$$

If δ decreases, the number of possible coverings will reduce and $\mathcal{H}^\alpha_\delta(A)$ will increase and it makes sense to define the α-dimensional (outer) Hausdorff measure

$$\mathcal{H}^\alpha(A) := \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(A).$$

It can be shown that $\mathcal{H}^\alpha$ is an outer measure (cf. [11], section 2.1), but the existence of a critical value for $\mathcal{H}^{(\cdot)}$ is more important.

Lemma 1.3. *Let $\alpha > \beta > 0$. Then $\mathcal{H}^\beta(A) < \infty$ implies $\mathcal{H}^\alpha(A) = 0$.*

Proof. Cf. section 2.2 in [11]. □

Lemma 1.3 motivates the following

Definition 1.4. Let $A \subseteq \mathbb{R}$. The value

$$\dim_H A := \inf\{\alpha : \mathcal{H}^\alpha(A) = 0\} = \sup\{\alpha : \mathcal{H}^\alpha(A) = \infty\}$$

is called Hausdorff dimension of A.

1.2.2 Packing dimension

Definition 1.5. Let $A \subseteq \mathbb{R}$ and $\delta > 0$. A $\delta-$ball packing for A is a sequence $(B_{r_i}(x_i))$ of balls such that

1. $x_i \in A$,
2. $B_{r_i}(x_i) \cap B_{r_j}(x_j) = \emptyset, i \neq j$,
3. $|B_{r_i}(x_i)| = 2r_i < \delta$.

Let $A \subseteq \mathbb{R}$, $\alpha \geqslant 0$ and $\delta > 0$. Define a set function by

$$\mathcal{P}^\alpha_\delta(A) := \sup\{\sum_{i=1}^{\infty}(2r_i)^\alpha : (B_{r_i}(x_i)) \text{ is a } \delta - \text{ball packing for } A\}.$$

If δ decreases, the number of possible packings will reduce and $\mathcal{P}^\alpha_\delta(A)$ will decrease, too. Thus,

$$\mathcal{P}^\alpha_0(A) := \lim_{\delta \to 0} \mathcal{P}^\alpha_\delta(A)$$

is well-defined.

Now, define

$$\mathcal{P}^\alpha(A) := \inf\{\sum_{i=1}^{\infty} \mathcal{P}^\alpha_0(A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i\}.$$

As in the Hausdorff dimension case, there is a critical value for α, where $\mathcal{P}^\alpha(A)$ changes from zero to infinity (cf. [11], section 3.4). This leads to

Definition 1.6. Let $A \subseteq \mathbb{R}$. The value

$$\dim_P A := \sup\{\alpha : \mathcal{P}^\alpha(A) = \infty\} = \inf\{\alpha : \mathcal{P}^\alpha(A) = 0\}$$

is called packing dimension of A.

Remark 1.7. *In the packing dimension case as well as in the Hausdorff dimension case, the values $\mathcal{P}^{\dim_P A}(A)$ and $\mathcal{H}^{\dim_H A}(A)$ can be 0, ∞ or finite and strictly positive. This situation is the reason for further studies. One can substitute the functions*

$x \mapsto x^\alpha$ being used above by more general dimension functions g, e.g., $x \mapsto x^\alpha (\log \frac{1}{x})^\beta$, mapping from $\mathbb{R}^+$ to $\mathbb{R}^+$. Given a set A, one can define set functions

$$\mathcal{H}^g_\delta(A) := \inf\{\sum_{i=1}^{\infty} g(|A_i|) : A \subseteq \bigcup_{i=1}^{\infty} A_i, |A_i| \leqslant \delta\}$$

and $\mathcal{H}^g(A) = \lim_{\delta \to 0} \mathcal{H}^g_\delta(A)$. Now, the task is to find a g such that $\mathcal{H}^g(A)$ is a strictly positive real number. The same can be done to find a refined packing function.

It is a consequence of the next result that in Hausdorff and packing dimension contexts countable sets are hardly interesting.

Corollary 1.8 ([11] p. 33, 53 and 56)**.** *Let $A \subseteq \mathbb{R}$ be countable. Then*

$$\dim_H A = \dim_P A = 0.$$

1.2.3 Entropy dimension

The third dimension is based on the covering numbers of a bounded set. Recall the definitions from section 1.1. Contrary to the first two dimensions, the following dimension is only useful for bounded sets.

Definition 1.9. Let $A \subseteq \mathbb{R}$ be bounded. The values

$$\overline{\dim}_E A = \limsup_{\varepsilon \to 0} \frac{\log N(A, |\cdot|, \varepsilon)}{-\log \varepsilon},$$
$$\underline{\dim}_E A = \liminf_{\varepsilon \to 0} \frac{\log N(A, |\cdot|, \varepsilon)}{-\log \varepsilon}$$

are called upper and lower entropy dimension of A. If both numbers are equal, the common value will be called entropy dimension and will be denoted by $\dim_E A$.

Remark 1.10. *To compute the entropy dimension of a set A, it is sufficient to have knowledge about the polynomial part of $N(A, |\cdot|, \varepsilon)$. Knowing the exact rate of the covering numbers is a finer type of dimension. Similar relations exist for Hausdorff and packing dimension (cf. Remark 1.7).*

Lemma 1.2 shows that we could also use the numbers $\overline{N}$ and M to compute the entropy dimension. In fact, there are even more quantities which could be used for the computation (cf.[11], section 3.1) and there are some other names for the entropy dimension, e.g., metric dimension or box (-counting) dimension.

1.2.4 Dimension relations

In this section, we briefly present the most important relations between the three dimensions.

Theorem 1.11 ([11] section 3)**.** *Let* $A \subseteq \mathbb{R}$*. Then*

- $\dim_H A \leqslant \dim_P A$.

Let $A \subseteq \mathbb{R}$ *be bounded. Then*

- $\dim_H A \leqslant \underline{\dim}_E A \leqslant \overline{\dim}_E A$,
- $\dim_P A \leqslant \overline{\dim}_E A$.

Because of the last estimates, covering numbers are a popular tool to gain upper bounds for the Hausdorff and packing dimensions (for example, cf. [13] or [38]). The example of the middle third cantor set C (cf. section 1.4.3.1) shows that all dimensions can concur (here, their common value is $\frac{\log 2}{\log 3}$).
On the other hand, the example $T = ((\log n)^{-1})$ shows that $\dim_H T = \dim_P T = 0$ and $\underline{\dim}_E T = \overline{\dim}_E T = 1$ is possible.
Exercise 3.8 in [11] gives a hint how a set with $\underline{\dim}_E T < \overline{\dim}_E T$ can be constructed.

There is a further connection between entropy and packing dimension. Let us define the modified upper entropy dimension of a real set A:

$$\overline{\dim}_{ME} A = \inf\{\sup_i \overline{\dim}_E A_i : A \subseteq \bigcup_{i=1}^{\infty} A_i\},$$

where the infimum is taken with respect to possible coverings of A. Sometimes, there is no difference to the usual definition, but in general, the new definition yields an additional possibility to compute packing dimensions.

Proposition 1.12 ([11], Behauptung 3.3)**.** *Let* $A \subseteq \mathbb{R}$ *be compact and assume that* $\overline{\dim}_E (A \cap V) = \overline{\dim}_E A$ *holds for every open, real set* V *with* $A \cap V \neq \emptyset$*. Then*

$$\overline{\dim}_E A = \overline{\dim}_{ME} A.$$

Proposition 1.13 ([11], Behauptung 3.4)**.** *Let* A *be a subset of* $\mathbb{R}$*. Then*

$$\dim_P A = \overline{\dim}_{ME} A.$$

1.3 Probability theory basics

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space and let $T \subseteq \mathbb{R}$ be a nonempty index set. We call $X : T \times \Omega \longrightarrow \mathbb{R}$ a stochastic process (short: process) if for every $t \in T$ the mapping $\omega \longmapsto X(t, \omega)$ will be measurable with respect to $\mathcal{A}$ and $\mathcal{B}(\mathbb{R})$, where $\mathcal{B}$ denotes the sigma-algebra of the Borel sets. We write $X_t(\omega) := X(t, \omega)$ and for $A \subseteq T$ we define $X(A) := \{X_t : t \in A\}$. A famous theorem of Kolmogoroff answers the question of existence of stochastic processes (cf. [33], section 1). The existence of the processes being considered in this thesis is ensured by the cited theorem and

in the following, we do not deal with existence questions.
Two processes X and Y will be considered as equal (stochastically equivalent) if they have the same finite dimensional distributions, i.e., for every choice of $t_1, \dots, t_n \in T$ and $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$ follows

$$\mathbb{P}\{X_{t_1} \in B_1, \dots, X_{t_n} \in B_n\} = \mathbb{P}\{Y_{t_1} \in B_1, \dots, Y_{t_n} \in B_n\}.$$

A process Y is a modification of a process X if $\mathbb{P}\{X_t = Y_t\} = 1$ for every $t \in T$. In our setting (real-valued processes), Y being a modification of X implies that the processes are stochastically equivalent (they are versions). In the following, we always assume that we work with a version of a process having the properties we need, e.g., continuity.
We say that a random variable Y will have a property p almost surely (short: a.s.), if there is a measurable set N such that $\mathbb{P}\{N\} = 0$ and $Y(\omega)$ has property p for each $\omega \in N^c$. If the underlying probability space is complete, this will be equivalent to $\{Y \text{ has property } p\} \in \mathcal{A}$ and $\mathbb{P}\{Y \text{ has property } p\} = 1$. We also say that Y has property p with probability one.

Next, we introduce the processes being considered in this thesis.

1.3.1 Fractional Brownian motion

A stochastic process $X = (X_t)_{t \in T}$ will be called a Gaussian process if for every finite sequence of indices $t_1, t_2, \dots, t_n$ and real numbers $\alpha_1, \dots, \alpha_n$ the random variable $\sum_{k=1}^{n} \alpha_k X_{t_k}$ is normally distributed, i.e., either there is a real number a and a strictly positive σ^2 such that we have

$$\mathbb{P}\left\{\sum_{k=1}^{n} \alpha_k X_{t_k} \leqslant s\right\} = \frac{1}{\sqrt{2\pi}|\sigma|} \int_{-\infty}^{s} e^{-\frac{(y-a)^2}{2\sigma^2}} dy$$

for each real s or there is some real number b such that $\mathbb{P}\left\{\sum_{k=1}^{n} \alpha_k X_{t_k} = b\right\} = 1$ is valid. Gaussian processes are uniquely characterized by their mean value function $m(t) = \mathbb{E}X_t$ and their covariance function $R(t,s) = \mathbb{E}(X_t - m(t))(X_s - m(s))$, i.e., for given m and R there is only one (up to versions) Gaussian process with mean function m and covariance function R , where R must be symmetric and positive semidefinite. This is a consequence of Kolmogoroff's existence theorem. If $m(t) \equiv 0$, the process will be called centered. A centered Gaussian process induces a (pseudo) metric on its index set T by $d_X(t,s) := \sqrt{\mathbb{E}(X_t - X_s)^2}$, where $s, t \in T$.
The fractional Brownian motion (fBm) B_H with index $0 < H < 1$ is a centered Gaussian process on $\mathbb{R}^+$ with covariance function $R(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$ for $s, t \in \mathbb{R}^+$. Especially, $B_H(t)$ is normally distributed with mean 0 and variance

t^{2H}. Consult [22] to get further information about Gaussian processes. More details about the fractional Brownian motion can be found in [29].

1.3.2 Lévy processes

We use the definition given in [33], section 1. A real-valued stochastic process $X = (X_t)_{t\in\mathbb{R}^+}$ will be called a Lévy process, if

1. for any choice of $n \geqslant 1$ and $0 \leqslant t_0 < t_1 < \cdots < t_n$, the random numbers $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent,
2. $X_0 = 0$ a.s.,
3. the distribution of $X_{t+s} - X_s$ does not depend on s,
4. X is continuous in probability, i.e., for every $t \geqslant 0$ and $\varepsilon > 0$ we have $\lim\limits_{s\to t} \mathbb{P}\{|X_s - X_t| > \varepsilon\} = 0$,
5. there is some $\Omega_0 \in \mathcal{A}$ with $\mathbb{P}\{\Omega_0\} = 1$ such that, for every $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous in $t \geqslant 0$ and has left limits in $t > 0$.

A Lévy process is parametrized by a characteristic triplet. The link is as follows.

Theorem 1.14 ([33], sections 7.,8.)**.** *Let $X = (X_t)_{t\in\mathbb{R}^+}$ be a Lévy process. Then we can find $a \in \mathbb{R}$, $\sigma^2 \geqslant 0$ and a measure μ on $\mathbb{R}\setminus\{0\}$ with*

$$\int\limits_{\mathbb{R}\setminus\{0\}} \min\{x^2, 1\} d\mu(x) < \infty$$

such that

$$\mathbb{E}\exp(izX_t) = \exp(t(iaz - \frac{1}{2}\sigma^2 z^2 + \int\limits_{\mathbb{R}\setminus\{0\}} (e^{izx} - 1 - izx\mathbb{1}_{\{|x|<1\}}) d\mu(x))) \tag{3}$$

$$=: \exp(t\Psi(z))$$

is true for every $t \in \mathbb{R}$. Conversely, if (a, σ^2, μ) fulfills the stated conditions, then there will be a Lévy process X such that (3) is satisfied. The function Ψ is called characteristic exponent of the Lévy process X.

A subordinator is a Lévy process which is almost surely increasing. If X is a subordinator, its Laplace exponent $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ will be defined by the equation

$$\mathbb{E}e^{-xA_t} = e^{-t\Phi(x)}.$$

A Lévy process will be said to be α- stable with parameters (a, β, σ) if the characteristic exponent fulfills

$$\Psi(z) = \begin{cases} iaz - \sigma^{\alpha}|z|^{\alpha}(1 - i\beta(\mathrm{sgn}z)\tan\frac{\pi\alpha}{2}), \text{ if } \alpha \neq 1 \\ iaz - \sigma|z|(1 + i\beta\frac{2}{\pi}(\mathrm{sgn}z)\log|z|), \text{ if } \alpha = 1 \end{cases}$$

for some $\alpha \in (0, 2]$, $\sigma \geqslant 0$, $|\beta| \leqslant 1$ and $a \in \mathbb{R}$.

1.3.3 Random sequences

Let (α_n) be a sequence of real numbers and let (ξ_n) be a sequence of independent and identically distributed random numbers explained on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then $X := (\alpha_n \xi_n)_{n \in \mathbb{N}}$ is a stochastic process. We call X a standard random sequence.
Assume the existence of a further sequence (U_n) of random numbers, where the components are independent of each other and independent of (ξ_n) and it holds $\mathbb{P}\{U_n \in \{0, 1\}\} = 1$. Then $Y := (\alpha_n U_n \xi_n)_{n \in \mathbb{N}}$ is a stochastic process. We call Y a random sequence with deletion factor.

1.4 Further objects, concepts and properties

1.4.1 Convex sequences

Let (α_n) be a strictly decreasing sequence of positive real numbers converging to zero. The sequence will be called convex if $\alpha_n - \alpha_{n+1} \leqslant \alpha_{n-1} - \alpha_n$ is valid for each $n \geqslant 2$. It will be called strictly convex if the inequality is strict. Examples of convex sequences are given by $(g(n))$, where $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a convex and strictly decreasing function with $\lim_{x \to \infty} g(x) = 0$. The opposite is also true: For each (strictly) convex sequence (α_n) there is a differentiable, (strictly) convex and strictly decreasing function
$g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\lim_{x \to \infty} g(x) = 0$ such that $g(n) = \alpha_n$ (cf. [31], page 395). We call g associated function to the sequence (α_n). We denote the collection of all strictly convex sequences by $\mathcal{C}$ and for $\gamma > 0$ define

$$\mathcal{C}^{\gamma} := \{(\alpha_n) \in \mathcal{C} : (\alpha_n^{\gamma}) \in \mathcal{C}\} \text{ and } \mathcal{C}_{\gamma} := \{(\alpha_n) \in \mathcal{C} : (\gamma\alpha_n) \in \mathcal{C}\}.$$

By definition, the relations $\mathcal{C}_{\gamma} \subseteq \mathcal{C}$ and $\mathcal{C}^{\gamma} \subseteq \mathcal{C}$ are true. The respective converse relations are considered in the following

Lemma 1.15. *Let $\gamma > 0$. Then*

1. *$\mathcal{C}_{\gamma} \supseteq \mathcal{C}$.*
2. *$\mathcal{C}^{\gamma} \supseteq \mathcal{C}$ if $\gamma > 1$.*

3. *If* $\gamma < 1$, *then*

$$\{(g(n)) \in \mathcal{C} : g(x) \cdot g^{''}(x) - (1-\gamma) \cdot (g^{'}(x))^2 > 0 \ \forall x \geqslant 1\} \subseteq \mathcal{C}^{\gamma}.$$

4. *If* $\gamma < 1$, *then* $\{(\alpha_n) \in \mathcal{C} : \alpha_n \leqslant \sqrt{\alpha_{n-1}\alpha_{n+1}} \ \forall n > 1\} \subseteq \mathcal{C}^{\gamma}$.

Proof.

1. The result follows directly from the definition.

2.-4. Let $(\alpha_n) \in \mathcal{C}$ be associated with g. The function g is differentiable, strictly convex and strictly decreasing, i.e., its first derivative is strictly negative and its second derivative is strictly positive. Because $x \longmapsto x^{\gamma}$ is differentiable, the same is true for g^{γ}. The first derivative of g^{γ} is given by $(g^{\gamma})' = \gamma \cdot g^{\gamma-1} \cdot g'$ which is strictly negative. The second derivative is given by

$$(g^{\gamma})^{''} = \gamma((\gamma-1)g^{\gamma-2} \cdot (g^{'})^2 + g^{\gamma-1} \cdot g^{''}) = \gamma g^{\gamma-2}((\gamma-1)(g^{'})^2 + g \cdot g^{''}).$$

Obviously, the last term is strictly positive for $\gamma > 1$ and for $\gamma < 1$ the positivity follows by the additional condition. Consequently, $g^{\gamma} : \mathbb{R}^+ \to \mathbb{R}^+$ is differentiable, strictly convex and strictly decreasing and $g^{\gamma}(n) = \alpha_n^{\gamma}$ holds for all $n \in \mathbb{N}$. We may conclude that (α_n^{γ}) is strictly convex with associated function g^{γ}.

The sufficiency of the second condition can be seen as follows. The assumption $\alpha_n \leqslant \sqrt{\alpha_{n-1}\alpha_{n+1}}$ and the inequality between geometric and arithmetic mean imply

$$\alpha_n^{\gamma} \leqslant \sqrt{\alpha_{n-1}^{\gamma}\alpha_{n+1}^{\gamma}} < \frac{\alpha_{n-1}^{\gamma} + \alpha_{n+1}^{\gamma}}{2}.$$

In other words, $\alpha_n^{\gamma} - \alpha_{n+1}^{\gamma} < \alpha_{n-1}^{\gamma} - \alpha_n^{\gamma}$, which is the convexity condition.

□

Remark 1.16. *Because of* $\mathcal{C} = \mathcal{C}_{\gamma}$, *we can restrict ourselves to sequences with values in* $(0,1]$, *if necessary.*

Corollary 1.17. *Let* $\gamma > 0$. *The following mappings are associated functions of strictly convex decreasing sequences staying strictly convex under* $x \longmapsto x^{\gamma}$.

- $g(x) = x^{-\beta}$, *where* $\beta > 0$ *(polynomial sequence).*
- $g(x) = (1 + \log x)^{-c}$, *where* $c > 0$ *(logarithmic sequence).*
- $g(x) = x^{-\beta}(1 + \log x)^{-c}$, *where* $\beta, c > 0$ *(polynomial sequence with logarithmic correction).*
- $g(x) = q^x$, *where* $q \in (0,1)$ *(exponential sequence).*

- *With restrictions:* $g(x) = q^{x^\alpha}$, *where* $q \in (0,1)$ *and* $\alpha > 0$ *(generalized exponential sequence).*

Proof. Compute the first and second derivatives. The restriction in the generalized exponential case means that it could be necessary to start the sequence in some n_0 and not in 1. The second derivative is

$$\log q \cdot \alpha \cdot x^{\alpha-2} \cdot q^{x^\alpha}[\alpha - 1 + \alpha x^\alpha \log q],$$

which is larger than 0 if and only if $x^\alpha > \frac{1-\alpha}{\alpha \log q}$. The inequality will always be fulfilled if $\alpha < 1$, but for $\alpha > 1$ it is only true for $x \geqslant x(\alpha, q)$. The transformation $x \longmapsto x^\gamma$ changes only the q and the same argumentation can be applied. □

Lemma 1.18. *Let* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *be strictly convex, strictly decreasing towards 0 and differentiable. Then* $\lim_{x\to\infty} x(-g'(x)) = 0$.

Proof. Without loss of generality, we can choose $x \geqslant 2$. We know that g is associated with $(g(n)) \in \mathcal{C}$. First, we show that $(x-1)(-g'(2x))$ converges to 0. This claim follows from the following chain of estimates, where we use the definition of convexity and the mean value theorem:

$$\begin{aligned}(x-1)(-g'(2x)) &\leqslant \lfloor x\rfloor(-g'(2\lfloor x\rfloor)) \leqslant \lfloor x\rfloor(g(2\lfloor x\rfloor - 1) - g(2\lfloor x\rfloor)) \\ &\leqslant \sum_{k=\lfloor x\rfloor}^{2\lfloor x\rfloor-1}(g(k) - g(k+1)) = g(\lfloor x\rfloor) - g(2\lfloor x\rfloor) \leqslant g(\lfloor x\rfloor).\end{aligned}$$

The last expression converges to 0, thus, $(x-1)(-g'(2x))$ goes to 0, too. Because $x - 1 \geqslant \frac{x}{2} = \frac{2x}{4}$ is true, the result follows by well-known convergence rules. □

Convex sequences are interesting because their covering rate is known.

Proposition 1.19 ([31], Lemma 3, Corollary 1)**.** *Let* $(\alpha_n) \in \mathcal{C}$ *with associated function* g. *Then*

$$N((\alpha_n), |\cdot|, \varepsilon) \approx \max\{(-g')^{-1}(\varepsilon), \frac{g((-g')^{-1}(\varepsilon))}{\varepsilon}\} \approx (-g')^{-1}(\varepsilon) + \frac{g((-g')^{-1}(\varepsilon))}{\varepsilon}$$

and, consequently,

$$\overline{\dim}_E(\alpha_n) = \limsup_{\varepsilon\to 0} \frac{\log\max\{(-g')^{-1}(\varepsilon), \frac{g((-g')^{-1}(\varepsilon))}{\varepsilon}\}}{|\log\varepsilon|} = \limsup_{x\to\infty} \frac{\log\max\{x, \frac{g(x)}{-g'(x)}\}}{|\log(-g'(x))|},$$

$$\underline{\dim}_E(\alpha_n) = \liminf_{\varepsilon\to 0} \frac{\log\max\{(-g')^{-1}(\varepsilon), \frac{g((-g')^{-1}(\varepsilon))}{\varepsilon}\}}{|\log\varepsilon|} = \liminf_{x\to\infty} \frac{\log\max\{x, \frac{g(x)}{-g'(x)}\}}{|\log(-g'(x))|}.$$

An improved dimension formula is also available.

Proposition 1.20 ([32], Theorem 2)**.** *Let $(\alpha_n) \in \mathcal{C}$ with associated function g. Then*

$$\dim_E(\alpha_n) = \lim_{x\to\infty} \frac{\log x}{\log \frac{1}{-g'(x)}}$$

if the limit exists.

Consider two decreasing null sequences of real numbers (α_n) and (β_n). In general, it is not true that the relation $\alpha_n \leqslant \beta_n$ being true for (almost) all n implies

$$N((\alpha_n), |\cdot|, \varepsilon) \preccurlyeq N((\beta_n), |\cdot|, \varepsilon).$$

In fact, we found the following result:

Proposition 1.21 ([30], Theorem 2)**.** *Let (α_n) be a strictly decreasing null sequence and let $\alpha \in [0,1]$. Then one can find another strictly decreasing null sequence (β_n) such that the cardinality of set $\{k : \alpha_k > \beta_k\}$ is finite and $\overline{\dim}_E(\beta_n) = \alpha$.*

If strictly convex sequences are involved, the situation will not be that complicated.

Lemma 1.22. *Let (α_n) be a decreasing sequence of positive real numbers and let $(\beta_n) \in \mathcal{C}$. Assume that there are some n_0 and some positive constant c_1 such that $\alpha_n \leqslant c_1\beta_n$ is valid for all $n \geqslant n_0$. Then*

$$N((\alpha_n), |\cdot|, \varepsilon) \preccurlyeq N((\beta_n), |\cdot|, \varepsilon).$$

Proof. Let $N-1 \geqslant n_0$ and $\varepsilon > 0$. Let h be associated to (β_n). Then

$$N((\alpha_n), |\cdot|, \varepsilon) \leqslant N((\alpha_n)_{n=N}^{\infty}, |\cdot|, \varepsilon) + N((\alpha_n)_{n=1}^{N-1}, |\cdot|, \varepsilon) \leqslant \frac{\alpha_N}{2\varepsilon} + 1 + N - 1$$

$$\leqslant \frac{c_1\beta_N}{\varepsilon} + N \leqslant c_2(\frac{h(N)}{\varepsilon} + N).$$

The choice $N = \lceil (-h')^{-1}(\varepsilon) \rceil$ yields $N((\alpha_n), |\cdot|, \varepsilon) \leqslant c_2(\frac{h((-h')^{-1}(\varepsilon))}{\varepsilon} + (-h')^{-1}(\varepsilon) + 1)$ and because (β_n) is convex, we know $\frac{h((-h')^{-1}(\varepsilon))}{\varepsilon} + (-h')^{-1}(\varepsilon) \approx N((\beta_n), |\cdot|, \varepsilon)$, what finishes the proof. □

1.4.2 Dimension profiles

We adapt the presentations given in [19] and [12].

1.4.2.1 Box dimension profiles

Given a finite Borel measure μ on $\mathbb{R}$ and a number $s \in (0, \infty]$, define

$$F_s^\mu(x, r) := \int_{\mathbb{R}} w_s(\frac{x-y}{r}) d\mu(y),$$

where

$$w_s(x) = \begin{cases} \min(1, |x|^{-s}) : & s \in (0, \infty), x \in \mathbb{R} \\ \mathbb{1}_{[-1,1]}(x) : & s = \infty, x \in \mathbb{R}. \end{cases}$$

Given a set $A \subseteq \mathbb{R}$, define $\mathcal{P}^F(A)$ to be the collection of all probability measures that are supported on a finite number of points in A and define

$$Z_s(r, A) := \inf_{\mu \in \mathcal{P}^F(A)} \int_{\mathbb{R}} F_s^\mu(x, r) d\mu(x).$$

The s-dimensional upper box dimension profile of A is given by

$$B - \overline{\dim}^s A := \limsup_{r \to 0} \frac{\log Z_s(r, A)}{\log r}. \tag{4}$$

The lower box dimension profile is defined by using the lower limit in (4).
The relation to the entropy dimension is as follows:

Lemma 1.23 ([12], Prop. 8)**.** *Let $A \subseteq \mathbb{R}$ and $s \geqslant 1$. Then*

$$\overline{\dim}_E A = B - \overline{\dim}^s A \ , \ \underline{\dim}_E A = B - \underline{\dim}^s A.$$

A further tool is given by general dimension profiles as introduced in [17].
We assume that we have a time-continuous stochastic process $X = (X_t)_{t \in \mathbb{R}^+}$ or a time-discrete process $X = (X_n)_{n \in \mathbb{N}}$. In the first case, let $\mathcal{A}_T$ be the Borel σ-Algebra on $\mathbb{R}^+$ and in the second case, let $\mathcal{A}_T$ be the power set of $\mathbb{N}$. Given $\varepsilon > 0$, assume that the sets $\{(\omega, t, s) : |X_t(\omega) - X_s(\omega)| \leqslant \varepsilon\}$ are elements of $\mathcal{A} \otimes \mathcal{A}_T \otimes \mathcal{A}_T$.
Now, let E be a bounded Borel subset of $\mathbb{R}^+$ or let $E \subseteq \mathbb{N}$ and let $t, s \in E$. Denote by $\mathcal{P}(E)$ the collection of all Borel probability measures on $\mathbb{R}^+$ with $\mu(E) = 1$ or denote by $\mathcal{P}(E)$ the collection of all discrete probability measures on $\mathbb{N}$ with

$\mu(E) = 1$. Then, by Fubini's theorem,

$$F(E, \varepsilon, \mu) := \int_E \int_E \mathbb{P}\{|X_t - X_s| \leqslant \varepsilon\} \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) \tag{5}$$

is well-defined. The box dimension profile $B - \overline{\dim} E$ of E with respect to the kernel $\kappa(\varepsilon, t, s) := \mathbb{P}\{|X_t - X_s| \leqslant \varepsilon\}$ is defined as follows:

$$B - \overline{\dim} E := \sup\{\alpha > 0 : \liminf_{\varepsilon \to 0} \varepsilon^{-\alpha} \inf_{\mu \in \mathcal{P}(E)} F(E, \varepsilon, \mu) = 0\}. \tag{6}$$

We use the abbreviation $F(E, \varepsilon) := \inf_{\mu \in \mathcal{P}(E)} F(E, \varepsilon, \mu)$.

Remark 1.24. *Our notation is different from the notation in [17]. The authors there use a kernel κ which is defined for a single time parameter t. This is possible because they consider Lévy processes and for this type of processes the distribution of $|X_t - X_s|$ only depends on $|t - s|$. A two parameter kernel allows to apply the methods to further kinds of processes. Furthermore, we use $\mathbb{P}\{|X_t - X_s| \leqslant \varepsilon\}$ instead of $\mathbb{P}\{|X_t - X_s| < \varepsilon\}$. Before we took note of [17], the expression $F(E, \varepsilon)$ was shown to us by F. Aurzada and S. Dereich as a possibility to compute lower bounds in $\overset{\mathbb{P}}{\succcurlyeq}$ (cf. section 3).*

The product measurability condition does not pose a problem for our purposes. In fact, it is well-known (cf. [8], Theorem 6.2.3 for a proof of the case $T = [0, 1]$, [9], Theorem 2.6) that a process $X = (X_t)_{t \in T}$ (where T is a separable topological space), considered as a random variable mapping from the product space $(\Omega \times T, \mathcal{A} \otimes \mathcal{A}_T)$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, will have a measurable version if the process is continuous in probability. Here, $\mathcal{A}_T$ is the Borel σ-Algebra which is induced by the topology on T. In particular, it follows that every process with countable index set has a measurable version because we can choose the discrete topology in this case.

Assume that we have given a real-valued time-continuous process $X = (X_t)_{t \in \mathbb{R}^+}$, which is continuous in probability (we choose the absolute value metric on $\mathbb{R}$). Define a map $Y : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $(\omega, s, t) \longmapsto X_s(\omega) - X_t(\omega)$. Because of

$$\mathbb{P}\{\omega : |Y(\omega, s, t) - Y(\omega, u, v)| > \varepsilon\}$$

$$\leqslant \mathbb{P}\{\omega : |X_s(\omega) - X_u(\omega)| + |X_t(\omega) - X_v(\omega)| > \varepsilon\}$$
$$\leqslant \mathbb{P}\left\{\omega : |X_s(\omega) - X_u(\omega)| > \frac{\varepsilon}{2}\right\} + \mathbb{P}\left\{\omega : |X_t(\omega) - X_v(\omega)| > \frac{\varepsilon}{2}\right\},$$

the process Y is also continuous in probability. Thus, Y is $\Omega \otimes \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^+) - \mathcal{B}(\mathbb{R})$-measurable. By $\mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^+) = \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^+)$ and the continuity of $f(x) = |x|$, it follows that the sets $\{(\omega, s, t) : |X_s(\omega) - X_t(\omega)| \leqslant \varepsilon\}$ are elements of the product sigma-algebra $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^+)$. The same can be done for time-discrete processes because the index set $\mathbb{N}^2$ is again discrete and $2^{\mathbb{N}^2} = 2^{\mathbb{N}} \otimes 2^{\mathbb{N}}$.

If $X = (X_n)_{n\in\mathbb{N}}$ is discrete and f is $\mathcal{B}(\mathbb{R}^2)$–$\mathcal{B}(\mathbb{R})$-measurable (here: $f(x, y) = |x-y|$), the representation

$$\{(\omega, m, n) : f(X_m(\omega), X_n(\omega)) \in A\} = \bigcup_{k=1}^{\infty}\bigcup_{l=1}^{\infty}(X_k, X_l)^{-1}(f^{-1}(A)) \times \{k\} \times \{l\}$$

will yield the desired measurability, too.
The time-continuous processes considered in this thesis are continuous in probability, which is the reason why we can use the function defined by (5) without having measurability troubles.

1.4.2.2 Packing dimension profiles

The s- dimensional packing dimension profile of $A \subseteq \mathbb{R}$ is given by

$$P - \dim^s A := \inf\{\sup_{k\,\geqslant\,1} B - \overline{\dim}^s A_k : A \subseteq \bigcup_{k=1}^{\infty} A_k\}.$$

The relation to the packing dimension is as follows:

Proposition 1.25 ([12], Cor. 27). *Let $A \subseteq \mathbb{R}$ and $s \geqslant 1$. Then*

$$P - \dim^s A = \dim_P A.$$

For time-continuous processes the general packing dimension profile (with respect to the kernel κ) is given by

$$P - \dim A := \inf\{\sup_{n\,\geqslant\,1} B - \overline{\dim} A_n : A \subseteq \bigcup_{n=1}^{\infty} A_n, (A_n) \text{ bounded real Borel sets }\}.$$

1.4.3 Self-similarity

Self-similarity is a property which is interesting for sets and processes. We briefly introduce the necessary terminology. Additional information can be found in [11] (sets) and [33] and [34] (processes).

1.4.3.1 Self-similar sets

Let $S_1, \ldots, S_N$ be similarity mappings from $[0, 1]$ to $[0, 1]$, i.e., we have

$$|S_i(x) - S_i(y)| = r_i|x - y|$$

for some $r_i \in (0, 1)$, $i = 1, \ldots, N$ and for all $x, y \in [0, 1]$. Given a compact set $K \subseteq [0, 1]$, define $S(K) = \bigcup_{i=1}^{N} S_i(K)$ and $S^{k+1}(K) = S(S^k(K))$ for $k \geqslant 1$, where $S^1(K) = S(K)$.

Theorem 1.26 ([11], Satz 9.1)**.** *There is a unique compact set* $\emptyset \neq F \subseteq [0,1]$ *such that* $F = \bigcup_{i=1}^{N} S_i(F)$ *and* $F = \bigcap_{k=1}^{\infty} S^k([0,1])$.

The sets generated as above are called strictly self-similar sets. We can compute their dimensions if we make a further assumption.

Theorem 1.27 ([11], Satz 9.2)**.** *Let* F *be generated by* $S_1, \ldots, S_N$ *with contraction factors* $r_1, \ldots, r_N$ *and assume that the similarities* $(S_i)_{i=1}^N$ *fulfill the open set condition, i.e., there is a nonempty bounded open set* V *so that* $\bigcup_{i=1}^{N} S_i(V) \subseteq V$ *and the union is disjoint. Then* $\dim_H F = \dim_P F = \dim_E F$ *and the common value is given by the unique number* D *defined by* $\sum_{i=1}^{N} r_i^D = 1$.

Remark 1.28. *Given the situation of Theorem 1.27, it is known that*

$$N(F, |\cdot|, \varepsilon) \approx \varepsilon^{-D}$$

is true (cf. [21], Theorem 1).

A well-known example is the (middle third) Cantor set. Here, $N = 2$ and the similarities are given by $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{2}{3} + \frac{1}{3}x$. The open set condition is fulfilled with $V = (0, 1)$ and the dimension D is equal to $\frac{\log 2}{\log 3}$.

1.4.3.2 Self-similar processes

A stochastic process $X = (X_t)_{t \geqslant 0}$ will be called self-similar if there is a positive real number γ such that for every positive real number a, the processes X and $Y = (a^{-\gamma} X_{at})_{t \geqslant 0}$ are equal in distribution, i.e., they have the same finite dimensional distributions. The number γ is called self-similarity index.

Examples: The fractional Brownian motion with index H is self-similar with $\gamma = H$. α-stable processes with parameters $(0, \beta, \sigma)$ are self-similar for $\alpha \neq 1$. The additional condition $\beta = 0$ is required for $\alpha = 1$. In both cases, the self-similarity index is given by $\gamma = \frac{1}{\alpha}$. Self-similar α-stable processes are called strictly α-stable processes.

1.4.4 Hölder continuity

A function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ will be called locally α-Hölder continuous if for every $K > 0$ there is a positive constant $c(K)$ such that $|f(x) - f(y)| \leqslant c(K)|x - y|^\alpha$ is valid for all $x, y \in [0, K]$.

An example: Given $0 < \varepsilon < H$, the fractional Brownian motion of index H is almost everywhere locally $(H - \varepsilon)$- Hölder continuous. This is a well-known result. A short proof of this claim can be found in [36] (Proposition 3.2). The cited source

mentions the paper [14], where the question of Hölder continuity has been considered earlier, but indirectly.
Hölder continuity is useful because it can be used to get a bound for the image dimension.

Lemma 1.29. *Let $T \subseteq \mathbb{R}^+$ be bounded and assume that $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is locally α-Hölder continuous. Then*

$$N(f(T), |\cdot|, \varepsilon) \leqslant N(T, |\cdot|, c_1 \varepsilon^{\frac{1}{\alpha}}) \quad \text{and} \quad \overline{\dim}_E f(T) \leqslant \frac{\overline{\dim}_E T}{\alpha}.$$

Proof. The boundedness of T implies the existence of some $K > 0$ such that $T \subseteq [0, K]$. Then there is some constant c such that $|f(x) - f(y)| \leqslant c|x - y|^\alpha$ for $x, y \in [0, K]$. Assume $n = N(T, |\cdot|, \varepsilon)$. Then there are $x_1, \ldots, x_n \in \mathbb{R}^+$ such that $T \subseteq \bigcup_{i=1}^{n} B_\varepsilon(x_i)$. Consequently, $\bigcup_{i=1}^{n} B_{c\varepsilon^\alpha}(f(x_i))$ covers $f(T)$. In fact, for $t \in T$ we can find a number i_0 such that $|t - x_{i_0}| \leqslant \varepsilon$. Then $|f(t) - f(x_{i_0})| \leqslant c|t - x_{i_0}|^\alpha \leqslant c\varepsilon^\alpha$. It follows

$$N(f(T), |\cdot|, c\varepsilon^\alpha) \leqslant n = N(T, |\cdot|, \varepsilon),$$

what can be rewritten to get the first part of the assertion. It remains to prove the bound for the image dimension. We have

$$\overline{\dim}_E f(T) = \limsup_{\varepsilon \to 0} \frac{\log N(f(T), |\cdot|, \varepsilon)}{\log \varepsilon^{-1}} \leqslant \limsup_{\varepsilon \to 0} \frac{\log N(T, |\cdot|, \frac{\varepsilon^{\frac{1}{\alpha}}}{c^{\frac{1}{\alpha}}})}{\log \varepsilon^{-1}}$$

$$= \limsup_{\varepsilon \to 0} \frac{\log N(T, |\cdot|, \frac{\varepsilon^{\frac{1}{\alpha}}}{c^{\frac{1}{\alpha}}})}{\alpha \log(\frac{\varepsilon}{c})^{-\frac{1}{\alpha}} - \log c} = \frac{\overline{\dim}_E T}{\alpha},$$

where the additional log term vanishes in the limit. □

A similar result holds for the Hausdorff dimension (cf. [11], Behauptung 2.3).

1.4.5 Some remarks on the measurability of $N(X(T), |\cdot|, \varepsilon)$

In order to compute probabilities such as $\mathbb{P}\{N(X(T), |\cdot|, \varepsilon) > f(\varepsilon)\}$, we must ensure that the sets $\{\omega \in \Omega : N(X(T), |\cdot|, \varepsilon)(\omega) = n\}, n \in \mathbb{N}, \varepsilon > 0$ are measurable, i.e., they must be elements of $\mathcal{A}$. In other words, for every $\varepsilon > 0$ $N(X(T), |\cdot|, \varepsilon)$ must be a random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{N} \cup \{+\infty\}$.
Unfortunately, this problem is hardly discussed in the available literature. To justify our studies, we briefly consider some important cases. During the following sections, we do not mention these problems any longer. If the measurability of some covering number should not be justified by the following lemmas, we will tacitly assume that the measurability is ensured.

If the index set T is countable, we can make use of the fact that the rational numbers are dense in $\mathbb{R}$. This ensures the measurability in the case of random sequences and in the case of time-continuous processes considered on a countable index set.

Lemma 1.30. *Let A be a nonempty subset of $\mathbb{R}$ and let $\varepsilon > 0$. Then A can be covered by a 2ε-interval with real center if and only if for every $m \in \mathbb{N}$, there is a rational number q_m such that A can be covered by $B_{\varepsilon+\frac{1}{m}}(q_m)$.*

Proof.

(i) Assume that A can be covered by $B_\varepsilon(x)$, where $x \in \mathbb{R}$. Let $m \in \mathbb{N}$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find a rational number q_m with $|x - q_m| \leqslant \frac{1}{m}$. The triangle inequality gives $B_\varepsilon(x) \subseteq B_{\varepsilon+\frac{1}{m}}(q_m)$. This proves the first part of the claim.

(ii) Assume that we can find coverings of A as claimed in the lemma. Define the set $I := \bigcap_{m=1}^{\infty} B_{\varepsilon+\frac{1}{m}}(q_m)$. Then I is a closed interval which covers A. The monotonicity of the Lebesgue measure yields that the diameter of I is at most 2ε. If the diameter is strictly smaller than 2ε, I can be enlarged to get an interval of length 2ε which contains I and A. This proves the second part of the claim.

□

Lemma 1.31. *Let T be countable and $\varepsilon > 0$. If $X = (X_t)_{t\in T}$ is a real-valued stochastic process, then $N(X(T), |\cdot|, \varepsilon)$ is a random number.*

Proof. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. We want to show that $\{\omega \in \Omega : N(X(T)(\omega), |\cdot|, \varepsilon) \leqslant n\}$ is measurable. Let $q_1, q_2, \ldots, q_n$ be real numbers and set $q := (q_1, \ldots, q_n)$ and $A(q, \varepsilon) := \bigcup_{j=1}^{n} B_\varepsilon(q_j)$. For $t \in T$ and $q \in \mathbb{R}^n$ define

$$B(t, q, \varepsilon) := \{\omega \in \Omega : X_t(\omega) \in A(q, \varepsilon)\} = X_t^{-1}(A(q, \varepsilon)).$$

Thus, $X(T)(\omega) \subseteq A(q, \varepsilon)$, if and only if $\omega \in \bigcap_{t\in T} B(t, q, \varepsilon)$. Together with Lemma 1.30, we get

$$\{\omega \in \Omega : N(X(T)(\omega), |\cdot|, \varepsilon) \leqslant n\} = \bigcup_{q\in\mathbb{R}^n} \{\omega : X(T)(\omega) \subseteq A(q, \varepsilon)\}$$

$$= \bigcap_{m=1}^{\infty} \bigcup_{q\in\mathbb{Q}^n} \{\omega : X(T)(\omega) \subseteq A(q, \varepsilon + \frac{1}{m})\} = \bigcap_{m=1}^{\infty} \bigcup_{q\in\mathbb{Q}^n} \bigcap_{t\in T} B(t, q, \varepsilon + \frac{1}{m})$$

$$= \bigcap_{m=1}^{\infty} \bigcup_{q\in\mathbb{Q}^n} \bigcap_{t\in T} X_t^{-1}(A(q, \varepsilon + \frac{1}{m})).$$

Because X_t is a random variable, $X_t^{-1}(A(q, \varepsilon + \frac{1}{m}))$ is measurable. The unions and intersections are countable, showing that $\{\omega \in \Omega : N(X(T)(\omega), |\cdot|, \varepsilon) \leqslant n\}$ is measurable, too. $\square$

The situation will also be clear if the process X is almost surely continuous and T is separable. This result ensures the measurability for the fractional Brownian motion with arbitrary (positive) real index set.

Lemma 1.32. *Assume that X is almost surely continuous and let $T \subseteq \mathbb{R}^+$. If $\varepsilon > 0$, then $N(X(T), |\cdot|, \varepsilon)$ is a random number.*

Proof. Assume the existence of a measurable set N with $\mathbb{P}\{N\} = 0$ such that for $\omega \in N^c$, the map $t \mapsto X_t(\omega)$ is $(\mathbb{R}^+, |\cdot|) - (\mathbb{R}, |\cdot|)$-continuous. Furthermore, T is separable because it is a subset of the separable metric space $(\mathbb{R}^+, |\cdot|)$, i.e., there is a set $M = \{s_1, s_2, \dots\} \subseteq T$ and for every $t \in T$ there is a sequence $(t_l) \subseteq M$ with $\lim_{l \to \infty} |t - t_l| = 0$. First, we can repeat the computations from the last lemma and get

$$\{\omega \in \Omega : N(X(T)(\omega), |\cdot|, \varepsilon) \leqslant n\} = \bigcap_{m=1}^{\infty} \bigcup_{q \in \mathbb{Q}^n} \bigcap_{t \in T} B(t, q, \varepsilon + \frac{1}{m}).$$

Now, our task is to use the additional assumptions on T and X to make the second intersection countable, what would yield the measurability. For every m, let us show

$$N^c \cap \bigcap_{t \in T} B(t, q, \varepsilon + \frac{1}{m}) = N^c \cap \bigcap_{k=1}^{\infty} B(s_k, q, \varepsilon + \frac{1}{m}).$$

The inclusion $N^c \cap \bigcap_{t \in T} B(t, q, \varepsilon + \frac{1}{m}) \subseteq N^c \cap \bigcap_{k=1}^{\infty} B(s_k, q, \varepsilon + \frac{1}{m})$ is obvious, it remains to show the remaining direction.
Let $\omega \in N^c \cap \bigcap_{k=1}^{\infty} B(s_k, q, \varepsilon + \frac{1}{m})$, meaning $\min_{1 \leqslant j \leqslant n} |X_{s_k}(\omega) - q_j| \leqslant \varepsilon + \frac{1}{m}$ for every $k \in \mathbb{N}$.
Let $t \in T$ and choose a sequence $(t_l) \subseteq M$ with $\lim_{l \to \infty} |t_l - t| = 0$.
The choice of ω yields $\min_{1 \leqslant j \leqslant n} |X_{t_l}(\omega) - q_j| \leqslant \varepsilon + \frac{1}{m}$ for every l.
The continuity property implies that we have $\min_{1 \leqslant j \leqslant n} |X_t(\omega) - q_j| \leqslant \varepsilon + \frac{1}{m}$, too. This can be seen as follows. Assume $\min_{1 \leqslant j \leqslant n} |X_t(\omega) - q_j| > \varepsilon + \frac{1}{m}$ and choose $l_{\varepsilon,m}$ so that

$$|X_{t_l}(\omega) - X_t(\omega)| < \frac{\min_{1 \leqslant j \leqslant n} |X_t(\omega) - q_j| - \varepsilon - \frac{1}{m}}{2} \text{ for every } l \geqslant l_{\varepsilon,m}.$$

The choice is possible because of the continuity. But then

$$\min_{1 \leqslant j \leqslant n} |X_{t_l}(\omega) - q_j| \geqslant \min_{1 \leqslant j \leqslant n} |X_t(\omega) - q_j| - |X_{t_l}(\omega) - X_t(\omega)|$$

$$> \min_{1 \leqslant j \leqslant n} |X_t(\omega) - q_j| - \frac{\min_{1 \leqslant j \leqslant n} |X_t(\omega) - q_j|}{2} + \frac{\varepsilon + \frac{1}{m}}{2} > \frac{\varepsilon}{2} + \frac{1}{2m} + \frac{\varepsilon}{2} + \frac{1}{2m} = \varepsilon + \frac{1}{m}$$

for $l \geqslant l_{\varepsilon,m}$, what is a contradiction.

This shows $\omega \in N^c \cap B(t, q, \varepsilon + \frac{1}{m})$ and because t was arbitrary, it follows

$$\omega \in N^c \cap \bigcap_{t \in T} B(t, q, \varepsilon + \frac{1}{m}),$$

yielding $N^c \cap \bigcap_{t \in T} B(t, q, \varepsilon + \frac{1}{m}) = N^c \cap \bigcap_{k=1}^{\infty} B(s_k, q, \varepsilon + \frac{1}{m})$. The right- hand side is the countable intersection of measurable sets, thus $N^c \cap \bigcap_{t \in T} B(t, q, \varepsilon + \frac{1}{m})$ is measurable. The set N is measurable and by the assumption of completeness, this is also true for $N \cap \bigcap_{t \in T} B(t, q, \varepsilon + \frac{1}{m})$. Because of

$$\bigcap_{t \in T} B(t, q, \varepsilon + \frac{1}{m}) = (N \cap \bigcap_{t \in T} B(t, q, \varepsilon + \frac{1}{m})) \bigcup (N^c \cap \bigcap_{t \in T} B(t, q, \varepsilon + \frac{1}{m})),$$

$\bigcap_{t \in T} B(t, q, \varepsilon + \frac{1}{m})$ must be measurable, too. Finally, by executing the countable unions and intersections, the measurability of $\{\omega \in \Omega : N(X(T)(\omega), |\cdot|, \varepsilon) \leqslant n\}$ follows. □

We conclude this section with a result from the literature.

Lemma 1.33 ([19], Lemma 3.1)**.** *Let $T \subseteq \mathbb{R}^+$ be bounded, let $\varepsilon > 0$ and let $0 < H < 1$. Then $M(B_H(T), |\cdot|, \varepsilon)$ is a non-negative random number.*

2 Dimension results for stochastic processes

In this section, we present known and new dimension and covering results for some types of stochastic processes. The proofs of the cited results can be found by following the references. For reasons of clarity, the proofs of our own results are postponed to section 3.

2.1 Fractional Brownian motion

2.1.1 Hausdorff dimension

Let B_H be the fractional Brownian motion with index $0 < H < 1$. The case of Hausdorff dimension is clear:

Theorem 2.1 ([15], corollary following Theorem 18.1, Theorem 18.2)**.** *Let $A \subseteq \mathbb{R}^+$ be a Borel set. Then*

$$\dim_H B_H(A) = \min\{1, \frac{\dim_H A}{H}\}, \quad a.s. \tag{7}$$

2.1.2 Packing dimension

In [38] it is remarked that for a Borel set $A \subseteq \mathbb{R}^+$ with $\dim_H A = \dim_P A$ a similar relation as equation (7) is true (with $\dim_H$ substituted by $\dim_P$). In the same paper, the authors showed that this relation is not true in general.

Theorem 2.2 ([38], Corollary 4.1)**.** *For every $\beta \in (0,1)$ there is a compact set $E_\beta \subseteq [0,1]$ with $\dim_P E_\beta = \beta$ such that, with probability 1,*

$$\dim_P B_H(E_\beta) = \frac{\dim_P E_\beta}{H + (1-H)\dim_P E_\beta}. \tag{8}$$

The lower bound is valid in a more general form.

Theorem 2.3 ([38], Theorem 4.1)**.** *Let $A \subseteq \mathbb{R}^+$ be compact. Then*

$$\dim_P B_H(A) \geqslant \frac{\dim_P A}{H + (1-H)\dim_P A}, \quad a.s. \tag{9}$$

The Cantor set C fulfills $\dim_P C = \dim_H C = \frac{\log 2}{\log 3}$. Thus,

$$\dim_P B_H(C) = \min\{1, \frac{1}{H}\frac{\log 2}{\log 3}\}, a.s.$$

This example shows that, in general, inequality (9) is not sharp.

A further bound can be derived if one combines equation (7) with inequality (9).

Corollary 2.4 ([38], Remark 4.1). *Let $A \subseteq \mathbb{R}^+$ be a Borel set. Then*

$$\dim_P B_H(A) \geqslant \max\{\min\{1, \frac{\dim_H A}{H}\}, \frac{\dim_P A}{H + (1-H)\dim_P A}\}, \ a.s.$$

Another formula holds in terms of dimension profiles (cf. section 1.4.2.2).

Theorem 2.5 ([19], Theorem 1.2, [39] Theorem 4.1). *Let $A \subseteq \mathbb{R}^+$ be analytic. Then*

$$\dim_P B_H(A) = \frac{1}{H} \cdot P - \dim^H A, \ a.s.$$

2.1.3 Entropy dimension and covering numbers

The following results are new. Their proofs can be found in section 3.1. The third statement is inspired by [19].

Theorem 2.6. *Let $A \in \mathcal{D}$ and let $(\alpha_n) \in \mathcal{C}^H$ with associated function g. Then*

1. $\lim_{\varepsilon \to 0} \mathbb{P}\left\{N(B_H((\alpha_n)), |\cdot|, \varepsilon) > A(\varepsilon) N((\alpha_n^H), |\cdot|, \varepsilon)\right\} = 0.$
2. (a) *If* $\lim_{x\to\infty} \frac{-g'(x)\cdot x}{g(x)} = 0$ *or* $-g'(x)x \approx g(x)$, *then*

$$\lim_{\varepsilon \to 0} \mathbb{P}\left\{N(B_H((\alpha_n)), |\cdot|, \varepsilon) < \frac{1}{A(\varepsilon)} N((\alpha_n^H), |\cdot|, \varepsilon)\right\} = 0.$$

   (b) *If* $\lim_{x\to\infty} \frac{g(x)}{-g'(x)\cdot x} = 0$, *then*

$$\lim_{\varepsilon \to 0} \mathbb{P}\left\{N(B_H((\alpha_n)), |\cdot|, \varepsilon) < \frac{1}{A(\varepsilon)} N((\alpha_n), |\cdot|, \varepsilon)\right\} = 0.$$

3. (a) *Assume* $\lim_{x\to\infty} \frac{-g'(x)\cdot x}{g(x)} = 0$ *or* $-g'(x)x \approx g(x)$. *For almost all ω there is a sequence $(\varepsilon_k)(\omega)$ with* $\lim_{k\to\infty} \varepsilon_k(\omega) = 0$ *such that*

$$N(B_H((\alpha_n)), |\cdot|, \varepsilon_k)(\omega) \succcurlyeq \frac{1}{A(\varepsilon_k)} N((\alpha_n^H), |\cdot|, \varepsilon_k)(\omega)$$

   is valid. Consequently, $\overline{\dim}_E B_H((\alpha_n)) \geqslant \frac{\dim_E(\alpha_n)}{H+(1-H)\dim_E(\alpha_n)}$, *almost surely, if* $\dim_E(\alpha_n)$ *exists.*

   (b) *Assume* $\lim_{x\to\infty} \frac{g(x)}{-g'(x)\cdot x} = 0$. *For almost all ω there is a sequence $(\varepsilon_k)(\omega)$ with* $\lim_{k\to\infty} \varepsilon_k(\omega) = 0$ *such that*

$$N(B_H((\alpha_n)), |\cdot|, \varepsilon_k)(\omega) \succcurlyeq \frac{1}{A(\varepsilon_k)} N((\alpha_n), |\cdot|, \varepsilon_k)(\omega)$$

is valid. Consequently, $\overline{\dim}_E B_H((\alpha_n)) \geqslant \dim_E(\alpha_n)$*, almost surely, if* $\dim_E(\alpha_n)$ *exists.*

Remark 2.7. *Estimates in the sense of Theorem 2.6 can be derived for more general sets. To get a result in the sense of 1., assume that* T *is a countable index set having one cluster point* x*. Let* $\rho \in (0,1)$ *and assume that the cardinality of the set* $\{T \cap (x-\rho, x+\rho)\}$ *is bounded by a multiple (independent of* ρ*) of some function* $f(\rho)$*. This bound transforms into a bound for* $N(B_H(T), |\cdot|, \varepsilon)$*. In the case of the theorem, we have* $f(\rho) = g^{-1}(\rho)$*, where* g *is the associated function to* (α_n)*. These considerations can be expanded to the case where* T *is countable with a finite number of cluster points.*
To get a result in the sense of 2. or 3., a lower bound for $N(T, |\cdot|, \varepsilon)$ *is needed. The structure of the index set or its cardinality are not important in this case (cf. the discussion in section 4).*

Results as in Theorem 2.6 appear later in this thesis. To shorten the statements, we introduce some further notation.

Definition 2.8. Let $f : (0,1] \times \Omega \to \mathbb{R}^+$ and $g : (0,1] \times \Omega \to \mathbb{R}^+$ be two (in the second argument) measurable functions with $\lim_{\varepsilon\to 0} f(\varepsilon,\omega) = \lim_{\varepsilon \to 0} g(\varepsilon, \omega) = \infty$ being valid for almost all $\omega \in \Omega$. We say that $f \overset{\mathbb{P}}{\preccurlyeq} g$ if for every $A \in \mathcal{D}$,

$$\lim_{\varepsilon\to 0} \mathbb{P}\{\omega : f(\varepsilon,\omega) > A(\varepsilon) g(\varepsilon,\omega)\} = 0$$

will be valid. If $f \overset{\mathbb{P}}{\preccurlyeq} g$ and $g \overset{\mathbb{P}}{\preccurlyeq} f$, we will write $f \overset{\mathbb{P}}{\approx} g$ and $f \overset{\mathbb{P}}{\succcurlyeq} g$ means $g \overset{\mathbb{P}}{\preccurlyeq} f$.
Assume that $B = B(\omega) \subseteq \mathbb{R}$ is chosen in such a way that $B(\omega)$ is bounded for almost all ω (in our context, we have $B = X(T)$, where X is a process and T an index set). Then we define

$$\overline{\dim}_E^{\mathbb{P}} B = \inf\{\alpha \in (0,1] : N(B, |\cdot|, \varepsilon) \overset{\mathbb{P}}{\preccurlyeq} \varepsilon^{-\alpha}\}$$

and

$$\underline{\dim}_E^{\mathbb{P}} B = \sup\{\alpha \in (0,1] : N(B, |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} \varepsilon^{-\alpha}\}.$$

If both numbers concur, we will simply write $\dim_E^{\mathbb{P}}$.

Let us reformulate Theorem 2.6 in terms of the new definition.

Corollary 2.9. *Let* $(\alpha_n) \in \mathcal{C}^H$ *with associated function* g*.*
If $\lim_{x\to\infty} \frac{-g'(x)\cdot x}{g(x)} = 0$ *or* $-g'(x)x \approx g(x)$*, then*

$$N(B_H((\alpha_n)), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\approx} N((\alpha_n^H), |\cdot|, \varepsilon).$$

If $\lim_{x\to\infty} \frac{g(x)}{-g'(x)\cdot x} = 0$, *then*

$$N(B_H((\alpha_n)), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\preccurlyeq} N((\alpha_n^H), |\cdot|, \varepsilon) \text{ and } N(B_H((\alpha_n)), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} N((\alpha_n), |\cdot|, \varepsilon).$$

Corollary 2.10. *Let* $(\alpha_n) \in \mathcal{C}$ *with associated function* g*, where* g *is of one of the five types from Corollary 1.17. Then*

$$N(B_H((\alpha_n)), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\approx} N((\alpha_n^H), |\cdot|, \varepsilon) \tag{10}$$

and

$$\dim_E^{\mathbb{P}}((B_H(\alpha_n))) = \dim_E((\alpha_n^H)) = \frac{\dim_E((\alpha_n))}{H + (1-H)\dim_E((\alpha_n))}. \tag{11}$$

Remark 2.11. *Observe the similarity to (8) and (9).*

Lemma 2.12. *Let* $T \subseteq \mathbb{R}^+$ *be bounded and assume* $N(T, |\cdot|, \varepsilon) \succcurlyeq \varepsilon^{-d}$*. Let* $A \in \mathcal{D}$*. For almost all* $\omega \in \Omega$ *there is a null sequence* $(\varepsilon_k)(\omega)$ *such that*

$$N(B_H(T), |\cdot|, \varepsilon_k)(\omega) \geqslant \frac{1}{A(\varepsilon_k)} \varepsilon_k^{-\frac{d}{H+(1-H)d}},$$

is true for large k*. Similar results can be derived if one chooses logarithmical or mixed bounds for* $N(T, |\cdot|, \varepsilon)$*.*

Corollary 2.13. *Assume that* $\dim_E T$ *exists. Then* $\overline{\dim}_E B_H(T) \geqslant \frac{\dim_E T}{H+(1-H)\dim_E T}$*, almost surely.*

A result similar to (11) is not true for arbitrary index sets. Because B_H fulfills an almost surely Hölder condition (cf. section 1.4.4), one can show that

$$\overline{\dim}_E B_H(A) \leqslant \min\{\frac{\overline{\dim}_E A}{H}, 1\},$$

almost surely. On the other hand, we have

$$\underline{\dim}_E B_H(A) \geqslant \dim_H B_H(A) = \min\{\frac{\dim_H A}{H}, 1\},$$

almost surely. If A fulfills $\overline{\dim}_E A = \dim_H A$, then $\dim_E B_H(A) = \min\{\frac{\dim_E A}{H}, 1\}$, almost surely. For example, sets of this type are given by self-similar sets (cf. section 1.4.3.1).

The idea for the almost surely lower bounds comes from the proof of the following result.

Proposition 2.14 ([19], Lemma 3.3)**.** *Let $T \subseteq \mathbb{R}^+$. Then*

$$\overline{\dim}_E B_H(T) \geqslant \frac{1}{H} B - \overline{\dim}^H T, a.s.$$

2.2 Lévy processes

2.2.1 Hausdorff dimension

In [7] Blumenthal and Getoor defined a lower index β and upper indices β', β'' for general Lévy processes. Moreover, they defined an index σ for subordinators. The indices can be used to state lower and upper bounds for the image dimension. Recall that a Lévy process is characterized by a triplet (a, σ^2, μ) or by its characteristic exponent Ψ and for a subordinator the Laplace exponent Φ is important. The definition of the indices is as follows:

- $\beta = \inf\{\alpha > 0 : \int\limits_{(-1,1)} |x|^\alpha \,\mathrm{d}\mu(x) < \infty\}$,
- $\beta'' = \sup\{\alpha \geqslant 0 : |y|^{-\alpha} \Re\Psi(y) \overset{|y|\to\infty}{\to} = \infty\}$,
- $\beta' = \sup\{\alpha \geqslant 0 : \int\limits_{\mathbb{R}} |x|^{\alpha-1} \frac{1-\exp(-\Re\Psi(x))}{\Re\Psi(x)} \,\mathrm{d}x < \infty\}$,
- $\sigma = \sup\{\alpha \leqslant 1 : \int\limits_1^\infty \frac{x^{\alpha-1}}{\Phi(x)} \,\mathrm{d}x < \infty\}$,

where $\Re z$ is the real part of a complex number z.

Theorem 2.15 ([7], Th. 8.1, Th. 8.5, [27] Th. 5.1)**.** *Let $X = (X_t)_{t\in\mathbb{R}^+}$ be a Lévy process and let $A \in \mathcal{B}([0,1])$. The following estimates hold almost surely:*

1. $\dim_H X(A) \geqslant \beta' \cdot \dim_H A$ *if* $\beta' \leqslant 1$,
2. $\dim_H X(A) \geqslant \min\{1, \beta'' \cdot \dim_H A\}$ *if* $\beta' > 1$,
3. $\dim_H X(A) \leqslant \beta \cdot \dim_H A$,
4. $\dim_H X(A) \geqslant \sigma \cdot \dim_H A$ *if X is a subordinator.*

Remark 2.16. *The condition $A \in \mathcal{B}([0,1])$ can be replaced by $A \in \mathcal{B}(\mathbb{R}^+)$ (cf. the introduction of [18]).*

Additionally, there are some "exact" results, but a computation seems to be difficult for most processes. Define

$$W(r) := \int\limits_{\mathbb{R}} \frac{\Re \frac{1}{1+\Psi(\frac{x}{r})}}{1+x^2} \,\mathrm{d}x \tag{12}$$

for a Lévy process with characteristic exponent Ψ.

Theorem 2.17 ([20], eq. (1.4)). *Let $X = (X_t)_{t\in\mathbb{R}^+}$ be a Lévy process with characteristic exponent Ψ. Then, almost surely,*

$$\dim_H X([0,1]) = \liminf_{r\to 0} \frac{\log W(r)}{\log r}.$$

Theorem 2.18 ([18], Cor. 2.6). *Let $X = (X_t)_{t\in\mathbb{R}^+}$ be a Lévy process and let $A \subseteq \mathbb{R}^+$ be a Borel set. Then, almost surely,*

$$\dim_H X(A) = \sup\{\beta \in (0,1) : \inf_{\mu\in\mathcal{P}(A)} \int_{\mathbb{R}} \mathbb{E}|\int e^{izX_t}\,\mathrm{d}\mu(t)|^2 |z|^{\beta-1}\,\mathrm{d}z < \infty\},$$

where $\mathcal{P}(A)$ is used as above.

For α- stable processes the situation is clear.

Theorem 2.19 ([6], section 3). *Let $A \subseteq [0,1]$ be a Borel set and let X be α- stable. Then*

$$\dim_H X(A) = \min\{1, \alpha \cdot \dim_H A\}\ ,\ a.s.$$

Remark 2.20. *The restriction $A \subseteq [0,1]$ can be dropped (cf. [18]).*

Another result is expressed in terms of the Laplace exponent Φ of a subordinator.

Theorem 2.21 ([5], section 5). *Let X be a subordinator and let Φ be its Laplace exponent. Then, almost surely,*

$$\dim_H X([0,1]) = \liminf_{x\to\infty} \frac{\log \Phi(x)}{\log x}.$$

2.2.2 Packing dimension

Theorem 2.22 ([37], Th. 6, [20], Th. 1.1). *Let $X = (X_t)_{\{t\,\geqslant\,0\}}$ be a Lévy process with characteristic exponent Ψ. Then, almost surely,*

$$\dim_P X([0,1]) = \sup\{\alpha > 0 : \liminf_{r\to 0} \int_0^1 \frac{\mathbb{P}\{|X_t| \leqslant r\}}{r^\alpha}\,\mathrm{d}t = 0\} = \limsup_{r\to 0} \frac{\log W(r)}{\log r},$$

where the definition of W is given by equation (12).

If X is a subordinator, then there is a dimension formula in terms of the Laplace exponent.

Theorem 2.23 ([5], section 5). *Let X be a subordinator and let Φ be its Laplace exponent. Then, almost surely,*

$$\dim_P X([0,1]) = \limsup_{x\to\infty} \frac{\log \Phi(x)}{\log x}.$$

Khoshnevisan, Schilling and Xiao [17] express the image dimension in terms of dimension profiles.

Theorem 2.24 ([17], Theorem 2.7). *Let* $X = (X_t)_{t\in\mathbb{R}^+}$ *be a Lévy process and let* $A \subseteq \mathbb{R}^+$ *be a nonrandom bounded Borel set. Then, almost surely,*

$$\dim_P X(A) = P - \dim A.$$

Corollary 2.25 ([17], Corollary 1.1). *Let* X *be a strictly real-valued* α*-stable Lévy process with parameters* $(0, 0, \sigma)$ *and let* $A \subseteq \mathbb{R}^+$ *be a nonrandom Borel set. Then*

$$\dim_P X(A) = \alpha \cdot P - \dim^{\frac{1}{\alpha}} A, \ \ a.s.$$

We know that $P - \dim^s A$ and $\dim_P A$ concur for $s \geqslant 1$, thus, the following (older) result is not surprising.

Corollary 2.26 ([13], Cor. 3.4). *Let* X *be a strictly stable Lévy process with parameters* $(0, 0, \sigma)$ *and index* $\alpha \in (0, 1]$ *and let* $A \subseteq \mathbb{R}^+$ *be a Borel set. Then*

$$\dim_P X(A) = \alpha \cdot \dim_P A, \ \ a.s.$$

Remark 2.27. *Some steps towards generality are taken in [35]. The authors give Hausdorff and packing dimension estimates for more general processes. The results are applicable to the fractional Brownian motion and to self-similar Lévy processes.*

2.2.3 Entropy dimension and covering numbers

One source of motivation for our work came from [25] and [26]. Take a process with "nice" properties and some index set and have a look at the covering numbers of the image. We state the results, but we remark that we cannot use the techniques developed in [25] and [26] for more general processes because the proofs there use that subordinators are increasing.

Theorem 2.28 ([25], Cor. 3.2). *Let* X *be a subordinator with Laplace exponent* Φ*. Assume* $\liminf\limits_{x\to\infty} \frac{\Phi(x)}{\log x} > 0$*. Then, almost surely,*

$$N(X([0, 1]), |\cdot|, \varepsilon) \approx \Phi(\varepsilon^{-1}).$$

A more general version can be found in [26].

Theorem 2.29 ([26], Th. 1.2). *Let* $K \subseteq \mathbb{R}^+$ *be compact and let* X *be a subordinator with Laplace exponent* Φ *such that*

$$\liminf_{x\to\infty} \frac{N(K, |\cdot|, \frac{1}{\Phi(x)})}{(\log x)^\beta} > 0$$

is true for some $\beta > 0$. Then for almost all paths of X there is some random ε_0 such that for $0 < \varepsilon < \varepsilon_0$, it follows that

$$\frac{1}{14}N(K, |\cdot|, \frac{2}{\Phi(\varepsilon^{-1})}) \leqslant N(X(K), |\cdot|, \varepsilon) \leqslant 100N(K, |\cdot|, \frac{1}{2\Phi(\varepsilon^{-1})}).$$

In particular, if $N(K, |\cdot|, \varepsilon)$ fulfills a doubling condition, then, almost surely,

$$N(X(K), |\cdot|, \varepsilon) \approx N(K, |\cdot|, \frac{1}{\Phi(\varepsilon^{-1})}).$$

A result for general Lévy processes is given by

Theorem 2.30 ([16], Th. 3.4)**.** *Let $X = (X_t)_{t\in\mathbb{R}^+}$ be a Lévy process with characteristic exponent Ψ and let $b > 0$. Then, almost surely,*

$$\overline{\dim}_E X([0, b]) = \limsup_{r\to 0} \frac{\log W(r)}{\log r},$$
$$\underline{\dim}_E X([0, b]) = \liminf_{r\to 0} \frac{\log W(r)}{\log r},$$

where the definition of W is given by (12).

Remark 2.31. *Compare the results of Theorem 2.30 with Theorems 2.22 and 2.17 to see that some dimensions concur.*

A quite new result gives the image dimension in terms of the generalized box dimension profile of the index set (cf. 1.4.2).

Theorem 2.32 ([17], Theorem 2.7)**.** *Let $X = (X_t)_{t\in\mathbb{R}^+}$ be a Lévy process and let $A \subseteq \mathbb{R}^+$ be a nonrandom bounded Borel set. Then, almost surely,*

$$\overline{\dim}_E X(A) = B - \overline{\dim} A.$$

Corollary 2.33 ([17], Corollary 1.2)**.** *Let X be a subordinator with Laplace exponent Φ and let $A \subseteq \mathbb{R}^+$ be a nonrandom Borel set. Then, almost surely,*

$$\overline{\dim}_E X(A) = \sup\{\eta > 0 : \limsup_{x\to\infty} x^\eta \inf_{\mu\in\mathcal{P}(A)} \int\int e^{-|t-s|\Phi(x)}\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(s) = 0\}.$$

Theorem 2.34. *Let X be strictly α-stable with $\alpha \in (0, 2]$ and let $T = (\alpha_n) \in \mathcal{C}^{\frac{1}{\alpha}}$ with associated function g. Let $A \in \mathcal{D}$. Then*

1. $N(X(T), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\preccurlyeq} N(T^{\frac{1}{\alpha}}, |\cdot|, \varepsilon)$.
2. *Assume $\alpha \in (1, 2]$.*
   (a) *If $\lim_{x\to\infty} \frac{-g'(x)\cdot x}{g(x)} = 0$ or $-g'(x)x \approx g(x)$, then*

$$N(X(T), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} N(T^{\frac{1}{\alpha}}, |\cdot|, \varepsilon).$$

(b) *If* $\lim_{x\to\infty} \frac{g(x)}{-g'(x)\cdot x} = 0$, *then* $N(X(T), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} N(T, |\cdot|, \varepsilon)$.

3. *If* $\alpha \in (0,1)$, *then* $N(X(T), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} N(T, |\cdot|, \varepsilon^{\alpha})$.

4. *If* $\alpha = 1$, *then* $N(X(T), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} N(T, |\cdot|, \varepsilon)\frac{1}{\log N(T, |\cdot|, \varepsilon)}$.

5. *Let* $\alpha \in (1,2]$.

   (a) *Assume* $\lim_{x\to\infty} \frac{-g'(x)\cdot x}{g(x)} = 0$ *or* $-g'(x)x \approx g(x)$. *For almost all* $\omega \in \Omega$ *there is a null sequence* $(\varepsilon_k) = (\varepsilon_k(\omega))$ *such that*

   $$N(X(T), |\cdot|, \varepsilon_k)(\omega) \succcurlyeq \frac{1}{A(\varepsilon_k(\omega))} N(T^{\frac{1}{\alpha}}, |\cdot|, \varepsilon_k)(\omega).$$

   (b) *Assume* $\lim_{x\to\infty} \frac{g(x)}{-g'(x)\cdot x} = 0$. *For almost all* $\omega \in \Omega$ *there is a null sequence* $(\varepsilon_k) = (\varepsilon_k(\omega))$ *such that*

   $$N(X(T), |\cdot|, \varepsilon_k)(\omega) \succcurlyeq \frac{1}{A(\varepsilon_k)} N(T, |\cdot|, \varepsilon_k).$$

6. *Let* $\alpha \in (0,1)$. *For almost all* $\omega \in \Omega$ *there is a null sequence* $(\varepsilon_k) = (\varepsilon_k(\omega))$ *such that*

   $$N(X(T), |\cdot|, \varepsilon_k)(\omega) \succcurlyeq \frac{1}{A(\varepsilon_k)} N(T, |\cdot|, \varepsilon_k^{\alpha}).$$

7. *Let* $\alpha = 1$. *For almost all* $\omega \in \Omega$ *there is a null sequence* $(\varepsilon_k) = (\varepsilon_k(\omega))$ *such that*

   $$N(X(T), |\cdot|, \varepsilon_k)(\omega) \succcurlyeq \frac{1}{A(\varepsilon_k)} N(T, |\cdot|, \varepsilon_k) \frac{1}{\log N(T, |\cdot|, \varepsilon_k)}.$$

8. *Let* $\alpha \in (1,2]$ *and assume* $\lim_{x\to\infty} \frac{-g'(x)\cdot x}{g(x)} = 0$ *or* $-g'(x)x \approx g(x)$. *Then*

   $$\overline{\dim}_E X(T) \geqslant \limsup_{\varepsilon\to 0} \frac{\log N(T^{\frac{1}{\alpha}}, |\cdot|, \varepsilon)}{|\log \varepsilon|}, \ \text{a.s.}$$

9. *Let* $\alpha \in (1,2]$ *and assume* $\lim_{x\to\infty} \frac{g(x)}{-g'(x)\cdot x} = 0$. *Then*

   $$\overline{\dim}_E X(T) \geqslant \limsup_{\varepsilon\to 0} \frac{\log N(T, |\cdot|, \varepsilon)}{|\log \varepsilon|}, \ \text{a.s.}$$

10. *Let* $\alpha \in (0,1)$. *Then*

    $$\overline{\dim}_E X(T) \geqslant \limsup_{\varepsilon\to 0} \frac{\log N(T, |\cdot|, \varepsilon^{\alpha})}{|\log \varepsilon|}$$

    $$= \alpha \cdot \limsup_{\varepsilon\to 0} \frac{\log N(T, |\cdot|, \varepsilon)}{|\log \varepsilon|} = \alpha \cdot \overline{\dim}_E T, \ \text{a.s.}$$

11. Assume $\alpha = 1$*. Then*

$$\overline{\dim}_E X(T) \geqslant \limsup_{\varepsilon \to 0} \frac{\log N(T, |\cdot|, \varepsilon)}{|\log \varepsilon|} = \overline{\dim}_E T, \text{ a.s.}$$

Remark 2.35. *The results of Theorem 2.34 imply that for* $\alpha \in (1, 2]$ *the problem of finding the order of the covering numbers of the image set is solved in most cases. But for* $\alpha \in (0, 1)$*, the situation is not clear. For example, consider* $T = (n^{-\beta})$ *with* $\beta > 0$*. Then* $N(T^{\frac{1}{\alpha}}, |\cdot|, \varepsilon) \approx \varepsilon^{-\frac{1}{1+\frac{\beta}{\alpha}}}$*, but* $N(T, |\cdot|, \varepsilon^\alpha) \approx \varepsilon^{-\frac{\alpha}{1+\beta}}$*. A result from [13] can clarify the situation.*

Theorem 2.36 ([13], Th. 2.2)**.** *Let* X *be strictly* α*-stable with parameters* $(0, 0, \sigma)$ *and index* $\alpha \in (0, 1]$ *and let* $A \subseteq \mathbb{R}^+$ *be a bounded Borel set. Then*

$$\overline{\dim}_E X(A) \leqslant \alpha \cdot \overline{\dim}_E A, \text{ a.s.}$$

In fact, the proof shows that, almost surely,

$$N(X(A), |\cdot|, (2\varepsilon)^{\frac{1}{\alpha'}}) \leqslant M(X(A), |\cdot|, (2\varepsilon)^{\frac{1}{\alpha'}}) \leqslant cN(A, |\cdot|, \varepsilon)$$

holds, where c is some constant and $\alpha' > \alpha$. For the example $T = (n^{-\beta})$ this means $N(X(T), |\cdot|, \varepsilon) \preceq \varepsilon^{-\frac{\alpha'}{1+\beta}}$, almost surely, for every $\alpha' > \alpha$.

A second upper bound result can be found in the new paper [17] (cf. proofs of Corollary 1.1 and Theorem 2.7). We reformulate the results in our notation.

Proposition 2.37. *Let* $X = (X_t)_{t \in \mathbb{R}^+}$ *be a Lévy process and* $T \subseteq \mathbb{R}^+$ *be a nonrandom Borel set. Then*

$$N(X(T), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\preccurlyeq} \varepsilon^{-s},$$

for every $s > B - \overline{\dim} T$*. In particular, if* X *is strictly* α*-stable with parameters* $(0, 0, \sigma)$*, then*

$$N(X(T), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\preccurlyeq} \varepsilon^{-\alpha s},$$

for every $s > B - \overline{\dim}^{\frac{1}{\alpha}} T$*. For* $\alpha \in (0, 1]$ *this means*

$$N(X(T), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\preccurlyeq} \varepsilon^{-\alpha s},$$

for every $s > \overline{\dim}_E T$*.*

Corollary 2.38. *Let* X *be a strictly* α*-stable Lévy process with parameters* $(0, 0, \sigma)$*. Then*

$$\overline{\dim}_E^{\mathbb{P}} X([0, 1]) \leqslant \begin{cases} 1 & : \alpha > 1 \\ \alpha & : \alpha \leqslant 1. \end{cases}$$

Lower bound results are independent of the special structure of T. Our methods can be used to verify the following

Corollary 2.39. *Let X be α-stable and let $T \subseteq \mathbb{R}^+$ so that $N(T, |\cdot|, \varepsilon) \succcurlyeq \varepsilon^{-d}$ is valid for some $d \in (0, 1]$.*

1. *Let $\alpha \in (0, 1)$. Then $N(X(T), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} \varepsilon^{-\alpha d}$.*
2. *Let $\alpha = 1$. Then $N(X(T), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} \varepsilon^{-d} |\log \varepsilon|^{-d}$.*
3. *Let $\alpha \in (1, 2]$. Then $N(X(T), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} \varepsilon^{-\frac{d}{\frac{1}{\alpha} + (1 - \frac{1}{\alpha})d}}$.*
4. *Let $\alpha \in (0, 1)$ and $A \in \mathcal{D}$. Then there is a full measure set F and for $\omega \in F$ there is a null sequence $(\varepsilon_k)(\omega)$ such that*
$$N(X(T), |\cdot|, \varepsilon_k)(\omega) \succcurlyeq \frac{1}{A(\varepsilon_k)} \varepsilon_k^{-\alpha d}$$
is valid.
5. *Let $\alpha = 1$ and $A \in \mathcal{D}$. Then there is a full measure set F and for $\omega \in F$ there is a null sequence $(\varepsilon_k)(\omega)$ such that*
$$N(X(T), |\cdot|, \varepsilon_k)(\omega) \succcurlyeq \frac{1}{A(\varepsilon_k)} \varepsilon_k^{-d} |\log \varepsilon_k|^{-d}$$
is valid.
6. *Let $\alpha \in (1, 2]$ and $A \in \mathcal{D}$. Then there is a full measure set F and for $\omega \in F$ there is a null sequence $(\varepsilon_k)(\omega)$ such that*
$$N(X(T), |\cdot|, \varepsilon_k)(\omega) \succcurlyeq \frac{1}{A(\varepsilon_k)} \varepsilon_k^{-\frac{d}{\frac{1}{\alpha} + (1 - \frac{1}{\alpha})d}}$$
is valid.
7. *Assume that $\dim_E T$ exists. Then, almost surely,*
$$\overline{\dim}_E X(T) \geqslant \begin{cases} \alpha \dim_E T : & \alpha \leqslant 1 \\ \frac{\dim_E T}{\frac{1}{\alpha} + (1 - \frac{1}{\alpha}) \dim_E T} : & \alpha > 1. \end{cases}$$

Corollary 2.40. *Let X be α-stable. Then*
$$\underline{\dim}_E^{\mathbb{P}} X([0, 1]) \geqslant \begin{cases} 1 & : \alpha > 1 \\ \alpha & : \alpha \leqslant 1. \end{cases}$$

Determining the rate of $N(X([0, 1]), |\cdot|, \varepsilon)$, where X is α-stable, is also the topic of the unpublished manuscript [24]. The author uses small deviations, local times, sums of independent random variables and the Markov property to derive his results. The result is that $N(X([0,1]), |\cdot|, \varepsilon)$ behaves like $\varepsilon^{-\alpha}$ for $\alpha < 1$ and it behaves like ε^{-1} for $\alpha > 1$. But the meaning of "behaves like" deviates from our formulation. It is neither "almost surely", nor $\overset{\mathbb{P}}{\approx}$.

2.3 Random sequences

2.3.1 From (right-) continuous processes to random sequences

Let B_H be the fractional Brownian motion with index $H \in (0,1)$ and let $T = (\alpha_n)$ be convex. In the last section, we have seen that the order of $N(B_H(T), |\cdot|, \varepsilon)$ is often determined by $N(T^H, |\cdot|, \varepsilon)$. In the first instance, this is surprising because the Hausdorff dimension results have a strong Hölderian influence and there does not seem to be any reason why the situation should change for the covering numbers. But a second view can explain the changed situation. Due to the self-similarity of B_H, we know that $B_H(\alpha_n)$ has the same distribution as $\alpha_n^H B_H(1)$, where $B_H(1)$ is standard normal. Thus, we can interpret $B_H((\alpha_n))$ as a sequence

$$(\alpha_n^H B_H^{(n)}(1)), \tag{13}$$

where $(B_H^{(n)}(1))$ is a dependent sequence of standard normal random numbers, whose correlation can be computed by the fBm correlation function. This representation could explain where the transformation $x \longmapsto x^H$ comes into play (cf. Corollary 2.10). The same can be done for strictly α-stable Lévy processes.
Representation (13) motivates to consider the problem of determining the covering number rates for random sequences in general. We restrict to the case of independent components, but one result is also valid for dependent components. The case of dependent components could be a topic for future research (cf. section 4).

2.3.2 Hausdorff and packing dimension

Because $(\alpha_n U_n \xi_n)(\omega)$ and $(\alpha_n \xi_n)(\omega)$ are countable sequences of real numbers for (almost) all $\omega \in \Omega$, we have, by Corollary 1.8,

$$\dim_H(\alpha_n U_n \xi_n) = \dim_P(\alpha_n U_n \xi_n) = \dim_H(\alpha_n \xi_n) = \dim_P(\alpha_n \xi_n) = 0, \text{ a.s.}$$

2.3.3 Entropy dimension and covering numbers

In order to get covering number results, we determine the behavior of the decreasing orders of the random sequences $(\alpha_n \xi_n)$ and $(\alpha_n U_n \xi_n)$. A typical result is given by the following theorem. In the first instance, the statement seems to be complicated, but for "nice" functions g and h and distributions $\mathbb{P}_{\xi_1}$ the result can be very useful.

Theorem 2.41. *Let (α_n) be a real null sequence such that $c_1 g(n) \leqslant \alpha_n \leqslant c_2 g(n)$, where $(g(n)) \in \mathcal{C}$ and $c_1, c_2 > 0$, let (ξ_n) be a sequence of independent and identically distributed random numbers with $\mathbb{P}\{\xi_1 \geqslant 0\} = 1$ and let (U_n) be a sequence of independent random numbers with $\mathbb{P}\{U_n = 1\} = 1 - \mathbb{P}\{U_n = 0\} = s_n$, where $\lim_{n\to\infty} s_n = 0$, $c_3 h(n) \leqslant s_n \leqslant c_4 h(n)$ for some $c_3, c_4 > 0$, $(h(n)) \in \mathcal{C}$ and $\sum_{n=1}^{\infty} h(n) = \infty$. Assume the existence of a positive function H with $H' = h$ and, for every $n \in \mathbb{N}$, assume the finiteness of $\mathbb{E}H(g^{-1}(\frac{g(H^{-1}(n))}{c_2\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_2 g(1)},\infty)}(\xi_1)$. Furthermore, let $\delta > 0$ and assume*

$$\sum_{n=1}^{\infty} \exp(-\frac{\delta^2}{4} c_3 \mathbb{E}H(g^{-1}(\frac{g(H^{-1}(n))}{c_1\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_1 g(1)},\infty)}(\xi_1)) < \infty. \tag{14}$$

Then $(\alpha_n U_n \xi_n)$ is bounded, almost surely, and there is a measurable set F_δ with full measure and for $\omega \in F_\delta$ and $n \geqslant n(\omega)$ we have

$$\begin{aligned}(\alpha_. U_. \xi_.)^*_{\left\lceil (1+\delta)(c_4 \mathbb{E}H(g^{-1}(\frac{g(H^{-1}(n))}{c_2\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_2 g(1)},\infty)}(\xi_1)+s_1)\right\rceil+1}(\omega) &\leqslant g(H^{-1}(n)) \\ &\leqslant (\alpha_. U_. \xi_.)^*_{\left\lfloor (1-\delta)(c_3 \mathbb{E}H(g^{-1}(\frac{g(H^{-1}(n))}{c_1\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_1 g(1)},\infty)}(\xi_1)-c_3 H(1))\right\rfloor}(\omega).\end{aligned}$$

Theorem 2.42. *Let (α_n) be a real null sequence such that $c_1 g(n) \leqslant \alpha_n \leqslant c_2 g(n)$, where $(g(n)) \in \mathcal{C}$ and $c_1, c_2 > 0$, and let (ξ_n) be a sequence of independent and identically distributed random numbers with $\mathbb{P}\{\xi_1 \geqslant 0\} = 1$. For every $n \in \mathbb{N}$, assume the finiteness of $\mathbb{E}g^{-1}(\frac{g(n)}{c_2\xi_1})\mathbb{1}_{[\frac{g(n)}{c_2 g(1)},\infty)}(\xi_1)$. Furthermore, let $\delta > 0$ and assume*

$$\sum_{n=1}^{\infty} \exp(-\frac{\delta^2}{4} \mathbb{E}g^{-1}(\frac{g(n)}{c_1\xi_1})\mathbb{1}_{[\frac{g(n)}{c_1 g(1)},\infty)}(\xi_1)) < \infty.$$

Then $(\alpha_n \xi_n)$ is bounded, almost surely, and there is a measurable set F_δ with full measure and for $\omega \in F_\delta$ and $n \geqslant n(\omega)$ we have

$$\begin{aligned}(\alpha_. \xi_.)^*_{\left\lceil (1+\delta)(\mathbb{E}g^{-1}(\frac{g(n)}{c_2\xi_1})\mathbb{1}_{[\frac{g(n)}{c_2 g(1)},\infty)}(\xi_1)+1)\right\rceil+1}(\omega) &\leqslant g(n) \\ &\leqslant (\alpha_. \xi_.)^*_{\left\lfloor (1-\delta)(\mathbb{E}g^{-1}(\frac{g(n)}{c_1\xi_1})\mathbb{1}_{[\frac{g(n)}{c_1 g(1)},\infty)}(\xi_1)-1)\right\rfloor}(\omega).\end{aligned}$$

Theorem 2.43. *Use the setting of Theorem 2.41. If g and h are polynomial, i.e., $g(x) = x^{-\beta}$ for some $\beta > 0$ and $h(x) = x^{-a}$ for some $a \in (0,1)$, and $\mathbb{E}\xi_1^{\frac{1-a}{\beta}} < \infty$, then*

$$N((\alpha_n U_n \xi_n), |\cdot|, \varepsilon) \preccurlyeq N((g(H^{-1}(n))), |\cdot|, \varepsilon) \approx ((-g \circ H^{-1})')^{-1}(\varepsilon), \text{ a.s.}$$

Theorem 2.44. *Use the setting of Theorem 2.42. If g is polynomial, i.e., $g(x) = x^{-\beta}$ for some $\beta > 0$, and $\mathbb{E}\xi_1^{\frac{1}{\beta}} < \infty$, then*

$$N((\alpha_n \xi_n), |\cdot|, \varepsilon) \preccurlyeq N((g(n)), |\cdot|, \varepsilon) \approx (-g')^{-1}(\varepsilon), \text{ a.s.}$$

Theorem 2.45. *If the distribution of ξ_1 has compact support, then*

$$N((\alpha_n \xi_n), |\cdot|, \varepsilon) \preccurlyeq N((g(n)), |\cdot|, \varepsilon), \text{a.s.}$$

The result is also true for negative random numbers, dependent random numbers and even for nonidentically distributed random numbers (as long as the (ξ_n) do not leave a common compact support).

The following two $\overset{\mathbb{P}}{\succcurlyeq}$ results can be applied to random numbers having support on the whole real line.

Theorem 2.46. *Let $(g(n)) \in \mathcal{C}$ and assume the existence of a real constant c such that*

$$\mathbb{P}\{|g(n)\xi_n - g(m)\xi_m| \leqslant \varepsilon\} \leqslant c \frac{\varepsilon}{\min\{g(n), g(m)\}} \tag{15}$$

holds for all $m \neq n$ and all $\varepsilon > 0$. Then

$$N((g(n)\xi_n), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} \min\{\frac{g(N((g(n)), |\cdot|, \varepsilon))}{\varepsilon}, N((g(n)), |\cdot|, \varepsilon)\}.$$

Some distributions fulfilling condition (15) can be found in section 3, following the proof of Theorem 2.46.

Proposition 2.47. *Let F be the distribution function of $|\xi_1|$ and assume that*

$$\lim_{\varepsilon \to 0} \prod_{n > f(\varepsilon)} F(f(\varepsilon)\varepsilon A(\varepsilon)\frac{1}{g(n)}) = 1$$

holds for all $A \in \mathcal{D}$ and some function f. Then $N((g(n)\xi_n), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\preccurlyeq} f(\varepsilon)$.

If the influence of chance is reduced to the sequence (U_n), we can give a lower bound.

Theorem 2.48. *Let $(g(n))$ and $(s(n))$ be elements of $\mathcal{C}$, where $(s(n)) \subseteq (0, 1]$. Furthermore, let (U_n) be a sequence of independent random numbers with distribution $\mathbb{P}\{U_n = 1\} = s(n) = 1 - \mathbb{P}\{U_n = 0\}$, where the sequence (s_n) fulfills $\sum_{n=1}^{\infty} s_n = \infty$. Assume the existence of a function S with $S' = s$ and assume $(g \circ S^{-1}(n)) \in \mathcal{C}$. If $\delta > 0$ and $\lim_{k\to\infty}(S(S^{-1}(k) + 1) - k) = 0$, then, almost surely,*

1. $N((g(n)U_n), |\cdot|, \varepsilon) \preccurlyeq N((g(S^{-1}(\frac{n}{1+\delta} - s_1 - 1)))_{n > (1+\delta)(1+s_1)}, |\cdot|, \varepsilon)$,

2. $N((g(n)U_n), |\cdot|, \varepsilon) \succcurlyeq [-g(S^{-1}(\varepsilon))']^{-1}$.

If the relation $g(S^{-1}(\frac{n}{1+\delta} - s_1 - 1)) \leqslant cg \circ S^{-1}(n)$ *is true for some* $c > 0$*, the upper bound simplifies to* $N((g \circ S^{-1}(n)), |\cdot|, \varepsilon)$.

3 Proofs and examples

First, we state and prove a helpful result to gain lower bounds for $N(X(T), |\cdot|, \varepsilon)$ in the $\overset{\mathbb{P}}{\preccurlyeq}$- sense. It was first shown to us by F. Aurzada and it is related to general dimension profiles (cf. section 1.4.2.1, recall the definition of $F(T, \varepsilon)$ from there).

Proposition 3.1. *Let $X = (X_t)_{t\in T}$, where T is a Borel subset of $\mathbb{R}^+$ or $T = \mathbb{N}$, be a stochastic process and assume that $X(T)$ is bounded, almost surely, and let $\varepsilon > 0$. Then*

$$\mathbb{P}\left\{N(X(T), |\cdot|, \varepsilon) \leqslant n\right\} \leqslant nF(T, 2\varepsilon).$$

Proof. We can neglect the zero set where $X(T)$ is not bounded. Choose ω from $\{X(T)$ is bounded$\}$ such that $N(X(T), |\cdot|, \varepsilon)(\omega) \leqslant n$ is true for some natural number n. Then we can cover $X(T)(\omega)$ by n intervals of radius 2ε and we can partition T into n disjoint sets $T_1, T_2, \ldots, T_n$ such that $|X_s(\omega) - X_t(\omega)| \leqslant 2\varepsilon$ is true for $s, t \in T_j$. We can compute

$$\int_T \int_T \mathbb{1}_{\{|X_t - X_s| \leqslant 2\varepsilon\}} \, d\mu(t) \, d\mu(s) = \sum_{i=1}^n \int_{T_i} \sum_{j=1}^n \int_{T_j} \mathbb{1}_{\{|X_t - X_s| \leqslant 2\varepsilon\}} \, d\mu(t) \, d\mu(s)$$

$$\geqslant \sum_{i=1}^n \int_{T_i} \int_{T_i} \mathbb{1}_{\{|X_t - X_s| \leqslant 2\varepsilon\}} \, d\mu(t) \, d\mu(s) = \sum_{i=1}^n \mu(T_i)^2 = \sum_{i=1}^n \mu(T_i)^2 \sum_{j=1}^n \left(\frac{1}{\sqrt{n}}\right)^2$$

$$\geqslant \left(\sum_{i=1}^n \mu(T_i) \cdot \frac{1}{\sqrt{n}}\right)^2 = \frac{1}{n},$$

where we used the Cauchy-Schwarz inequality in order to get the last estimate. Therefore,

$$\mathbb{P}\left\{N(X(T), |\cdot|, \varepsilon) \leqslant n\right\} \leqslant \mathbb{P}\left\{\int_T \int_T \mathbb{1}_{\{|X_t - X_s| \leqslant 2\varepsilon\}} d\mu(t) d\mu(s) \geqslant \frac{1}{n}\right\}$$

$$\leqslant n\mathbb{E} \int_T \int_T \mathbb{1}_{\{|X_t - X_s| \leqslant 2\varepsilon\}} \, d\mu(t) \, d\mu(s) = n \int_T \int_T \mathbb{E}\mathbb{1}_{\{|X_t - X_s| \leqslant 2\varepsilon\}} \, d\mu(t) \, d\mu(s)$$

$$= n \int_T \int_T \mathbb{P}\left\{|X_t - X_s| \leqslant 2\varepsilon\right\} \, d\mu(t) \, d\mu(s),$$

where we used Markov's inequality and Fubini's theorem. Taking the infimum over all μ gives the result. □

Corollary 3.2. *Consider the situation given in Proposition 3.1. Then*

$$N(X(T), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} \frac{1}{F(T, 2\varepsilon)}.$$

Proof. Let $A \in \mathcal{D}$. Firstly, assume $\lim_{\varepsilon \to 0} \frac{1}{A(\varepsilon)F(T,2\varepsilon)} = \infty$. Given $\varepsilon > 0$, choose n in such a way that $n \geqslant \frac{1}{A(\varepsilon)F(T,2\varepsilon)} > n-1$ holds. Then

$$\begin{aligned}\mathbb{P}\left\{N(X(T), |\cdot|, \varepsilon) < \frac{1}{A(\varepsilon)F(T,2\varepsilon)}\right\} &\leqslant \mathbb{P}\left\{N(X(T), |\cdot|, \varepsilon) < n\right\} \\ &\leqslant nF(T,2\varepsilon) \leqslant 2(n-1)F(T,2\varepsilon) \leqslant \frac{2}{A(\varepsilon)},\end{aligned}$$

what goes to zero.
Secondly, assume $\lim_{\varepsilon \to 0} \frac{1}{A(\varepsilon)F(T,2\varepsilon)} = 0$. The set $X(T)$ is nonempty and must be covered by one interval, at least. Thus,

$$\mathbb{P}\left\{N(X(T), |\cdot|, \varepsilon) < \frac{1}{A(\varepsilon)F(T,2\varepsilon)}\right\} \leqslant \mathbb{P}\left\{1 < \frac{1}{A(\varepsilon)F(T,2\varepsilon)}\right\} = \mathbb{P}\left\{\varnothing\right\} = 0,$$

if ε is small enough.
The case $\frac{1}{A(\varepsilon)F(T,2\varepsilon)} \approx 1$ can be transformed to the the first case by

$$\mathbb{P}\left\{N(X(T), |\cdot|, \varepsilon) < \frac{1}{A(\varepsilon)F(T,2\varepsilon)}\right\} \leqslant \mathbb{P}\left\{N(X(T), |\cdot|, \varepsilon) < \frac{A(\varepsilon)^{\frac{1}{2}}}{A(\varepsilon)F(T,2\varepsilon)}\right\}.$$

□

In view of applications of Corollary 3.2, the task is to find an upper bound $f(\varepsilon)$ for $F(T,2\varepsilon)$. Then $\frac{1}{f(\varepsilon)}$ is a $\overset{\mathbb{P}}{\succcurlyeq}$ lower bound for $N(X(T), |\cdot|, \varepsilon)$.

From time to time, we use the following result.

Lemma 3.3. *Let $H > 0$. Then, as $M \to \infty$,*

$$\sum_{i=1}^{M} \sum_{j=1, j\neq i}^{M} \frac{1}{|i-j|^H} \approx \begin{cases} M^{2-H} & : H < 1 \\ M \log M & : H = 1 \\ M & : H > 1. \end{cases}$$

Proof. Recall that $\int_1^{n+1} f(x)\,\mathrm{d}x \leqslant \sum_{k=1}^{n} f(k) \leqslant f(1) + \int_1^{n} f(x)\,\mathrm{d}x$ is true if f is a positive and decreasing function.

1. Assume $H < 1$. Then

$$\sum_{i=1}^{M} \sum_{j=1, j\neq i}^{M} \frac{1}{|i-j|^H} = 2 \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \frac{1}{(j-i)^H} = 2 \sum_{j=2}^{M} \sum_{i=1}^{j-1} \frac{1}{(j-i)^H}$$

$$= 2\sum_{j=2}^{M}\sum_{i=1}^{j-1}\frac{1}{i^H}\begin{cases}\leqslant \frac{2}{1-H}\sum\limits_{j=2}^{M}(j-1)^{1-H}\leqslant c_1 M^{2-H}\\ \geqslant \frac{2}{1-H}\sum\limits_{j=2}^{M}(j^{1-H}-1)\geqslant c_2 M^{2-H}.\end{cases}$$

2. Assume $H = 1$. Then

$$\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M}\frac{1}{|i-j|^1} = 2\sum_{j=2}^{M}\sum_{i=1}^{j-1}\frac{1}{i}\begin{cases}\leqslant c_1\sum\limits_{j=2}^{M}\log(j-1)\leqslant c_2 M\log M\\ \geqslant c_3\sum\limits_{j=2}^{M}(\log j-1)\geqslant c_4 M\log M.\end{cases}$$

3. Assume $H > 1$. Then

$$\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M}\frac{1}{|i-j|^H} = 2\sum_{j=2}^{M}\sum_{i=1}^{j-1}\frac{1}{i^H}\begin{cases}\leqslant \sum\limits_{j=2}^{M}c_1\leqslant c_1 M\\ \geqslant \sum\limits_{j=2}^{M}c_2\geqslant c_3 M.\end{cases}$$

□

Remark 3.4. *The same techniques can be applied to get results for more complicated double sums, i.e., we can substitute the denominator by* $|i^\beta - j^\beta|^H$*, where* $\beta \in \mathbb{R}$*, by* $|(\log i)^\nu - (\log j)^\nu|^H$*, where* $\nu \in \mathbb{R}$*, or by* $|q^i - q^j|^H$*, where* $q \in \mathbb{R}^+$*. For these examples, the computation of the behavior of the double sum is joint work with F. Aurzada.*

3.1 Fractional Brownian motion

Because B_H is almost surely continuous (cf. 1.4.4) on $\mathbb{R}^+$, the image $B_H(K)$ of a compact set $K \subseteq \mathbb{R}^+$ is again a compact (i.e., bounded) set (a.s.) and it makes sense to compute $N(B_H(K), |\cdot|, \varepsilon)$. Often, we do not assume K to be compact but only bounded. But in this case, the closure $\overline{K}$ is compact and therefore, $B_H(\overline{K})$ is a.s. compact (i.e., bounded), too. Thus, it follows the a.s. boundedness of the smaller set $B_H(K)$.

An application of Proposition 3.1 will be simplified if one has a good estimate for $\mathbb{P}\{|B_H(t) - B_H(s)| \leqslant 2\varepsilon\}$.

Lemma 3.5. *Let* $\varepsilon > 0$*. Then* $\mathbb{P}\{|B_H(t) - B_H(s)| \leqslant \varepsilon\} \leqslant \min\{\frac{\varepsilon}{\sqrt{\frac{\pi}{2}}|t-s|^H}, 1\}$.

Proof. If $t = s$, then $B_H(t) = B_H(s)$ and we have

$$\mathbb{P}\{|B_H(t) - B_H(s)| \leqslant \varepsilon\} = \mathbb{P}\{0 \leqslant \varepsilon\} = 1.$$

If $t \neq s$ then,

$$\mathbb{P}\{|B_H(t) - B_H(s)| \leqslant \varepsilon\} = \mathbb{P}\{|B_H(|t-s|)| \leqslant \varepsilon\}$$

$$= \frac{2}{\sqrt{2\pi}}|t-s|^{-H} \int_0^{\varepsilon} \exp(-\frac{x^2}{2|t-s|^{2H}})\,\mathrm{d}x \leqslant \sqrt{\frac{2}{\pi}} \frac{\varepsilon}{|t-s|^H}.$$

Of course, $\mathbb{P}\{|B_H(t) - B_H(s)| \leqslant \varepsilon\} \leqslant 1$ is valid, too, what proves the assertion. □

Proof of Theorem 2.6. By assumption, (α_n^H) is an element of $\mathcal{C}$ with associated function g^H. We know the orders of the covering numbers of (α_n) and (α_n^H) (cf. Prop. 1.19). The order of $N((\alpha_n), |\cdot|, \varepsilon)$ is given by $\max\{(-g')^{-1}(\varepsilon), \frac{g((-g')^{-1}(\varepsilon))}{\varepsilon}\}$. The proof requires to distinguish three cases:

(i) $\lim_{\varepsilon \to 0} \frac{\frac{g((-g')^{-1}(\varepsilon))}{\varepsilon}}{(-g')^{-1}(\varepsilon)} = 0$,

(ii) $\lim_{\varepsilon \to 0} \frac{(-g')^{-1}(\varepsilon)}{\frac{g((-g')^{-1}(\varepsilon))}{\varepsilon}} = 0$,

(iii) $(-g')^{-1}(\varepsilon) \approx \frac{g((-g')^{-1}(\varepsilon))}{\varepsilon}$, for $\varepsilon \to 0$.

To avoid inversion, we can rewrite the conditions by substituting $x = (-g')^{-1}(\varepsilon)$:

(i) $\lim_{x \to \infty} \frac{\frac{g(x)}{(-g')(x)}}{x} = 0$,

(ii) $\lim_{x \to \infty} \frac{x}{\frac{g(x)}{(-g')(x)}} = 0$,

(iii) $x \approx \frac{g(x)}{(-g')(x)}$, for $x \to \infty$.

First, observe that g and g^H belong to the same case. In fact, for case (i),

$$0 = \lim_{\varepsilon \to 0} \frac{\frac{g((-g')^{-1}(\varepsilon))}{\varepsilon}}{(-g')^{-1}(\varepsilon)} = \lim_{x \to \infty} \frac{g(x)}{(-g')(x)x} = H \lim_{x \to \infty} \frac{g^H(x)}{Hg^{H-1}(x)(-g')(x)x}$$

$$= H \lim_{x \to \infty} \frac{g^H(x)}{(-g^H)'(x)x} = H \lim_{\varepsilon \to 0} \frac{\frac{g^H(((-g^H)')^{-1}(\varepsilon))}{\varepsilon}}{((-g^H)')^{-1}(\varepsilon)}.$$

The two remaining cases can be treated similarly, but in case (iii) one has to work with lim sup instead of lim. In particular, we can deduce from the last computation that $g(x) \leqslant (-g')(x)x$ holds for large x.

We start with the proof of the first assertion and we consider case (i). Because $-(g^H)'$ is a strictly decreasing function on $\mathbb{R}^+$, we can substitute occurrences of ε by $-(g^H)'(x)$. This means that we have to change the limits from $\varepsilon \to 0$ to $x \to \infty$. Let $A \in \mathcal{D}$. For convenience, we neglect the argument of A. Then

$$\limsup_{\varepsilon\to 0} \mathbb{P}\left\{N(B_H((\alpha_n)),|\cdot|,\varepsilon) > A\cdot N((\alpha_n^H),|\cdot|,\varepsilon)\right\}$$

$$\leqslant \limsup_{\varepsilon\to 0} \mathbb{P}\left\{N(B_H((\alpha_n)),|\cdot|,\varepsilon) > c_1 A\cdot(-(g^H)')^{-1}(\varepsilon)\right\}$$

$$\leqslant \limsup_{\varepsilon\to 0} \mathbb{P}\left\{N((B_H(\alpha_n))_{n=\lceil(-(g^H)')^{-1}(\varepsilon)\rceil}^{\infty},|\cdot|,\varepsilon)\right.$$

$$\left.+N((B_H(\alpha_n))_1^{n=\lceil(-(g^H)')^{-1}(\varepsilon)\rceil-1},|\cdot|,\varepsilon) > c_1 A\cdot(-(g^H)')^{-1}(\varepsilon)\right\}$$

$$\leqslant \limsup_{\varepsilon\to 0} \mathbb{P}\left\{N((B_H(\alpha_n))_{n=\lceil(-(g^H)')^{-1}(\varepsilon)\rceil}^{\infty},|\cdot|,\varepsilon) > c_2 A\cdot(-(g^H)')^{-1}(\varepsilon)\right\}$$

$$\leqslant \limsup_{\varepsilon\to 0} \mathbb{P}\left\{\frac{\sup\limits_{n\geqslant\lceil(-(g^H)')^{-1}(\varepsilon)\rceil}|B_H(\alpha_n)|}{\varepsilon} > c_2 A\cdot(-(g^H)')^{-1}(\varepsilon)\right\}$$

$$\leqslant \limsup_{\varepsilon\to 0} \mathbb{P}\left\{\sup_{t\in[0,g((-(g^H)')^{-1}(\varepsilon))]}|B_H(t)| > c_2 A\cdot(-(g^H)')^{-1}(\varepsilon)\varepsilon\right\},$$

where we used that a covering of a finite set M consists of at most $|M|$ balls (for a finite set, $|\cdot|$ denotes the number of its elements) and we used that a bounded set B can be covered by $\left\lceil\frac{|B|}{2\varepsilon}\right\rceil$ balls. During the next steps, we use the self-similarity of B_H, the chain rule for derivatives and the properties of case (i), i.e., $g(x)\leqslant -g'(x)x$ for large x. This leads to

$$\limsup_{\varepsilon\to 0} \mathbb{P}\left\{N(B_H((\alpha_n)),|\cdot|,\varepsilon) > A\cdot N((\alpha_n^H),|\cdot|,\varepsilon)\right\}$$

$$\leqslant \limsup_{\varepsilon\to 0} \mathbb{P}\left\{g^H((-(g^H)')^{-1}(\varepsilon))\sup_{t\in[0,1]}|B_H(t)| > c_2 A\cdot(-(g^H)')^{-1}(\varepsilon)\varepsilon\right\}$$

$$=\limsup_{x\to\infty} \mathbb{P}\left\{\sup_{t\in[0,1]}|B_H(t)| > c_2 A\cdot x(-g^H)'(x)\cdot g^{-H}(x)\right\}$$

$$=\limsup_{x\to\infty} \mathbb{P}\left\{\sup_{t\in[0,1]}|B_H(t)| > c_2 A\cdot Hxg^{H-1}(x)(-g'(x))\cdot g^{-H}(x)\right\}$$

$$=\limsup_{x\to\infty} \mathbb{P}\left\{\sup_{t\in[0,1]}|B_H(t)| > c_2 A\cdot H\frac{x}{g(x)}(-g'(x))\right\}$$

$$\leqslant \limsup_{x\to\infty} \mathbb{P}\left\{\sup_{t\in[0,1]}|B_H(t)| > c_2 A\cdot H\right\} \leqslant \lim_{x\to\infty}\frac{c_3}{A((-g^H)'(x))} = 0.$$

The last step is an application of Markov's inequality, where the constant c_3 depends on c_2 and $\mathbb{E}\sup\limits_{t\in[0,1]}|B_H(t)|$, which is finite (cf. the discussion at the beginning of this section). The result implies that the limit superiors are in fact limits, what finishes case (i).

In case (ii) we use similar arguments, but this time, we have $g^H(x) \geqslant (-g^H)'(x)x$ for large x.

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} \mathbb{P}\left\{N(B_H((\alpha_n)),|\cdot|,\varepsilon) > A\cdot N((\alpha_n^H),|\cdot|,\varepsilon)\right\} \\
&\leqslant \limsup_{\varepsilon\to 0} \mathbb{P}\left\{N(B_H((\alpha_n)),|\cdot|,\varepsilon) > c_1 A\cdot \frac{g^H(((-g^H)')^{-1}(\varepsilon))}{\varepsilon}\right\} \\
&\leqslant \limsup_{\varepsilon\to 0} \mathbb{P}\left\{N((B_H(\alpha_n))_{n=\lceil \frac{g^H(((-g^H)')^{-1}(\varepsilon))}{\varepsilon}\rceil}^{\infty},|\cdot|,\varepsilon) > c_2 A\cdot \frac{g^H(((-g^H)')^{-1}(\varepsilon))}{\varepsilon}\right\} \\
&\leqslant \limsup_{\varepsilon\to 0} \mathbb{P}\left\{\frac{\sup_{n\geqslant \lceil \frac{g^H(((-g^H)')^{-1}(\varepsilon))}{\varepsilon}\rceil} |B_H(\alpha_n)|}{\varepsilon} > c_2 A\cdot \frac{g^H(((-g^H)')^{-1}(\varepsilon))}{\varepsilon}\right\} \\
&\leqslant \limsup_{\varepsilon\to 0} \mathbb{P}\left\{\sup_{t\in[0,g(\frac{g^H(((-g^H)')^{-1}(\varepsilon))}{\varepsilon})]} |B_H(t)| > c_2 A\cdot g^H(((-g^H)')^{-1}(\varepsilon))\right\} \\
&= \limsup_{\varepsilon\to 0} \mathbb{P}\left\{g^H(\frac{g^H(((-g^H)')^{-1}(\varepsilon))}{\varepsilon})\sup_{t\in[0,1]} |B_H(t)| > c_2 A\cdot g^H(((-g^H)')^{-1}(\varepsilon))\right\} \\
&= \limsup_{x\to\infty} \mathbb{P}\left\{\sup_{t\in[0,1]} |B_H(t)| > c_2 A\cdot g^H(x) g^{-H}(\frac{g^H(x)}{(-g^H)'(x)})\right\} \\
&\leqslant \limsup_{x\to\infty} \mathbb{P}\left\{\sup_{t\in[0,1]} |B_H(t)| > c_2 A\cdot g^H(x) g^{-H}(x)\right\} \\
&\leqslant \lim_{x\to\infty} \frac{c_3}{A((-g^H)'(x))} = 0.
\end{aligned}
$$

Case (iii) can be handled as case (i), but there is a small difference: We do not have $g(x) \leqslant (-g')(x)x$ for large x, but we have $g(x) \leqslant c_4(-g')(x)x$ for some positive constant c_4. This leads to an additional constant in the last steps of the proof of case (i). Thus, the proof of the first assertion is finished.

We continue with the proof of assertion 2.b), meaning that we are situated in case (i). If necessary, we use the convention $\frac{1}{0} = \infty$. We have to show that $\frac{1}{(-g')^{-1}(\varepsilon)}$ is an upper bound for $F((\alpha_n), 2\varepsilon)$.

To this end, let $\varepsilon > 0$ and define $\varepsilon_0 := \varepsilon^{\frac{1}{H}}((-g')^{-1}(\varepsilon))^{\frac{1-H}{H}}$ and $x = (-g')^{-1}(\varepsilon)$. Because of $N((\alpha_n),|\cdot|,\varepsilon_0) \approx (-g')^{-1}(\varepsilon_0)$, there are $M := \lfloor c_1(-g')^{-1}(\varepsilon_0)\rfloor$ points in (α_n) having at least the distance ε_0 (pairwise). We denote the collection of these points by D. Let μ be the uniform distribution on D, then μ is an element of $\mathcal{P}((\alpha_n))$. Using Lemma 3.5, we can estimate

$$F((\alpha_n),2\varepsilon) \leqslant \int\limits_{(\alpha_n)}\int\limits_{(\alpha_n)} \min\{\frac{2\varepsilon}{\sqrt{\frac{\pi}{2}}|t-s|^H},1\}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)$$

$$= \sum_{s\in D}\sum_{t\in D}\min\{\frac{2\varepsilon}{\sqrt{\frac{\pi}{2}}|t-s|^H},1\}\frac{1}{M^2}.$$

The elements of D can be ordered from left to right and are denoted by $(b_i)_{i=1}^M$. Because we know that neighbored points (i.e. b_i and b_{i+1}) have at least the distance ε_0, we may conclude that the estimate $|b_i - b_j| \geqslant |i-j|\varepsilon_0$ is true, too. Then, by Lemma 3.3,

$$F((\alpha_n),2\varepsilon) \leqslant \sum_{i=1}^{M}\sum_{j=1}^{M}\min\{\frac{2\varepsilon}{\sqrt{\frac{\pi}{2}}|i-j|^H\varepsilon_0^H},1\}\frac{1}{M^2}$$

$$\leqslant 2\sum_{i=1}^{M-1}\sum_{j=i+1}^{M}\frac{2\varepsilon}{\sqrt{\frac{\pi}{2}}(j-i)^H\varepsilon_0^H}\frac{1}{M^2}+\frac{1}{M}$$

$$\leqslant c_2 M^{-H}\varepsilon\varepsilon_0^{-H} + M^{-1} \leqslant c_3((-g')^{-1}(\varepsilon_0))^{-H}\varepsilon_0^{-H}\varepsilon + c_4((-g')^{-1}(\varepsilon_0))^{-1}$$

$$= c_3((-g')^{-1}(\varepsilon^{\frac{1}{H}}((-g')^{-1}(\varepsilon))^{\frac{1-H}{H}}))^{-H}\varepsilon^{-1}((-g')^{-1}(\varepsilon))^{H-1}\varepsilon$$

$$+c_4((-g')^{-1}(\varepsilon^{\frac{1}{H}}((-g')^{-1}(\varepsilon))^{\frac{1-H}{H}}))^{-1}.$$

Again, we substitute $\varepsilon = -g'(x)$. Because g is convex, Lemma 1.18 implies that $-g'(x) \leqslant \frac{1}{x}$ is valid for large x. Observe that $[(-g')^{-1}(y)]^{-1}$ is increasing in y. Then

$$F((\alpha_n),2\varepsilon) = F((\alpha_n),-2g'(x))$$

$$\leqslant c_3((-g')^{-1}((-g'(x))^{\frac{1}{H}}x^{\frac{1-H}{H}}))^{-H}x^{H-1} + c_4((-g')^{-1}((-g'(x))^{\frac{1}{H}}x^{\frac{1-H}{H}}))^{-1}$$

$$\leqslant c_3((-g')^{-1}((-g'(x))^{\frac{1}{H}}(-g'(x))^{-\frac{1-H}{H}}))^{-H}x^{H-1}$$

$$+c_4((-g')^{-1}((-g'(x))^{\frac{1}{H}}(-g'(x))^{-\frac{1-H}{H}}))^{-1}$$

$$= c_3x^{-H}\cdot x^{H-1} + c_4x^{-1} = c_5x^{-1} = c_5[(-g')^{-1}(\varepsilon)]^{-1},$$

what finishes the proof of assertion 2.(b).

Next, we consider assertion 2.(a). Let $\varepsilon > 0$ and define $\varepsilon_0 = \frac{1}{H}\varepsilon g^{1-H}(((-g^H)')^{-1}(\varepsilon))$. Because of case (ii) or (iii), we know that there are $M := \lfloor c_1 g((-g')^{-1}(\varepsilon_0))\varepsilon_0^{-1}\rfloor$ points having pairwise distances larger than ε_0. Repeating the computation from case (b) yields

$$F((\alpha_n),2\varepsilon) \leqslant c_2 g((-g')^{-1}(\varepsilon_0))^{-H}\varepsilon_0^H\frac{\varepsilon}{\varepsilon_0^H} + c_3 g((-g')^{-1}(\varepsilon_0))^{-1}\varepsilon_0$$

$$= c_2 g((-g')^{-1}(\frac{1}{H}\varepsilon g^{1-H}(((-g^H)')^{-1}(\varepsilon))))^{-H}\varepsilon$$

$$+c_4 g((-g')^{-1}(\frac{1}{H}\varepsilon g^{1-H}(((-g^H)')^{-1}(\varepsilon))))^{-1}\varepsilon g^{1-H}(((-g^H)')^{-1}(\varepsilon)).$$

The substitution $\varepsilon = (-g^H)'(x)$ and the formula for $(-g^H)'(x)$ lead to

$$\begin{aligned}
F((\alpha_n), 2(-g^H)'(x)) &\leqslant c_2 g^{-H}((-g')^{-1}(\frac{1}{H}(-g^H)'(x)g^{1-H}(x)))(-g^H)'(x) \\
&+c_4 g((-g')^{-1}(\frac{1}{H}(-g^H)'(x)g^{1-H}(x)))^{-1}(-g^H)'(x)g^{1-H}(x) \\
&= c_2 g^{-H}((-g')^{-1}(g^{H-1}(x)(-g'(x))g^{1-H}(x)))(-g^H)'(x) \\
&+c_4 g((-g')^{-1}(g^{H-1}(x)(-g'(x))g^{1-H}(x)))^{-1}(-g^H)'(x)g^{1-H}(x) \\
&= c_2 g^{-H}(x)(-g^H)'(x) + c_4 g^{-H}(x)(-g^H)'(x) = c_5 g^{-H}(((-g^H)')^{-1}(\varepsilon))\varepsilon,
\end{aligned}$$

what gives the result in this case.

Next, we prove the third assertion. The idea of the following proof is the same as in [19], Lemmas 3.2 and 3.3. We start with some notation from there. For $\varepsilon > 0$ and $E \subseteq \mathbb{R}^+$, we define

$$Z_\infty(\varepsilon, E) := \inf_{\mu \in \mathcal{P}(E)} \int_E \int_E \mathbb{1}_{[0,\varepsilon]}(|x-y|)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y),$$

where $\mathcal{P}(E)$ is the set of all probability measures on $\mathbb{R}^+$ being concentrated on E. It is also possible to work with the set of all probability measures which are supported by a finite number of points in E (cf. [19]). It is known (cf. [12], Lemma 2, Lemma 5) that $M(E, |\cdot|, \frac{\varepsilon}{2}) = \frac{1}{Z_\infty(\varepsilon, E)}$, what gives the measurability of $Z_\infty(\varepsilon, B_H(E))$ (cf. Lemma 1.33.

Note that $\mu \circ B_H^{-1}$ is a probability measure on $B_H(E)$ whenever μ is a probability measure on E. Here, we have $E = (\alpha_n)$ and again the computation depends on the rate of $N((\alpha_n), |\cdot|, \varepsilon)$. We start with 3.b). Let $\varepsilon > 0$ and define $\varepsilon_0 = \varepsilon^{\frac{1}{H}}((-g')^{-1}(\varepsilon))^{\frac{1-H}{H}}$. Being in case (i) implies the existence of $M := \lfloor c_1(-g')^{-1}(\varepsilon_0) \rfloor$ points having pairwise distances larger than ε_0. Let μ be the uniform distribution on these points. Then, by Fatou's lemma, transformation theorem and Fubini's theorem,

$$\begin{aligned}
\mathbb{E} \liminf_{\varepsilon \to 0} \frac{Z_\infty(4\varepsilon, B_H((\alpha_n)))}{A(\varepsilon)[(-g')^{-1}(\varepsilon)]^{-1}} &\leqslant \liminf_{\varepsilon \to 0} \frac{\mathbb{E} Z_\infty(4\varepsilon, B_H((\alpha_n)))}{A(\varepsilon)[(-g')^{-1}(\varepsilon)]^{-1}} \\
\leqslant \liminf_{\varepsilon \to 0} & \frac{\mathbb{E} \int_{B_H((\alpha_n))} \int_{B_H((\alpha_n))} \mathbb{1}_{[0,4\varepsilon]}(|x-y|)\, \mathrm{d}\mu \circ B_H^{-1}(x)\, \mathrm{d}\mu \circ B_H^{-1}(y)}{A(\varepsilon)[(-g')^{-1}(\varepsilon)]^{-1}} \\
\leqslant \liminf_{\varepsilon \to 0} & \frac{\mathbb{E} \int_{(\alpha_n)} \int_{(\alpha_n)} \mathbb{1}_{[0,4\varepsilon]}(|B_H(x) - B_H(y)|)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y)}{A(\varepsilon)[(-g')^{-1}(\varepsilon)]^{-1}} \\
\leqslant \liminf_{\varepsilon \to 0} & \frac{\int_{(\alpha_n)} \int_{(\alpha_n)} \mathbb{P}\{|B_H(x) - B_H(y)| \leqslant 4\varepsilon\}\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y)}{A(\varepsilon)[(-g')^{-1}(\varepsilon)]^{-1}}
\end{aligned}$$

$$\leqslant \liminf_{\varepsilon\to 0} \frac{c_2[(-g')^{-1}(\varepsilon)]^{-1}}{A(\varepsilon)[(-g')^{-1}(\varepsilon)]^{-1}} = 0,$$

where the last estimate can be obtained by repeating the computation made during deriving the second assertion. We may conclude that

$$\liminf_{\varepsilon\to 0} \frac{Z_\infty(4\varepsilon, B_H((\alpha_n)))}{A(\varepsilon)[(-g')^{-1}(\varepsilon)]^{-1}} = 0,$$

almost surely. Given ω from a full measure set, the definition of the limit inferior implies that there is a sequence $(\varepsilon_k) = (\varepsilon_k(\omega))$, going to 0, such that

$$\frac{Z_\infty(4\varepsilon_k, B_H((\alpha_n)))(\omega)}{A(\varepsilon_k)[(-g')^{-1}(\varepsilon_k)]^{-1}} \leqslant 1$$

is true for every k. This can be reformulated as

$$\frac{(-g')^{-1}(\varepsilon_k)}{A(\varepsilon_k)} \leqslant \frac{1}{Z_\infty(4\varepsilon_k, B_H((\alpha_n)))(\omega)} = M(B_H((\alpha_n)), |\cdot|, 2\varepsilon_k)(\omega)$$
$$\leqslant N(B_H((\alpha_n)), |\cdot|, \varepsilon_k)(\omega),$$

what finishes the proof of the covering part of assertion 3.(b).
Assertion 3.(a), i.e., the cases (ii) and (iii), can be handled similarly.
To derive the dimension result (3.(a)), choose A in such a way that $\lim_{\varepsilon\to 0} \frac{\log \frac{1}{A(\varepsilon)}}{\log \varepsilon^{-1}} = 0$. Now, for ω from a full measure set, choose a sequence (ε_k) according to the first part. Then, by the definition of the limit superior and case (ii) and (iii),

$$\overline{\dim}_E B_H((\alpha_n))(\omega) = \limsup_{\varepsilon\to 0} \frac{\log N(B_H((\alpha_n)), |\cdot|, \varepsilon)(\omega)}{|\log \varepsilon|}$$

$$\geqslant \limsup_{\varepsilon_k\to 0} \frac{\log N(B_H((\alpha_n)), |\cdot|, \varepsilon_k)(\omega)}{|\log \varepsilon_k|} \geqslant \limsup_{\varepsilon_k\to 0} \frac{\log N((\alpha_n^H), |\cdot|, \varepsilon_k) + \log \frac{1}{A(\varepsilon_k)}}{|\log \varepsilon_k|}$$

$$= \lim_{\varepsilon\to 0} \frac{\log N((\alpha_n^H), |\cdot|, \varepsilon)}{|\log \varepsilon|} = \lim_{n\to\infty} \frac{\log \frac{g^H(n)}{-(g^H)'(n)}}{-\log(-g^H)'(n)}$$

$$= \lim_{n\to\infty} \frac{\log \frac{g(n)}{-g'(n)H}}{-\log g^{H-1}(n)(-g)'(n)} = \lim_{n\to\infty} \frac{\log \frac{g(n)}{-g'(n)}}{(1-H)\log g(n) - \log(-g)'(n)}$$

$$= \lim_{n\to\infty} \frac{\log \frac{g(n)}{-g'(n)}}{(1-H)\log g(n) - (1-H)\log(-g)'(n) - H\log(-g)'(n)}$$

$$= \lim_{n\to\infty} \frac{\frac{\log \frac{g(n)}{-g'(n)}}{-\log(-g)'(n)}}{(1-H)\frac{\log \frac{g(n)}{-g'(n)}}{-\log(-g)'(n)} + H} = \frac{\dim_E(\alpha_n)}{(1-H)\dim_E(\alpha_n) + H}.$$

The dimension result of part (b) is clear and the proof is finished. ☐

Proof of Corollary 2.10. We compute the necessary functions.

- $g(x) = x^{-\beta}$, $\beta > 0$, induces a case (iii) sequence. Then $-g'(x) = \beta x^{-(\beta+1)}$ and $(-g')^{-1}(\varepsilon) = \beta^{\frac{1}{1+\beta}}\varepsilon^{-\frac{1}{1+\beta}}$. Thus,

$$N((\alpha_n), |\cdot|, \varepsilon) \approx \max\{\frac{\beta^{-\frac{\beta}{1+\beta}}\varepsilon^{\frac{\beta}{1+\beta}}}{\varepsilon}, \beta^{\frac{1}{1+\beta}}\varepsilon^{-\frac{1}{1+\beta}}\} \approx \varepsilon^{-\frac{1}{1+\beta}}.$$

$(-g^H)'(x) = H\beta x^{-(H\beta+1)}$ and $((-g^H)')^{-1}(\varepsilon) = (H\beta)^{\frac{1}{1+H\beta}}\varepsilon^{-\frac{1}{1+H\beta}}$. Thus,

$$N((\alpha_n^H), |\cdot|, \varepsilon) \approx \max\{\frac{(H\beta)^{-\frac{H\beta}{1+H\beta}}\varepsilon^{\frac{H\beta}{1+H\beta}}}{\varepsilon}, (H\beta)^{\frac{1}{1+H\beta}}\varepsilon^{-\frac{1}{1+H\beta}}\} \approx \varepsilon^{-\frac{1}{1+H\beta}}.$$

It follows $\dim_E(\alpha_n) = \frac{1}{1+\beta}$ and $\dim_E(\alpha_n^H) = \frac{1}{1+H\beta}$. The theorem implies

$$N(B_H((\alpha_n)), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\approx} \varepsilon^{-\frac{1}{1+H\beta}}$$

and the definition of $\dim_E^{\mathbb{P}}$ gives

$$\dim_E^{\mathbb{P}} B_H((\alpha_n)) = \frac{1}{1+H\beta} = \frac{\frac{1}{1+\beta}}{H + (1-H)\frac{1}{1+\beta}} = \frac{\dim_E(\alpha_n)}{H + (1-H)\dim_E(\alpha_n)}.$$

- $g(x) = (1 + \log x)^{-c}$, $c > 0$, induces a case (ii) sequence. Then

$$-g'(x) = c(1 + \log x)^{-(c+1)}\frac{1}{x} \text{ and } (-g')^{-1}(\varepsilon) \sim c\varepsilon^{-1}|\log\varepsilon|^{-(c+1)}.$$

For the covering numbers this means

$$N((\alpha_n), |\cdot|, \varepsilon) \approx \max\{\frac{|\log\varepsilon|^{-c}}{\varepsilon}, \varepsilon^{-1}|\log\varepsilon|^{-(c+1)}\} \approx \varepsilon^{-1}|\log\varepsilon|^{-c}.$$

The same computation and the theorem give

$$N(B_H((\alpha_n)), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\approx} N((\alpha_n^H), |\cdot|, \varepsilon) \approx \varepsilon^{-1}|\log\varepsilon|^{-cH}.$$

Consequently,

$$\dim_E(\alpha_n) = \dim_E(\alpha_n^H) = \dim_E^{\mathbb{P}} B_H(\alpha_n) = 1 = \frac{1}{H + (1-H)\cdot 1}$$

$$= \frac{\dim_E((\alpha_n))}{H + (1-H)\dim_E((\alpha_n))}.$$

- $g(x) = x^{-\beta}(1 + \log x)^{-c}$, $\beta, c > 0$, induces a case (iii) sequence. Then

$$-g'(x) = x^{-\beta-1}(1+\log x)^{-c}(\beta + \frac{c}{1+\log x}) \text{ and } (-g')^{-1}(\varepsilon) \sim b\varepsilon^{-\frac{1}{1+\beta}}|\log\varepsilon|^{-\frac{c}{1+\beta}}$$

with some constant b. This means

$$N((\alpha_n), |\cdot|, \varepsilon) \approx \varepsilon^{-\frac{1}{1+\beta}} |\log \varepsilon|^{-\frac{c}{1+\beta}}$$

and

$$N(B_H((\alpha_n)), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\approx} N((\alpha_n^H), |\cdot|, \varepsilon) \approx \varepsilon^{-\frac{1}{1+H\beta}} |\log \varepsilon|^{-\frac{Hc}{1+H\beta}}.$$

Consequently,

$$\dim_E^{\mathbb{P}} B_H((\alpha_n)) = \frac{1}{1+H\beta} = \frac{\frac{1}{1+\beta}}{H + (1-H)\frac{1}{1+\beta}} = \frac{\dim_E(\alpha_n)}{H + (1-H)\dim_E(\alpha_n)}.$$

- $g(x) = q^x$, $q \in (0,1)$, induces a case (i) sequence. Then $-g'(x) = |\log q| q^x$ and $(-g')^{-1}(\varepsilon) = \log_q \frac{\varepsilon}{|\log q|} = \frac{1}{|\log q|} \log \frac{\varepsilon^{-1}}{|\log q|^{-1}} \approx |\log \varepsilon|$. The covering number rate is

$$N((\alpha_n), |\cdot|, \varepsilon) \approx \max\{\frac{q^{\log_q \frac{\varepsilon}{|\log q|}}}{\varepsilon}, \frac{1}{|\log q|} \log \frac{\varepsilon^{-1}}{|\log q|^{-1}}\} \approx |\log \varepsilon|.$$

The same rate is valid for $N((\alpha_n^H), |\cdot|, \varepsilon)$ because q^H is again an element of $(0,1)$. The results $N((\alpha_n), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\preccurlyeq} N(B_H((\alpha_n)), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\preccurlyeq} N((\alpha_n^H), |\cdot|, \varepsilon)$ imply

$$N(B_H((\alpha_n)), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\approx} |\log \varepsilon|.$$

The dimension formula is also true:

$$\dim_E^{\mathbb{P}}(B_H(\alpha_n)) = 0 = \frac{0}{H + (1-H)\cdot 0} = \frac{\dim_E(\alpha_n)}{H + (1-H)\dim_E(\alpha_n)}.$$

- $g(x) = q^{x^\alpha}$, $q \in (0,1)$, $\alpha > 0$, induces a case (i) sequence. Then $-g'(x) = |\log q| \alpha x^{\alpha-1} q^{x^\alpha}$ is strictly decreasing for $x \geqslant x(\alpha, q)$. A direct inversion is not possible, so we must use asymptotics (cf. Lemma 3.6). The rate

$$N((\alpha_n), |\cdot|, \varepsilon) \approx (-g')^{-1}(\varepsilon) \sim [\log_q(\frac{1}{\alpha |\log q|} \varepsilon (\log_q \varepsilon)^{\frac{1-\alpha}{\alpha}})]^{\frac{1}{\alpha}} \approx |\log \varepsilon|^{\frac{1}{\alpha}}$$

is also valid for $N((\alpha_n^H), |\cdot|, \varepsilon)$. Thus,

$$N(B_H((\alpha_n)), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\approx} |\log \varepsilon|^{\frac{1}{\alpha}}$$

and the dimension formula is fulfilled, too.

□

Lemma 3.6. *1. Let $f(x) = |\log q|\alpha x^{\alpha-1}q^{x^\alpha}$. Then f is strictly decreasing for $x \geqslant x(\alpha, q)$ and the following asymptotic is valid for f^{-1} and $\varepsilon \to 0$:*

$$f^{-1}(\varepsilon) \sim [\log_q(\frac{1}{\alpha|\log q|}\varepsilon(\log_q \varepsilon)^{\frac{1-\alpha}{\alpha}})]^{\frac{1}{\alpha}}.$$

2. Let $f(x) = c(1+\log x)^{-(c+1)}\frac{1}{x}$. Then f is strictly decreasing for $x > 0$ and

$$f^{-1}(\varepsilon) \sim c\varepsilon^{-1}|\log \varepsilon|^{-(c+1)},\ \textit{for } \varepsilon \to 0.$$

3. Let $f(x) = x^{-\beta-1}(1+\log x)^{-c}(\beta + \frac{c}{1+\log x})$. Then f is strictly decreasing for $x > 0$ and

$$f^{-1}(\varepsilon) \sim \beta^{\frac{1}{1+\beta}}(\beta+1)^{\frac{c}{1+\beta}}\varepsilon^{-\frac{1}{1+\beta}}|\log \varepsilon|^{-\frac{c}{1+\beta}},\ \textit{for } \varepsilon \to 0.$$

Proof. The monotonicity was discussed during the proof of Corollary 1.17. It remains to verify the asymptotics. The idea is from Aurzada's ([1]) proofs of Theorems 2.7 and 2.8. During the following computations, we subsitute $\varepsilon = f(x)$.

1.

$$\lim_{\varepsilon\to 0} \frac{[\log_q(\frac{1}{\alpha|\log q|}\varepsilon(\log_q \varepsilon)^{\frac{1-\alpha}{\alpha}})]^{\frac{1}{\alpha}}}{f^{-1}(\varepsilon)}$$

$$= \lim_{x\to\infty} \frac{[\log_q(\frac{1}{\alpha|\log q|}|\log q|\alpha x^{\alpha-1}q^{x^\alpha}(\log_q(|\log q|\alpha x^{\alpha-1}q^{x^\alpha}))^{\frac{1-\alpha}{\alpha}})]^{\frac{1}{\alpha}}}{x}$$

$$= \lim_{x\to\infty}[\frac{\log_q(x^{\alpha-1}q^{x^\alpha}(\log_q(|\log q|\alpha x^{\alpha-1}q^{x^\alpha}))^{\frac{1-\alpha}{\alpha}})}{x^\alpha}]^{\frac{1}{\alpha}}$$

$$= \lim_{x\to\infty}[\frac{(\alpha-1)\log_q x}{x^\alpha} + 1 + \frac{(\frac{1-\alpha}{\alpha})\log_q(\log_q(|\log q|\alpha x^{\alpha-1}q^{x^\alpha}))}{x^\alpha}]^{\frac{1}{\alpha}}$$

$$= \lim_{x\to\infty}[1 + \frac{(\frac{1-\alpha}{\alpha})\log_q(\log_q(|\log q|\alpha) + (\alpha-1)\log_q x + x^\alpha)}{x^\alpha}]^{\frac{1}{\alpha}} = 1$$

because $\lim_{x\to\infty}\frac{\log x^\alpha}{x^\alpha} = 0$.

2.

$$\lim_{\varepsilon\to 0}\frac{c\varepsilon^{-1}|\log\varepsilon|^{-(c+1)}}{f^{-1}(\varepsilon)} = \lim_{x\to\infty}\frac{(1+\log x)^{(c+1)}x(\log(\frac{1}{c}(1+\log x)^{(c+1)}x))^{-(c+1)}}{x}$$

$$= \lim_{x\to\infty}[\frac{\log(\frac{1}{c}) + (c+1)\log(1+\log x) + \log x}{1+\log x}]^{-(c+1)} = 1.$$

3.

$$\lim_{\varepsilon\to 0}\frac{\beta^{\frac{1}{1+\beta}}(\beta+1)^{\frac{c}{1+\beta}}\varepsilon^{-\frac{1}{1+\beta}}|\log\varepsilon|^{-\frac{c}{1+\beta}}}{f^{-1}(\varepsilon)}$$

$$= \lim_{x\to\infty} \frac{\beta^{\frac{1}{1+\beta}} x(1+\log x)^{\frac{c}{1+\beta}} (\log(x^{+\beta+1}(1+\log x)^c(\beta+\frac{c}{1+\log x})^{-1}))^{-\frac{c}{1+\beta}}}{(\beta+1)^{-\frac{c}{1+\beta}}(\beta+\frac{c}{1+\log x})^{\frac{1}{1+\beta}} x}$$

$$= \lim_{x\to\infty} \frac{(\beta+1)^{\frac{c}{1+\beta}} (\log(x^{+\beta+1}(1+\log x)^c(\beta+\frac{c}{1+\log x})^{-1}))^{-\frac{c}{1+\beta}}}{(1+\log x)^{-\frac{c}{1+\beta}}}$$

$$= \lim_{x\to\infty} (\beta+1)^{\frac{c}{1+\beta}} [\frac{(\beta+1)\log x + c\log(1+\log x) - \log(\beta+\frac{c}{1+\log x})}{1+\log x}]^{-\frac{c}{1+\beta}} = 1.$$

□

Proof of Lemma 2.12. We apply the techniques developed during the proof of the last theorem. Let $\varepsilon > 0$ and define $\varepsilon_0 := \varepsilon^{\frac{1}{H+d(1-H)}}$. By assumption, we may derive that there are $M := \lfloor c_1 \varepsilon_0^{-d} \rfloor$ points with minimal distance $\varepsilon_0 > 0$, where c_1 is some positive constant. Denote the collection of these points by D and let μ be the uniform distribution on D. Then

$$\int_T \int_T \mathbb{P}\{|B_H(s) - B_H(t)| \leqslant 4\varepsilon\} \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) \leqslant c_2 \sum_{s\in D} \sum_{t\in D} \min\{\frac{\varepsilon}{|s-t|^H}, 1\} \frac{1}{M^2}$$

$$\leqslant c_3 \sum_{i=1}^{M} \sum_{j=1, j\neq i}^{M} \frac{\varepsilon}{|i-j|^H \varepsilon_0^H} \frac{1}{M^2} + c_4 M^{-1} \leqslant c_5 M^{-H} \varepsilon \varepsilon_0^{-H} + c_4 M^{-1}$$

$$= c_6 \varepsilon \varepsilon_0^{dH-H} + c_7 \varepsilon_0^d = c_8 \varepsilon^{\frac{d}{H+d(1-H)}}.$$

In order to get the assertion, follow the argumentation from above. To get results for non-polynomial covering rates, modify the choice of M and ε_0, where $M \approx N(T, |\cdot|, \varepsilon_0)$. □

3.2 α-stable processes

Let $X = (X_t)_{t\in\mathbb{R}^+}$ be a Lévy process. We know that $X_0 = 0$, almost surely, and X is right-continuous on $\mathbb{R}^+$ and it has left limits in each $t \in \mathbb{R}^+ \setminus \{0\}$, almost surely. Firstly, we must answer the question, if it makes sense to try to compute covering numbers, i.e., is the image of X bounded?

Lemma 3.7. *Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be right-continuous on $\mathbb{R}^+$ and let f have left limits in each $t \in \mathbb{R}^+ \setminus \{0\}$. Let $K \subseteq \mathbb{R}^+$ be compact. Then $f(K)$ is bounded.*

Proof. Assume that $f(K)$ is not bounded. Then, for every $n \in \mathbb{N}$, we can find an element $x_n \in K$ such that $|f(x_n)| \geqslant n$. Consider the sequence (x_n). It is well-known that (x_n) has a monotone subsequence (x_{n_k}). Because K is compact, x_{n_k} converges to an element $x \in K$. Assume first that the subsequence is decreasing. By the right-continuity, $f(x_{n_k})$ converges to $f(x) \in \mathbb{R}$. A convergent sequence is always bounded, thus, $(f(x_{n_k}))$'s convergence contradicts $|f(x_{n_k})| \geqslant n_k$. Secondly, assume that (x_{n_k}) is increasing. Then the limit $\lim\limits_{k\to\infty} f(x_{n_k})$ exists in $\mathbb{R}$ and because $(f(x_{n_k}))$ converges, it must be bounded, too. Consequently, the assumption that $f(K)$ is not bounded must be false, what proves the lemma. □

Corollary 3.8. *Let $K \subseteq \mathbb{R}^+$ be bounded and let X be a Lévy process. Then $X(K)$ is bounded, almost surely, and $N(X(K), |\cdot|, \varepsilon)$ is finite for all $\varepsilon > 0$.*

Secondly, we prepare the application of Proposition 3.1 and Corollary 3.2.

Lemma 3.9. *1. Let X be α-stable. There is a constant $c = c(\alpha)$ such that*

$$\mathbb{P}\{|X_t - X_s| \leqslant \varepsilon\} \leqslant \min\{\frac{c\varepsilon}{|t-s|^{\frac{1}{\alpha}}}, 1\}$$

is true for every $s, t, \varepsilon > 0$.

2. Let X be strictly α-stable with parameters $(0, 0, \sigma)$ and let $K > 0$. Then there is a constant $c_1(K, \alpha)$ such that

$$\mathbb{P}\{|X_t - X_s| \leqslant \varepsilon\} \geqslant c_1 \min\{\frac{\varepsilon}{|t-s|^{\frac{1}{\alpha}}}, 1\}$$

is true for every $\varepsilon \in (0, 1)$ and $s, t \geqslant 0$ with $|s - t| \leqslant K$.

Proof.

1. Because of the stationarity, we may assume $s = 0$ and it suffices to estimate $\mathbb{P}\{|X_t| \leqslant \varepsilon\}$. Because $\mathbb{P}$ is a probability measure, $\mathbb{P}\{|X_t| \leqslant \varepsilon\} \leqslant 1$ is true. The random number X_t has the characteristic function

$$\mathbb{E}e^{iX_t z} = \begin{cases} \exp(iatz - (t^{\frac{1}{\alpha}}\sigma)^\alpha |z|^\alpha (1 - i\beta(\operatorname{sgn} z)\tan\frac{\pi\alpha}{2})), & \text{if } \alpha \neq 1 \\ \exp(iatz - t\sigma|z|(1 + i\beta\frac{2}{\pi}(\operatorname{sgn} z)\log|z|)), & \text{if } \alpha = 1. \end{cases}$$

If $\alpha \neq 1$, the density of X_t is given by (cf. [4], section 2)

$$p(x) = \frac{1}{\sigma t^{\frac{1}{\alpha}}} \int_0^\infty \exp(-y^\alpha) \cos(y \cdot \frac{x - at}{\sigma t^{\frac{1}{\alpha}}} - \beta y^\alpha \tan\frac{\pi\alpha}{2})\, \mathrm{d}y.$$

This leads to

$$\mathbb{P}\{|X_t| \leqslant \varepsilon\} = \int_{-\varepsilon}^{\varepsilon} p(x)\, \mathrm{d}x$$

$$\leqslant \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sigma t^{\frac{1}{\alpha}}} \int_0^\infty \exp(-y^\alpha) \cdot |\cos(y \cdot \frac{x - at}{\sigma t^{\frac{1}{\alpha}}} - \beta y^\alpha \tan\frac{\pi\alpha}{2})|\, \mathrm{d}y\, \mathrm{d}x$$

$$\leqslant \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sigma t^{\frac{1}{\alpha}}} \int_0^\infty \exp(-y^\alpha)\, \mathrm{d}y\, \mathrm{d}x = \frac{\varepsilon}{t^{\frac{1}{\alpha}}} \cdot \frac{2}{\sigma} \int_0^\infty \exp(-y^\alpha)\, \mathrm{d}y.$$

The assertion follows because the remaining integral is finite. The same is true for $\alpha = 1$, but the density must be modified (cf.[4], section 3.1).

2. Cf. [17], proof of Corollary 1.1.

□

Proof of Theorem 2.34.

1.+2. The proof goes along the same lines as the respective parts of the proof of Theorem 2.6 (use Lemma 3.9). Substitute $H = \frac{1}{\alpha}$. The upper bound proof is also valid for $\alpha = 1$, but occurrences of g^H must be substituted by g and the chain rule is not needed to compute the derivative of $g^H = g$.

3. The corresponding part in the proof of Theorem 2.6 must be modified because the double sum changes its limit behavior (cf. Lemma 3.3). If $\varepsilon > 0$ is given, let $\varepsilon_0 := \varepsilon^\alpha$. There are $M := N((\alpha_n), |\cdot|, \varepsilon_0)$ points having pairwise distances larger than ε_0. Let μ be the uniform distribution on these points. Then (cf. Lemma 3.9),

$$F((\alpha_n), 2\varepsilon) \leqslant \int\limits_{(\alpha_n)} \int\limits_{(\alpha_n)} \min\{\frac{c\varepsilon}{|s-t|^{\frac{1}{\alpha}}}, 1\} \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t)$$

$$\leqslant \frac{1}{M^2} \sum_{i=1}^{M} \sum_{j=1, j\neq i}^{M} \frac{c\varepsilon}{|i-j|^{\frac{1}{\alpha}} \varepsilon_0^{\frac{1}{\alpha}}} + \frac{1}{M} \leqslant c_1 \frac{1}{M} = c_1 \frac{1}{N((\alpha_n), |\cdot|, \varepsilon^\alpha)}.$$

It follows $\frac{1}{F((\alpha_n),2\varepsilon)} \succcurlyeq N((\alpha_n), |\cdot|, \varepsilon^\alpha)$, meaning

$$N(X((\alpha_n)), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} N((\alpha_n), |\cdot|, \varepsilon^\alpha).$$

4. Also in this case, the changed limit behavior of the double sum (cf. Lemma 3.3) requires a modification of the proof of Theorem 2.6.
Given $\varepsilon > 0$, there are $N((\alpha_n), |\cdot|, \varepsilon)$ points having pairwise distances larger than ε. Let μ be the uniform distribution on these points. Then

$$F((\alpha_n), 2\varepsilon) \leqslant \frac{1}{M^2} \sum_{i=1}^{M} \sum_{j=1, j\neq i}^{M} \frac{c\varepsilon}{|i-j|\varepsilon} + \frac{1}{M}$$

$$\leqslant c_1 \frac{\log M}{M} + \frac{1}{M} \leqslant c_2 \frac{\log N((\alpha_n), |\cdot|, \varepsilon)}{N((\alpha_n), |\cdot|, \varepsilon)}.$$

The assertion follows.

5.-7. Modify the proof of the third part of the proof of Theorem 2.6. Be careful with the different behavior of the double sum for $\alpha < 1$,$\alpha = 1$ and $\alpha > 1$.

8.-11. The results follow from the definition of the upper entropy dimension and 5. to 7., respectively. Choose $A \in \mathcal{D}$ in such a way that $A(\varepsilon) \preccurlyeq |\log \varepsilon|$, for $\varepsilon \to 0$. The additional log term in the case $\alpha = 1$ does not influence the result because $N((\alpha_n), |\cdot|, \varepsilon)$ can be estimated by $c\varepsilon^{-1}$ and $\frac{\log(\log(c\varepsilon^{-1}))}{\log \varepsilon^{-1}}$ goes to zero for $\varepsilon \to 0$.

□

3.3 Random sequences

Let $g : \mathbb{R}^+ \setminus \{0\} \longrightarrow \mathbb{R}^+ \setminus \{0\}$ and $h : \mathbb{R}^+ \setminus \{0\} \longrightarrow \mathbb{R}^+ \setminus \{0\}$ be strictly decreasing and continuously differentiable. In particular, the inverse functions g^{-1} and h^{-1} exist. Let c_1, c_2, c_3, c_4 be strictly positive constants and let $(s_n) \subseteq (0, 1]$ and (α_n) be real sequences so that we have $c_1 g(n) \leqslant \alpha_n \leqslant c_2 g(n)$ and $c_3 h(n) \leqslant s_n \leqslant c_4 h(n)$ for all $n \geqslant 1$.

Assume that there is a differentiable function H with $c_5 h(x) \leqslant H'(x) \leqslant c_6 h(x)$, where $c_5, c_6 > 0$.

Furthermore, let (ξ_n) be a sequence of independent and identically distributed random numbers and let (U_n) be a sequence of independent random numbers with $\mathbb{P}\{U_n = 1\} = 1 - \mathbb{P}\{U_n = 0\} = s_n$. Here, the independence of (U_n) means independence of the components of (U_n) and independence of (ξ_n). We consider the standard random sequence $X = (\alpha_n \xi_n)$ and the random sequence with deletion factor $Y = (\alpha_n U_n \xi_n)$.

Especially, we have in mind the case where $c_i = 1$ for $i \in \{1, \dots, 6\}$.

3.3.1 Boundedness

Firstly, we must check under which conditions X and Y are (a.s.) bounded subsets of $\mathbb{R}$, then it makes sense to compute their covering numbers. The idea of the following proof is the same as in Aurzada([1])'s Theorem 1.1.

Lemma 3.10. *1. Assume the finiteness of* $\mathbb{E}[g^{-1}(\frac{C}{|\xi_1|})\mathbb{1}_{[\frac{C}{g(1)},+\infty)}(|\xi_1|)]$ *for some* $C > 0$. *Then* $(\alpha_n \xi_n)$ *is bounded, almost surely.*

2. Assume the finiteness of $\mathbb{E}[H(g^{-1}(\frac{C}{|\xi_1|})\mathbb{1}_{[\frac{C}{g(1)},+\infty)}(|\xi_1|)]$ *for some* $C > 0$. *Then* $(\alpha_n U_n \xi_n)$ *is bounded, almost surely.*

Proof. If K and c_2 are strictly positive constants, $c_2 g(n)|\xi_1(\omega)| \geqslant K$ implies $|\xi_1(\omega)| > 0$. Thus, $g(n) \geqslant \frac{K}{c_2|\xi_1(\omega)|}$ is well-defined for $\omega \in \{c_2 g(n)|\xi_1| \geqslant K\}$.

1. Clearly,

$$\{\sup_{n \in \mathbb{N}} |\alpha_n \xi_n| < \infty\} = \bigcup_{K=1}^{\infty} \{|\alpha_n \xi_n| < K \text{ eventually}\}.$$

By Kolmogorov's Zero-One-Law, we have $\mathbb{P}\left\{\sup_{n\in\mathbb{N}} |\alpha_n \xi_n| < \infty\right\} \in \{0,1\}$ and by Borel's and Cantelli's Lemma, we have

$$\mathbb{P}\left\{\sup_{n\in\mathbb{N}} |\alpha_n \xi_n| = \infty\right\} = \lim_{K\to\infty} \mathbb{P}\left\{\alpha_n|\xi_n| \geqslant K \text{ infinitely often (i.o.)}\right\} = 0$$

if and only if $\sum_{n=1}^{\infty} \mathbb{P}\left\{\alpha_n|\xi_n| \geqslant K\right\}$ is finite for some K. Using the assumptions and Fubini's Theorem, we can estimate

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\alpha_n|\xi_n| \geqslant K\right\} \leqslant \sum_{n=2}^{\infty} \mathbb{P}\left\{c_2 g(n)|\xi_1| \geqslant K\right\} + 1$$

$$= \sum_{n=2}^{\infty} \mathbb{P}\left\{n \leqslant g^{-1}(\frac{K}{c_2|\xi_1|})\right\} + 1 \leqslant \int_1^{\infty} \mathbb{P}\left\{x \leqslant g^{-1}(\frac{K}{c_2|\xi_1|})\right\} \mathrm{d}x + 1$$

$$= \int_1^{\infty}\int_x^{\infty} \mathrm{d}\mathbb{P}_{g^{-1}(\frac{K}{c_2|\xi_1|})}(t)\,\mathrm{d}x + 1 = \int_{[1,\infty)}\int_1^{t} \mathrm{d}x\,\mathrm{d}\mathbb{P}_{g^{-1}(\frac{K}{c_2|\xi_1|})}(t) + 1$$

$$\leqslant \mathbb{E}[g^{-1}(\frac{K}{c_2|\xi_1|})\mathbb{1}_{[1,\infty)}(g^{-1}(\frac{K}{c_2|\xi_1|}))] + 1.$$

By assumption, the last term is finite if K is sufficiently large, what proves the assertion.

2. Again,

$$\{\sup_{n\in\mathbb{N}} |\alpha_n U_n \xi_n| = \infty\} = \bigcap_{K=1}^{\infty} \{|\alpha_n U_n \xi_n| \geqslant K \text{ i.o.}\}.$$

Due to Kolmogorov's and Borel's and Cantelli's results, the sequence $(\alpha_n U_n \xi_n)$ is almost surely bounded if and only if $\sum_{n=1}^{\infty} \mathbb{P}\left\{\alpha_n U_n|\xi_n| \geqslant K\right\}$ is finite for some K. Taking the independence of (ξ_n) and (U_n) into account, a similar computation as above can be carried out:

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\alpha_n U_n|\xi_n| \geqslant K\right\} = \sum_{n=1}^{\infty} \mathbb{P}\left\{\alpha_n|\xi_1| \geqslant K\right\} s_n$$

$$\leqslant c_4 \sum_{n=2}^{\infty} \mathbb{P}\left\{c_2 g(n)|\xi_1| \geqslant K\right\} h(n) + s_1 \leqslant c_4 \int_1^{\infty} \mathbb{P}\left\{c_2 g(x)|\xi_1| \geqslant K\right\} h(x)\,\mathrm{d}x + s_1$$

$$\leqslant c_4 \int_1^{\infty}\int_{\frac{K}{c_2 g(x)}}^{\infty} h(x)\,\mathrm{d}\mathbb{P}_{|\xi_1|}(t)\,\mathrm{d}x + s_1 \leqslant c_7 \int_{\frac{K}{c_2 g(1)}}^{\infty}\int_1^{g^{-1}(\frac{K}{c_2 t})} H'(x)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}_{|\xi_1|}(t) + s_1$$

$$\leqslant c_7 \int\limits_{\frac{K}{c_2 g(1)}}^{\infty} H(g^{-1}(\frac{K}{c_2 t}))\,\mathrm{d}\mathbb{P}_{|\xi_1|}(t) + s_1 = c_7 \mathbb{E}[H(g^{-1}(\frac{K}{c_2|\xi_1|})\mathbb{1}_{[\frac{K}{c_2 g(1)},+\infty)}(|\xi_1|)] + s_1.$$

The last term is finite for large K and the assertion follows.

□

3.3.2 Chernoff bounds

In the following, we need knowledge about the sums of independent 0-1-random numbers. Chernoff type results allow the conclusion that a good approximation for (finite) sums of the mentioned type is given by the corresponding expectation. To be exact, we use the following result.

Lemma 3.11 ([28], Th. 4.1, Th.4.2, Th. 4.3). *Let (D_n) be a sequence of independent random variables with $p_n = \mathbb{P}\{D_n = 1\} = 1 - \mathbb{P}\{D_n = 0\}$ and let $\delta < 2e - 1$. Then, for every $n \in \mathbb{N}$,*

1. $\mathbb{P}\left\{\sum_{k=1}^{n} D_k > (1+\delta)\mathbb{E}\sum_{k=1}^{n} D_k\right\} \leqslant \exp(-\frac{\delta^2}{4}\mathbb{E}\sum_{k=1}^{n} D_k)$,
2. $\mathbb{P}\left\{\sum_{k=1}^{n} D_k < (1-\delta)\mathbb{E}\sum_{k=1}^{n} D_k\right\} \leqslant \exp(-\frac{\delta^2}{2}\mathbb{E}\sum_{k=1}^{n} D_k)$.

We need a version for infinite sums.

Lemma 3.12. *Consider the situation given in Lemma 3.11 with the additional assumption $\mathbb{E}\sum_{k=1}^{\infty} D_k < \infty$. Then, for $\delta < 2e - 1$,*

1. $\mathbb{P}\left\{\sum_{k=1}^{\infty} D_k > (1+\delta)\mathbb{E}\sum_{k=1}^{\infty} D_k\right\} \leqslant \exp(-\frac{\delta^2}{4}\mathbb{E}\sum_{k=1}^{\infty} D_k)$,
2. $\mathbb{P}\left\{\sum_{k=1}^{\infty} D_k < (1-\delta)\mathbb{E}\sum_{k=1}^{\infty} D_k\right\} \leqslant \exp(-\frac{\delta^2}{2}\mathbb{E}\sum_{k=1}^{\infty} D_k)$.

Proof. To derive the first inequality, we show how the proof of Lemma 3.11 can be adapted. During the first steps, we use the sum's finite expectation, Markov's inequality, the exponential function's continuity, the independence of (D_n) and the Monotone Convergence Theorem. Let $t > 0$, then

$$\mathbb{P}\left\{\sum_{k=1}^{\infty} D_k > (1+\delta)\mathbb{E}\sum_{k=1}^{\infty} D_k\right\} = \mathbb{P}\left\{\exp(t\sum_{k=1}^{\infty} D_k) > \exp(t(1+\delta)\mathbb{E}\sum_{k=1}^{\infty} D_k)\right\}$$

$$\leqslant \frac{\mathbb{E}\exp(t\sum_{k=1}^{\infty} D_k)}{\exp(t(1+\delta)\mathbb{E}\sum_{k=1}^{\infty} D_k)} = \frac{\mathbb{E}\lim_{n\to\infty}\prod_{k=1}^{n}\exp(tD_k)}{\exp(t(1+\delta)\mathbb{E}\sum_{k=1}^{\infty} D_k)}$$

$$= \frac{\lim\limits_{n\to\infty} \prod\limits_{k=1}^{n} \mathbb{E}\exp(tD_k)}{\exp(t(1+\delta)\mathbb{E}\sum\limits_{k=1}^{\infty} D_k)} = \frac{\lim\limits_{n\to\infty} \prod\limits_{k=1}^{n} (p_k \cdot e^t + (1-p_k))}{\exp(t(1+\delta)\mathbb{E}\sum\limits_{k=1}^{\infty} D_k)}.$$

For the next steps, we use $1 + y < e^y$, what is valid for $y > 0$, once more the exponential function's continuity and $\sum\limits_1^\infty p_k = \mathbb{E}\sum\limits_1^\infty D_k$. We continue with

$$\mathbb{P}\left\{\sum_{k=1}^{\infty} D_k > (1+\delta)\mathbb{E}\sum_{k=1}^{\infty} D_k\right\} \leqslant \frac{\lim\limits_{n\to\infty} \prod\limits_{k=1}^{n} \exp(p_k(e^t-1))}{\exp(t(1+\delta)\mathbb{E}\sum\limits_{k=1}^{\infty} D_k)}$$

$$= \frac{\lim\limits_{n\to\infty} \exp(\sum\limits_{k=1}^{n} p_k(e^t-1))}{\exp(t(1+\delta)\mathbb{E}\sum\limits_{k=1}^{\infty} D_k)} = \frac{\exp(\mathbb{E}\sum\limits_{k=1}^{\infty} D_k(e^t-1))}{\exp(t(1+\delta)\mathbb{E}\sum\limits_{k=1}^{\infty} D_k)} = \left(\frac{\exp(\delta)}{(1+\delta)^{1+\delta}}\right)^{\mathbb{E}\sum\limits_{k=1}^{\infty} D_k}$$

if we choose $t = \log(1+\delta)$. For $\delta < 2e - 1$ the last expression can be estimated by the claimed term. The computation can be found in [28] and because there is no difference to the finite sum case, we omit the proof.
A similar proof yields the second estimate. Here, one starts with

$$\mathbb{P}\left\{\sum_{k=1}^{\infty} D_k < (1-\delta)\mathbb{E}\sum_{k=1}^{\infty} D_k\right\} = \mathbb{P}\left\{\exp(-t\sum_{k=1}^{\infty} D_k) > \exp(-t(1-\delta)\mathbb{E}\sum_{k=1}^{\infty} D_k)\right\}.$$

□

3.3.3 Random sequences with deletion factor

3.3.3.1 Fast decreasing deletion probabilities

An important assumption of Theorem 2.41 is given by $\sum\limits_{n=1}^{\infty} h(n) = \infty$. If this condition is not fulfilled, the problem of determining the covering number rate for a random sequence with deletion factor will be solved quite easily.

Proposition 3.13. *Let $Y = (\alpha_n U_n \xi_n)$ be a random sequence with deletion factor. Assume $\sum\limits_{n=1}^{\infty} \mathbb{P}\{U_n = 1\} < \infty$. Then $|\{\alpha_k U_k \xi_k : k \in \mathbb{N}\}|$ is finite, almost surely. Thus, $N((\alpha_n U_n \xi_n), |\cdot|, \varepsilon) \preccurlyeq f(\varepsilon)$, almost surely, for every $f \in \mathcal{D}$.*

Proof. Because of $\sum\limits_{n=1}^{\infty} \mathbb{P}\{U_n = 1\} < \infty$, the event $\{U_n = 1 \text{ i.o.}\}$ has the probability zero and its complement $\{U_n = 0 \text{ eventually}\}$ has the probability one. Consequently, for $\omega \in \{U_n = 0 \text{ eventually}\}$ we can find a number $n(\omega)$ such that $\alpha_n U_n(\omega)\xi_n(\omega) = 0$

for $n \geqslant n(\omega)$. This means, for every $\varepsilon > 0$,

$$\begin{aligned} N((\alpha_k U_k \xi_k)_1^\infty, |\cdot|, \varepsilon)(\omega) &\leqslant N((\alpha_k U_k \xi_k)_1^{n(\omega)-1}, |\cdot|, \varepsilon)(\omega) + N(\{0\}, |\cdot|, \varepsilon) \\ &\leqslant |\{\alpha_k U_k \xi_k : k \leqslant n(\omega) - 1\} + 1 \leqslant n(\omega) - 1 + 1 = n(\omega). \end{aligned}$$

The bound is independent of ε and the assertion follows. □

3.3.3.2 Slowly decreasing deletion probabilities

In some cases, we must ensure that we do not leave the class of convex sequences.

Lemma 3.14. *Let $(g(n)), (s(n)) \in \mathcal{C}$ with associated functions g and s. Assume that S is chosen in such a way that $S' = s$.*

1. *Then $(g \circ S^{-1}(n)) \in \mathcal{C}$ if $g'' \cdot s > g' \cdot s'$.*
2. *If h is an increasing linear transformation, i.e., $h(x) = ax + b$, $a > 0$, then $(g \circ h(n))_{n > -\frac{b}{a}}$ will be an element of $\mathcal{C}$.*

Proof.

1. The associated function of $(g \circ S^{-1}(n))$ is $g \circ S^{-1}$. Its first derivative is negative and the second derivative is positive. This can be checked by a direct computation. We use the derivative rule for inverse functions.

$$(g \circ S^{-1})'(x) = g' \circ S^{-1}(x) \cdot (S^{-1})'(x) = g' \circ S^{-1}(x) \cdot \frac{1}{s \circ S^{-1}(x)} < 0,$$

$$(g \circ S^{-1})''(x)$$

$$= \frac{g'' \circ S^{-1}(x) \cdot \frac{1}{s \circ S^{-1}(x)} \cdot s \circ S^{-1}(x) - g' \circ S^{-1}(x) \cdot s'(S^{-1}(x)) \frac{1}{s \circ S^{-1}(x)}}{s^2(S^{-1}(x))}.$$

   The assumption yields the result.

2. We compute the derivatives.

$$(g(ax+b))' = g'(ax+b)a < 0 \ , \ (g(ax+b))'' = g''(ax+b)a^2 > 0.$$

   The restriction $n > -\frac{b}{a}$ is necessary because g is defined only on $\mathbb{R}^+$.

□

Proof of Theorem 2.41. The almost surely finiteness of $(\alpha_n U_n \xi_n)$ is implied by Lemma 3.10, e.g., choose $C = \frac{g(H^{-1}(1))}{c_2}$.
Let $\delta \in (0, 1)$. For every n define the random number (we omit the ω)

$$Z(n) := \sum_{k=1}^{\infty} \mathbb{1}_{\{\alpha_k U_k \xi_k \,\geqslant\, g(H^{-1}(n))\}}.$$

We want to use Lemma 3.12 to compare $Z(n)$ with its expectation. Set

$$D_k^n := \mathbb{1}_{\{\alpha_k U_k \xi_k \geqslant g(H^{-1}(n))\}}.$$

Because of $\mathbb{P}\{D_k^n \in \{0,1\}\} = 1$, it remains to verify $\mathbb{E}\sum_{k=1}^{\infty} D_k^n < \infty$. During the first steps of the following computation, we use Beppo Levi's theorem and the independence of the given random numbers. Observe that $c_2 g(k)\xi_1(\omega) \geqslant g(H^{-1}(n))$ implies $\xi_1(\omega) > 0$, what allows us to divide by $\xi_1(\omega)$.

$$\mathbb{E}\sum_{k=1}^{\infty} D_k^n = \sum_{k=1}^{\infty} \mathbb{P}\left\{\alpha_k U_k \xi_k \geqslant g(H^{-1}(n))\right\} = \sum_{k=1}^{\infty} \mathbb{P}\left\{\alpha_k \xi_k \geqslant g(H^{-1}(n))\right\} s_k$$

$$\leqslant c_4 \sum_{k=2}^{\infty} \mathbb{P}\left\{c_2 g(k)\xi_1 \geqslant g(H^{-1}(n))\right\} h(k) + s_1$$

$$= c_4 \sum_{k=2}^{\infty} \mathbb{P}\left\{k \leqslant g^{-1}(\frac{g(H^{-1}(n))}{c_2\xi_1})\right\} h(k) + s_1$$

$$\leqslant c_4 \int_1^{\infty} \mathbb{P}\left\{x \leqslant g^{-1}(\frac{g(H^{-1}(n))}{c_2\xi_1})\right\} h(x)\,\mathrm{d}x + s_1$$

$$= c_4 \int_1^{\infty} \mathbb{P}\left\{\xi_1 \geqslant \frac{g(H^{-1}(n))}{c_2 g(x)}\right\} h(x)\,\mathrm{d}x + s_1$$

$$= c_4 \int_1^{\infty} \int_{\frac{g(H^{-1}(n))}{c_2 g(x)}}^{\infty} h(x)\,\mathrm{d}\mathbb{P}_{\xi_1}(t)\,\mathrm{d}x + s_1 = c_4 \int_{\frac{g(H^{-1}(n))}{c_2 g(1)}}^{\infty} \int_1^{g^{-1}(\frac{g(H^{-1}(n))}{c_2 t})} h(x)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}_{\xi_1}(t) + s_1$$

$$\leqslant c_4 \int_{\frac{g(H^{-1}(n))}{c_2 g(1)}}^{\infty} H(g^{-1}(\frac{g(H^{-1}(n))}{c_2 t}))\,\mathrm{d}\mathbb{P}_{\xi_1}(t) + s_1$$

$$= c_4 \mathbb{E}\left[H(g^{-1}(\frac{g(H^{-1}(n))}{c_2\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_2 g(1)},\infty)}(\xi_1)\right] + s_1,$$

which is finite by assumption. During the last steps, we used Fubini's Theorem to change the order of integration and we used the integral's monotonicity. Before we will apply Lemma 3.12, we compute a lower bound for $\mathbb{E}\sum_{k=1}^{\infty} D_k^n$, which is needed later. The computation is as above, but this time, we omit some steps.

$$\mathbb{E}\sum_{k=1}^{\infty} D_k^n = \sum_{k=1}^{\infty} \mathbb{P}\left\{\alpha_k \xi_k \geqslant g(H^{-1}(n))\right\} s_k \geqslant c_3 \sum_{k=1}^{\infty} \mathbb{P}\left\{c_1 g(k)\xi_1 \geqslant g(H^{-1}(n))\right\} h(k)$$

$$\geqslant c_3 \int_1^\infty \mathbb{P}\left\{c_1 g(x)\xi_1 \geqslant g(H^{-1}(n))\right\} h(x)\,\mathrm{d}x = c_3 \int_1^\infty \int_{\frac{g(H^{-1}(n))}{c_1 g(x)}}^\infty h(x)\,\mathrm{d}\mathbb{P}_{\xi_1}(t)\,\mathrm{d}x$$

$$= c_3 \int_{\frac{g(H^{-1}(n))}{c_1 g(1)}}^\infty H(g^{-1}(\frac{g(H^{-1}(n))}{c_1 t})) - H(1)\,\mathrm{d}\mathbb{P}_{\xi_1}(t)$$

$$\geqslant c_3 \mathbb{E}\left[H(g^{-1}(\frac{g(H^{-1}(n))}{c_1 \xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_1 g(1)},\infty)}(\xi_1)\right] - c_3 H(1).$$

Now, an application of Lemma 3.12 and the lower estimate from above yield, for every $n \in \mathbb{N}$,

$$\mathbb{P}\{Z(n) > (1+\delta)\mathbb{E}Z(n)\} = \mathbb{P}\left\{\sum_{k=1}^\infty D_k^n > (1+\delta)\mathbb{E}\sum_{k=1}^\infty D_k^n\right\} \leqslant \exp(-\frac{\delta^2}{4}\mathbb{E}\sum_{k=1}^\infty D_k^n)$$

$$\leqslant \exp(-\frac{\delta^2}{4}c_3(\mathbb{E}\left[H(g^{-1}(\frac{g(H^{-1}(n))}{c_1\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_1 g(1)},\infty)}(\xi_1)\right] - H(1))).$$

Using the assumptions, we may conclude that $\sum_{n=1}^\infty \mathbb{P}\{Z(n) > (1+\delta)\mathbb{E}Z(n)\}$ is finite. Therefore, by Borel's and Cantelli's Lemma, the event $\{Z(n) > (1+\delta)\mathbb{E}Z(n) \text{ i.o.}\}$ has the probability zero and the complement $\{Z(n) \leqslant (1+\delta)\mathbb{E}Z(n) \text{ eventually}\}$ has the probability one. A second application of Lemma 3.12 yields

$$\mathbb{P}\{Z(n) < (1-\delta)\mathbb{E}Z(n)\}$$

$$\leqslant \exp(-\frac{\delta^2}{2}c_3(\mathbb{E}\left[H(g^{-1}(\frac{g(H^{-1}(n))}{c_1\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_1 g(1)},\infty)}(\xi_1)\right] - H(1))),$$

showing that the event $\{Z(n) \geqslant (1-\delta)\mathbb{E}Z(n) \text{ eventually}\}$ has the probability one, too. Set

$$F_\delta := \{Z(n) \leqslant (1+\delta)\mathbb{E}Z(n) \text{ eventually}\} \cap \{Z(n) \geqslant (1-\delta)\mathbb{E}Z(n) \text{ eventually}\},$$

then $\mathbb{P}\{F_\delta\} = 1$. For $\omega \in F_\delta$ we can find an index $n(\omega)$ such that

$$(1-\delta)\mathbb{E}Z(n) \leqslant Z(n)(\omega) \leqslant (1+\delta)\mathbb{E}Z(n)$$

is valid for $n \geqslant n(\omega)$. $Z(n)(\omega)$ is the number of all components $\alpha_k U_k(\omega)\xi_k(\omega)$ which are larger than or equal to $g(H^{-1}(n))$. For $n \geqslant n(\omega)$ our computations show that this number is smaller than or equal to

$$(1+\delta)(c_4\mathbb{E}\left[H(g^{-1}(\frac{g(H^{-1}(n))}{c_2\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_2 g(1)},\infty)}(\xi_1)\right] + s_1),$$

i.e., at most $\left\lceil (1+\delta)(c_4\mathbb{E}\left[H(g^{-1}(\frac{g(H^{-1}(n))}{c_2\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_2 g(1)},\infty)}(\xi_1)\right]+s_1)\right\rceil$ components of the sequence $(\alpha_k U_k(\omega)\xi_k(\omega))$ are larger than or equal to $g(H^{-1}(n))$. We may conclude that the

$$(\left\lceil (1+\delta)(c_4\mathbb{E}\left[H(g^{-1}(\frac{g(H^{-1}(n))}{c_2\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_2 g(1)},\infty)}(\xi_1)\right]+s_1)\right\rceil+1)\text{ -th}$$

component of the decreasing sequence $(\alpha_k U_k(\omega)\xi_k(\omega))^*$ is smaller than or equal to $g(H^{-1}(n))$. To get the second estimate, observe that at least

$$(1-\delta)c_3(\mathbb{E}\left[H(g^{-1}(\frac{g(H^{-1}(n))}{c_1\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_1 g(1)},\infty)}(\xi_1)\right]-H(1))$$

components of $(\alpha_k U_k(\omega)\xi_k(\omega))$ are greater than or equal to $g(H^{-1}(n))$. It follows that the

$$\left\lfloor (1-\delta)c_3(\mathbb{E}\left[H(g^{-1}(\frac{g(H^{-1}(n))}{c_1\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_1 g(1)},\infty)}(\xi_1)\right]-H(1))\right\rfloor\text{ -th}$$

component of the decreasing sequence $(\alpha_k U_k(\omega)\xi_k(\omega))^*$ is greater than or equal to $g(H^{-1}(n))$. The proof is finished. □

3.3.4 Standard random sequences

Proof of Theorem 2.42. The proof goes along the same lines as the proof of Theorem 2.41. Therefore, we skip some details in the following argumentation. By choosing $C=\frac{g(1)}{c_2}$, the sequence's boundedness follows by Lemma 3.10. For $\delta>0$ and $n\in\mathbb{N}$ define the random number

$$Z(n):=\sum_{k=1}^{\infty}\mathbb{1}_{\{\alpha_k\xi_k\,\geqslant\,g(n)\}}.$$

Next, we compute bounds for $\mathbb{E}Z(n)$.

$$\mathbb{E}Z(n)=\sum_{k=1}^{\infty}\mathbb{P}\left\{\alpha_k\xi_k\geqslant g(n)\right\}\leqslant\sum_{k=2}^{\infty}\mathbb{P}\left\{c_2 g(k)\xi_k\geqslant g(n)\right\}+1$$

$$\leqslant\int_1^{\infty}\mathbb{P}\left\{c_2 g(x)\xi_1\geqslant g(n)\right\}\,\mathrm{d}x+1=\int_1^{\infty}\int_{\frac{g(n)}{c_2 g(x)}}^{\infty}\mathrm{d}\mathbb{P}_{\xi_1}(t)\,\mathrm{d}x+1$$

$$=\int_{\frac{g(n)}{c_2 g(1)}}^{\infty}\int_1^{g^{-1}(\frac{g(n)}{c_2 t})}\mathrm{d}x\,\mathrm{d}\mathbb{P}_{\xi_1}(t)+1\leqslant\int_{\frac{g(n)}{c_2 g(1)}}^{\infty}g^{-1}(\frac{g(n)}{c_2 t})\,\mathrm{d}\mathbb{P}_{\xi_1}(t)+1$$

$$= \mathbb{E}\left[g^{-1}(\frac{g(n)}{c_2\xi_1})\mathbb{1}_{[\frac{g(n)}{c_2 g(1)},\infty)}(\xi_1)\right] + 1,$$

$$\mathbb{E}Z(n) \geqslant \sum_{k=1}^{\infty} \mathbb{P}\left\{c_1 g(k)\xi_k \geqslant g(n)\right\} \geqslant \int\limits_{1}^{\infty} \int\limits_{\frac{g(n)}{c_1 g(x)}}^{\infty} \mathrm{d}\mathbb{P}_{\xi_1}(t)\,\mathrm{d}x$$

$$= \int\limits_{\frac{g(n)}{c_1 g(1)}}^{\infty} (g^{-1}(\frac{g(n)}{c_1 t}) - 1)\,\mathrm{d}\mathbb{P}_{\xi_1}(t) \geqslant \mathbb{E}\left[g^{-1}(\frac{g(n)}{c_1\xi_1})\mathbb{1}_{[\frac{g(n)}{c_1 g(1)},\infty)}(\xi_1)\right] - 1.$$

In particular, the assumptions yield the finiteness of $\mathbb{E}Z(n)$ and an application of Lemma 3.12 is possible. That shows

$$\mathbb{P}\left\{Z(n) > (1+\delta)\mathbb{E}Z(n)\right\} \leqslant \exp(-\frac{\delta^2}{4}\mathbb{E}Z(n))$$

$$\leqslant \exp(-\frac{\delta^2}{4}(\mathbb{E}\left[g^{-1}(\frac{g(n)}{c_1\xi_1})\mathbb{1}_{[\frac{g(n)}{c_1 g(1)},\infty)}(\xi_1)\right] - 1))$$

and

$$\mathbb{P}\left\{Z(n) < (1-\delta)\mathbb{E}Z(n)\right\} \leqslant \exp(-\frac{\delta^2}{2}\mathbb{E}Z(n))$$

$$\leqslant \exp(-\frac{\delta^2}{4}(\mathbb{E}\left[g^{-1}(\frac{g(n)}{c_1\xi_1})\mathbb{1}_{[\frac{g(n)}{c_1 g(1)},\infty)}(\xi_1)\right] + 1)).$$

As above, we may conclude that the event

$$F_\delta = \{Z(n) \leqslant (1+\delta)\mathbb{E}Z(n) \text{ eventually}\} \cap \{Z(n) \geqslant (1-\delta)\mathbb{E}Z(n) \text{ eventually}\}$$

has full measure. Given $\omega \in F_\delta$, we can find a number $n(\delta,\omega)$ such that

$$(1-\delta)\mathbb{E}Z(n) \leqslant Z(n)(\omega) \leqslant (1+\delta)\mathbb{E}Z(n)$$

holds for all $n \geqslant n(\delta,\omega)$. This means, together with the bounds for $\mathbb{E}Z(n)$, that at most

$$\left\lceil (1+\delta)(\mathbb{E}\left[g^{-1}(\frac{g(n)}{c_2\xi_1})\mathbb{1}_{[\frac{g(n)}{c_2 g(1)},\infty)}(\xi_1)\right] + 1)\right\rceil$$

and at least

$$\left\lfloor (1-\delta)(\mathbb{E}\left[g^{-1}(\frac{g(n)}{c_1\xi_1})\mathbb{1}_{[\frac{g(n)}{c_1 g(1)},\infty)}(\xi_1)\right] - 1)\right\rfloor$$

components of $(\alpha_k\xi_k)(\omega)$ are larger than or equal to $g(n)$. It follows

$$(\alpha_{\cdot}\xi_{\cdot})^*_{\left\lceil (1+\delta)(\mathbb{E}\left[g^{-1}(\frac{g(n)}{c_2\xi_1})\mathbb{1}_{[\frac{g(n)}{c_2 g(1)},\infty)}(\xi_1)\right]+1)\right\rceil+1}(\omega) \leqslant g(n)$$

$$\leqslant (\alpha_{\cdot}\xi_{\cdot})^*_{\left\lfloor (1-\delta)(\mathbb{E}\left[g^{-1}(\frac{g(n)}{c_1\xi_1})\mathbb{1}_{[\frac{g(n)}{c_1 g(1)},\infty)}(\xi_1)\right]-1)\right\rfloor}(\omega),$$

what completes the proof. □

3.3.5 Examples

This section contains the proofs of some examples in order to demonstrate the usefulness of Theorem 2.41 and Theorem 2.42.

Proof of Theorem 2.43. Assume that g and h are polynomial, i.e., we have $g(x) = x^{-\beta}$, where $\beta > 0$, and we have $h(x) = x^{-a}$. To exclude the trivial case (cf. section 3.3.3.1), we restrict to $a \in (0,1)$. We can choose $H(x) = \frac{1}{1-a}x^{1-a}$. Then,

$$\mathbb{E}\left[H(g^{-1}(\frac{g(H^{-1}(n))}{c_i\xi_1}))\mathbb{1}_{[\frac{g(H^{-1}(n))}{c_i g(1)},\infty)}(\xi_1)\right]$$

$$= c_i^{\frac{1-a}{\beta}} n\mathbb{E}\left[\xi_1^{\frac{1-a}{\beta}}\mathbb{1}_{[n^{-\frac{\beta}{1-a}}(1-a)^{-\frac{\beta}{1-a}}\cdot\frac{1}{c_i},\infty)}(\xi_1)\right] \leqslant c_i^{\frac{1-a}{\beta}} n\mathbb{E}\xi_1^{\frac{1-a}{\beta}}.$$

An application of Theorem 2.41 requires $\mathbb{E}\left[\xi_1^{\frac{1-a}{\beta}}\mathbb{1}_{[n^{-\frac{\beta}{1-a}}(1-a)^{-\frac{\beta}{1-a}}\cdot\frac{1}{c_i},\infty)}(\xi_1)\right] < \infty$. Here, this will be satisfied if $\mathbb{E}\xi_1^{\frac{1-a}{\beta}}$ is finite. Because of the Monotone Convergence Theorem, we have $\lim_{n\to\infty}\mathbb{E}\left[\xi_1^{\frac{1-a}{\beta}}\mathbb{1}_{[n^{-\frac{\beta}{1-a}}(1-a)^{-\frac{\beta}{1-a}}\cdot\frac{1}{c_i},\infty)}(\xi_1)\right] = \mathbb{E}\xi_1^{\frac{1-a}{\beta}}$.
Thus, for $\delta > 0$ we can find a number $n(\delta)$ such that

$$(1-\delta)\mathbb{E}\xi_1^{\frac{1-a}{\beta}} \leqslant \mathbb{E}\left[\xi_1^{\frac{1-a}{\beta}}\mathbb{1}_{[n^{-\frac{\beta}{1-a}}(1-a)^{-\frac{\beta}{1-a}}\cdot\frac{1}{c_i},\infty)}(\xi_1)\right]$$

is valid for all $n \geqslant n(\delta)$. Now, the assertion of Theorem 2.41 reduces to, for $\omega \in F_\delta$ and large n,

$$(\alpha_. U_. \xi_.)^*_{\left\lceil(1+\delta)(c_4 c_2^{\frac{1-a}{\beta}} n\mathbb{E}\xi_1^{\frac{1-a}{\beta}}+s_1)\right\rceil+1}(\omega) \leqslant n^{-\frac{\beta}{1-a}}(1-a)^{-\frac{\beta}{1-a}}$$

$$\leqslant (\alpha_. U_. \xi_.)^*_{\left\lfloor(1-\delta)^2 c_3 c_1^{\frac{1-a}{\beta}} n\mathbb{E}\xi_1^{\frac{1-a}{\beta}}-c_3 H(1)\right\rfloor}(\omega).$$

In particular, the first estimate is valid for $n(m) := \left\lfloor(\frac{m-1}{1+\delta}-s_1)c_4^{-1}c_2^{-\frac{1-a}{\beta}}(\mathbb{E}\xi_1^{\frac{1-a}{\beta}})^{-1}\right\rfloor$ if m is large enough. This gives

$$(\alpha_. U_. \xi_.)^*_m(\omega) \leqslant (\alpha_. U_. \xi_.)^*_{\left\lceil(1+\delta)(c_4 c_2^{\frac{1-a}{\beta}} n(m)\mathbb{E}\xi_1^{\frac{1-a}{\beta}}+s_1)\right\rceil+1}(\omega) \leqslant n(m)^{-\frac{\beta}{1-a}}(1-a)^{-\frac{\beta}{1-a}}$$

$$\leqslant ((\frac{m-1}{1+\delta}-s_1)c_4^{-1}c_2^{-\frac{1-a}{\beta}}(\mathbb{E}\xi_1^{\frac{1-a}{\beta}})^{-1}-1)^{-\frac{\beta}{1-a}}(1-a)^{-\frac{\beta}{1-a}}$$

$$\leqslant (1+\delta)^{\frac{4\beta}{1-a}} c_4^{\frac{\beta}{1-a}} c_2 (\mathbb{E}\xi_1^{\frac{1-a}{\beta}})^{\frac{\beta}{1-a}} (1-a)^{-\frac{\beta}{1-a}} m^{-\frac{\beta}{1-a}},$$

where in the last step we used (three times) that $Cm - b \geqslant \frac{Cm}{1+\delta}$ is valid for large m and fixed positive constants C and b. To get a second estimate, choose

$$n(m) := \left\lceil \frac{m + \frac{c_3}{1-a}}{(1-\delta)^2} c_3^{-1} c_1^{-\frac{1-a}{\beta}} (\mathbb{E}\xi_1^{\frac{1-a}{\beta}})^{-1} \right\rceil .$$

The result is

$$(\alpha_. U_. \xi_.)^*_m(\omega) \geqslant n(m)^{-\frac{\beta}{1-a}} (1-a)^{-\frac{\beta}{1-a}}$$

$$\geqslant (1-\delta)^{\frac{4\beta}{1-a}} c_3^{\frac{\beta}{1-a}} c_1 (\mathbb{E}\xi_1^{\frac{1-a}{\beta}})^{\frac{\beta}{1-a}} (1-a)^{-\frac{\beta}{1-a}} m^{-\frac{\beta}{1-a}}.$$

In conclusion, for every $\kappa > 0$ there is a full measure set F_κ and for every $\omega \in F_\kappa$ there is a number $N = N(\omega, \kappa, c_1, c_2, c_3, c_4, a, \beta, \mathbb{P}_\xi)$ such that

$$(1-\kappa) c_3^{\frac{\beta}{1-a}} c_1 (\mathbb{E}\xi_1^{\frac{1-a}{\beta}})^{\frac{\beta}{1-a}} (1-a)^{-\frac{\beta}{1-a}} m^{-\frac{\beta}{1-a}} \leqslant (\alpha_. U_. \xi_.)^*_m(\omega)$$
$$\leqslant (1+\kappa) c_4^{\frac{\beta}{1-a}} c_2 (\mathbb{E}\xi_1^{\frac{1-a}{\beta}})^{\frac{\beta}{1-a}} (1-a)^{-\frac{\beta}{1-a}} m^{-\frac{\beta}{1-a}}$$

holds for every $m \geqslant N$. An application of Lemma 1.22 yields

$$N((\alpha_m U_m \xi_m)(\omega), |\cdot|, \varepsilon) = N((\alpha_. U_. \xi_.)^*_m(\omega), |\cdot|, \varepsilon)$$

$$\preccurlyeq N(((1-a)^{-\frac{\beta}{1-a}} m^{-\frac{\beta}{1-a}}), |\cdot|, \varepsilon) \approx \varepsilon^{-\frac{1}{1+\frac{\beta}{1-a}}}.$$

Observe that $g(H^{-1}(n)) = n^{-\frac{\beta}{1-a}} (1-a)^{-\frac{\beta}{1-a}}$. □

Proof of Theorem 2.44. The proof is similar to the proof of Theorem 2.43. Therefore, in the following, we do not go too much into details. Assume $g(x) = x^{-\beta}$, where $\beta > 0$, and choose $\delta > 0$. Then,

$$(1-\delta) c_2^{\frac{1}{\beta}} n \mathbb{E}[\xi_1^{\frac{1}{\beta}}] \leqslant \mathbb{E} g^{-1}(\frac{g(n)}{c_2 \xi_1}) \mathbb{1}_{[\frac{g(n)}{c_2 g(1)}, \infty)}(\xi_1) = c_2^{\frac{1}{\beta}} n \mathbb{E}[\xi_1^{\frac{1}{\beta}} \mathbb{1}_{[\frac{n^{-\beta}}{c_2}, \infty)}(\xi_1)] \leqslant c_2^{\frac{1}{\beta}} n \mathbb{E}[\xi_1^{\frac{1}{\beta}}],$$

where the lower bound is valid for large n (Monotone Convergence Theorem). The estimates show that an application of Theorem 2.42 will be possible if the $\frac{1}{\beta}$-th moment of ξ_1 is finite. If this is ensured, the assertion of Theorem 2.42 will be reduced to

$$(\alpha_. \xi_.)^*_{\left\lceil (1+\delta)(c_2^{\frac{1}{\beta}} n \mathbb{E}[\xi_1^{\frac{1}{\beta}}]+1) \right\rceil + 1}(\omega) \leqslant n^{-\beta} \leqslant (\alpha_. \xi_.)^*_{\left\lfloor (1-\delta)((1-\delta) c_2^{\frac{1}{\beta}} n \mathbb{E}[\xi_1^{\frac{1}{\beta}}]-1) \right\rfloor}(\omega),$$

what is valid for $\omega \in F_\delta$ and large n, where F_δ is a full measure set. The choice

$$n(m) := \left\lfloor (\frac{m-1}{1+\delta} - 1) c_2^{-\frac{1}{\beta}} (\mathbb{E}\xi_1^{\frac{1}{\beta}})^{-1} \right\rfloor$$

leads to $(\alpha_\cdot\xi_\cdot)^*_m(\omega) \leqslant (1+\kappa)c_2(\mathbb{E}\xi_1^{\frac{1}{\beta}})^\beta m^{-\beta}$, for every $\kappa > 0$ and large m. The result follows by Lemma 1.22 and Proposition 1.19. □

Remark 3.15. *The ideas of the last proofs can be used to approximate moments of ξ_1. One needs the possibility to simulate independent realisations of ξ_1 and one needs a sorting method. How the scaling must be done can be seen by analyzing the proofs above.*

3.3.6 Further proofs and examples

Let us consider a non-polynomial example of a standard random sequence. We cannot expect a general result as nice as in the purely polynomial case but by detailed computation, results are also possible in this case. Let $g(x) = q^x$ with $q \in (0,1)$ and let $c_1 = c_2 = 1$, i.e., $\alpha_n = q^n$, and assume that ξ_1 is uniformly distributed on $[0,1]$. Given $\delta > 0$, Theorem 2.42 ensures the existence of a full measure set F_δ such that

$$(\alpha_\cdot\xi_\cdot)^*_{\left\lceil(1+\delta)(\mathbb{E}g^{-1}(\frac{g(n)}{\xi_1})\mathbb{1}_{[\frac{g(n)}{g(1)},\infty)}(\xi_1)+1)\right\rceil+1}(\omega) \leqslant g(n) \tag{16}$$
$$\leqslant (\alpha_\cdot\xi_\cdot)^*_{\left\lfloor(1-\delta)(\mathbb{E}g^{-1}(\frac{g(n)}{\xi_1})\mathbb{1}_{[\frac{g(n)}{g(1)},\infty)}(\xi_1)-1)\right\rfloor}(\omega)$$

is valid for $\omega \in F_\delta$ and large (in dependence of ω) n, put the case that

$$\sum_{n=1}^{\infty} \exp(-\frac{\delta^2}{4}\mathbb{E}g^{-1}(\frac{g(n)}{\xi_1})\mathbb{1}_{[\frac{g(n)}{g(1)},\infty)}(\xi_1)) < \infty \tag{17}$$

holds and $\mathbb{E}g^{-1}(\frac{g(n)}{\xi_1})\mathbb{1}_{[\frac{g(n)}{g(1)},\infty)}(\xi_1)$ is finite. Here,

$$\mathbb{E}g^{-1}(\frac{g(n)}{\xi_1})\mathbb{1}_{[\frac{g(n)}{g(1)},\infty)}(\xi_1) = n \cdot \mathbb{P}\left\{\xi_1 \geqslant q^{n-1}\right\} - \int_{q^{n-1}}^{1} \log_q x \,\mathrm{d}x$$
$$= n(1-q^{n-1}) - \frac{1\cdot\log 1 - 1 - q^{n-1}\log q^{n-1} + q^{n-1}}{\log q}$$
$$= n(1-q^{n-1}) + \frac{1}{\log q} + (n-1)q^{n-1} - \frac{q^{n-1}}{\log q} = n + q^{n-1}(1-\frac{1}{\log q}) + \frac{1}{\log q}.$$

For large n the last expression is smaller than $(1+\delta)n$ and greater than $(1-\delta)n$. Thus, the convergence of (17) is ensured and (16) simplifies to

$$(\alpha_\cdot\xi_\cdot)^*_{\lceil(1+\delta)^4 n\rceil}(\omega) \leqslant q^n \leqslant (\alpha_\cdot\xi_\cdot)^*_{\lfloor(1-\delta)^3 n\rfloor}(\omega).$$

As above, this can be reformulated as

$$q^{(1+\kappa)n} \leqslant (\alpha_{.}\xi_{.})^*_n(\omega) \leqslant q^{(1-\kappa)n}$$

for $\kappa > 0$ and large n. By Lemma 1.22, it follows that $|\log \varepsilon|$ is an almost surely upper bound for $N((\alpha_n \xi_n), |\cdot|, \varepsilon)$.

Proof of Theorem 2.45. Assume that the distribution of ξ_1 has compact support, i.e., there is some positive constant K such that $\mathbb{P}\{\xi_1 \in [-K, K]\} = 1$.
Let $N \in \mathbb{N}$. Observe that, in general, covering two sets separately requires more balls than covering the union of the two sets. Then the following is true, almost surely,

$$N((\alpha_n \xi_n), |\cdot|, \varepsilon) \leqslant N((\alpha_n \xi_n)_{n=1}^{N-1}, |\cdot|, \varepsilon) + N((\alpha_n \xi_n)_{n=N}^{\infty}, |\cdot|, \varepsilon)$$

$$\leqslant N - 1 + \frac{\sup\limits_{k \geqslant N} \alpha_k |\xi_k|}{\varepsilon} + 1 \leqslant N + \frac{K \alpha_N}{\varepsilon} \leqslant N + \frac{c_7 g(N)}{\varepsilon}.$$

Finally, the choice $\varepsilon = (-g')(N)$ and the convexity of $(g(n))$ yield

$$N((\alpha_n \xi_n), |\cdot|, \varepsilon) \leqslant c_8 \max\{(-g')^{-1}(\varepsilon), \frac{g((-g')^{-1}(\varepsilon))}{\varepsilon}\} \approx N((g(n)), |\cdot|, \varepsilon).$$

□

Proof of Theorem 2.46. We use Proposition 3.1 and Corollary 3.2. Let $\varepsilon > 0$ and $M := N((g(n)), |\cdot|, \varepsilon)$. The task is to find an upper bound for $F(\mathbb{N}, 2\varepsilon)$. Choose μ as uniform distribution on $\{1, 2, \ldots, M\}$. Then

$$F(\mathbb{N}, 2\varepsilon) \leqslant \int\limits_{\mathbb{N}} \int\limits_{\mathbb{N}} \mathbb{P}\{|g(n)\xi_n - g(m)\xi_m| \leqslant 2\varepsilon\} \, \mathrm{d}\mu(n) \, \mathrm{d}\mu(m)$$

$$\leqslant c_1 \frac{1}{M^2} \sum_{n=1}^{M-1} \sum_{m=n+1}^{M} \frac{\varepsilon}{\min\{g(n), g(m)\}} + \frac{1}{M} \leqslant c_1 \frac{\varepsilon}{g(M)} + \frac{1}{M} \leqslant c_2 \max\{\frac{\varepsilon}{g(M)}, \frac{1}{M}\}$$

$$= c_2 \max\{\frac{\varepsilon}{g(N((g(n)), |\cdot|, \varepsilon))}, \frac{1}{N((g(n)), |\cdot|, \varepsilon)}\}.$$

It follows

$$N((g(n)\xi_n), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} \frac{1}{\max\{\frac{\varepsilon}{g(N((g(n)), |\cdot|, \varepsilon))}, \frac{1}{N((g(n)), |\cdot|, \varepsilon)}\}}$$

$$= \min\{\frac{g(N((g(n)), |\cdot|, \varepsilon))}{\varepsilon}, N((g(n)), |\cdot|, \varepsilon)\}.$$

□

Lemma 3.16. *Assume that ξ_1 has a Lebesgue density p and assume that there are positive constants c_1, c_2 and γ such that $p(x) \leqslant c_1 \cdot e^{-c_2|x|^\gamma}$ is valid. Then*

$$\mathbb{P}\{|\alpha_n\xi_n - \alpha_m\xi_m| \leqslant \varepsilon\} \leqslant \frac{4c_1^2\Gamma(\frac{1}{\gamma})}{\gamma c_2^{\frac{1}{\gamma}}} \cdot \frac{\varepsilon}{\min\{\alpha_n, \alpha_m\}}.$$

Proof. If Y is a random variable with Lebesgue density, we will denote its density by p_Y. For the following computation we use the fact that aY has a density if Y has a density and that in this case, the formula $p_{aY}(x) = p_Y(\frac{x}{a})\frac{1}{a}$ is valid (a being strictly positive).

$$\mathbb{P}\{|\alpha_n\xi_n - \alpha_m\xi_m| \leqslant \varepsilon\} = \int\limits_{-\varepsilon}^{\varepsilon}\int\limits_{-\infty}^{\infty} p_{\alpha_n\xi_n}(x+y)p_{\alpha_m\xi_m}(y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int\limits_{-\varepsilon}^{\varepsilon}\int\limits_{-\infty}^{\infty} \frac{1}{\alpha_n\alpha_m} p_{\xi_n}(\frac{x+y}{\alpha_n})p_{\xi_m}(\frac{y}{\alpha_m})\,\mathrm{d}y\,\mathrm{d}x \leqslant \int\limits_{-\varepsilon}^{\varepsilon}\int\limits_{-\infty}^{\infty} \frac{1}{\alpha_n\alpha_m} c_1^2 e^{-c_2\frac{|x+y|^\gamma}{\alpha_n^\gamma}} e^{-c_2\frac{|y|^\gamma}{\alpha_m^\gamma}}\,\mathrm{d}y\,\mathrm{d}x$$

$$\leqslant \int\limits_{-\varepsilon}^{\varepsilon}\int\limits_{-\infty}^{\infty} \frac{1}{\alpha_n\alpha_m} c_1^2 e^{-c_2\frac{|y|^\gamma}{\alpha_m^\gamma}}\,\mathrm{d}y\,\mathrm{d}x = \int\limits_{0}^{\infty} \frac{4\varepsilon c_1^2}{\alpha_n\alpha_m} e^{-c_2\frac{y^\gamma}{\alpha_m^\gamma}}\,\mathrm{d}y.$$

Substituting $z = c_2\frac{y^\gamma}{\alpha_m^\gamma}$ leads to

$$\mathbb{P}\{|\alpha_n\xi_n - \alpha_m\xi_m| \leqslant \varepsilon\} \leqslant \frac{4\varepsilon c_1^2}{\alpha_n\alpha_m} \cdot \frac{\alpha_m}{\gamma c_2^{\frac{1}{\gamma}}} \int\limits_{0}^{\infty} z^{\frac{1}{\gamma}-1}e^{-z}\,\mathrm{d}z \leqslant \frac{4\varepsilon c_1^2\Gamma(\frac{1}{\gamma})}{\min\{\alpha_n, \alpha_m\}\gamma c_2^{\frac{1}{\gamma}}}.$$

□

Corollary 3.17. *The following distributions fulfill the assumptions of Lemma 3.16:*

- *Uniform distribution on $[0, K]$: Choose $c_1 = \frac{1}{K} \cdot e, c_2 = \frac{1}{K}, \gamma = 1$.*
- *Centered normal distribution with variance σ^2: Choose $c_1 = \frac{1}{\sqrt{2\pi}\sigma}, c_2 = \frac{1}{2\sigma^2}$, $\gamma = 2$.*
- *Exponential distribution with $\lambda > 0$: Choose $c_1 = c_2 = \lambda, \gamma = 1$.*
- *Distributions with bounded support $[a, b] \subseteq \mathbb{R}^+$ and bounded density p, where $\sup_x p(x) \leqslant K$: Choose $c_1 = K \cdot e$, $c_2 = \frac{1}{b}$, $\gamma = 1$.*

Proof of Proposition 2.47. Let $A \in \mathcal{D}$. Observe that $N((g(n)\xi_n), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\preccurlyeq} f(\varepsilon)$ and $N((g(n)\xi_n), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\preccurlyeq} 2f(\varepsilon)$ are equivalent conditions. During the following computation, we use lim instead of lim inf and lim sup because the result shows that the limits really exist.

$$\lim_{\varepsilon\to 0}\mathbb{P}\{N((g(n)\xi_n),|\cdot|,\varepsilon) > 2A(\varepsilon)f(\varepsilon)\}$$

$$\leqslant \lim_{\varepsilon\to 0}\mathbb{P}\{N((g(n)\xi_n),|\cdot|,\varepsilon) > (A(\varepsilon)+1)f(\varepsilon)\}$$

$$\leqslant \lim_{\varepsilon\to 0}\mathbb{P}\left\{N((g(n)\xi_n)_{n>f(\varepsilon)},|\cdot|,\varepsilon) > A(\varepsilon)f(\varepsilon)\right\}$$

$$\leqslant \lim_{\varepsilon\to 0}\mathbb{P}\left\{\frac{\sup\limits_{n>f(\varepsilon)}|g(n)\xi_n|}{\varepsilon} > A(\varepsilon)f(\varepsilon)\right\} = 1-\lim_{\varepsilon\to 0}\mathbb{P}\left\{\frac{\sup\limits_{n>f(\varepsilon)}|g(n)\xi_n|}{\varepsilon} \leqslant A(\varepsilon)f(\varepsilon)\right\}$$

$$= 1-\lim_{\varepsilon\to 0}\mathbb{P}\left\{\bigcap_{n>f(\varepsilon)}\{|g(n)\xi_n|\leqslant A(\varepsilon)f(\varepsilon)\varepsilon\}\right\}$$

$$= 1-\lim_{\varepsilon\to 0}\prod_{n>f(\varepsilon)}\mathbb{P}\{|g(n)\xi_n|\leqslant A(\varepsilon)f(\varepsilon)\varepsilon\} = 1-\lim_{\varepsilon\to 0}\prod_{n>f(\varepsilon)}F(\frac{A(\varepsilon)f(\varepsilon)\varepsilon}{g(n)}) = 0,$$

by assumption. □

We demonstrate a possible application of Proposition 2.47.

Corollary 3.18. *Assume that ξ_1 has bounded support, i.e., there is a number $t_0 > 0$ such that $F(t) = 1$ for all $t \geqslant t_0$. Then*

$$N((n^{-\beta}\xi_n),|\cdot|,\varepsilon) \overset{\mathbb{P}}{\preccurlyeq} \varepsilon^{-\frac{1}{1+\beta}}.$$

Proof. We analyze $\lim\limits_{\varepsilon\to 0}\prod\limits_{n>f(\varepsilon)}F(\frac{A(\varepsilon)f(\varepsilon)\varepsilon}{g(n)})$, where $f(\varepsilon) = \varepsilon^{-\frac{1}{1+\beta}}$ and $g(n) = n^{-\beta}$. The product can be rewritten as

$$\prod_{n>\varepsilon^{-\frac{1}{1+\beta}}}F(A(\varepsilon)\varepsilon^{\frac{\beta}{1+\beta}}n^{\beta}) = \prod_{k\geqslant 1}F(A(\varepsilon)\varepsilon^{\frac{\beta}{1+\beta}}(\left\lfloor\varepsilon^{-\frac{1}{1+\beta}}\right\rfloor+k)^{\beta})$$

$$\geqslant \prod_{k\geqslant 1}F(A(\varepsilon)\varepsilon^{\frac{\beta}{1+\beta}}(\varepsilon^{-\frac{1}{1+\beta}}+k-1)^{\beta}) = \prod_{k\geqslant 1}F(A(\varepsilon)(1+(k-1)\varepsilon^{\frac{1}{1+\beta}})^{\beta})$$

$$\geqslant \prod_{k\geqslant 1}F(A(\varepsilon)).$$

If ε is small enough, $A(\varepsilon)$ will be larger than t_0 and $F(A(\varepsilon)) = 1$. The result follows. □

It is interesting that in the case of random sequences with deletion factor there is no useful application of Proposition 3.1. This makes Theorem 2.48 very valuable. Recall that $\mathbb{P}\{U_n = 1\} = s_n = 1-\mathbb{P}\{U_n = 0\}$ and $\lim\limits_{n\to\infty}s_n = 0$. Because of the independence, we get

$$\mathbb{P}\{|\alpha_n U_n \xi_n - \alpha_m U_m \xi_m| \leqslant 2\varepsilon\} = (1-s_n)(1-s_m) + \mathbb{P}\{\alpha_n|\xi_n| \leqslant 2\varepsilon\}(1-s_m)s_n$$

$$+\mathbb{P}\{\alpha_m|\xi_m| \leqslant 2\varepsilon\} s_m(1-s_n) + \mathbb{P}\{|\alpha_n\xi_n - \alpha_m\xi_m| \leqslant 2\varepsilon\} s_n s_m.$$

For large n and m the last expression is almost equal to 1. In difference to bounds like $\min\{\frac{\varepsilon}{|s-t|^H}, 1\}$, where the minimum can be very small for small ε and large $|s-t|$, the value of $\mathbb{P}\{|\alpha_n\xi_n - \alpha_m\xi_m| \leqslant 2\varepsilon\}$ hardly depends on ε. To illustrate the situation, assume that s_n is decreasing in n and assume that $s_1 < 1$. Then $\mathbb{P}\{|\alpha_n U_n \xi_n - \alpha_m U_m \xi_m| \leqslant 2\varepsilon\} \geqslant (1-s_1)^2$ and

$$\inf_{\mu\in\mathcal{P}(\mathbb{N})} \int_{\mathbb{N}}\int_{\mathbb{N}} \mathbb{P}\{|\alpha_n U_n \xi_n - \alpha_m U_m \xi_m| \leqslant 2\varepsilon\} \,\mathrm{d}\mu(n)\,\mathrm{d}\mu(m)$$

$$\geqslant \inf_{\mu\in\mathcal{P}(\mathbb{N})} \int_{\mathbb{N}}\int_{\mathbb{N}} (1-s_1)^2 \,\mathrm{d}\mu(n)\,\mathrm{d}\mu(m) = (1-s_1)^2 > 0.$$

It follows $(1-s_1)^2 \leqslant F(\mathbb{N}, 2\varepsilon) \leqslant 1$ and $N((\alpha_n U_n \xi_n), |\cdot|, \varepsilon) \overset{\mathbb{P}}{\succcurlyeq} \frac{1}{F(\mathbb{N},2\varepsilon)} \approx 1$, meaning that at least a finite constant number of balls is needed to cover $(\alpha_n U_n \xi_n)$. But this is clear (if $(\alpha_n U_n \xi_n)$ is not empty) and does not require a computation.

Proof of Theorem 2.48. Firstly, we apply Theorem 2.41 with $c_1 = c_2 = c_3 = c_4 = 1$ and $\xi_1(\omega) = 1$ for every $\omega \in \Omega$. Checking condition (14) reduces to verifying the convergence of $\sum_{n\geqslant 1} q^n$, where $q \in (0,1)$, and this is true. If $\delta > 0$, n is large and $\omega \in F_\delta$, we can deduce

$$(\alpha_. U_.)^*_{\lceil(1+\delta)(n+s_1)\rceil+1}(\omega) \leqslant g(S^{-1}(n)) \leqslant (\alpha_. U_.)^*_{\lfloor(1-\delta)(n-S(1))\rfloor}(\omega).$$

Using the result for $n(m) := \left\lfloor \frac{m-1}{1+\delta} - s_1 \right\rfloor$, where $m \in \mathbb{N}$, gives $(\alpha_. U_.)^*_m \leqslant g(S^{-1}(\frac{m-1}{1+\delta} - s_1 - 1))$. The upper bound result follows with Proposition 1.19, Lemma 1.22 and Lemma 3.14.

The idea of the first part of the following considerations is to locate non-zero components of $(\alpha_n U_n)$. Because $\sum_1^\infty \mathbb{P}\{U_n = 1\} = \infty$ holds, there is an infinite number of non-zero components (a.s.). But to get a lower bound, we must gain knowledge about their distances. Define, for $k \in \mathbb{N}$, independent random numbers Y_k by

$$Y_k := \mathbb{1}_{\{\sum_{l=\lfloor S^{-1}(k)\rfloor+1}^{\lfloor S^{-1}(k+1)\rfloor} U_l > 0\}}.$$

Let $\delta > 0$. Chernoff's bound yields

$$\mathbb{P}\left\{\sum_{k=1}^{n} Y_k < (1-\delta)\mathbb{E}\sum_{k=1}^{n} Y_k\right\} \leqslant \exp(-\frac{\delta^2}{2}\mathbb{E}\sum_{k=1}^{n} Y_k).$$

We compute a lower bound for the expectation. The first inequality will be an application of the inequality between geometric and arithmetic mean. Furthermore, we use the relationship between sums and integrals and for $x > 0$ we use the inequality $1 - x \leqslant \exp(-x)$.

$$\mathbb{E}\sum_{k=1}^{n} Y_k = \sum_{k=1}^{n} \mathbb{P}\left\{\sum_{l=\lfloor S^{-1}(k)\rfloor+1}^{\lfloor S^{-1}(k+1)\rfloor} U_l > 0\right\} = \sum_{k=1}^{n}(1 - \prod_{l=\lfloor S^{-1}(k)\rfloor+1}^{\lfloor S^{-1}(k+1)\rfloor} \mathbb{P}\left\{U_l = 0\right\})$$

$$= \sum_{k=1}^{n}(1 - \prod_{l=\lfloor S^{-1}(k)\rfloor+1}^{\lfloor S^{-1}(k+1)\rfloor} (1 - s(l)))$$

$$\geqslant \sum_{k=1}^{n} 1 - (\frac{\lfloor S^{-1}(k+1)\rfloor - \lfloor S^{-1}(k)\rfloor - \sum_{l=\lfloor S^{-1}(k)\rfloor+1}^{\lfloor S^{-1}(k+1)\rfloor} s(l)}{\lfloor S^{-1}(k+1)\rfloor - \lfloor S^{-1}(k)\rfloor})^{\lfloor S^{-1}(k+1)\rfloor - \lfloor S^{-1}(k)\rfloor}$$

$$\geqslant \sum_{k=1}^{n} 1 - (1 - \frac{\int_{\lfloor S^{-1}(k)\rfloor+1}^{\lfloor S^{-1}(k+1)\rfloor+1} s(x)\,\mathrm{d}x}{\lfloor S^{-1}(k+1)\rfloor - \lfloor S^{-1}(k)\rfloor})^{\lfloor S^{-1}(k+1)\rfloor - \lfloor S^{-1}(k)\rfloor}$$

$$= \sum_{k=1}^{n} 1 - (1 - \frac{S(\lfloor S^{-1}(k+1)\rfloor + 1) - S(\lfloor S^{-1}(k)\rfloor + 1)}{\lfloor S^{-1}(k+1)\rfloor - \lfloor S^{-1}(k)\rfloor})^{\lfloor S^{-1}(k+1)\rfloor - \lfloor S^{-1}(k)\rfloor}$$

$$\geqslant \sum_{k=1}^{n} 1 - \exp(-S(\lfloor S^{-1}(k+1)\rfloor + 1) + S(\lfloor S^{-1}(k)\rfloor + 1))$$

$$\geqslant \sum_{k=1}^{n} 1 - \exp(-k - 1 + S(S^{-1}(k) + 1)).$$

By assumption, $S(S^{-1}(k)+1) - k$ converges to 0. Thus, we can choose k_0 such that $S(S^{-1}(k) + 1) \leqslant k + \frac{1}{2}$ holds for $k \geqslant k_0$. For $n > k_0$ it follows

$$\mathbb{E}\sum_{k=1}^{n} Y_k \geqslant \mathbb{E}\sum_{k=k_0}^{n} Y_k \geqslant \sum_{k=k_0}^{n} 1 - \exp(-\frac{1}{2})$$

$$= (n - k_0 + 1)(1 - \exp(-\frac{1}{2})) \geqslant \frac{1}{2}n(1 - \exp(-\frac{1}{2})),$$

where the last estimate is valid for $n \geqslant n_1$. Set $n_2 := \max\{k_0, n_1\}$. Because of

$$\sum_{n=1}^{\infty} \exp(-\frac{\delta^2}{4}\mathbb{E}\sum_{k=1}^{n} Y_k) = \sum_{n=1}^{n_2} \exp(-\frac{\delta^2}{4}\mathbb{E}\sum_{k=1}^{n} Y_k) + \sum_{n=n_2+1}^{\infty} \exp(-\frac{\delta^2}{4}\mathbb{E}\sum_{k=1}^{n} Y_k)$$

$$\leqslant \sum_{n=1}^{n_2} \exp(-\frac{\delta^2}{4}\mathbb{E}\sum_{k=1}^{n} Y_k) + \sum_{n=n_2+1}^{\infty} \exp(-\frac{\delta^2}{8}(1-\exp(-\frac{1}{2}))n),$$

the sum $\sum_{n=1}^{\infty} \exp(-\frac{\delta^2}{4}\mathbb{E}\sum_{k=1}^{n} Y_k)$ is finite. Thus, by Borel-Cantelli, the event

$$F = \{\sum_{k=1}^{n} Y_k \geqslant (1-\delta)\mathbb{E}\sum_{k=1}^{n} Y_k \text{ eventually}\}$$

has the probability 1. Choose $\omega \in F$ and $n(\omega)$ such that

$$\sum_{k=1}^{n} Y_k \geqslant (1-\delta)\mathbb{E}\sum_{k=1}^{n} Y_k \geqslant (1-\delta)\frac{1}{2}(1-\exp(-\frac{1}{2}))n$$

is valid for $n \geqslant n(\omega)$. In other words, we can divide $(\alpha_k U_k)_1^{\lfloor S^{-1}(n)\rfloor}$ into n groups and at least $m := \lfloor (1-\delta)\frac{1}{2}(1-\exp(-\frac{1}{2}))n \rfloor$ of these groups have non-zero elements. To be exact, we have only $m - m_0$ groups with non-zero elements. It can happen that $\lfloor S^{-1}(n) \rfloor$ is 0 if n is too small. The corresponding groups are empty in that case. But because of $m - m_0 \sim m$, we can make the computation without m_0, without loss of generality.
The last part of the proof deals with finding points with large distances. Observe that $(-g \circ S^{-1})'$ is decreasing because $g \circ S^{-1}$ is convex. Choose $\varepsilon = g(S^{-1}(n-1)) - g(S^{-1}(n)) \geqslant (-g \circ S^{-1})'(n)$. We show that at least $\frac{m}{2}$ elements of $(\alpha_k U_k)_1^{\lfloor S^{-1}(n)\rfloor}$ have at least the distance ε. To derive this result, we estimate the minimal distances of points being elements of the sections

$$(\alpha_l U_l)_{\lfloor S^{-1}(1)\rfloor+1}^{\lfloor S^{-1}(2)\rfloor}, (\alpha_l U_l)_{\lfloor S^{-1}(3)\rfloor+1}^{\lfloor S^{-1}(4)\rfloor}, (\alpha_l U_l)_{\lfloor S^{-1}(5)\rfloor+1}^{\lfloor S^{-1}(6)\rfloor}, \ldots$$

and

$$(\alpha_l U_l)_{\lfloor S^{-1}(2)\rfloor+1}^{\lfloor S^{-1}(3)\rfloor}, (\alpha_l U_l)_{\lfloor S^{-1}(4)\rfloor+1}^{\lfloor S^{-1}(5)\rfloor}, (\alpha_l U_l)_{\lfloor S^{-1}(6)\rfloor+1}^{\lfloor S^{-1}(7)\rfloor}, \ldots.$$

Without loss of generality, we may assume that at least $\lfloor \frac{m}{2} \rfloor$ non-zero points are part of the sequence finishing with $(\alpha_l U_l)_{\lfloor S^{-1}(n-1)\rfloor+1}^{\lfloor S^{-1}(n)\rfloor}$. Let $x \in (\alpha_l U_l)_{\lfloor S^{-1}(k-1)+1\rfloor}^{\lfloor S^{-1}(k)\rfloor}$ and $y \in (\alpha_l U_l)_{\lfloor S^{-1}(k-3)\rfloor+1}^{\lfloor S^{-1}(k-2)\rfloor}$ so that $x, y > 0$. We estimate their distance.

$$y - x \geqslant g(\lfloor S^{-1}(k-2)\rfloor) - g(\lfloor S^{-1}(k-1)\rfloor + 1) \geqslant g(S^{-1}(k-2)) - g(S^{-1}(k-1))$$

$$= (-g \circ S^{-1})'(\xi)_{|\xi \in (k-2,k-1)} \geqslant (-g \circ S^{-1})'(k-1) \geqslant (-g \circ S^{-1})'(n-1) \geqslant \varepsilon,$$

whenever $k \leqslant n$. In conclusion, we have ensured that there are $\lfloor \frac{m}{2} \rfloor$ non-zero points in the set $(\alpha_k U_k)_1^{\lfloor S^{-1}(n) \rfloor}(\omega)$ having pairwise distances larger than or equal to ε. It follows that at least $\lfloor \frac{1}{2} \cdot \lfloor \frac{m}{2} \rfloor \rfloor$ points have pairwise distances larger than 2ε. Thus,

$$N((\alpha_n U_n), |\cdot|, \varepsilon)(\omega) \geqslant N((\alpha_n U_n)_1^{\lfloor S^{-1}(n) \rfloor}, |\cdot|, \varepsilon)(\omega) \geqslant c_0 m$$

$$\geqslant cn \geqslant c((-g \circ S^{-1})')^{-1}(\varepsilon).$$

□

Corollary 3.19. *The following asymptotics are true, almost surely:*

1. *Given* $s(n) = n^{-a}, a \in (0,1)$, $\alpha_n = n^{-\beta}, \beta > 0$, *then*

$$(\alpha_. U_.)_n^* \sim (1-a)^{-\frac{\beta}{1-a}} n^{-\frac{\beta}{1-a}} \text{ and } N((\alpha_n U_n), |\cdot|, \varepsilon) \approx \varepsilon^{-\frac{1}{1+\frac{\beta}{1-a}}}.$$

2. *Given* $s(n) = n^{-a}, a \in (0,1)$, $\alpha_n = (1 + \log n)^{-b}, b > 0$, *then*

$$(\alpha_. U_.)_n^* \sim (1+\log n)^{-b} \text{ and } \varepsilon^{-1} |\log \varepsilon|^{-(b+1)} \preccurlyeq N((\alpha_n U_n), |\cdot|, \varepsilon) \preccurlyeq \varepsilon^{-1} |\log \varepsilon|^{-b}.$$

3. *Given* $s(n) = n^{-1}(1 + \log n)^{-a}, a \in (0,1)$, $\alpha_n = (1 + \log n)^{-b}, b > 0$, *then*

$$(\alpha_. U_.)_n^* \sim (1-a)^{-\frac{b}{1-a}} n^{-\frac{b}{1-a}} \text{ and } N((\alpha_n U_n), |\cdot|, \varepsilon) \approx \varepsilon^{-\frac{1}{1+\frac{b}{1-a}}}.$$

4. *Given* $s(n) = n^{-1}(1 + \log n)^{-a}, a \in (0,1)$, $\alpha_n = n^{-\beta}, \beta > 0$, *then*

$$q_1^{n^{\frac{1}{1-a}}} \leqslant (\alpha_. U_.)_n^* \leqslant q_2^{n^{\frac{1}{1-a}}} \text{ and } N((\alpha_n U_n), |\cdot|, \varepsilon) \approx |\log \varepsilon|^{1-a},$$

   where $0 < q_1 < q_2 < 1$.

5. *Given* $s(n) = n^{-1}(1 + \log n)^{-1}(1 + \log(1 + \log n))^{-a}, a \in (0,1)$,
   $\alpha_n = (1 + \log n)^{-b}, b > 0$, *then*

$$q_1^{n^{\frac{1}{1-a}}} \leqslant (\alpha_. U_.)_n^* \leqslant q_2^{n^{\frac{1}{1-a}}} \text{ and } N((\alpha_n U_n), |\cdot|, \varepsilon) \approx |\log \varepsilon|^{1-a},$$

   where $0 < q_1 < q_2 < 1$.

Proof. Apply a reformulation of Theorem 2.41 to derive that

$$g(S^{-1}(\frac{n}{1-\delta} + S(1))) \leqslant (\alpha_. U_.)_n^*(\omega) \leqslant g(S^{-1}(\frac{n-1}{1+\delta} - s_1 - 1))$$

is valid for $\omega \in F_\delta$, $\mathbb{P}\{F_\delta\} = 1$, and large n. Verify the regularity conditions and compute S^{-1} for the examples and subsitute the results into the formula. Use Theorem 2.48 to get the bounds for the covering numbers. □

Observe how the deletion factor changes the type of the original sequence (α_n). Example 3 shows how a logarithmical sequence becomes polynomial, example 4 shows how a polynomial sequence becomes a generalized exponential sequence and example 5 shows how a logarithmical sequence becomes generalized exponential. The first example shows how the deletion factor changes the sequence without leaving the original class (here: polynomial). And the second example shows that the influence of deletion factors vanishing too slowly can be very limited.

4 Conclusion, discussion and open problems

The last chapters revealed what kind of results can be expected for covering numbers of random processes' images. However, the techniques we used leave space for generalizations and, of course, there remain unanswered questions.

4.1 Open problems

1. There are few upper bound results for time-continuous processes indexed by arbitrary sets. Only some results are available for Lévy processes (cf. Theorem 2.36 and Proposition 2.37) and for the fractional Brownian motion the Hölder condition implies a bound (cf. the statement following Corollary 2.13).

2. Are there more results which are comparable to 2.43 and 2.44? Find conditions making it possible to gain knowledge about the covering numbers of $(\alpha_n \xi_n)$ and $(\alpha_n U_n \xi_n)$, but which require only limited knowledge of ξ_1.

3. Is it possible to transfer the ideas of Theorem 2.48 to more general random sequences with deletion factor, to get a.s. lower bounds? Close the gap between lower and upper bound which sometimes appears, e.g., in the logarithmical case.

4. Soften the regularity conditions given in Theorem 2.46.

5. Allow dependence among the components of $(\alpha_n \xi_n)$ and $(\alpha_n U_n \xi_n)$. Then Chernoff bounds cannot be used, but perhaps Chebycheff's estimate could help. Results in this more difficult situation could be of great interest in related fields. For example, in [3] a task is to compute entropy numbers (These numbers are related to the covering numbers. Covering numbers give the minimal number of balls with a fixed radius covering a bounded set, entropy numbers fix the number of intervals and ask for the minimal possible radius being necessary to cover a set.) of diagonal operators with random diagonal. To this purpose, it is necessary to determine the order of the decreasing rearrangement of the random diagonal elements. But in difference to our studies in this thesis, the diagonal elements used there are dependent and they have a more complicated structure. Nevertheless, a further application of random order results could be studying compactness properties of random operators.

6. Convex sequences: Find conditions making it possible to deduce $(g(n)^\gamma) \in \mathcal{C}$ if $\gamma < 1$ and $(g(n)) \in \mathcal{C}$.
   If $(f(n))$ is a null sequence and $(g(n)) \in \mathcal{C}$ and if we know $f(n) \succcurlyeq g(n)$, is there a possibility to gain a lower bound for $N((f(n)), |\cdot|, \varepsilon)$ using $N((g(n)), |\cdot|, \varepsilon)$?

7. Consider the multidimensional case (index and image dimension). Can we find index sets being as useful as the convex sequences?

4.2 Generalizations

1. Upper $\overset{\mathbb{P}}{\preccurlyeq}$ bounds for time-continuous processes indexed by convex sequences only require selfsimilarity properties and the (a.s.) boundedness of the image. Furthermore, we can relax the convexity assumption: Firstly, the index sets do not have to be null sequences. There is no problem in formulating convexity conditions for non-null sequences. Secondly, the monotonicity of the index sequence T is not necessary. In fact, to get results, it is necessary to know how many points of T can be found outside every ball around T's limit x_0, i.e., an upper estimate (in dependence of ρ) for the cardinality of $T \setminus B_\rho(x_0)$ transforms into an upper $\overset{\mathbb{P}}{\preccurlyeq}$ bound. It is also allowed that T has a finite number of cluster points.

2. Lower $\overset{\mathbb{P}}{\succcurlyeq}$ bounds for arbitrary processes only require useful estimates for

$$\mathbb{P}\left\{|X_t - X_s| \leqslant \varepsilon\right\}.$$

   Then Proposition 3.1 and Corollary 3.2 can help.

3. Consider random numbers ξ_1 with $\mathbb{P}\left\{\xi_1 \in \mathbb{R}^+\right\} < 1$. Then it is necessary to introduce two rearranged sequences, one containing the positive realisations and one containing the negative realisations. The techniques introduced above can be used to derive convergence and covering results for both parts.

References

[1] Aurzada, F.: *Small deviation probabilities of some stochastic processes.* Dissertation, Friedrich-Schiller-Universität Jena, 2006.

[2] Aurzada, F.; Lifshits, M. A.: *On the small deviation problem for some iterated processes.* Electronic Journal of Probability 14, 1992-2010, 2009.

[3] Aurzada, F.; Lifshits, M. A.; Linde, W.: *Small deviations of stable processes and entropy of the associated random operators*, Bernoulli 15(4), 1305-1334, 2009.

[4] Belov, I.A.: *On the computation of the probability density function of α-stable distributions.* Mathematical Modelling and Analysis 2005, Proceedings of the 10^{th} International Conference MMa2005&CMAM2, Trakai, 2005.

[5] Bertoin, J.: *Subordinators: Examples and Applications.* Ecole d'été de Probabilités de St-Flour, 1997.

[6] Blumenthal, R.M.; Getoor, R.K.: *A dimension theorem for sample functions of stable processes.* Ill. Journ. of Math. 4, 370-375, 1960.

[7] Blumenthal, R.M.; Getoor, R.K.: *Sample functions of stochastic processes with stationary independent increments.* J. Math. Mech. 10, 493-516, 1961.

[8] Borkar, V.S.: *Probability theory: an advanced course.* Springer, New York, 1995.

[9] Doob, J.L. *Stochastic processes.* Wiley, New York, NY, 1990.

[10] Ehm, W.: *Sample function properties of multi-parameter stable processes.* Probability Theory and Related Fields, Vol.56, No.2, 195-228, 1981.

[11] Falconer, K.: *Fraktale Geometrie: Mathematische Grundlagen und Anwendungen.* Spektrum, Akademischer Verlag, Heidelberg, 1993.

[12] Howroyd, J.D.: *Box and packing dimensions of projections and dimension profiles.* Math. Proc. Cambridge Philo. Soc., 130, 135-160, 2001.

[13] Hu, D.; Zhong, Y.: *Uniform packing dimension results for multiparameter stable processes.* Acta Math. Scientia 27B(1), 1-10, 2007.

[14] Hunt, G.A.: *Random Fourier transforms.* Trans. Amer. Math. Soc. 71, 38-69, 1951.

[15] Kahane, J.-P.: *Some Random Series of Functions.* 2nd Ed. Cambridge University Press, 1985.

[16] Khoshnevisan, D.: *From fractals and probability to Lévy processes and stochastic PDEs*. Progress in Probability, Vol.61, 111-141, 2009.

[17] Khoshnevisan, D.; Schilling, R.; Xiao, Y.: *Packing dimension profiles and Lévy processes*. 2010, Preprint. Available under http://www.math.utah.edu/~davar/PPT/KSX/

[18] Khoshnevisan, D.; Xiao, Y.: *Lévy processes: Capacity and Hausdorff dimension*. The Annals of Probability, Vol.33, No.3, 841-878, 2005.

[19] Khoshnevisan, D.; Xiao, Y.: *Packing dimension profiles and fractional Brownian motion*. Math. Proc. Cambridge Philo. Soc., 145, 205-213, 2008.

[20] Khoshnevisan, D.; Xiao, Y.: *Packing dimension of the range of a Lévy process*. Proc. Amer. Math. Soc. 136, 2597-2607, 2008.

[21] Lalley, S.P.: *The packing and covering functions of some self-similar fractals*. Indiana Univ. Math. J., 37, 699-709, 1988.

[22] Lifshits, M.A.:*Gaussian Random Functions*. Kluwer Academic Publishers, Dordrecht, 1995.

[23] Lifshits, M.A.: *unpublished manuscript*, 2008.

[24] Lifshits, M.A.: *unpublished manuscript*, 2009.

[25] Linde, W.; Shi, Z.: *Evaluating the small deviation probabilities for subordinated Lévy processes*. Stoch. Proc. Appl. 113, 273-287, 2004.

[26] Linde, W.; Zipfel, P.: *Small deviation of subordinated processes over compact sets*. Probab. Math. Statist. 28, 281-304, 2008.

[27] Millar, P.W.: *Path behavior of processes with stationary independent increments*. Z. Wahrsch. Verw. Gebiete 17, 53-73, 1971.

[28] Motwani, R.; Raghavan, P.: *Randomized Algorithms*. Cambridge University Press, Cambridge, 1995.

[29] Mandelbrot, B.B.; van Ness, J.W.: *Fractional Brownian Motions, Fractional Brownian Noises and Applications*. SIAM Review, Vol. 10, No. 4 (Oct., 1968), pp. 422-437, 1968.

[30] Mišík, L.; Žáčik, T.: *On the metric dimension of converging sequences*. Comment. Math. Univ. Carolinae 34,2, 367-373, 1993.

[31] Mišík, L.; Žáčik, T.: *A formula for calculation of metric dimension of converging sequences*. Comment. Math. Univ. Carolinae 40,2, 393-401, 1999.

[32] Mišík, L.; Žáčik, T.: *A simplified formula for calculation of metric dimension of converging sequences.* Mathematica Slovaca, Vol. 55, No. 3, 363-372, 2005.

[33] Sato, K.: *Lévy Processes and Infinitely Divisible Distributions.* Cambridge Studies in Advanced Mathematics 68, Cambridge University Press, Cambridge, 1999.

[34] Samorodnitsky, G.; Taqqu, M.S.: *Stable Non-Gaussian Random Processes.* Stochastic Modeling. Chapman & Hall, New York, 1994.

[35] Shieh, N.-R.; Xiao, Y.: *Hausdorff and packing dimensions of the images of random fields.* Bernoulli 16(4), 926-952, 2010.

[36] Sottinen, T.: *Fractional Brownian motion in Finance and Queueing.* Dissertation, University of Helsinki, 2003.

[37] Taylor, S.J.: *The use of packing measure in the analysis of random sets.* Stochastic Processes and their Applications. Lecture Notes in Mathematics, vol. 1203, Springer-Verlag, Berlin, 1986.

[38] Talagrand, M.; Xiao, Y.: *Fractional Brownian Motion and Packing Dimension.* Journal of Theoretical Probability, Vol. 9, No. 3, 1996.

[39] Xiao, Y.: *Packing dimension of the image of fractional Brownian motion.* Statist. Probab. Lett. 33, 379-387, 1997.

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