Róbert Vrábeľ

Nonlinear Dynamical Systems with High-Speed Feedback

Scientific Monographs in Automation and Computer Science

Edited by
Prof. Dr. Peter Husar (Ilmenau University of Technology) and
Dr. Kvetoslava Resetova (Slovak University of Technology in Bratislava)

Vol. 4

NONLINEAR DYNAMICAL SYSTEMS WITH HIGH-SPEED FEEDBACK

Róbert Vrábeľ

Universitätsverlag Ilmenau
2012

Impressum

Bibliographic information of the German National Library
The German National Library lists this publication in the German national bibliography, with detailed bibliographic information on the Internet at http://dnb.d-nb.de.

Author's acknowledgement to Dagmar Rusková for translation.

This scientific monograph originated from the author's habilitation thesis defended at the Slovak University of Technology in Bratislava, Faculty of Materials Science and Technology in Trnava.

Reviewers:
Dušan Krokavec, Professor, Ph.D
Dušan Mudrončík, Professor, Ph.D.
Juraj Spalek, Professor, Ph.D.

Author's contact address:
Róbert Vrábeľ, Assoc. Professor, Ph.D.
Slovak University of Technology in Bratislava
Faculty of Materials Science and Technology in Trnava

Ilmenau Technical University / University Library
Universitätsverlag Ilmenau
Postfach 10 05 65
98684 Ilmenau
www.tu-ilmenau.de/universitaetsverlag

Production and delivery
Verlagshaus Monsenstein und Vannerdat OHG
Am Hawerkamp 31
48155 Münster
www.mv-verlag.de

ISSN 2193-6439 (Print)
ISBN 978-3-86360-016-7 (Print)
URN urn:nbn:de:gbv:ilm1-2012100043

Titelfoto: photocase.com

Abstract

The aim of the work is to contribute to the qualitative and quantitative analysis of nonlinear dynamical systems of the second order with high-speed feedback, which can also be generally introduced into higher order systems which at present are gaining an increasing amount of popularity in use of the high-frequency oscillators in electronic circuits. Singular Perturbation Theory is the mathematical framework that yields the tools to explore the complicated dynamical behavior of these systems. The work gives an overview of some methods used to investigate the dynamics of singularly perturbed nonlinear systems, for which does not exist a comprehensive theory that would give us an overall view of their behavior.

This issue is currently happening world wide, not only in the mathematical community when dealing with the theory and application of nonlinear dynamical systems, but also in the field of electrical engineering and automation (contributions in research seminars and conferences in the Centre for Intelligent Control, National University of Singapore, IFAC Workshop on Singular Solutions and Perturbations in Control Systems, Russia, 1997; IEEE Conference Decision and Control, New Orleans, USA, 2007; The 2009 IEEE International Conference on Networking, Sensing and Control, Okayama University, Okayama, Japan;. . .).

Key words

nonlinear dynamical system of the second order with high-speed feedback, singularly perturbed nonlinear system, boundary layer

MOTIVATION AND INTRODUCTION

The work deals with the continuous nonlinear dynamical systems with high-speed feedback which can be described by differential equations of the second order

$$\epsilon y'' = f(t, y, y', \epsilon) \tag{$*$}$$

with a small parameter $\epsilon > 0$ at the highest derivative and a continuous function f. After rewriting into the system of differential equations

$$\begin{aligned} y_1' &= y_2 \\ \epsilon y_2' &= f(t, y_1, y_2, \epsilon) \end{aligned}$$

we are given a system of two differential equations of the first order. This system we can reformulate as follows

$$\begin{aligned} \dot{y}_1 &= \epsilon y_2 \\ \dot{y}_2 &= f(\epsilon\tau, y_1, y_2, \epsilon), \end{aligned}$$

where $\dot{} = \frac{\mathrm{d}}{\mathrm{d}\tau}$ and $t = \epsilon\tau$. Both scalings agree on the level of the structure of phase space system for $\epsilon \neq 0$ but they are completely different in the limit when $\epsilon \to 0^+$. It is important to realize that $\tau \to \infty$ for $\epsilon \to 0^+$, which means that the small time interval for t (slow time) corresponds to a large time interval for τ (fast time) which goes to infinity asymptotically. When examining global properties of systems described by a system of differential equations with a small parameter at the highest derivative, this often uses a combination of both approaches ([3], [4]).

Naming such systems as dynamical systems with high-speed feedback follows due to the fact that for $\epsilon \to 0^+$, with a finite right hand side of a mathematical model in the expression $\epsilon y_2'$ is allowed $|y_2'| \to \infty$ at any point of the time interval. This situation is typical for many cases interested in practice (Appendices A and B) in connection with the phenomena arising in nonlinear systems such as auto-oscillations, resonance, hysteresis, oscillations with large frequencies, ... ([10], [14]).

Differential equations (systems) with a small parameter at the highest derivative in professional literature depicts as a singularly perturbed system respecting the non validity of theorem about continuous dependence of solutions on a parameter and resulting problems in the approximation of solutions of differential equations in the right neighborhood of $\epsilon = 0$. The subject of exploration is the asymptotic behavior of these systems, when the value of the parameter ϵ is close to zero.

In the 1st chapter we refer to inappropriateness of the use of approximate methods for nonlinear dynamical systems with high-speed feedback and motivation to other parts of the work, where the used methods of the theory of dynamical systems are suitable for the study of nonlinear systems with high-speed feedback and especially those that allow high-frequency oscillations.

The 2nd chapter summarizes the methods used in other parts of the work and in Subsection 2.2.1 we describe the basic types of auto-oscillations which may occur in autonomous systems of the 2nd order.

Appendices A and B are devoted to the existence and analysis of high-frequency oscillations of nonlinear systems, namely for singular perturbed oscillators of Duffing's type. The systems considered here constitute a structural mathematical model to describe and simulate the nonlinear high-frequency circuits. For a review and for further references, we refer to the article of Ping Mei, Chenxiao Cai, Yun Zou: A Generalized KYP Lemma Based Approach for H_∞ control of Singularly Perturbed Systems published in the journal *Circuits, Systems, and Signal Processing*, 2009, where the autonomous linear system

$$\begin{aligned} x_1'(t) &= A_{11}x_1 + A_{12}x_2 + B_1u + F_1w \\ \epsilon x_2'(t) &= A_{21}x_1 + A_{22}x_2 + B_2u + F_2w \\ z &= C_1x_1 + C_2x_2, \end{aligned}$$

is investigated, where (x_1, x_2) is the state vector, A_{ij}, B_i, C_i, F_i are matrices of the system, u is a control input, w is outside interference and z is the measured output.

It is a system with one low frequency (in the first equation $\epsilon = 1$) and one high frequency (in the second equation $\epsilon \to 0^+$) regulator. A small perturbation parameter ϵ reflects the degree of separation between a „slow“ and „fast“ channel system.

Experimental equipment from the field of high-energy physics (e.g. nuclear electronics) often consists of subsystems that contain similar elements. Similarly, high frequency amplifiers which work in the region of frequencies that lie above the low frequency region, i.e. above 1 MHz, are used so that their input and output circuit include a resonant circuit or a system of resonant circuits. Unlike the mentioned article, we deal with mathematical models of high-frequency oscillators with an asymptotically negligible damping and nonlinear stiffness. We have derived the exact relationship defining the scope of the band considered perturbed nonlinear circuit depending on the resonance parameters of the subsystem (p. 31 and p. 42), which allows the desired frequency range and amplitude of the output to be reached. In addition, in Appendix A, where we investigate oscillators with nonlinear stiffness of the cubic type and where derived conditions on the parameters of the system, in compliance with the declining value of the parameter ϵ leads to damped oscillations ($\epsilon-$damped oscillations, p. 31).

Appendices C and D analyze the system of the 2nd order on the time interval of the finite length, with an emphasis on determining the conditions guaranteeing the existence of non-oscillatory solutions and their approximation up to order $O(\epsilon)$ in the right neighborhood of $\epsilon = 0$. Here and in Appendix E is a description of the so-called boundary layer phenomenon, which is typical for singularly perturbed systems. We show that the solutions, in general, start with fast transient and after decay of this transient they remain close to the solution of reduced problem with an arising new fast transient of solution at the end of interval. Boundary layers are formed due to the nonuniform convergence of the exact solution to the degenerate solution in the neighborhood of the ends of the considered interval ([4]). This is crucial for an approximation of these systems (p. 59).

Appendix F is a preparatory study (linear case) to Appendices G and H, where we explore the existence of damped nonlinear oscillations of a system with a high frequency and amplitude $A_\epsilon \to 0$ for $\epsilon \to 0^+$. The problem is transformed into ill-posed Fredholm's integral inequality of the 2nd order with a semiseparable kernel. Mathematical analysis of integral equations of this type which are a major problem in the theory of integral equations and their investigations which are devoted to specialized institutions, e.g. Sobolev Institute of Mathematics, Johann Radon Institute for Computational and Applied Mathematics.

All appendices are separate in the sense that the nomenclatures are used as defined therein and valid only for a particular appendix.

1 METHOD OF HARMONIC LINEARIZATION AND SYSTEMS WITH HIGH-SPEED FEEDBACK

1.1 Introductory considerations

Dynamical systems are found wherever there are interrelated elements that are producing and dissipating energy. The mathematical models of differential equations are used for continuous systems. This mathematical model can be addressed by conventional means of mathematical analysis until the system is free of nonlinear elements, which can not be neglected. It may be, for example, nonlinear resistance, which varies depending on temperature and on a nonlinear inductance and nonlinear capacitor. The mechanical elements include, for example, dry friction and nonlinear dampers. Physical quantities, numerically characterizing these elements enter into the model as parameters. In addition to these predictable parameters in the model enter even further, „parasitic“ elements such as capacitance, induction of the connecting wires, small time constants, mass, ... which are usually represented by a small parameter at the highest derivative of differential equation. As an example we can indicate a general model of DC motors with anchor that converts electrical energy into mechanical energy, or generators that convert mechanical energy into electrical energy. The mathematical model consists of a mechanical torque equation and the equation of the electrical circuit's device

$$J\omega' = ki$$
$$Li' = -k\omega - Ri + U,$$

where i, U, R and L are current, voltage, resistance and inductance of the anchor circuit, J is the moment of inertia, ω the angular velocity, ki is the torque and $k\omega$ is the induced back electromotive force (back e.m.f.). For well-designed engines L is very small and may play the role of our parameter ϵ. It can be shown mathematically that the description of the activity of the motor's singular direction can be approximated by the reduced differential equation of the 1st order, which we get if we put $L = 0$ ([4]).

In the idealized model, it is natural not to consider these „parasitic" elements, but it is known that in many cases this omission can lead not only inaccurate but even to an incorrect qualitative description of the operation of the device. If we make the values of these parameters equal to zero we obtain differential equations of the lower order, often unsolvable with regard to the derivative that appears implicitly (e.g. for an equation (*) we get, in general, an implicit equation $f(t, y, y', 0) = 0$). In these circumstances, non consideration of such parameters leads to an incomplete description of the physical phenomena ([7], [9], [11], [12]).

Another reason why the examination of differential equations with a small parameter at the highest derivative gives increased attention is the fact that they are a good model for the description of systems with strong nonlinearities and the high frequency circuits.

In general, the problem of solving mathematical models of nonlinear dynamical systems in their complexity is very difficult. We can solve only some special cases. Therefore, in practice, the approximate methods are used. There have been several methods developed to find a solution for nonlinear models. Each of them has its specific area of application in which its benefits when applied and where it can not be ignored. Therefore it is one of main tasks chosen of the most appropriate methods for analyzing nonlinear models. A common drawback of all these approximative approaches is their ineffectiveness in investigating singularly perturbed systems. We demonstrate this on the equivalent transfer method, belonging to one method of harmonic linearization (MHL), which allows us to specify the amplitudes and angular frequencies of auto-oscillations arising in some nonlinear dynamical systems. The problems of using the method of harmonic linearization for continuous nonlinear circuits is dealt with in e.g. work [8].

1.2 Nonlinear autonomous circuits and MHL

Consider a nonlinear autonomous oscillation circuit without damping with one stationary nonlinearity $\epsilon^{-2}y^3$ and with a linear term $-\epsilon^{-2}y$. If there are steady oscillations in the circuit, in some places the circuit will generally periodic but not be harmonic (at the output of nonlinearity). Assuming that the higher harmonic parts of this non-sinusoidal process are filtered by the linear part of the circuit, on the input of nonlinearity only the first harmonic from the output signal of nonlinearity is present. Then based on the principle of harmonic linearization we define the so-called equivalent transfer $F_N(A,\omega)$ as the ratio of the first harmonic of output to the sinusoidal signal $A\sin\omega t$ on the input of nonlinearity. In the case of nonlinear circuit with high-speed feedback, modeled by the singularly perturbed differential equation

$$\epsilon^2 y'' + y^3 - y = 0, \tag{1.2.1}$$

its linear part is described by frequency transfer $F_L(i\omega) = -\epsilon^{-2}$ and the nonlinear part is described by equivalent transfer

$$\begin{aligned} F_N(A,\omega) &= \frac{\epsilon^{-2} \int\limits_0^{\frac{2\pi}{\omega}} (A \sin \omega t)^3 \, e^{i\omega t} \mathrm{d}t}{\int\limits_0^{\frac{2\pi}{\omega}} (A \sin \omega t) \, e^{i\omega t} \mathrm{d}t} \\ &= \frac{\epsilon^{-2} i \int\limits_0^{\frac{2\pi}{\omega}} (A \sin \omega t)^3 \sin \omega t \mathrm{d}t}{i \int\limits_0^{\frac{2\pi}{\omega}} (A \sin \omega t) \sin \omega t \mathrm{d}t}. \end{aligned} \tag{1.2.2}$$

After little algebraic manipulation we obtain

$$F_N(A,\omega) = \frac{\epsilon^{-2} A^3 i \int\limits_0^{\frac{2\pi}{\omega}} \sin^4 \omega t \mathrm{d}t}{A i \int\limits_0^{\frac{2\pi}{\omega}} \sin^2 \omega t \mathrm{d}t}. \tag{1.2.3}$$

By using relationships

$$\begin{aligned} \sin^2 \omega t &= \frac{1}{2} - \frac{\cos 2\omega t}{2} \\ \sin^4 \omega t &= \frac{3}{8} - \frac{\cos 2\omega t}{2} + \frac{\cos 4\omega t}{8} \end{aligned}$$

and after integrating in the fraction (1.2.3) we get

$$F_N(A,\omega) = \frac{\frac{3}{4}\epsilon^{-2} A^3 \frac{\pi}{\omega} i}{A \frac{\pi}{\omega} i} = \frac{3}{4} A^2 \epsilon^{-2}. \tag{1.2.4}$$

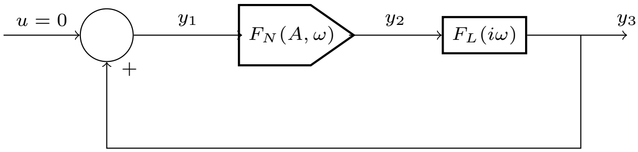

Fig. 1.2.1 *Scheme of nonlinear autonomous circuits*

Block scheme of examined circuit is in Figure ***1.2.1***.

From the amplitudinal condition for keeping constant oscillations

$$|F_N(A, \omega)| \cdot |F_L(i\omega)| = 1 \tag{1.2.5}$$

for the amplitude of auto-oscillations we will get the relation ([5], [6], [13])

$$A_\epsilon = \frac{2}{\sqrt{3}}\epsilon^2, \tag{1.2.6}$$

i.e $A_\epsilon \to 0$ for $\epsilon \to 0^+$.

Appendices A and B prove that the system described by the differential equation (1.2.1) except for auto-oscillations with a small amplitude, the existence of which we derived by MHL, allows the auto-oscillations of the 3rd type (p. 22) with arbitrarily high frequencies depending on the choice of parameter ϵ and with amplitude $A_\epsilon \geq \sqrt{2}$ for $\epsilon \to 0^+$, for a sufficiently large initial kinetic energy $(y'(0) = O\left(\epsilon^{-1}\right))$ to overcome potential barriers. This proves that the method based on system linearization around the equilibria is giving only a local characterization of the trajectories of a mathematical model system. When analyzing the global properties of nonlinear systems, the relevance of the nonlinearity of circuit elements are manifested, where we can observe phenomena such as auto-oscillations around several equilibrium states. The problem of the existence of the 3rd type of auto-oscillations with a large amplitude (1.2.1) was examined in the paper by B.S. Wu, W.P. Sun and C.W. Lim: Analytical approximations to the double-well Duffing oscillator in large amplitude oscillations, *Journal of Sound and Vibration,* Volume 307, Issues 3-5, (2007), pp. 953-960.

We recall that if we put $\epsilon = 0$, in the differential equation (1.2.1), the reduced equation $y^3 - y = 0$ has only a constant solution $y = -1$, 0, 1, i.e. ignoring the parameter ϵ we get an incorrect mathematical model of behavior of the considered nonlinear circuit.

In the next chapter we will describe the methods suitable for analysis of auto-oscillations of nonlinear dynamical systems.

2 MATHEMATICAL METHODS OF INVESTIGATION OF NONLINEAR DYNAMICAL SYSTEMS

In this chapter we will briefly describe the main analytical methods used at work. These are discussed in more detail in specific sections (appendices), in which they are applied.

2.1 Method of differential inequalities

The method of lower and upper solutions belongs to the so-called methods of a priori estimates of the solutions of nonlinear dynamical systems and is used in Appendices D, C and G as an effective tool to the study of asymptotic behavior of nonlinear dynamical systems with high-speed feedback on intervals of finite length where the system allows non-resonant solutions. The only principal restriction for the systems described by differential equations of the following form

$$\epsilon y'' = f(t, y, y'), \quad 0 < \epsilon << 1$$

is that the function f cannot have in the variable y' a growth greater than quadratic, i.e. $f(t, y, z) = O(z^2)$.

The method of lower and upper solutions is applicable to a broad class of dynamical systems, not only of the 2nd order, but also of the higher orders as well as for systems described by partial differential equations.

2.2 Method of phase-state space

This method can be used to analyze different types of resonances, oscillations and auto-oscillations arising in some types of nonlinear systems and in combination with other procedures is applied in Appendices A a B.

We apply the phase-state space method to the autonomous dynamical system described by the singularly perturbed differential equation

$$\epsilon^2 y'' + f(y) = 0, \tag{2.2.1}$$

which we may transform into the system

$$\epsilon y' = w \tag{2.2.2}$$

$$\epsilon w' = -f(y). \tag{2.2.3}$$

The equations of phase space trajectories can be obtained as the first integral of the system (2.2.2), (2.2.3) in the form

$$\frac{1}{2}w^2 + \int f(y)\mathrm{d}y = H^\epsilon, \quad H^\epsilon \in \mathbb{R}. \tag{2.2.4}$$

Then after simple algebraic manipulation,

$$w = \pm \left(2(H^\epsilon - V(y))\right)^{\frac{1}{2}}.$$

The physical interpretation of the equation (2.2.4) complies with the law of conservation of energy of the conservative system (2.2.2), (2.2.3) Term

$\frac{1}{2}w^2$ we interpret as the kinetic energy and the term $V(y) = \int f(y)\mathrm{d}y$ as a potential energy of the system. It is well known that the total energy of the system H^ϵ is considered as a Lyapunov function for investigating the stability of the trivial equilibria of the Hamiltonian systems as well as for determining the stability region. To what region of stability of the system can be found depends on the choice of optimal Lyapunov function. Generally applicable methods for choosing this function have not been developed ([1], [2]).

If the points $[y_k, 0]$ are equilibrium points of (2.2.2), (2.2.3) we investigate their stability by using the function $V(y)$. If the function V has in y_k nondegenerate local minimum, then the equilibrium point $[y_k, 0]$ is stable (center) and if y_k is a nondegenerate local maximum of the function V, then the corresponding equilibrium point is unstable (saddle).

Specifically, for the nonlinear autonomous circuits analyzed in Section 1.2 is $f(y) = y^3 - y$. In the absence of a term $\delta y'$, it is an undamped oscillation circuit, which is the basic assumption of the existence of oscillations with steady amplitude. Analogously to p. 28 we can deduce that

$$\frac{\mathrm{d}H^\epsilon}{\mathrm{d}t} \equiv 0 \tag{2.2.5}$$

along the trajectories of the system, i.e. the solutions of mathematical model are parameterized by a constant total energy. In view of (2.2.5) the function $\frac{\mathrm{d}H^\epsilon}{\mathrm{d}t}$ is negatively semi-definite and therefore is a suitable Lyapunov function for investigating stability (not asymptotic) of equilibrium states of the considered system (2.2.2), (2.2.3) with the function $f(y) = y^3 - y$ ([1], p. 337). Their analysis shows that the points $C_1[-1, 0]$ and $C_2[1, 0]$ of the phase-state space are the centra and point $S[0, 0]$ is the saddle. Systems having the equilibria of type C_1, C_2 are sometimes referred to as the systems at the border of stability, because small changes in the initial conditions do not cause significant changes in response to the system but the system does not return to equilibrium C_1, C_2, respectively, for $t \to \infty$ (stable oscillations are formed around the points C_1 or C_2 with a constant amplitude). The region of stability of the equilibria C_1 and C_2 can be expressed by the condition $0 \leq H^\epsilon < V(0)$ with $V(y) = \int_1^y f(s)\mathrm{d}s$, $i = 1, 2$. The equilibrium S is unstable.

The following subsection shows the basic types of auto-oscillations that may arise in the autonomous dynamical systems (2.2.1) depending on the nature of equilibrium points.

2.2.1 *Types of auto-oscillations in the autonomous 2nd order dynamical systems*

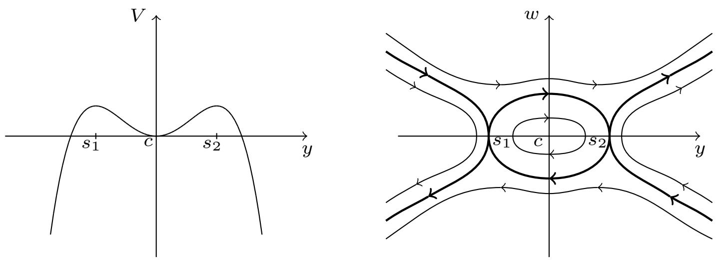

Fig. 2.2.1 *Potential V and phase portrait (Type 1)*

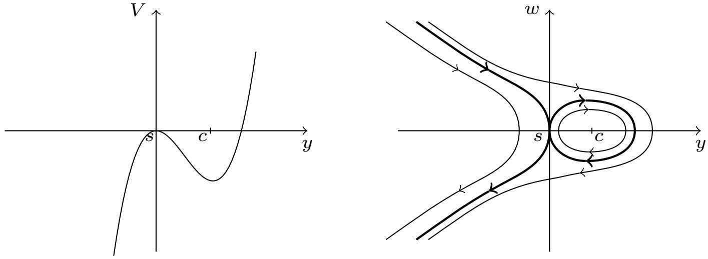

Fig. 2.2.2 *Potential* V *and phase portrait (Type 2)*

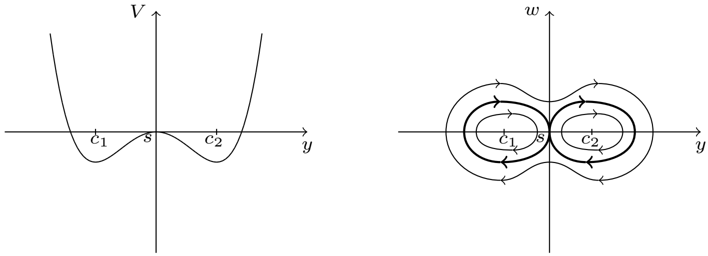

Fig. 2.2.3 *Potential* V *and phase portrait (Type 3)*

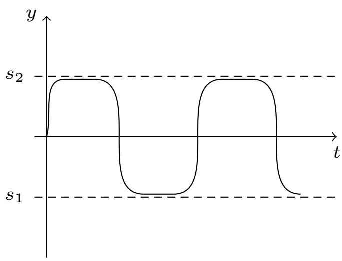

Fig. 2.2.4 *Auto-oscillations – Type 1 (for* $y_\epsilon(0) = 0,\ y'_\epsilon(0) > 0$*)*

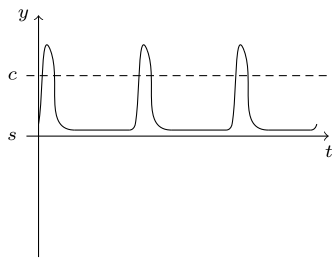

Fig. 2.2.5 *Auto-oscillations – Type 2* (*for* $y_\epsilon(0) > 0,\ y'_\epsilon(0) > 0$)

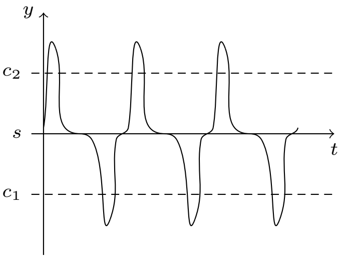

Fig. 2.2.6 *Auto-oscillations – Type 3* (*for* $y_\epsilon(0) > 0,\ y'_\epsilon(0) > 0$)

Bibliography

1. M. Gopal: *Modern Control System Theory*, New Age International, New Delhi, ISBN 8122405037, 1993.
2. L. Gvozdjak, M. Boršč and M. Vitko: *Základy kybernetiky (Fundamentals of Cybernetics)*, Alfa, Bratislava, 1990.
3. Ch. K. R. T. Jones: *Geometric Singular Perturbation Theory*, C.I.M.E. Lectures, Montecatini Terme, June 1994, Lecture Notes in Mathematics 1609, Springer-Verlag, Heidelberg, 1995.
4. P. Kokotovic, H. K. Khali and J. O'Reilly: *Singular Perturbation Methods in Control, Analysis and Design*, Philadelphia: Society for Industrial and Applied Mathematics, ISBN 0898714443, 1999.
5. S. Kubík *et al.*: *Teorie automatického řízení I. (Theory of Automatic Control I)*, SNTL, Praha, 1982.
6. S. Kubík *et al.*: *Teorie automatického řízení II. (Theory of Automatic Control II)*, SNTL, Praha, 1982.
7. P. Mei, Ch. Cai and Y. Zou: *A Generalized KYP Lemma-Based Approach for H_∞ Control of Singularly Perturbed Systems*, Circuits, Systems, and Signal Processing, 2009.
8. G. A. Leonov: *On the Method of Harmonic Linearization*, Automation and Remote Control, Vol. 70, No. 5 (2009), pp. 800—810.
9. E. P. Mishchenko and L. S. Pontryagin: *Differential equations with a small parameter attached to the higher derivatives and some problems in the theory of oscillation*, IRE Trans.CT-7 (1960), pp. 527–535.
10. M. P. Mortell: *Singular perturbations and hysteresis*, Cambridge University Press, ISBN 9780898715972, 2006.
11. L. V. Rodygin and L. S. Pontryagin: *Approximate solution of a system of ordinary differential equations with a small parameter in the derivatives*, Dokl. Akad. Nauk SSSR, 131 (1960), pp. 255–258.

12. A. S. Sidorkin, A. M. Solodukha and L. L. Nesterenko: *Nonlinear dynamics in an electrical circuit containing a ferroelectric*, Fiz. Tverd. Tela (St. Petersburg) Vol. 39 (1997), pp. 918—919.

13. A. Vrban, J. Halama and B. Husárová: *Základy teórie automatického riadenia (Fundamentals of automatic control)*, scriptum, STU, Bratislava, 1999.

14. Y. Yamamoto, T. Yamaguchi and K. Imaeda: *Hysteresis and Relaxation Phenomena near a Bifurcation Point of the Oscillation Mode of the Forced Oregonator*, J. Phys. Soc. Jpn. 58 (1989), pp. 1474–1481.

APPENDICES

A ANHARMONIC QUARTIC POTENTIAL OSCILLATOR PROBLEM AT OCCURRENCE OF SINGULAR PERTURBATIONS

A.1 Introduction

Consider a singularly perturbed quartic potential oscillator problem described by a differential equation of the form

$$\epsilon^2 \left(a^2(t)y'\right)' + f(y) = 0, \tag{A.1.1}$$

where $a(t)$ is a positive C^1 (i.e. continuous up to first derivative) nondecreasing function, ϵ is a small positive parameter and $f(y) = 2Ay + 3By^2 + 4Cy^3,\ A, B, C \in \mathbb{R}$.

For the singular problem, that is, when $\epsilon \to 0^+$, considerations as below may be relevant the study of many phenomena in physics (motion of a nonlinear spring with spring constant large compared to the mass, critical paths of Feynman path integrals [5], [6], theory of diffusion and reaction in permeable catalysts [1], [2], for example). Moreover, this class of equations has special significance in connection with applications involving nonlinear vibrations and chaos (e.g. references [4], [7], [9]).

Martin Sanches *et al.* [8] have considered problems of the above form with $\epsilon^2 = 1$, $a \equiv 1$; the solutions for the different types of quartic potentials are given in terms of Jacobi elliptic functions.

In this chapter we study in particular the appearance of ϵ-undamped oscillations, and our considerations are based upon Prüffer's transformation of coordinates [3], [11].

A.2 Prüffer's transformation and ϵ-undamped oscillations

Consider the problem (A.1.1) with initial conditions

$$y(0,\epsilon) = 0,\ y'(0,\epsilon) = c \neq 0 \tag{A.2.1}$$

on the finite interval $[0,\tau]$.

Following the standard approach for the study of the solutions of (A.1.1) (see, e.g., [10]) we introduce the variable $w = \epsilon a y'$ and write this equation in the system form

$$y' = \frac{1}{\epsilon a} w \tag{A.2.2}$$

$$w' = -\frac{1}{\epsilon a} f(y) - \frac{a'}{a} w. \tag{A.2.3}$$

If we consider the function

$$H(y,w) = \frac{1}{2}w^2 + V(y), \qquad V(y) = \int_0^y f(s)\mathrm{d}s$$

and compute its derivative along the solutions of (A.2.2), (A.2.3), we have

$$\dot{H}(y,w) = ww' + f(y)y' = w\left[-\frac{1}{\epsilon a}f(y) - \frac{a'}{a}w\right] + f(y)y' = -\frac{a'}{a}w^2.$$

We use the level curve of H to characterize the trajectories of (A.2.2), (A.2.3). It is clear that the orbits of (A.2.2), (A.2.3) point to the right in the upper half-plane of the phase space (y, w) and to the left in the lower half-plane.

Now we will assume that $C > 0$ and $D = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 < 0$ where $p = \frac{A}{2C} - \frac{3B^2}{16C^2}$ and $q = \frac{B^3}{32C^3} - \frac{AB}{8C^2}$. The level curve corresponding to $H = V(0) = 0$ connect the points $\left(\tilde{y}_0^{(1)}, 0\right)$, $(0, 0)$ and $\left(\tilde{y}_0^{(2)}, 0\right)$ where $\tilde{y}_0^{(i)}$, $i = 1, 2$ are the roots of equation $H(y, 0) = 0$. For later reference we also remark that if $B = 0$ then $\tilde{y}_0^{(1)} + \tilde{y}_0^{(2)} = 0$. Let us consider the nearby level curves with $H > 0$. The function $H\left(y(t, \epsilon), \epsilon a(t) y'(t, \epsilon)\right)$ is a monotone nonincreasing function and

$$
\begin{aligned}
0 < H\left(y(t, \epsilon), \epsilon a(t) y'(t, \epsilon)\right) &\leq H\left(y(0, \epsilon), \epsilon a(0) y'(0, \epsilon)\right) \\
&= \frac{1}{2}(\epsilon a(0) c)^2
\end{aligned}
$$

for every $t \in [0, \tau]$.

Hence, $|y(t, \epsilon)| \leq \max\left\{\left|\tilde{y}_{\epsilon_0}^{(i)}\right|, i = 1, 2\right\}$ for $0 < \epsilon < \epsilon_0$ where $\tilde{y}_{\epsilon_0}^{(i)}$, $i = 1, 2$ are the roots of the equation $H(y, 0) = \frac{1}{2}\left(\epsilon_0 a(0) c\right)^2$.

Let $z_{[0,\tau]}(y)$ denote the number of zeros of the nontrivial solution y of (A.1.1) on $(0, \tau)$, briefly reffered to as the „zero number“ of y on $(0, \tau)$.

In order to apply the standard technique of performing Prüffer's transformation of coordinates, we introduce the variable $v = \epsilon a^2 y'$ and write (A.1.1) in the following system form:

$$
\begin{aligned}
y' &= \frac{1}{\epsilon a^2} v \\
v' &= -\frac{1}{\epsilon} f(y).
\end{aligned}
$$

Then, we let $y = r \cos \gamma$ and $v = -r \sin \gamma$, and we obtain the following differential equation for γ:

$$
\gamma' = \frac{1}{\epsilon}\left[\frac{1}{a^2} \sin^2 \gamma + \bar{f}(y(t)) \cos^2 \gamma\right],
$$

$\gamma(0) = \frac{\pi}{2}$(for $c < 0$) or $\gamma(0) = \frac{3\pi}{2}$(for $c > 0$) where $\bar{f}(y) = \frac{f(y)}{y}$ for $y \neq 0$ and $\bar{f}(0) = f'(0)$. After little arrangements we get

$$\gamma' = \frac{1}{\epsilon}\left[\frac{1}{a^2} + \cos^2\gamma\left(\bar{f}(y) - \frac{1}{a^2}\right)\right].$$

From the equation $y = r\cos\gamma$ we obtain

$$\left|\cos^2\gamma\left(\bar{f} - \frac{1}{a^2}\right)\right| = \left|\frac{y^2\left(\bar{f} - \frac{1}{a^2}\right)}{y^2 + v^2}\right| = \left|\frac{y^2\left(\bar{f} - \frac{1}{a^2}\right)}{y^2 + 2a^2(H - V)}\right|$$

for $y \in [y_0, y_1]$, where $y_0 < 0 < y_1$ are the roots of the equation $f(y) = 0$. Thus, there is a positive constant $\kappa_1 < \min\{-y_0, y_1\}$ such that

$$\left|\cos^2\gamma\left(\bar{f} - \frac{1}{a^2}\right)\right| < \frac{1}{2a^2(t)}$$

for $t \in [0, \tau]$ and $|y| < \kappa_1$. Furthermore, for $|y| \geq \kappa_1$

$$\left|\cos^2\gamma\left(\bar{f} - \frac{1}{a^2}\right)\right| \leq \left|\frac{y^2\left(a^{-2}(t) - \bar{f}(y)\right)}{y^2 - 2a^2(t)V}\right|.$$

Let $B = 0$. Therefore

$$0 < \frac{a^{-2} - \bar{f}(y)}{a^{-2} - \frac{2V(y)}{y^2}} < 1,$$

and simple algebraic manipulation leads to inequality

$$0 < \frac{y^2\left(a^{-2}(t) - \bar{f}(y)\right)}{y^2 - 2a^2(t)V(y)} < \frac{1}{a^2(t)}$$

for $t \in [0, \tau]$ and $\kappa_1 \leq |y| \leq y_1 + \kappa_2 < -\tilde{y}_0^{(1)}\left(= \tilde{y}_0^{(2)}\right)$, where κ_2 is a sufficiently small positive constant. Thus, taking into consideration the fact that y is uniformly bounded, we obtain that $\gamma' \geq \frac{1}{\epsilon}\tilde{c}$ where $\tilde{c}$ is a positive constant such that $\tilde{c} = \min\{\tilde{c}_1, \tilde{c}_2\}$, and

$$\tilde{c}_1 = \min\left\{a^{-2}(t) - \frac{y^2\left(a^{-2}(t) - \bar{f}(y)\right)}{y^2 - 2a^2(t)V(y)}, t \in [0, \tau], |y| \leq y_1 + \kappa_2\right\},$$

$$\tilde{c}_2 = \min\left\{\frac{1}{a^2(t)}\sin^2\gamma + \bar{f}(y)\cos^2\gamma, t \in [0, \tau], |y| \geq y_1 + \kappa_2,\right.$$

$$\gamma \in \mathbb{R}\Big\}.$$

Since $\tilde{c}$ is independent of ϵ we conclude that, by taking ϵ sufficiently small, γ can be made arbitrarily large. Moreover, setting

$$b_{\epsilon_0} = \max\left\{a^{-2}(t), t \in [0, \tau]\right\} + \max\left\{\left|\bar{f}(y)\right|, |y| \le -\tilde{y}^{(1)}_{\epsilon_0}\right\}$$

we get $\gamma' \le \frac{1}{\epsilon} b_{\epsilon_0}$ for $|y| \le \tilde{y}^{(2)}_{\epsilon_0}$ and $0 < \epsilon \le \epsilon_0$. This implies that for any n_0 there is $\epsilon_0 = \epsilon_0(\tau, n_0)$ such that $\gamma(\tau, \epsilon) > \frac{\pi}{2} + \pi n_0$ for every $0 < \epsilon < \epsilon_0$. The standard theory then implies that $z_{[0,\tau]}(y(t,\epsilon)) \ge n_0$ for $\epsilon \in (0, \epsilon_0)$. Notice that if $C < 0, D < 0$ and $B = 0$ then the problem (A.1.1),(A.2.1) has for every sufficiently small ϵ a solution y, so-called $\epsilon-$damped oscillations, with arbitrarily large zero number $z_{[0,\tau]}(y)$ depending on ϵ and an amplitude tending to zero for $\epsilon \to 0^+$.

Finally, let us denote by s the spacing between two successive zeros of y on $(0, \tau)$, then from inequalities $\frac{\tilde{c}}{\epsilon} \le \gamma' \le \frac{b_{\epsilon_0}}{\epsilon}$ we obtain an estimate of s of the form

$$\epsilon\left(\frac{\pi}{b_{\epsilon_0}}\right) \le s \le \epsilon\left(\frac{\pi}{\tilde{c}}\right).$$

As an illustrative example, we consider the equation

$$\epsilon^2 y'' + y^3 - y = 0, \qquad 0 < t < 1 \tag{A.2.4}$$

with initial conditions

$$y(0,\epsilon) = 0, y'(0,\epsilon) = c < 0. \tag{A.2.5}$$

There the trajectories are parametrized by their constant energy $H(\epsilon)$,

$$\lim_{\epsilon \to 0^+} H(\epsilon) = 0$$

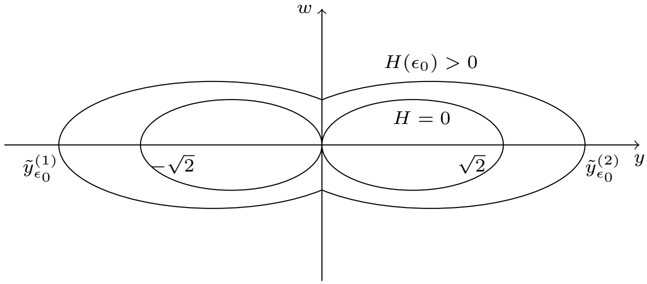

Fig. A.2.1 *Phase-plane portrait of the problem (A.2.4)*

while conservation of energy requires that $\frac{1}{2}w^2 + V(y) = H(\epsilon)$ (see Figure ***A.2.1***).

The constant becomes

$$b_{\epsilon_0} = \max\left\{a^{-2}(t), t \in [0,1]\right\} + \max\left\{\left|\bar{f}(y)\right|, |y| \le \tilde{y}^{(2)}_{\epsilon_0}\right\}$$

$$= 1 + \sqrt{1 + 2\left(\epsilon_0 c\right)^2}.$$

Therefore, $\gamma\left(1, \epsilon_0\right) < \frac{3\pi}{2}$, for instance, when

$$|c| < \frac{\sqrt{2}\pi}{4} \quad \text{and} \quad \epsilon_0 > \left[\left(\frac{\pi}{2}\right)^2 - 2c^2\right]^{-\frac{1}{2}}.$$

On the basis of the considerations above we conclude that for every $n_0 \in \mathbb{N}$ there is an ϵ_0 such that for $\epsilon \in (0, \epsilon_0)$ the corresponding solution has a zero number $z_{[0,1]}\left(y(t,\epsilon)\right) \ge n_0$.

References

1. R. Aris: *The mathematical theory of diffusion and reaction in permeable catalysts*, Clarendon Press, Oxford, 1975.

2. J. A. Boa: *A Model Biochemical Reaction*, Doctoral Dissertation, California Institute of Technology, 1974.
3. E. A. Coddington, N. Levinson: *Theory of Ordinary Differential Equations*, Krieger Publishing, Malabar, 1984.
4. N. Minorski : *Nonlinear Oscillations*, Van Nostrand, New Jersey, 1962.
5. D. W. McLaughlin: *Path integrals, asymptotics, and singular perturbations*, J. Math. Physics 12 (1972), pp. 786–796.
6. D. W. McLaughlin: *Complex time, contour independent path integral, and barrier penetration*, J. Math. Physics 12 (1972), pp. 1099–1108.
7. F. C. Moon: *Chaotic Vibrations*, Wiley, New York, 1987.
8. A. M. Sanches, J. D. Bejarano, D. C. Marzal: *Solution of the anharmonic quartic potential oscillator problem*, J. Sound and Vibration,161(1) (1993), pp. 19–31.
9. S. Timoshenko, D. H. Young, and W. Weaver, Jr: *Vibration Problems in Engineering*, Wiley, New York, 1990.
10. R. Vrábeľ: *Singularly pertubed anharmonic quartic potential oscillator problem*, Zeitschrift für angewandte Mathematik und Physik ZAMP. - ISSN 0044-2275. - Vol. 55 (2004), pp. 720–724.
11. J. Weidmann: *Spectral theory of ordinary differential operators*, Lecture Notes in Mathematics 1258, Springer-Verlag, 1987.

B SOLUTIONS WITH ARBITRARILY LARGE ZERO NUMBERS

B.1 Introduction

We consider the second order singularly perturbed differential equation

$$\epsilon^2 \left(a^2(t)y'\right)' + p(t)f(y) = 0, \qquad 0 < t < \tau, \qquad \epsilon > 0 \qquad \text{(B.1.1)}$$

with initial conditions

$$y(0,\epsilon) = 0,\ y'(0,\epsilon) = \frac{\bar{c}}{\epsilon}, \qquad \bar{c} \in \mathbb{R} \setminus \{0\} \qquad \text{(B.1.2)}$$

where

$$a : [0,\tau] \to (0,\infty)$$

is C^1 nonincreasing function,
p is C^1 decreasing function, $p(t) \geq 0$ on $[0,\tau]$,
f is a continuous odd function with exactly three simple zeros

$$f(-y_0) = f(0) = f(y_0) = 0,\ f'(0) < 0 \text{ and } \lim_{y\to\infty} \frac{f(y)}{y} < \infty.$$

Without loss of generality, we may consider that f is increasing on (y_0, ∞) and as a illustrative example, we can take $f(y) = y - 2 \arctan y$ (i.e. $(\pm y_0, 0)$ be the nonhyperbolic equilibria). For hyperbolic theory see [3] (in this work the author's consider the FitzHugh-Nagumo equation with a specific choice of f, $f(y) = y(1-y)(y-a)$, where $a > 0$ is a real parameter), [2], for example.

For the singular problem, that is, when $\epsilon \to 0^+$, the considerations as below may be relevant to the study of equilibrium solutions of the scalar parabolic reaction-diffusion equation with decreasing diffusion. Also, we remark that for $p \geq 0$ the corresponding equation is not dissipative. Moreover, this class of equations has special significance in connection with applications involving nonlinear vibrations and chaos (e.g. references [4], [5], [7]).

The problems of this form have been previously considered by Rocha [6] and Angenent *et al.* [1]. In [6], $p(t) \equiv -1$. However, the situation is more complicated when $p(t) \geq 0$ from the existence of the zeros of solutions point of view. It follows from the fact that the linearized version of (B.1.1) do not admit an oscillatory solution for $p(t) \geq 0$, unlike the case $p(t) < 0$.

The goal of this chapter is to prove the existence of decreasing sequence $\{\epsilon_n\}_{n=n_0}^{\infty}$, $\epsilon_n \to 0^+$ such that the corresponding solutions of (B.1.1),(B.1.2) have n zeros on $(0, \tau)$ for every $n \geq n_0$, $n \in \mathbb{N}$ and $y(\tau, \epsilon_n) = 0$.

Nomenclature

$\bar{f}(y) = \frac{f(y)}{y}$ for $y \neq 0$ and $\bar{f}(0) = f'(0)$;

$V(y) = \int\limits_0^y f(u)\mathrm{d}u;$

$z_{[0,\tau]}(y)$ denotes the zero number of the nontrivial solution y of (B.1.1) on $(0, \tau)$;

$-Y_{s,\epsilon}, Y_{s,\epsilon}$ $(-Y_{s,\epsilon} < Y_{s,\epsilon})$ denote the real roots of equation

$$p(s)V(y) = \hat{H}(s, y(s, \epsilon), w(s, \epsilon), \epsilon) - r(s, \epsilon)$$

for $s \in [0, \tau]$ (for definition of $\hat{H}$ and r see below);

$-Y_0, 0, Y_0$ $(-Y_0 < 0 < Y_0)$ denote the real roots of equation $V(y) = 0$;

$d_I(y)$ denotes spacing between two successive zero numbers of y on the interval I.

The rest of this chapter is organized as follows. In Section B.2 is explained technique necessary for the understanding of the chapter and in Section B.3 is formulated the theorem which is the main result.

B.2 Preliminaries

In order to apply the standard approach for the study of the solutions of (B.1.1) we introduce the variable $w = \epsilon a y'$ and write this equation in the system form

$$y' = \frac{1}{\epsilon a} w \tag{B.2.1}$$

$$w' = -\frac{1}{\epsilon a} p(t) f(y) - \frac{a'}{a} w. \tag{B.2.2}$$

If we consider the function

$$\hat{H}(t, y, w, \epsilon) = H(t, y, w) + r(t, \epsilon)$$

where

$$H(t,y,w) = \frac{1}{2}w^2 + p(t)V(y),$$

$$r(t,\epsilon) = -\int_0^t \left\{ p'(s) \int_0^{y(s,\epsilon)} f(u)\,\mathrm{d}u \right\} \mathrm{d}s$$

and compute its derivative along the solutions of (B.2.1), (B.2.2), we have

$$\dot{\hat{H}}(t,y,w,\epsilon) = ww' + p(t)f(y)y' + p'(t)\int_0^y f(u)\mathrm{d}u + r'(t,\epsilon)$$

$$= w\left[-\frac{1}{\epsilon a}p(t)f(y) - \frac{a'}{a}w\right] + p(t)f(y)y' + p'(t)\int_0^y f(u)\mathrm{d}u + r'(t,\epsilon)$$

$$= -\frac{a'}{a}w^2.$$

Hence, we conclude that $\hat{H}$ is monotone, nondecreasing function and $\hat{H}\,\big|_{t=0} = \frac{1}{2}\left(a(0)\bar{c}\right)^2 > 0$ implies that $\hat{H} > 0$ on $[0,\tau]$ for every ϵ. We use the level curves of $\hat{H}$ to characterize the trajectories of (B.2.1), (B.2.2). One first draws the $(t,y) - p(t)V(y)$ profile and then graphically draws the trajectory (y,w) with

$$w = \pm\left(2\left(\hat{H} - r(t,\epsilon) - p(t)V(y)\right)\right)^{\frac{1}{2}}$$

extending it as long as w remains real (i.e., until $p(t)V(y)$ exceeds $\hat{H}-r$). It is instructive for the future to keep in the mind the phase-portraits of the system (B.2.1), (B.2.2) with $\hat{H} - r \geq 0$ (see Figure ***B.2.1***).

Let us introduce the new variable $v = \epsilon a^2 y'$ and write (B.1.1) in the system form

$$y' = \frac{1}{\epsilon a^2}v \tag{B.2.3}$$

$$v' = -\frac{1}{\epsilon}p(t)f(y). \tag{B.2.4}$$

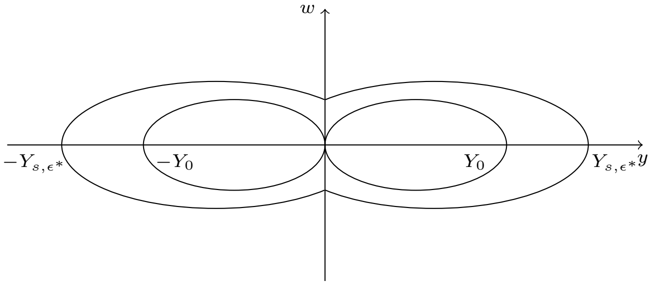

Fig. B.2.1 *Intersection a 3-dimensional manifold* $\frac{1}{2}w^2 + p(s)V(y) = \hat{H} - r$ *with the subspace* $(t, y) = (s, \epsilon^*)$ *of* (t, y, w, ϵ)*-space for* $\hat{H} - r > 0$

Then, expressing (y, v) in polar coordinates, $y = r\cos\gamma$, $v = -r\sin\gamma$ (obviously, $y^2 + v^2 > 0$ for every nontrivial solution of (B.1.1)) we obtain the following differential equation for γ

$$\gamma' = \frac{1}{\epsilon}\left[\frac{1}{a^2(t)}\sin^2\gamma + p(t)\bar{f}(y(t,\epsilon))\cos^2\gamma\right], \tag{B.2.5}$$

$\gamma(0) = \frac{\pi}{2}$(for $\bar{c} < 0$) or $\gamma(0) = \frac{3\pi}{2}$(for $\bar{c} > 0$). It is clear, that $\gamma(1, \epsilon) > (2k+1)\frac{\pi}{2}$, $k \in \mathbb{N}$ implies that $z_{[0,\tau]}(y(t, \epsilon) > k$ (for $\bar{c} < 0$).

B.3 Main result

We precede the main result of this chapter by important Lemma.

Lemma 1. $\hat{H} - r \geq 0$ *on* $[0, \tau]$, $\hat{H} - r > 0$ *for* $y \in (-Y_0, Y_0)$.

Proof. The statement follows immediately from the inequalities

$$\left(\hat{H}-r\right)' = p'V - \frac{a'}{a}w^2 \geq p'V > 0$$

for $y \in (-Y_0, Y_0)$ and

$$\hat{H} - r > 0$$

for $y \in [-Y_{t,\epsilon}, Y_{t,\epsilon}] \setminus [-Y_0, Y_0]$.

Now we formulate the main result of this chapter.

Theorem 1. (see, e.g., [8]) *Consider the problem (B.1.1), (B.1.2). Assume that*

$$\frac{f(y)}{y} > \frac{2V(y)}{y^2} \text{ for every } y \in [-y_0, y_0] \setminus \{0\}. \tag{B.3.1}$$

Then there is a decreasing sequence $\{\epsilon_n\}_{n=n_0}^{\infty}$, $\lim_{n\to\infty} \epsilon_n = 0$ *such that*

(i) *The corresponding solution of (B.1.1), (B.1.2) has*

$$z_{[0,\tau]}\left(y\left(t,\epsilon_n\right)\right) = n$$

.

(ii) $h_1\left(t,\epsilon_n\right) \leq d_I\left(y\left(t,\epsilon_n\right)\right) \leq h_2\left(t,\epsilon_n\right)$,

where $h_1\left(t,\epsilon_n\right) = O\left(\epsilon_n\right)$, $h_2\left(t,\epsilon_n\right) = O\left(\epsilon_n\right)$ *and* $I \subset [0,\tau]$ *is connected and closed.*

Proof. Using the identity $\cos^2\gamma + \sin^2\gamma = 1$, from (B.2.5) we obtain

$$\gamma' = \frac{1}{\epsilon}\left[\frac{1}{a^2} + \cos^2\gamma\left(p(t)\bar{f}(y) - \frac{1}{a^2}\right)\right].$$

Let $\xi \in [0,\tau]$. The definition of polar coordinates implies that $\cos^2\gamma = \frac{y^2}{y^2+v^2} = \frac{y^2}{y^2+a^2w^2}$. Thus,

$$\left|\cos^2\gamma\left(p\bar{f} - \frac{1}{a^2}\right)\right| = \left|\frac{y^2\left(p\bar{f} - \frac{1}{a^2}\right)}{y^2+v^2}\right| = \left|\frac{y^2\left(p\bar{f} - \frac{1}{a^2}\right)}{y^2 + 2a^2(\hat{H} - r - pV)}\right|$$

for $y \in [-Y_{\xi,\epsilon}, Y_{\xi,\epsilon}]$.
Because $\hat{H} - r > 0$, there is a positive constant κ_1 independent of ϵ, $\kappa_1 < y_0$ such that

$$\left|\cos^2\gamma\left(p(\xi)\bar{f} - \frac{1}{a^2(\xi)}\right)\right| < \frac{1}{2a^2(\xi)}$$

for $|y| < \kappa_1$. Further for $|y| \geq \kappa_1$, taking into consideration the conclusions of Lemma, we obtain

$$\left|\cos^2\gamma\left(p(\xi)\bar{f} - \frac{1}{a^2(\xi)}\right)\right| \leq \left|\frac{y^2\left(a^{-2}(\xi) - p(\xi)\bar{f}(y)\right)}{y^2 - 2a^2(\xi)p(\xi)V(y)}\right|.$$

From the condition (B.3.1) we conclude that

$$0 < \frac{a^{-2}(\xi) - p(\xi)\bar{f}(y)}{a^{-2}(\xi) - \frac{2p(\xi)V(y)}{y^2}} < 1$$

and after little arrangement we get

$$0 < \frac{y^2\left(a^{-2}(\xi) - p(\xi)\bar{f}(y)\right)}{y^2 - 2a^2(\xi)p(\xi)V(y)} < \frac{1}{a^2(\xi)}$$

for $\kappa_1 \leq |y| \leq y_0 + \kappa_2 < Y_0$, where κ_2 is a sufficiently small positive constant.
Let $\tilde{c}(\xi) = \min\{\tilde{c}_1(\xi), \tilde{c}_2(\xi)\}$, where

$$\tilde{c}_1(\xi) = \min\left\{\frac{1}{a^2(\xi)} - \frac{y^2\left(a^{-2}(\xi) - p(\xi)\bar{f}(y)\right)}{y^2 - 2a^2(\xi)p(\xi)V(y)}, |y| \leq y_0 + \kappa_2\right\},$$

$$\tilde{c}_2(\xi) = \min\left\{\frac{1}{a^2(\xi)}\sin^2\gamma + p(\xi)\bar{f}(y)\cos^2\gamma,\right.$$
$$\left. y_0 + \kappa_2 \leq |y| \leq Y_{\xi,\epsilon},\ \gamma \in \mathbb{R}\right\}$$

and we define for $t \in [0, \tau_0)$ the function

$$c(t) = \min\{\tilde{c}(\xi); \xi \in [0, t]\}.$$

Clearly, $c(t)$ is a positive nonincreasing function on $[0, \tau]$ except that $c(\tau) = 0$ if $p(\tau) = 0$. Hence, $\gamma' \geq \frac{c(t)}{\epsilon}$ for $t \in [0, \tau]$. Since $c(t)$ is independent of ϵ we conclude that, by taking ϵ sufficiently small, γ can be made arbitrarily large, $\gamma(\tau, \epsilon) \geq \frac{\pi}{2} + \frac{1}{\epsilon}\int_0^\tau c(u)\mathrm{d}u$ for $\bar{c} < 0$.
Moreover, setting

$$c^*(t) = a^{-2}(t) + \sup\left\{p(t)\bar{f}(y); t \in [0, \tau], y \in \mathbb{R}\right\}$$

on $[0, \tau]$, we get $\gamma' \leq \frac{c^*(t)}{\epsilon}$ for $t \in [0, \tau]$. The continuity of γ with respect to a parameter ϵ, $\epsilon \neq 0$ we obtain such values ϵ that $\gamma(\tau, \epsilon_n) = \frac{\pi}{2} + \pi n$. for every $n \geq n_0$. The corresponding solution of (B.1.1), (B.1.2) has a zero number $z_{[0,\tau]}(y(t, \epsilon_n)) = n - 1$. This completes the proof of statement (i). Integrating between two successive zero numbers of solution $y(t, \epsilon)$ on an interval $I \subset [0, \tau]$ (if $p(\tau) = 0$ then $I \subset [0, \tau)$), we obtain an estimate of $d_I(y(t, \epsilon_n))$ of the form

$$\epsilon_n\left(\frac{\pi}{c_I^*}\right) \leq d_I(y(t, \epsilon_n)) \leq \epsilon_n\left(\frac{\pi}{c_I}\right),$$

where $c_I^* = \sup_{t \in I} c^*(t)$ and $c_I = \inf_{t \in I} c(t)$. Theorem is proven.

Remark 1. Oddness of f is considered in order to avoid technicalities, and is not essential. Ones modify the theorem for a general function f with three undegenerate zeros and satisfying $0 \leq \limsup_{y \to \infty} \frac{f(y)}{y} < \infty$.

References

1. S. Angenent, J. Mallet-Paret and L. Peletier: *Stable transitions layers in a semilinear boundary value problem*, J. Differential Equations Vol. 67 (1989), pp. 212–242.
2. Ch. K.R.T. Jones: *Geometric Singular Perturbation Theory*, C.I.M.E Lectures, Session on Dynamical Systems, 1994.

3. C. Krupa, B. Sandstede and P. Szmolyan: *Fast and Slow Waves in the FitzHugh-Nagumo Equation*, J. Math. Anal. Appl., Vol. 133 (1997), pp. 49–97.

4. N. Minorski : *Nonlinear Oscillations*, Van Nostrand, New Jersey, 1962.

5. F. C. Moon: *Chaotic Vibrations*, Wiley, New York, 1987.

6. C. Rocha: *On the Singular problem for the Scalar Parabolic Equation with Variable Diffusion*, J. Math. Anal. Appl., Vol. 183 (1994), pp. 413–428.

7. S. Timoshenko, D. H. Young and W. Weaver, Jr.: *Vibration Problems in Engineering*, Wiley, New York, 1990.

8. R. Vrabel: *On the solutions of differential equation*

$$\epsilon^2 \left(a^2(t)y'\right)' + p(t)f(y) = 0$$

with arbitrarily large zero number, Journal of Computational Analysis and Applications. - ISSN 1521-1398. - Vol. 6, No. 2 (2004), pp. 139–146.

C THREE-POINT BOUNDARY VALUE PROBLEM

C.1 Introduction

We will consider the three point problem

$$\epsilon y'' + ky = f(t, y),\ t \in \langle a, b\rangle,\ k < 0,\ 0 < \epsilon << 1 \tag{C.1.1}$$

$$y'(a) = 0,\ y(b) - y(c) = 0,\ a < c < b \tag{C.1.2}$$

This is singular perturbation problem because the order of differential equation drops when ϵ becomes zero. The situation in the present case is complicated by the fact that there are the inner points in the boundary conditions, on the difference with the „standard" boundary conditions as Dirichlet problem, Neumann problem, Robin problem, periodic boundary value problem ([2], [3]), for example.
We apply the method of upper and lower solutions and the delicate estimates to prove the existence of a solution for problem (C.1.1), (C.1.2) which converges uniformly to the solution of reduced problem (i.e. if we let $\epsilon \to 0^+$ in (C.1.1)) on every compact subset of interval $\langle a, b)$ for $\epsilon \to 0^+$.
As usual, we say that $\alpha_\epsilon \in C^2(\langle a, b\rangle)$ is a lower solution for problem (C.1.1), (C.1.2) if $\epsilon\alpha_\epsilon''(t) + k\alpha_\epsilon(t) \geq f(t, \alpha_\epsilon(t))$ and $\alpha_\epsilon'(0) = 0$,

$\alpha_\epsilon(b) - \alpha_\epsilon(c) \le 0$ for every $t \in \langle a, b\rangle$. An upper solution $\beta_\epsilon \in C^2(\langle a, b\rangle)$ satisfies $\epsilon\beta_\epsilon''(t) + k\beta_\epsilon(t) \le f(t, \beta_\epsilon(t))$ and $\beta_\epsilon'(0) = 0$, $\beta_\epsilon(b) - \beta_\epsilon(c) \ge 0$ for every $t \in \langle a, b\rangle$.

Lemma 1 (cf. [1]). *If $\alpha_\epsilon, \beta_\epsilon$ are lower and upper solutions for (C.1.1), (C.1.2) such that $\alpha_\epsilon \le \beta_\epsilon$, then there exists solution y_ϵ of (C.1.1), (C.1.2) with $\alpha_\epsilon \le y_\epsilon \le \beta_\epsilon$.*

Denote $D(u) = \{(t, y)|\ a \le t \le b, |y - u(t)| < d(t)\}$, where $d(t)$ is the positive continuous function on $\langle a, b\rangle$ such that

$$d(t) = \begin{cases} \delta & \text{for } a \le t \le b - \delta \\ |u(b) - u(c)| + \delta & \text{for } b - \frac{\delta}{2} \le t \le b \end{cases}$$

δ is a small positive constant and $u \in C^2$ is a solution of reduced problem $ku = f(t, u)$.

C.2 Existence and asymptotic behavior of solutions

Theorem 1. (see, [4]) *Let $f \in C^1(D(u))$ satisfies the condition*

$$\left|\frac{\partial f(t, y)}{\partial y}\right| \le w < -k \text{ for every } (t, y) \in D(u). \text{ (hyperbolicity)}$$

Then there exists ϵ_0 such that for every $\epsilon \in (0, \epsilon_0\rangle$ the problem (C.1.1), (C.1.2) has a unique solution satisfying the inequality

$$\hat{v}_\epsilon(t) - C\epsilon \le y_\epsilon(t) - (u(t) + v_\epsilon(t)) \le -\hat{v}_\epsilon(t) + \tilde{C}\sqrt{\epsilon} \tag{C.2.1}$$

for $u'(a) \le 0$ and

$$\hat{v}_\epsilon(t) - \tilde{C}\sqrt{\epsilon} \le y_\epsilon(t) - (u(t) + v_\epsilon(t)) \le -\hat{v}_\epsilon(t) + C\epsilon \tag{C.2.2}$$

for $u'(a) \geq 0$ *on* $\langle a, b \rangle$ *where*

$$v_\epsilon(t) = u'(a) \cdot \frac{e^{\sqrt{\frac{m}{\epsilon}}(t-b)} - e^{\sqrt{\frac{m}{\epsilon}}(b-t)} + e^{\sqrt{\frac{m}{\epsilon}}(c-t)} - e^{\sqrt{\frac{m}{\epsilon}}(t-c)}}{\sqrt{\frac{m}{\epsilon}}\left(e^{\sqrt{\frac{m}{\epsilon}}(a-c)} - e^{\sqrt{\frac{m}{\epsilon}}(a-b)} + e^{\sqrt{\frac{m}{\epsilon}}(c-a)} - e^{\sqrt{\frac{m}{\epsilon}}(b-a)}\right)}$$

$$\hat{v}_\epsilon(t) = |u(b) - u(c)| \cdot \frac{e^{\sqrt{\frac{m}{\epsilon}}(t-a)} + e^{\sqrt{\frac{m}{\epsilon}}(a-t)}}{\left(e^{\sqrt{\frac{m}{\epsilon}}(a-c)} - e^{\sqrt{\frac{m}{\epsilon}}(a-b)} + e^{\sqrt{\frac{m}{\epsilon}}(c-a)} - e^{\sqrt{\frac{m}{\epsilon}}(b-a)}\right)},$$

$m = -k - w$ *and* C, $\tilde{C}$ *are the positive constants.*

Proof. The functions $v_\epsilon(t)$ and $\hat{v}_\epsilon(t)$ on $\langle a, b \rangle$ satisfy:

1. $\epsilon v_\epsilon'' - m v_\epsilon = 0$, $v_\epsilon'(a) = -u'(a)$, $v_\epsilon(b) - v_\epsilon(c) = 0$, $v_\epsilon \leq 0$ for $u'(a) \leq 0$ and $v_\epsilon \geq 0$ for $u'(a) \geq 0$

2. $\epsilon \hat{v}_\epsilon'' - m\hat{v}_\epsilon = 0$, $\hat{v}_\epsilon'(a) = 0$, $\hat{v}_\epsilon(b) - \hat{v}_\epsilon(c) = -|u(b) - u(c)|$, $\hat{v}_\epsilon \leq 0$.

For $u'(a) \leq 0$ we define the lower solutions by

$$\alpha_\epsilon(t) = u(t) + v_\epsilon(t) + \hat{v}_\epsilon(t) - \Gamma_\epsilon$$

and the upper solutions by

$$\beta_\epsilon(t) = u(t) + v_\epsilon(t) - \hat{v}_\epsilon(t) + \tilde{\Gamma}_\epsilon$$

(for $u'(a) \geq 0$ we proceed analogously.)

Here $\Gamma_\epsilon = \frac{\epsilon\Delta}{m}$ and $\tilde{\Gamma}_\epsilon = \frac{\sqrt{\epsilon}\tilde{\Delta}}{m}$ where Δ, $\tilde{\Delta}$ are the constants which shall be defined below. $\alpha \leq \beta$ on $\langle a, b \rangle$ and satisfy the boundary conditions prescribed for the lower and upper solutions of (C.1.1), (C.1.2).

Now we show that $\epsilon\alpha_\epsilon''(t) + k\alpha_\epsilon(t) \geq f(t, \alpha_\epsilon(t))$ and $\epsilon\beta_\epsilon''(t) + k\beta_\epsilon(t) \leq f(t, \beta_\epsilon(t))$. Denote $h(t, y) = f(t, y) - ky$. By the Taylor theorem

we obtain

$$h(t,\alpha_\epsilon(t)) = h(t,\alpha_\epsilon(t)) - h(t,u(t))$$
$$= \frac{\partial h(t,\theta_\epsilon(t))}{\partial y}(v_\epsilon(t) + \hat{v}_\epsilon(t) - \Gamma_\epsilon)$$

where $(t,\theta_\epsilon(t))$ is a point between $(t,\alpha_\epsilon(t))$ and $(t,u(t))$, and $(t,\theta_\epsilon(t)) \in D(u)$ for sufficiently small ϵ. Hence

$$\epsilon\alpha_\epsilon''(t) - h(t,\alpha_\epsilon(t)) \geq \epsilon u'' + \epsilon v_\epsilon''(t) + \epsilon\hat{v}_\epsilon''(t) - m(v_\epsilon(t) + \hat{v}_\epsilon(t) - \Gamma_\epsilon)$$

$$\geq -\epsilon|u''| + \epsilon\Delta.$$

If we choose a constant Δ such that $\Delta \geq |u''(t)|$, $t \in \langle a,b\rangle$ then $\epsilon\alpha_\epsilon''(t) \geq h(t,\alpha_\epsilon(t))$ in $\langle a,b\rangle$.

Inequality for $\beta_\epsilon(t))$:

$$h(t,\beta_\epsilon(t)) - \epsilon\beta_\epsilon''(t) = \frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y}(v_\epsilon(t) - \hat{v}_\epsilon(t) + \tilde{\Gamma}_\epsilon) - \epsilon\beta_\epsilon''(t)$$

$$= \frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y}(v_\epsilon(t) - \hat{v}_\epsilon(t) + \tilde{\Gamma}_\epsilon) - \epsilon(u'' + v_\epsilon''(t) - \hat{v}_\epsilon''(t))$$

$$\geq \frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y}v_\epsilon(t) + m\tilde{\Gamma}_\epsilon - \epsilon u'' - \epsilon v_\epsilon''(t)$$

$$= \left(\frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y} - m\right)v_\epsilon(t) + m\tilde{\Gamma}_\epsilon - \epsilon u''.$$

Let $L = \max\{|u''(t)|\ |t \in \langle a,b\rangle\}$ and denote $\tilde{v}_\epsilon = \frac{v_\epsilon}{\sqrt{\epsilon}}$. Then $\epsilon\beta_\epsilon''(t) \leq h(t,\beta_\epsilon(t))$ if

$$m\tilde{\Gamma}_\epsilon - \epsilon L \geq \left(\frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y} - m\right)|v_\epsilon(t)|$$

i.e.

$$\sqrt{\epsilon}\left(\tilde{\Delta} - \sqrt{\epsilon}L\right) \geq \left(\frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y} - m\right)\sqrt{\epsilon}|\tilde{v}_\epsilon(t)|$$

$$\tilde{\Delta} \geq \sqrt{\epsilon}L + \left(\frac{\partial h(t, \tilde{\theta}_\epsilon(t))}{\partial y} - m \right) |\tilde{v}_\epsilon(t)|$$

Thus, from the inequalities $m \leq \frac{\partial h(t, \tilde{\theta}_\epsilon(t))}{\partial y} \leq m + 2w$ in $D(u)$ and $v_\epsilon(t) \leq 0$ follows that it is sufficient to choose a constant $\tilde{\Delta}$ such that

$$\tilde{\Delta} \geq \sqrt{\epsilon}L + 2w\frac{|u'(a)|}{\sqrt{m}}.$$

The existence of a solution for (C.1.1), (C.1.2) satisfying the above inequality follows from Lemma 1.

Remark 2. Theorem 1 implies that $y_\epsilon(t) = u(t) + O(\sqrt{\epsilon})$ on every compact subset of $\langle a, b)$ and $\lim\limits_{\epsilon \to 0^+} y_\epsilon(b) = u(c)$. Boundary layer effect occurs at the point b in the case, when $u(c) \neq u(b)$.

References

1. J. Mawhin: *Points fixes, points critiques et problemes aux limites*, Semin. Math. Sup. no. 92, Presses Univ. Montreal, 1985.
2. R. Vrábeľ: *Asymptotic behavior of T-periodic solutions of singularly perturbed second-order differential equation*, Mathematica Bohemica 121 (1996), pp. 73—76.
3. R. Vrábeľ: *Semilinear singular perturbation*, Nonlinear analysis, TMA. - ISSN 0362-546X. - Vol. 25, No. 1 (1995), pp. 17–26.
4. R. Vrábeľ: *Three point boundary value problem for singularly perturbed semilinear differential equations*, E. J. Qualitative Theory of Diff. Equ. - ISSN: 1417–3875. - No. 70 (2009), pp. 1–4.

D FOUR-POINT BOUNDARY VALUE PROBLEM

D.1 Preliminaries

We will consider the four (or three) point boundary value problem

$$\epsilon y'' + ky = f(t, y), \ t \in \langle a, b \rangle, \ k < 0, \ 0 < \epsilon << 1 \tag{D.1.1}$$

$$y(c) - y(a) = 0, \ y(b) - y(d) = 0, \ a < c \leq d < b. \tag{D.1.2}$$

We can view this equation as the mathematical model of the nonlinear dynamical system with a high-speed feedback. Moreover, this class of equations has special significance in connection with applications involving nonlinear vibrations. We focus on the existence and asymptotic behavior of a solutions $y_\epsilon(t)$ for ϵ belonging to a non-resonant set and on an estimate of the difference between the solution $y_\epsilon(t)$ of (D.1.1), (D.1.2) and a singular solution $u(t)$ of the equation $ku = f(t, u)$.

The situation in the present case is complicated by the fact that there are the inner points in the boundary conditions, in contrast to the boundary conditions as the Dirichlet problem, Neumann problem, Robin problem, periodic boundary value problem ([1, 2, 5, 6, 9]), for example. In the problem considered there does not exist a positive solution $\tilde{v}_\epsilon$ of Diff. Eq. $\epsilon y'' - my = 0$, $m > 0$, $0 < \epsilon$ (i.e. $\tilde{v}_\epsilon$ is convex) such that $\tilde{v}_\epsilon(c) - \tilde{v}_\epsilon(a) = u(c) - u(a) > 0$ and $\tilde{v}_\epsilon(t) \to 0^+$ for $t \in (a, b\rangle$ and $\epsilon \to 0^+$, which could be used to solve this problem by the method of upper and lower solutions. We will define the correction function $v_\epsilon^{(corr)}(t)$ which will allow us to apply the method.

As was said before, we apply the method of upper and lower solutions and some delicate estimates to prove the existence of a solution for problem (D.1.1), (D.1.2) which converges uniformly to the solution u of the reduced problem (i.e. if we let $\epsilon \to 0^+$ in (D.1.1)) on every compact subset of interval (a, b) for $\epsilon \to 0^+$. Moreover, we give the accurate numerical approximation of solutions up to order $O(\epsilon)$. The regular case, when $\epsilon = 1$, for the difference equations with three-point time scale boundary value conditions was studied in [3].

As usual, we say that $\alpha_\epsilon \in C^2(\langle a, b\rangle)$ is a lower solution for problem (D.1.1), (D.1.2) if $\epsilon\alpha_\epsilon''(t) + k\alpha_\epsilon(t) \geq f(t, \alpha_\epsilon(t))$ and $\alpha_\epsilon(c) - \alpha_\epsilon(a) = 0$, $\alpha_\epsilon(b) - \alpha_\epsilon(d) \leq 0$ for every $t \in \langle a, b\rangle$. An upper solution $\beta_\epsilon \in C^2(\langle a, b\rangle)$ satisfies $\epsilon\beta_\epsilon''(t) + k\beta_\epsilon(t) \leq f(t, \beta_\epsilon(t))$ and $\beta_\epsilon(c) - \beta_\epsilon(a) = 0$, $\beta_\epsilon(b) - \beta_\epsilon(d) \geq 0$ for every $t \in \langle a, b\rangle$.

Lemma 1 ([4]). *If $\alpha_\epsilon, \beta_\epsilon$ are respectively lower and upper solutions for (D.1.1), (D.1.2) such that $\alpha_\epsilon \leq \beta_\epsilon$, then there exists solution y_ϵ of (D.1.1), (D.1.2) with $\alpha_\epsilon \leq y_\epsilon \leq \beta_\epsilon$.*

Denote $\mathcal{H}(u) = \{(t, y) |\ a \leq t \leq b, |y - u(t)| < d(t)\}$, where $d(t)$ is the positive continuous function on $\langle a, b\rangle$ such that

$$d(t) = \begin{cases} |u(c) - u(a)| + \delta & \text{for } a \leq t \leq a + \frac{\delta}{2} \\ \delta & \text{for } a + \delta \leq t \leq b - \delta \\ |u(b) - u(d)| + \delta & \text{for } b - \frac{\delta}{2} \leq t \leq b \end{cases}$$

δ is a small positive constant and $u \in C^2$ is a solution of the reduced problem $ku = f(t,u)$. We will write $s(\epsilon) = \mathcal{O}(r(\epsilon))$ when $0 < \lim_{\epsilon \to 0^+} \left| \frac{s(\epsilon)}{r(\epsilon)} \right| < \infty$.

D.2 Main result

Theorem 1. (see, e.g., [7, 8]) *Let $f \in C^1(\mathcal{H}(u))$ satisfies the condition*

$$\left| \frac{\partial f(t,y)}{\partial y} \right| \leq w < -k \text{ for every } (t,y) \in \mathcal{H}(u) \text{ (hyperbolicity).}$$

Then there exists ϵ_0 such that for every $\epsilon \in (0, \epsilon_0\rangle$ the problem (D.1.1), (D.1.2) has a unique solution satisfying the inequality

$$-v_\epsilon^{(corr)}(t) - \hat{v}_\epsilon(t) - C\epsilon \leq y_\epsilon(t) - (u(t) + v_\epsilon(t)) \leq \hat{v}_\epsilon(t) + C\epsilon$$

for $u(c) - u(a) \geq 0$ and

$$-\hat{v}_\epsilon(t) - C\epsilon \leq y_\epsilon(t) - (u(t) + v_\epsilon(t)) \leq v_\epsilon^{(corr)}(t) + \hat{v}_\epsilon(t) + C\epsilon$$

for $u(c) - u(a) \leq 0$ on $\langle a, b\rangle$ where

$$\begin{aligned}
v_\epsilon(t) &= \frac{u(c) - u(a)}{D} \\
&\quad \cdot \left(e^{\sqrt{\frac{m}{\epsilon}}(b-t)} - e^{\sqrt{\frac{m}{\epsilon}}(t-b)} + e^{\sqrt{\frac{m}{\epsilon}}(t-d)} - e^{\sqrt{\frac{m}{\epsilon}}(d-t)} \right), \\
\hat{v}_\epsilon(t) &= \frac{|u(b) - u(d)|}{D} \\
&\quad \cdot \left(e^{\sqrt{\frac{m}{\epsilon}}(t-a)} - e^{\sqrt{\frac{m}{\epsilon}}(a-t)} + e^{\sqrt{\frac{m}{\epsilon}}(c-t)} - e^{\sqrt{\frac{m}{\epsilon}}(t-c)} \right), \\
D &= \left(e^{\sqrt{\frac{m}{\epsilon}}(b-a)} + e^{\sqrt{\frac{m}{\epsilon}}(d-c)} + e^{\sqrt{\frac{m}{\epsilon}}(c-b)} + e^{\sqrt{\frac{m}{\epsilon}}(a-d)} \right) \\
&- \left(e^{\sqrt{\frac{m}{\epsilon}}(a-b)} + e^{\sqrt{\frac{m}{\epsilon}}(c-d)} + e^{\sqrt{\frac{m}{\epsilon}}(b-c)} + e^{\sqrt{\frac{m}{\epsilon}}(d-a)} \right),
\end{aligned}$$

$m = -k - w$, $C = \frac{1}{m} \max\{|u''(t)|\,; t \in \langle a, b\rangle\}$ *and the positive function*

$$v_\epsilon^{(corr)}(t) = \frac{w|u(c)-u(a)|}{\sqrt{m\epsilon}} \cdot \left[-O(1)\frac{v_\epsilon(t)}{(u(c)-u(a))} \right.$$
$$\left. + O\left(e^{\sqrt{\frac{m}{\epsilon}}(a-d)}\right)\frac{\hat{v}_\epsilon(t)}{|u(b)-u(d)|} + tO\left(e^{\sqrt{\frac{m}{\epsilon}}\chi(t)}\right)\right],$$

$\chi(t) < 0$ *for* $t \in (a, b\rangle$.

Remark 3. The function $v_\epsilon(t)$ satisfies

1. $\epsilon v_\epsilon'' - m v_\epsilon = 0$,
2. $v_\epsilon(c) - v_\epsilon(a) = -(u(c) - u(a))$, $v_\epsilon(b) - v_\epsilon(d) = 0$,
3. $v_\epsilon(t) \geq 0$ (≤ 0) is decreasing (increasing) for $a \leq t \leq \frac{b+d}{2}$ and increasing (decreasing) for $\frac{b+d}{2} \leq t \leq b$ if $u(c) - u(a) \geq 0$ (≤ 0),
4. $v_\epsilon(t)$ converges uniformly to 0 for $\epsilon \to 0^+$ on every compact subset of $(a, b\rangle$,
5. $v_\epsilon(t) = (u(c) - u(a))O\left(e^{\sqrt{\frac{m}{\epsilon}}\chi(t)}\right)$ where $\chi(t) = a - t$ for $a \leq t \leq \frac{b+d}{2}$ and $\chi(t) = t - b + a - d$ for $\frac{b+d}{2} < t \leq b$.

The function $\hat{v}_\epsilon(t)$ satisfies

1. $\epsilon \hat{v}_\epsilon'' - m\hat{v}_\epsilon = 0$,
2. $\hat{v}_\epsilon(c) - \hat{v}_\epsilon(a) = 0$, $\hat{v}_\epsilon(b) - \hat{v}_\epsilon(d) = |u(b) - u(d)|$,
3. $\hat{v}_\epsilon(t) \geq 0$ is decreasing for $a \leq t \leq \frac{a+c}{2}$ and increasing for $\frac{a+c}{2} \leq t \leq b$,
4. $\hat{v}_\epsilon(t)$ converges uniformly to 0 for $\epsilon \to 0^+$ on every compact subset of $\langle a, b)$,

5. $\hat{v}_\epsilon(t) = |u(b) - u(d)| O\left(e^{\sqrt{\frac{m}{\epsilon}}\hat{\chi}(t)}\right)$ where $\hat{\chi}(t) = c - b + a - t$ for $a \le t < \frac{a+c}{2}$ and $\hat{\chi}(t) = t - b$ for $\frac{a+c}{2} \le t \le b$.

The correction function $v_\epsilon^{(corr)}(t)$ will be determined precisely in the next section.

D.3 The correction function $v_\epsilon^{(corr)}(t)$

Consider the linear problem

$$\epsilon y'' - my = -2w\left|v_\epsilon(t)\right|, \ t \in \langle a, b\rangle, \ \epsilon > 0 \tag{D.3.1}$$

with the boundary condition (D.1.2).

We apply the method of upper and lower solutions. We define

$$\alpha_\epsilon(t) = 0$$

and

$$\beta_\epsilon(t) = \frac{2w}{m}\max\left\{\left|v_\epsilon(t)\right|, t \in \langle a, b\rangle\right\} = \frac{2w}{m}\left|v_\epsilon(a)\right|.$$

Obviously, $\left|v_\epsilon(a)\right| = \frac{2w}{m}|u(c) - u(a)|\left(1 + \mathcal{O}\left(e^{\sqrt{\frac{m}{\epsilon}}(a-c)}\right)\right)$ and the constant functions α, β satisfy the differential inequalities required on the upper and lower solutions for (D.3.1) and the boundary conditions for (D.1.2). Thus on the basis of Lemma 1 there exists unique solution y_ϵ^{Lin} of linear problem with (D.1.2) for every ϵ such that

$$0 \le y_\epsilon^{Lin}(t) \le \frac{2w}{m}\left|u(c) - u(a)\right|\left(1 + \mathcal{O}\left(e^{\sqrt{\frac{m}{\epsilon}}(a-c)}\right)\right)$$

on $\langle a, b\rangle$. The solution we denote by $v_\epsilon^{(corr)}(t)$, i.e. the function

$$v_\epsilon^{(corr)}(t) \stackrel{\text{def}}{=} y_\epsilon^{Lin}(t)$$

and we compute $v_\epsilon^{(corr)}(t)$ exactly:

$$v_\epsilon^{(corr)}(t) = -\frac{(\psi_\epsilon(a) - \psi_\epsilon(c))}{(u(c) - u(a))} v_\epsilon(t) + \frac{(\psi_\epsilon(d) - \psi_\epsilon(b))}{|u(b) - u(d)|} \hat{v}_\epsilon(t) + \psi_\epsilon(t)$$

where

$$\psi_\epsilon(t) = \frac{w|u(c) - u(a)|}{D\sqrt{m\epsilon}} t$$
$$\cdot \left(e^{\sqrt{\frac{m}{\epsilon}}(b-t)} + e^{\sqrt{\frac{m}{\epsilon}}(t-b)} - e^{\sqrt{\frac{m}{\epsilon}}(d-t)} - e^{\sqrt{\frac{m}{\epsilon}}(t-d)} \right).$$

Hence

$$\begin{aligned}
\psi_\epsilon(a) - \psi_\epsilon(c) = & \frac{w|u(c) - u(a)|}{D\sqrt{m\epsilon}} a \\
& \cdot \left(e^{\sqrt{\frac{m}{\epsilon}}(b-a)} + e^{\sqrt{\frac{m}{\epsilon}}(a-b)} - e^{\sqrt{\frac{m}{\epsilon}}(d-a)} - e^{\sqrt{\frac{m}{\epsilon}}(a-d)} \right) \\
& - \frac{w|u(c) - u(a)|}{D\sqrt{m\epsilon}} c \\
& \cdot \left(e^{\sqrt{\frac{m}{\epsilon}}(b-c)} + e^{\sqrt{\frac{m}{\epsilon}}(c-b)} - e^{\sqrt{\frac{m}{\epsilon}}(d-c)} - e^{\sqrt{\frac{m}{\epsilon}}(c-d)} \right) \\
& = \frac{w|u(c) - u(a)|}{\sqrt{m\epsilon}} \mathcal{O}(1),
\end{aligned}$$

$$\begin{aligned}
\psi_\epsilon(d) - \psi_\epsilon(b) = & \frac{w|u(c) - u(a)|}{D\sqrt{m\epsilon}} d \left(e^{\sqrt{\frac{m}{\epsilon}}(b-d)} + e^{\sqrt{\frac{m}{\epsilon}}(d-b)} - 2 \right) \\
& - \frac{w|u(c) - u(a)|}{D\sqrt{m\epsilon}} b \left(2 - e^{\sqrt{\frac{m}{\epsilon}}(d-b)} - e^{\sqrt{\frac{m}{\epsilon}}(b-d)} \right) \\
& = \frac{w|u(c) - u(a)|}{\sqrt{m\epsilon}} \mathcal{O}\left(e^{\sqrt{\frac{m}{\epsilon}}(a-d)} \right), \\
\psi_\epsilon(t) = & \frac{w|u(c) - u(a)|}{\sqrt{m\epsilon}} \mathcal{O}\left(e^{\sqrt{\frac{m}{\epsilon}}\chi(t)} \right).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
v_\epsilon^{(corr)}(t) = & \frac{w|u(c) - u(a)|}{\sqrt{m\epsilon}} \cdot \left[-O(1) \frac{v_\epsilon(t)}{(u(c) - u(a))} \right. \\
& \left. + O\left(e^{\sqrt{\frac{m}{\epsilon}}(a-d)} \right) \frac{\hat{v}_\epsilon(t)}{|u(b) - u(d)|} + tO\left(e^{\sqrt{\frac{m}{\epsilon}}\chi(t)} \right) \right].
\end{aligned}$$

D.4 Proof of Theorem

First we will consider the case $\mathbf{u}(\mathbf{c}) - \mathbf{u}(\mathbf{a}) \geq \mathbf{0}$. We define the lower solutions by

$$\alpha_\epsilon(t) = u(t) + v_\epsilon(t) - v_\epsilon^{(corr)}(t) - \hat{v}_\epsilon(t) - \Gamma_\epsilon$$

and the upper solutions by

$$\beta_\epsilon(t) = u(t) + v_\epsilon(t) + \hat{v}_\epsilon(t) + \Gamma_\epsilon.$$

Here $\Gamma_\epsilon = \frac{\epsilon\Delta}{m}$ where Δ is the constant which shall be defined below, $\alpha \leq \beta$ on $\langle a, b\rangle$ and satisfy the boundary conditions prescribed for the lower and upper solutions of (D.1.1), (D.1.2).

Now we show that $\epsilon\alpha_\epsilon''(t) + k\alpha_\epsilon(t) \geq f(t, \alpha_\epsilon(t))$ and $\epsilon\beta_\epsilon''(t) + k\beta_\epsilon(t) \leq f(t, \beta_\epsilon(t))$. Denote $h(t, y) = f(t, y) - ky$. By the Taylor theorem we obtain

$$h(t, \alpha_\epsilon(t)) = h(t, \alpha_\epsilon(t)) - h(t, u(t))$$

$$= \frac{\partial h(t, \theta_\epsilon(t))}{\partial y}(v_\epsilon(t) - v_\epsilon^{(corr)}(t) - \hat{v}_\epsilon(t) - \Gamma_\epsilon),$$

where $(t, \theta_\epsilon(t))$ is a point between $(t, \alpha_\epsilon(t))$ and $(t, u(t))$, and $(t, \theta_\epsilon(t)) \in \mathcal{H}(u)$ for sufficiently small ϵ. Hence, from the inequalities $m \leq \frac{\partial h(t,\theta_\epsilon(t))}{\partial y} \leq m + 2w$ in $\mathcal{H}(u)$ we have

$$\epsilon\alpha_\epsilon''(t) - h(t, \alpha_\epsilon(t)) \geq$$

$$\epsilon u''(t) + \epsilon v_\epsilon''(t) - \epsilon v_\epsilon^{(corr)''}(t) - \epsilon\hat{v}_\epsilon''(t) - (m + 2w)v_\epsilon(t)$$

$$+ m v_\epsilon^{(corr)}(t) + m\hat{v}_\epsilon(t) + m\Gamma_\epsilon.$$

Because $v_\epsilon(t) = |v_\epsilon(t)|$ we have

$$-\epsilon v_\epsilon^{(corr)''}(t) - 2wv_\epsilon(t) + mv_\epsilon^{(corr)}(t) = 0,$$

as follows from Diff. Eq. (D.3.1), we get

$$\epsilon\alpha_\epsilon''(t) - h(t,\alpha_\epsilon(t)) \geq \epsilon u''(t) + m\Gamma_\epsilon \geq -\epsilon|u''(t)| + \epsilon\Delta.$$

For $\beta_\epsilon(t))$ we have the inequality

$$h(t,\beta_\epsilon(t)) - \epsilon\beta_\epsilon''(t) = \frac{\partial h(t,\tilde{\theta}_\epsilon(t))}{\partial y}(v_\epsilon(t) + \hat{v}_\epsilon(t) + \Gamma_\epsilon) - \epsilon\beta_\epsilon''(t)$$

$$= m(v_\epsilon(t) + \hat{v}_\epsilon(t) + \Gamma_\epsilon) - \epsilon(u''(t) + v_\epsilon''(t) + \hat{v}_\epsilon''(t)) \geq \epsilon\Delta - \epsilon|u''(t)|$$

where $(t,\tilde{\theta}_\epsilon(t))$ is a point between $(t,u(t))$ and $(t,\beta_\epsilon(t))$ and $(t,\tilde{\theta}_\epsilon(t)) \in \mathcal{H}(u)$ for sufficiently small ϵ.

The case $\mathbf{u(c) - u(a) \leq 0}$:
The lower solutions

$$\alpha_\epsilon(t) = u(t) + v_\epsilon(t) - \hat{v}_\epsilon(t) - \Gamma_\epsilon$$

and the upper solutions

$$\beta_\epsilon(t) = u(t) + v_\epsilon(t) + v_\epsilon^{(corr)}(t) + \hat{v}_\epsilon(t) + \Gamma_\epsilon$$

satisfy

$$\begin{aligned}
&\epsilon\alpha_\epsilon'' - h(t,\alpha_\epsilon)\\
&= \epsilon u'' + \epsilon v_\epsilon'' - \epsilon\hat{v}_\epsilon'' - \frac{\partial h}{\partial y}(v_\epsilon - \hat{v}_\epsilon - \Gamma_\epsilon)\\
&= \epsilon u'' + \epsilon v_\epsilon'' - \epsilon\hat{v}_\epsilon'' + \frac{\partial h}{\partial y}(-v_\epsilon + \hat{v}_\epsilon + \Gamma_\epsilon)\\
&\geq \epsilon u'' + \epsilon v_\epsilon'' - \epsilon\hat{v}_\epsilon'' + m(-v_\epsilon + \hat{v}_\epsilon + \Gamma_\epsilon) = \epsilon u'' + \epsilon\Delta\\
&\geq \epsilon\Delta - \epsilon|u''|
\end{aligned}$$

$$h(t,\beta_\epsilon) - \epsilon\beta_\epsilon''$$

$$= \frac{\partial h}{\partial y} \left(v_\epsilon + v_\epsilon^{(corr)} + \hat{v}_\epsilon + \Gamma_\epsilon \right) - \epsilon u'' - \epsilon v_\epsilon'' - \epsilon v_\epsilon^{(corr)''} - \epsilon \hat{v}_\epsilon''$$

$$\geq (m + 2w) v_\epsilon + m \left(v_\epsilon^{(corr)} + \hat{v}_\epsilon + \Gamma_\epsilon \right) - \epsilon u'' - \epsilon v_\epsilon'' - \epsilon v_\epsilon^{(corr)''} - \epsilon \hat{v}_\epsilon''$$

$$= -2w \left| v_\epsilon \right| + m v_\epsilon^{(corr)} - \epsilon v_\epsilon^{(corr)''} + \epsilon \Delta - \epsilon u'' = \epsilon \Delta - \epsilon u''$$

$$\geq \epsilon \Delta - \epsilon |u''|.$$

Now, if we choose a constant Δ such that $\Delta \geq |u''(t)|$, $t \in \langle a, b \rangle$ then $\epsilon \alpha_\epsilon''(t) \geq h(t, \alpha_\epsilon(t))$ and $\epsilon \beta_\epsilon''(t) \leq h(t, \beta_\epsilon(t))$ in $\langle a, b \rangle$.

The existence of a solution for (D.1.1), (D.1.2) satisfying the above inequality follows from Lemma 1. The uniqueness of solutions follows from the fact that the function $h(t, y)$ is increasing in the variable y in $\mathcal{H}(u)$ (Peano's phenomenon).

Remark 4. Theorem 1 implies that $y_\epsilon(t) = u(t) + O(\epsilon)$ on every compact subset of (a, b) and $\lim_{\epsilon \to 0^+} y_\epsilon(a) = u(c)$, $\lim_{\epsilon \to 0^+} y_\epsilon(b) = u(d)$. The boundary layer effect occurs at the point a or/and b in the case when $u(a) \neq u(c)$ or/and $u(b) \neq u(d)$.

D.5 Approximation of the solutions

In this section the case $u(c) - u(a) \geq 0$ will be considered only. For $u(c) - u(a) \leq 0$ we proceed analogously. We define the approximate solution $\tilde{y}_\epsilon(t)$ of (D.1.1), (D.1.2) by

$$\tilde{y}_\epsilon(t) = \frac{1}{2} \left(\alpha_\epsilon(t) + \beta_\epsilon(t) \right) = u(t) + v_\epsilon(t) - \frac{v_\epsilon^{(corr)}(t)}{2}.$$

Taking into consideration the conclusions of Theorem 1, in both cases we obtain the following estimate for the solution y_ϵ of problem (D.1.1), (D.1.2)

$$|y_\epsilon(t) - \tilde{y}_\epsilon(t)| \le \hat{v}_\epsilon(t) + \frac{v_\epsilon^{(corr)}(t)}{2} + \frac{\epsilon}{m} \max\left\{|u''(t)|\,, t \in \langle a, b\rangle\right\}.$$

References

1. K. W. Chang and F. A. Howes: *Nonlinear Singular perturbation phenomena: Theory and Applications*, Springer-Verlag, New York, 1984.

2. C. De Coster and P. Habets: *Two-Point Boundary Value Problems: Lower and Upper Solutions*, Volume 205 (Mathematics in Science and Engineering), Elsevier Science; 1 edition, 2006.

3. R. A. Khan, J. Nieto and V. Otero-Espinar: *Existence and approximation of solution of three-point boundary value problems on time scales*, Journal of Difference Equations and Applications, Volume 14 (2008), Number 7, pp. 723–736(14).

4. J. Mawhin: *Points fixes, points critiques et problemes aux limites*, Semin. Math. Sup. no. 92, Presses Univ. Montreal, 1985.

5. R. Vrábeľ: *Asymptotic behavior of T-periodic solutions of singularly perturbed second-order differential equation*, Mathematica Bohemica 121 (1996), pp. 73-–76.

6. R. Vrábeľ: *Semilinear singular perturbation*, Nonlinear Analysis, TMA. - ISSN 0362-546X. - Vol. 25 (1995), No. 1, pp. 17–26.

7. R. Vrábeľ: *Nonlocal Four-Point Boundary Value Problem for the Singularly Perturbed Semilinear Differential Equations*, Boundary value problems. - ISSN 1687-2762. - Article ID 570493 (2011), 70493–70493.

8. R. Vrábeľ: *A priori estimates for the solutions of a four point boundary value problem for the singularly perturbed semilinear differential equations*, Electron. J. Diff. Equ. - ISSN 1072-6691. - Vol. 2011 (2011), No. 21, pp. 1–7.

9. B. Q. Yan, D. O'Regan and R. P. Agarwal: *Positive Solutions to Singular Boundary Value Problems with Sign Changing Nonlinearities on the Half-Line via Upper and Lower Solutions*, Acta Mathematica Sinica, English Series, Vol. 23 (2007), No. 8, pp. 1447-–1456.

E BOUNDARY LAYER PHENOMENON

E.1 Introduction

In various fields of science and engineering, systems with two-time-scale dynamics are often investigated. In state space, such systems are commonly modeled using the mathematical framework of singular perturbations, with a small parameter, say ϵ, determining the degree of separation between the "slow" and "fast" channels of the system. Singularly perturbed systems (SPSs) may occur also due to the presence of small "parasitic" parameters, armature inductance in a common model for most DC motors, small time constants, etc.

The literature on control of nonlinear SPSs is extensive, at least starting with the pioneering work [3] of P. Kokotovic *et al.* nearly 30 years ago and continuing to the present including authors such as Z. Artstein [1], V. Gaitsgory [2], etc.

In this chapter, we will study the nonlinear singularly perturbed systems described by differential equation of the form

$$\epsilon y'' + ky = f(x, y), \ x \in \langle a, b \rangle, \ k < 0 \tag{E.1.1}$$

subject to the three–point boundary value conditions

$$y(a) = y(c) = y(b),\ \ a < c < b \tag{E.1.2}$$

where ϵ is a small perturbation parameter ($0 < \epsilon \ll 1$). The dependence upon the variable x of the continuous function f represents the effects of outer disturbances.

Such boundary value problems can arise in the study of the steady–states of a heated bar with the thermostats described by scalar partial differential equation

$$\frac{\partial y}{\partial t} = \epsilon \frac{\partial^2 y}{\partial x^2} + ky - f(x, y),$$

with stationary condition $\partial y/\partial t = 0$, where the controllers at $x = a$ and $x = b$ maintain a temperature according to the temperature detected by a sensor at $x = c$. In this case, we consider a uniform bar of length $b - a$ with non-uniform temperature lying on the x-axis from $x = a$ to $x = b$. The parameter ϵ represents the thermal diffusivity. Thus, the singular perturbation problems are of common occurrence in modeling the heat-transport problems with large Peclet number. One of the typical behaviors of SPSs is the boundary layer phenomenon: the solutions vary rapidly within very thin layer regions near the boundary. The goal of this chapter is to analyze the thermal boundary layer phenomena arising in such singularly perturbed systems. We give an accurate estimate for determining the rate of boundary layer growth.

Recently, in the paper [4], it has been shown that for every $\epsilon > 0$ sufficiently small there is a unique solution y_ϵ of (E.1.1), (E.1.2) such that y_ϵ converges uniformly to the solution u of reduced problem $ku = f(x, u)$ for $\epsilon \to 0^+$ on every compact subset $K \subset (a, b)$.

In this chapter we focus our attention on the detailed analysis of the behavior of the solutions y_ϵ for (E.1.1), (E.1.2) in the endpoints

$x = a$ and $x = b$ when a small parameter ϵ tends to zero. We show that the solutions of (E.1.1), (E.1.2), in general, start with fast transient ($|y'_\epsilon(a)| \to \infty$ and after decay of this transient they remain close to $u(x)$ with an arising new fast transient of $y_\epsilon(x)$ from $u(x)$ to $y_\epsilon(b)$ ($|y'_\epsilon(b)| \to \infty$), which is the so-called boundary layer phenomenon.

E.2 Boundary layer analysis

The existence, uniqueness and asymptotic behavior of the solutions for (E.1.1), (E.1.2) on the compact subset $K \subset (a, b)$ has been proven in [4] using the method of lower and upper solutions. It remains to prove a boundary layer phenomenon at $x = a$ and $x = b$. We will proceed analogously as in the work [5].

The solution of the problem under consideration is equivalent to the integral equation

$$\begin{aligned} y_\epsilon(x) = \frac{A}{D} \Big[& e^{\sqrt{\frac{-k}{\epsilon}}(x-b)} - e^{\sqrt{\frac{-k}{\epsilon}}(x-c)} + e^{\sqrt{\frac{-k}{\epsilon}}(c-x)} - e^{\sqrt{\frac{-k}{\epsilon}}(b-x)} \\ & + e^{\sqrt{\frac{-k}{\epsilon}}(x-a)} - e^{\sqrt{\frac{-k}{\epsilon}}(x-b)} + e^{\sqrt{\frac{-k}{\epsilon}}(b-x)} - e^{\sqrt{\frac{-k}{\epsilon}}(a-x)} \Big] \\ & + \frac{I_1}{D} \Big[e^{\sqrt{\frac{-k}{\epsilon}}(x-a)} - e^{\sqrt{\frac{-k}{\epsilon}}(x-b)} + e^{\sqrt{\frac{-k}{\epsilon}}(b-x)} - e^{\sqrt{\frac{-k}{\epsilon}}(a-x)} \Big] \\ & + \int_a^x C(x, s) \cdot \frac{f(s, y_\epsilon(s))}{\epsilon} \mathrm{d}s \end{aligned}$$

with Cauchy function

$$C(x, s) = \frac{e^{\sqrt{\frac{-k}{\epsilon}}(x-s)} - e^{-\sqrt{\frac{-k}{\epsilon}}(x-s)}}{2\sqrt{\frac{-k}{\epsilon}}}$$

and

$$A = -\int_a^b C(b,s) \cdot \frac{f(s, y_\epsilon(s))}{\epsilon} \mathrm{d}s, \;\; I_1 = \int_a^c C(c,s) \cdot \frac{f(s, y_\epsilon(s))}{\epsilon} \mathrm{d}s,$$

$$D = e^{\sqrt{\frac{-k}{\epsilon}}(b-a)} + e^{\sqrt{\frac{-k}{\epsilon}}(c-b)} + e^{\sqrt{\frac{-k}{\epsilon}}(a-c)} - e^{\sqrt{\frac{-k}{\epsilon}}(a-b)} - e^{\sqrt{\frac{-k}{\epsilon}}(b-c)} - e^{\sqrt{\frac{-k}{\epsilon}}(c-a)}.$$

Then

$$\begin{aligned} y_\epsilon'(x) = & \frac{A\sqrt{\frac{-k}{\epsilon}}}{D} \\ & \cdot \Big[e^{\sqrt{\frac{-k}{\epsilon}}(x-b)} - e^{\sqrt{\frac{-k}{\epsilon}}(x-c)} - e^{\sqrt{\frac{-k}{\epsilon}}(c-x)} + e^{\sqrt{\frac{-k}{\epsilon}}(b-x)} \\ & + e^{\sqrt{\frac{-k}{\epsilon}}(x-a)} - e^{\sqrt{\frac{-k}{\epsilon}}(x-b)} - e^{\sqrt{\frac{-k}{\epsilon}}(b-x)} + e^{\sqrt{\frac{-k}{\epsilon}}(a-x)} \Big] \\ & + \frac{I_1\sqrt{\frac{-k}{\epsilon}}}{D} \\ & \cdot \Big[e^{\sqrt{\frac{-k}{\epsilon}}(x-a)} - e^{\sqrt{\frac{-k}{\epsilon}}(x-b)} - e^{\sqrt{\frac{-k}{\epsilon}}(b-x)} + e^{\sqrt{\frac{-k}{\epsilon}}(a-x)} \Big] \\ & + \int_a^x \frac{e^{\sqrt{\frac{-k}{\epsilon}}(x-s)} + e^{\sqrt{\frac{-k}{\epsilon}}(s-x)}}{2} \cdot \frac{f(s, y_\epsilon(s))}{\epsilon} \mathrm{d}s. \end{aligned}$$

Hence

$$\begin{aligned} y_\epsilon'(a) = & \frac{A\sqrt{\frac{-k}{\epsilon}}}{D} \Big[2 - e^{\sqrt{\frac{-k}{\epsilon}}(a-c)} - e^{\sqrt{\frac{-k}{\epsilon}}(c-a)} \Big] \\ & + \frac{I_1\sqrt{\frac{-k}{\epsilon}}}{D} \Big[2 - e^{\sqrt{\frac{-k}{\epsilon}}(a-b)} - e^{\sqrt{\frac{-k}{\epsilon}}(b-a)} \Big]. \end{aligned}$$

Similarly

$$
\begin{aligned}
y'_\epsilon(b) = & \frac{A\sqrt{\frac{-k}{\epsilon}}}{D} \\
& \cdot \left[e^{\sqrt{\frac{-k}{\epsilon}}(b-a)} + e^{\sqrt{\frac{-k}{\epsilon}}(a-b)} - e^{\sqrt{\frac{-k}{\epsilon}}(c-b)} - e^{\sqrt{\frac{-k}{\epsilon}}(b-c)}\right] \\
& + \frac{I_1\sqrt{\frac{-k}{\epsilon}}}{D}\left[e^{\sqrt{\frac{-k}{\epsilon}}(b-a)} + e^{\sqrt{\frac{-k}{\epsilon}}(a-b)} - 2\right] \\
& + \int_a^b \frac{e^{\sqrt{\frac{-k}{\epsilon}}(b-s)} + e^{\sqrt{\frac{-k}{\epsilon}}(s-b)}}{2} \cdot \frac{f(s, y_\epsilon(s))}{\epsilon} \mathrm{d}s. \qquad \text{(E.2.1)}
\end{aligned}
$$

Now compute the integrals A and I_1 :

$$
\begin{aligned}
A = & -\frac{1}{k} f(b, y_\epsilon(b)) + \frac{e^{\sqrt{\frac{-k}{\epsilon}}(b-a)} + e^{\sqrt{\frac{-k}{\epsilon}}(a-b)}}{2k} f(a, y_\epsilon(a)) \\
& + \frac{1}{2k}\int_a^b \left(e^{\sqrt{\frac{-k}{\epsilon}}(b-s)} + e^{\sqrt{\frac{-k}{\epsilon}}(s-b)}\right) \cdot \frac{df(s, y_\epsilon(s))}{\mathrm{d}s} \mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
I_1 = & \frac{1}{k} f(c, y_\epsilon(c)) - \frac{e^{\sqrt{\frac{-k}{\epsilon}}(c-a)} + e^{\sqrt{\frac{-k}{\epsilon}}(a-c)}}{2k} f(a, y_\epsilon(a)) \\
& - \frac{1}{2k}\int_a^c \left(e^{\sqrt{\frac{-k}{\epsilon}}(c-s)} + e^{\sqrt{\frac{-k}{\epsilon}}(s-c)}\right) \cdot \frac{df(s, y_\epsilon(s))}{\mathrm{d}s} \mathrm{d}s.
\end{aligned}
$$

The integral (E.2.1)

$$
\begin{aligned}
& \int_a^b \frac{e^{\sqrt{\frac{-k}{\epsilon}}(b-s)} + e^{\sqrt{\frac{-k}{\epsilon}}(s-b)}}{2} \cdot \frac{f(s, y_\epsilon(s))}{\epsilon} \mathrm{d}s \\
= & \frac{1}{2\sqrt{-k\epsilon}}\left[\left(e^{\sqrt{\frac{-k}{\epsilon}}(b-a)} - e^{\sqrt{\frac{-k}{\epsilon}}(a-b)}\right) f(a, y_\epsilon(a)) \right. \\
& \left. + \int_a^b \left(e^{\sqrt{\frac{-k}{\epsilon}}(b-s)} - e^{\sqrt{\frac{-k}{\epsilon}}(s-b)}\right) \cdot \frac{f(s, y_\epsilon(s))}{\mathrm{d}s} \mathrm{d}s\right].
\end{aligned}
$$

Analyzing the integrals (by similar way as in [5]) we obtain

$$y'_\epsilon(a) = -\frac{1}{\sqrt{-k\epsilon}}\left[\frac{e^{\sqrt{\frac{-k}{\epsilon}}(b-a)}}{D}\left(f(a, y_\epsilon(a)) - f(c, y_\epsilon(c))\right) + O(\sqrt{\epsilon})\right]$$

and

$$y'_\epsilon(b) = -\frac{1}{\sqrt{-k\epsilon}}\left[\frac{e^{\sqrt{\frac{-k}{\epsilon}}(b-a)}}{D}\left(f(c, y_\epsilon(c)) - f(b, y_\epsilon(b))\right) + O(\sqrt{\epsilon})\right],$$

i.e. $|y'_\epsilon(a)| \to \infty$ if $u(a) \neq u(c)$ and $|y'_\epsilon(b)| \to \infty$ if $u(b) \neq u(c)$, respectively, for $\epsilon \to 0^+$.

References

1. Z. Artstein: *Stability in the presence of singular perturbations.* Nonlinear Anal. **34(6)** (1998), 817–827.
2. V. Gaitsgory: *On a representation of the limit occupational measures set of a control system with applications to singularly perturbed control systems.* SIAM J. Control Optim. **43(1)** (2004), 325–340.
3. P. Kokotovic, H. K. Khali, and J. O'Reilly: *Singular Perturbation Methods in Control, Analysis and Design.* Academic Press, London 1986.
4. R. Vrabel: *Nonlocal Four-Point Boundary Value Problem for the Singularly Perturbed Semilinear Differential Equations.* Boundary Value Problems, Volume 2011, Article ID 570493, 9 pages doi:10.1155/2011/570493.
5. R. Vrabel: *Boundary layer phenomenon for three-point boundary value problem for the nonlinear singularly perturbed systems.* Kybernetika **47**, no. 4, 644-652, 2011.

F LINEAR PROBLEM WITH CHARACTERISTIC ROOTS ON IMAGINARY AXIS

F.1 Introduction

In this chapter we will study the singularly perturbed linear problem

$$\epsilon y'' + ky = f(t),\ k > 0,\ 0 < \epsilon << 1,\ f \in C^3(\langle a, b \rangle) \tag{F.1.1}$$

with Neumann boundary condition

$$y'(a) = 0,\ y'(b) = 0. \tag{F.1.2}$$

We can view this equation as the mathematical model of the dynamical systems with high–speed feedback. The situation considered here is complicated by the fact that a characteristic equation of this Diff. Eq. has roots on the imaginary axis i.e. the system is not hyperbolic. For hyperbolic ones the dynamics close critical manifold is well–know (see e.g. [2]), but for the non–hyperbolic systems the problem of existence and asymptotic behavior is open, in general, and leads to the substantial technical difficulties in nonlinear case [1]. The considerations below may be instructive for this ones.

We prove, that there exists infinitely many sequences $\{\epsilon_n\}_0^\infty$, $\epsilon_n \to 0^+$ such that $y_{\epsilon_n}(t)$ converges to $u(t)$ for all $t \in \langle a, b\rangle$ where y_ϵ is a solution of problem (F.1.1), (F.1.2) and u is a solution of reduced problem (when we put $\epsilon = 0$ in (F.1.1)) $ku = f(t)$ i.e. $u(t) = \frac{f(t)}{k}$.

We will consider for the parameter ϵ the set J_n only,

$$J_n = \left\langle k\left(\frac{b-a}{(n+1)\pi - \lambda}\right)^2, k\left(\frac{b-a}{n\pi + \lambda}\right)^2\right\rangle, \quad n = 0, 1, 2, \ldots$$

where $\lambda > 0$ is an arbitrarily small, but fixed constant which garantees the existence and uniqueness of solutions of (F.1.1), (F.1.2).

Example 1. Consider the linear problem

$$\epsilon y'' + ky = e^t, \;\; t \in \langle a, b\rangle, \;\; k > 0, \;\; 0 < \epsilon << 1$$

$$y'(a) = 0, \;\; y'(b) = 0$$

and its solution

$$y_\epsilon(t) = \frac{-e^a \cos\left[\sqrt{\frac{k}{\epsilon}}(b-t)\right] + e^b \cos\left[\sqrt{\frac{k}{\epsilon}}(t-a)\right]}{\sqrt{\frac{k}{\epsilon}}(k+\epsilon)\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]} + \frac{e^t}{k+\epsilon}.$$

Hence, for every sequence $\{\epsilon_n\}$, $\epsilon_n \in J_n$ the solution of considered problem

$$y_{\epsilon_n}(t) = \frac{e^t}{k+\epsilon_n} + \mathcal{O}(\sqrt{\epsilon_n})$$

converges uniformly for $n \to \infty$ to the solution $u(t) = \frac{e^t}{k}$ of the reduced problem on $\langle a, b\rangle$.

The main result of this chapter is following one.

F.2 Main result

Theorem 1. *For all* $f \in C^3(\langle a,b\rangle)$ *and for every sequence* $\{\epsilon_n\}_{n=0}^{\infty}$, $\epsilon_n \in J_n$ *there exists unique solution* y_ϵ *of problem (F.1.1), (F.1.2) satisfying*

$$y_{\epsilon_n} \to u \text{ uniformly on } \langle a,b\rangle \tag{F.2.1}$$

for $n \to \infty$.

More precisely,

$$y_{\epsilon_n}(t) = u(t) + \mathcal{O}\left(\sqrt{\epsilon_n}\right) \text{ on } \langle a,b\rangle. \tag{F.2.2}$$

Proof. As first we show that

$$y_\epsilon(t) = \frac{\cos\left[\sqrt{\frac{k}{\epsilon}}(t-a)\right]\int\limits_a^b \cos\left[\sqrt{\frac{k}{\epsilon}}(b-s)\right]\frac{f(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{k}{\epsilon}}\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]} + \int\limits_a^t \frac{\sin\left[\sqrt{\frac{k}{\epsilon}}(t-s)\right]\frac{f(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{k}{\epsilon}}}\mathrm{d}s \tag{F.2.3}$$

is a solution of (F.1.1), (F.1.2). Differentiating (F.2.3) twice, taking into cosideration that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_a^t H(t,s)f(s)\mathrm{d}s = \int\limits_a^t \frac{\partial H(t,s)}{\partial t}f(s)\mathrm{d}s + H(t,t)f(t)$$

we obtain

$$y_\epsilon'(t) = -\frac{\sqrt{\frac{k}{\epsilon}}\sin\left[\sqrt{\frac{k}{\epsilon}}(t-a)\right]\int\limits_a^b \cos\left[\sqrt{\frac{k}{\epsilon}}(b-s)\right]\frac{f(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{k}{\epsilon}}\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]}$$

$$+\int\limits_a^t \frac{\sqrt{\frac{k}{\epsilon}}\cos\left[\sqrt{\frac{k}{\epsilon}}(t-s)\right]\frac{f(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{k}{\epsilon}}}\mathrm{d}s, \qquad \text{(F.2.4)}$$

$$y_\epsilon''(t) = -\frac{\left(\sqrt{\frac{k}{\epsilon}}\right)^2\cos\left[\sqrt{\frac{k}{\epsilon}}(t-a)\right]\int\limits_a^b \cos\left[\sqrt{\frac{k}{\epsilon}}(b-s)\right]\frac{f(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{k}{\epsilon}}\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]}$$

$$-\int\limits_a^t \frac{\left(\sqrt{\frac{k}{\epsilon}}\right)^2\sin\left[\sqrt{\frac{k}{\epsilon}}(t-s)\right]\frac{f(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{k}{\epsilon}}}\mathrm{d}s + \frac{f(t)}{\epsilon}. \qquad \text{(F.2.5)}$$

From (F.2.5) and (F.2.3), after little algebraic arrangement we get

$$y_\epsilon'' = \frac{k}{\epsilon}(-y_\epsilon) + \frac{f(t)}{\epsilon}$$

i.e. y_ϵ is a solution of differential equation (F.1.1) and from (F.2.4) it is easy to verify that this solution satisfies (F.1.2).

Let $t_0 \in \langle a, b\rangle$ is arbitrary, but fixed. Denote by I_1 and I_2 the integrals

$$I_1 = \int\limits_a^b \cos\left[\sqrt{\frac{k}{\epsilon}}(b-s)\right]\frac{f(s)}{\epsilon}\mathrm{d}s$$

$$I_2 = \int\limits_a^{t_0} \sin\left[\sqrt{\frac{k}{\epsilon}}(t_0-s)\right]\frac{f(s)}{\epsilon}\mathrm{d}s.$$

Then

$$y_\epsilon(t_0) = \frac{\cos\left[\sqrt{\frac{k}{\epsilon}}(t_0 - a)\right] I_1}{\sqrt{\frac{k}{\epsilon}} \sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]} + \frac{I_2}{\sqrt{\frac{k}{\epsilon}}}. \tag{F.2.6}$$

Integrating I_1 and I_2 by parts we obtain

$$I_1 = \left| \begin{matrix} h' = \cos\left[\sqrt{\frac{k}{\epsilon}}(b-s)\right] & g = \frac{f(s)}{\epsilon} \\ h = -\sqrt{\frac{\epsilon}{k}} \sin\left[\sqrt{\frac{k}{\epsilon}}(b-s)\right] & g' = \frac{f'(s)}{\epsilon} \end{matrix} \right|$$

$$= \sqrt{\frac{\epsilon}{k}} \sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right] \frac{f(a)}{\epsilon}$$

$$+ \int_a^b \sqrt{\frac{\epsilon}{k}} \sin\left[\sqrt{\frac{k}{\epsilon}}(b-s)\right] \frac{f'(s)}{\epsilon} \mathrm{d}s,$$

$$I_2 = \left| \begin{matrix} h' = \sin\left[\sqrt{\frac{k}{\epsilon}}(t_0-s)\right] & g = \frac{f(s)}{\epsilon} \\ h = \sqrt{\frac{\epsilon}{k}} \cos\left[\sqrt{\frac{k}{\epsilon}}(t_0-s)\right] & g' = \frac{f'(s)}{\epsilon} \end{matrix} \right|$$

$$= \frac{\sqrt{\frac{\epsilon}{k}} f(t_0)}{\epsilon} - \sqrt{\frac{\epsilon}{k}} \cos\left[\sqrt{\frac{k}{\epsilon}}(t_0-a)\right] \frac{f(a)}{\epsilon}$$

$$- \int_a^{t_0} \sqrt{\frac{\epsilon}{k}} \cos\left[\sqrt{\frac{k}{\epsilon}}(t_0-s)\right] \frac{f'(s)}{\epsilon} \mathrm{d}s.$$

Also

$$y_\epsilon(t_0) = \frac{f(t_0)}{k} + \frac{\cos\left[\sqrt{\frac{k}{\epsilon}}(t_0-a)\right]}{\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]} \int_a^b \sin\left[\sqrt{\frac{k}{\epsilon}}(b-s)\right] \frac{f'(s)}{k} \mathrm{d}s$$

$$- \int_a^{t_0} \cos\left[\sqrt{\frac{k}{\epsilon}}(t_0-s)\right] \frac{f'(s)}{k} \mathrm{d}s. \tag{F.2.7}$$

Now we estimate the difference

$$y_\epsilon(t_0) - \frac{f(t_0)}{k}.$$

We obtain

$$\left| y_\epsilon(t_0) - \frac{f(t_0)}{k} \right| \le \frac{1}{k \sin \lambda} \left| \int_a^b \sin\left[\sqrt{\frac{k}{\epsilon}}(b-s) \right] f'(s) \mathrm{d}s \right|$$
$$+ \frac{1}{k} \left| \int_a^{t_0} \cos\left[\sqrt{\frac{k}{\epsilon}}(t_0 - s) \right] f'(s) \mathrm{d}s \right|. \qquad \text{(F.2.8)}$$

The integrals in (F.2.8) converge to zero for $\epsilon = \epsilon_n \in J_n$, $n \to \infty$.

Indeed, with respect to assumption on f we may integrate by parts in (F.2.8). Thus,

$$\int_a^b \sin\left[\sqrt{\frac{k}{\epsilon}}(b-s) \right] f'(s) \mathrm{d}s = \left[\sqrt{\frac{\epsilon}{k}} \cos\left[\sqrt{\frac{k}{\epsilon}}(b-s) \right] f'(s) \right]_a^b$$
$$- \int_a^b \sqrt{\frac{\epsilon}{k}} \cos\left[\sqrt{\frac{k}{\epsilon}}(b-s) \right] f''(s) \mathrm{d}s$$
$$\le \sqrt{\frac{\epsilon}{k}} \left(|f'(a)| + |f'(b)| + \left| \int_a^b \cos\left[\sqrt{\frac{k}{\epsilon}}(b-s) \right] f''(s) \mathrm{d}s \right| \right)$$
$$\le \sqrt{\frac{\epsilon}{k}} \left\{ |f'(a)| + |f'(b)| + \sqrt{\frac{\epsilon}{k}} \left(|f''(a)| + \mu_2(b-a) \right) \right\} \qquad \text{(F.2.9)}$$

and

$$\int_a^{t_0} \cos\left[\sqrt{\frac{k}{\epsilon}}(t_0-s)\right] f'(s)\mathrm{d}s$$
$$= \left[-\sqrt{\frac{\epsilon}{k}}\sin\left[\sqrt{\frac{k}{\epsilon}}(t_0-s)\right] f'(s)\right]_a^{t_0}$$
$$+\int_a^{t_0} \sqrt{\frac{\epsilon}{k}}\sin\left[\sqrt{\frac{k}{\epsilon}}(t_0-s)\right] f''(s)\mathrm{d}s$$

$$\leq \sqrt{\frac{\epsilon}{k}}\left(|f'(a)|+\left|\int_a^{t_0}\sin\left[\sqrt{\frac{k}{\epsilon}}(t_0-s)\right] f''(s)\mathrm{d}s\right|\right)$$
$$\leq \sqrt{\frac{\epsilon}{k}}\left\{|f'(a)|+\sqrt{\frac{\epsilon}{k}}\left(\mu_1+|f''(a)|+\mu_2(b-a)\right)\right\} \qquad \text{(F.2.10)}$$

where $\mu_1 = \sup\limits_{t\in\langle a,b\rangle} |f''(t)|$ and $\mu_2 = \sup\limits_{t\in\langle a,b\rangle} |f'''(t)|$.

Substituing (F.2.9) and (F.2.10) into (F.2.8) we obtain an a priori estimate of solutions of (F.1.1), (F.1.2) for all $t_0 \in \langle a, b\rangle$ of the form

$$\left|y_\epsilon(t_0)-\frac{f(t_0)}{k}\right|$$
$$\leq \frac{1}{k\sin\lambda}\sqrt{\frac{\epsilon}{k}}\left\{|f'(a)|+|f'(b)|+\sqrt{\frac{\epsilon}{k}}\left(|f''(a)|+\mu_2(b-a)\right)\right\}$$
$$+\frac{1}{k}\sqrt{\frac{\epsilon}{k}}\left\{|f'(a)|+\sqrt{\frac{\epsilon}{k}}\left(\mu_1+|f''(a)|+\mu_2(b-a)\right)\right\}. \qquad \text{(F.2.11)}$$

Because the right side of the inequality (F.2.11) is independent on t_0, the convergence is uniform on $\langle a, b\rangle$. Theorem 1 holds.

Remark 5. As remark we conclude that in the case $|f'(a)| = |f'(b)| = 0$, the convergence rate is $\mathcal{O}(\epsilon_n)$, $\epsilon_n \in J_n$, as follows from (F.2.11).

References

1. R. Vrábeľ: *Singularly perturbed semilinear Neumann problem with non-normally hyperbolic critical manifold*, E. J. Qualitative Theory of Diff. Equ. - ISSN 1417–3875. - No. 9 (2010), pp. 1–11.
2. R. Vrábeľ: *Upper and lower solutions for singularly perturbed semilinear Neumann's problem*, Mathematica Bohemica, 122, no.2 (1997), pp. 175–180.

G PROBLEM WITH NON-NORMALLY HYPERBOLIC MANIFOLD

G.1 Introduction

We will consider the Neumann problem

$$\epsilon y'' + ky = f(t, y),\ t \in \langle a, b\rangle,\ k > 0,\ 0 < \epsilon << 1 \tag{G.1.1}$$

$$y'(a) = 0,\ y'(b) = 0. \tag{G.1.2}$$

The dynamics near the normally hyperbolic manifold of critical point is well-known (Theorem on persistence of normally hyperbolic manifold, see [2, 3, 5, 9, 12], for reference). However, this theory is not valid for a set of critical points where normal hyperbolicity breaks down, and this problem remained a major obstacle to the theory of differential equations and leads to substantial technical difficulties. This work is an alternative to the geometric theory of singular perturbation and admits another sight on the singularly perturbed problems.

We apply the method of upper and lower solutions ([1]) and the delicate estimates to prove the existence of the solutions for problem

(G.1.1), (G.1.2), and by adding auxiliary condition, which converge uniformly to the solution u of reduced problem (i.e. by setting $\epsilon = 0$ in (G.1.1)) on the interval $\langle a, b\rangle$ for appropriate chosen $\{\epsilon_n\}_{n=0}^{\infty}$, $\epsilon_n \to 0^+$.

As usual, we say that $\alpha_\epsilon \in C^2(\langle a, b\rangle)$ is a lower solution for problem (G.1.1), (G.1.2) if $\epsilon\alpha_\epsilon''(t) + k\alpha_\epsilon(t) \geq f(t, \alpha_\epsilon(t))$ and $\alpha_\epsilon'(a) \geq 0$, $\alpha_\epsilon'(b) \leq 0$ for every $t \in \langle a, b\rangle$. An upper solution $\beta_\epsilon \in C^2(\langle a, b\rangle)$ satisfies $\epsilon\beta_\epsilon''(t) + k\beta_\epsilon(t) \leq f(t, \beta_\epsilon(t))$ and $\beta_\epsilon'(a) \leq 0$, $\beta_\epsilon'(b) \geq 0$ for every $t \in \langle a, b\rangle$. Then

Lemma 1 ([8])**.** *If the functions $\alpha_\epsilon, \beta_\epsilon$ are lower and upper solutions for (G.1.1), (G.1.2) such that $\alpha_\epsilon \leq \beta_\epsilon$, then there exists solution y_ϵ of (G.1.1), (G.1.2) with $\alpha_\epsilon \leq y_\epsilon \leq \beta_\epsilon$.*

Denote $\mathcal{D}_\delta(u) = \{(t, y)|\ a \leq t \leq b, |y - u(t)| < \delta\}$, δ is a positive constant and $u \in C^2$ is a solution of reduced problem $ku = f(t, u)$.

Let

$$v_{1,\epsilon}(t) = |u'(a)| \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(b - t)\right]}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b - a)\right]}$$

and

$$v_{2,\epsilon}(t) = -|u'(b)| \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t - a)\right]}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b - a)\right]}$$

where $m = k + w$ (for the constant w see Theorem 1 below).

Let

$$J_n(\lambda) = \left\langle m\left(\frac{b - a}{(n + 1)\pi - \lambda}\right)^2, m\left(\frac{b - a}{n\pi + \lambda}\right)^2\right\rangle, \quad n = 0, 1, 2, \ldots,$$

$\lambda > 0$ is an arbitrarily small, but fixed constant and

$$\mathcal{M} = \left\{\bigcup J_n, n = 0, 1, 2, \ldots\right\}.$$

The function $v_{1,\epsilon}(t)$ satisfies:

1. $\epsilon v''_{1,\epsilon} + m v_{1,\epsilon} = 0$

2. $v'_{1,\epsilon}(a) = |u'(a)|$, $v'_{1,\epsilon}(b) = 0$

3. $v_{1,\epsilon}(t)$ is periodic in the variable t with the period $\frac{2\pi\sqrt{\epsilon}}{\sqrt{m}} \to 0$

4. $v_{1,\epsilon_n}(t)$ converges uniformly to 0 for every sequence $\{\epsilon_n\}_{n=0}^{\infty}$ such that $\epsilon_n \in J_n$ and $|v_{1,\epsilon_n}(t)| \leq \frac{\sqrt{\epsilon_n}}{\sqrt{m}\sin\lambda}$, $t \in \langle a, b\rangle$.

The function $v_{2,\epsilon}(t)$ satisfies:

1. $\epsilon v''_{2,\epsilon} + m v_{2,\epsilon} = 0$

2. $v'_{2,\epsilon}(a) = 0$, $v'_{2,\epsilon}(b) = |u'(b)|$

3. $v_{2,\epsilon}(t)$ is periodic in the variable t with the period $\frac{2\pi\sqrt{\epsilon}}{\sqrt{m}} \to 0$

4. $v_{2,\epsilon_n}(t)$ converges uniformly to 0 for every sequence $\{\epsilon_n\}_{n=0}^{\infty}$ such that $\epsilon_n \in J_n$ and $|v_{2,\epsilon_n}(t)| \leq \frac{\sqrt{\epsilon_n}}{\sqrt{m}\sin\lambda}$, $t \in \langle a, b\rangle$.

Denote $\omega_{0,\epsilon}(t) = v_{2,\epsilon}(t) - v_{1,\epsilon}(t)$.
Let $\omega_{1,\epsilon,i}(t)$ is a solution of the linear problem

$$\epsilon y'' + my = \pm\epsilon u''(t), \;\; i = \alpha_\epsilon, \beta_\epsilon$$

with the Neumann boundary condition G.1.2, where the sign $+$ and $-$ is considered for $i = \alpha_\epsilon$ and $i = \beta_\epsilon$, respectively. These solutions may be computed exactly

$$\omega_{1,\epsilon,i}(t) = \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right] \int\limits_a^b \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right](\pm u''(s))\,\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]}$$
$$+ \int\limits_a^t \frac{\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right](\pm u''(s))\,\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}}\mathrm{d}s = \mathcal{O}(\epsilon), \epsilon \in \mathcal{M}.$$

Obviously, $\omega_{1,\epsilon,\alpha_\epsilon}(t) = -\omega_{1,\epsilon,\beta_\epsilon}(t)$ on $\langle a, b\rangle$.

Let $r_{\epsilon,i}(t)$ is a continuous solution of the Fredholm equation of the first kind

$$\Gamma(\epsilon)\int_a^b K_\epsilon(t,s) r_{\epsilon,i}(s)\mathrm{d}s$$

$$+\Omega_{\epsilon,i}(t) = z_{\epsilon,i}(t),\ z_{\epsilon,i}(t) \geq 0,\ i = \alpha_\epsilon, \beta_\epsilon \tag{G.1.3}$$

where $\Gamma(\epsilon) = \frac{1}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]}\cdot\frac{1}{\epsilon}$, $\Gamma^{-1}(\epsilon) = \mathcal{O}(\sqrt{\epsilon})$, $\epsilon \in \mathcal{M}$,

$$\Omega_{\epsilon,i}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t)$$

and the kernel

$$K_\epsilon(t,s) = \begin{cases} K_{1,\epsilon}(t,s), & a \leq s \leq t \leq b \\ K_{2,\epsilon}(t,s), & a \leq t \leq s \leq b, \end{cases}$$

$$K_{1,\epsilon}(t,s) = \cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]\cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right] +$$

$$\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right]$$

$$K_{2,\epsilon}(t,s) = \cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]\cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right]$$

for $\epsilon \in \mathcal{M}$ and a modulation function $z_{\epsilon,i}(t)$ is an appropriate continuous nonnegative function such that $r_{\epsilon,i}(t) \leq 0$.

This is an integral equation of the kernel $K_\epsilon(t,s)$ that is continuous on the square $\langle a, b\rangle \times \langle a, b\rangle$. The problem (G.1.3) is defined as ill-posed and, in general, may be described numerically with Tikhonov regularization ([6, 7, 10, 11]).

By substituing $z_{\epsilon,i}(t) = r_{\epsilon,i}(t) + \tilde{z}_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$ into (G.1.3) we obtain

$$\Gamma(\epsilon) \int_a^b K_\epsilon(t,s) r_{\epsilon,i}(s) \mathrm{d}s + \tilde{\Omega}_{\epsilon,i}(t) = r_{\epsilon,i}(t), \ i = \alpha_\epsilon, \beta_\epsilon,$$

i.e. $r_{\epsilon,i}(t)$ is a solution of Fredholm integral equation of second kind

$$\Gamma(\epsilon) \int_a^b K_\epsilon(t,s) y(s) \mathrm{d}s + \tilde{\Omega}_{\epsilon,i}(t) = y(t), \ i = \alpha_\epsilon, \beta_\epsilon, \tag{G.1.4}$$

where $\tilde{\Omega}_{\epsilon,i}(t) = \Omega_{\epsilon,i}(t) - \tilde{z}_{\epsilon,i}(t)$ and $\tilde{z}_{\epsilon,i}(t)$ is an appropriate chosen function such that

$$\tilde{z}_{\epsilon,i}(t) \geq -r_{\epsilon,i}(t), \tag{G.1.5}$$

$$r_{\epsilon,i}(t) \leq 0, \tag{G.1.6}$$

$t \in \langle a, b \rangle$, $i = \alpha_\epsilon, \beta_\epsilon$.

The kernel K_ϵ is semiseparable ([4]), therefore the equation (G.1.4) can be rewritten as

$$y(t) = \sum_{k=1}^{3} A_{k,\epsilon,a}(t) \int_a^t B_{k,\epsilon,a}(s) y(s) \mathrm{d}s$$

$$+ A_{1,\epsilon,b}(t) \int_t^b B_{1,\epsilon,b}(s) y(s) \mathrm{d}s + \tilde{\Omega}_{\epsilon,i}(t)$$

where

$$A_{1,\epsilon,a}(t) = \Gamma(\epsilon)\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]$$

$$A_{2,\epsilon,a}(t) = \Gamma(\epsilon)\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]\sin\left[\sqrt{\frac{m}{\epsilon}}t\right]$$

$$A_{3,\epsilon,a}(t) = -\Gamma(\epsilon)\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]\cos\left[\sqrt{\frac{m}{\epsilon}}t\right]$$

$$A_{1,\epsilon,b}(t) = \Gamma(\epsilon)\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]$$

$$B_{1,\epsilon,a}(s) = \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right]$$

$$B_{2,\epsilon,a}(s) = \cos\left[\sqrt{\frac{m}{\epsilon}}s\right]$$

$$B_{3,\epsilon,a}(s) = \sin\left[\sqrt{\frac{m}{\epsilon}}s\right]$$

$$B_{1,\epsilon,b}(s) = \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right]$$

or

$$y(t) = \sum_{k=1}^{3} A_{k,\epsilon,a}(t)X_{k,\epsilon,a,i}(t)$$

$$+A_{1,\epsilon,b}(t)X_{1,\epsilon,b,i}(t) + \tilde{\Omega}_{\epsilon,i}(t),\ i = \alpha_\epsilon, \beta_\epsilon \tag{G.1.7}$$

where

$$X_{k,\epsilon,a,i}(t) = \int_a^t B_{k,\epsilon,a}(s)y(s)\mathrm{d}s,\ k = 1,2,3,$$

$$X_{1,\epsilon,b,i}(t) = \int_t^b B_{1,\epsilon,b}(s)y(s)\mathrm{d}s.$$

Multiply both sides of the integral equation (G.1.7) by $B_{j,\epsilon,a}(t)$ and integrate from a to t and by $B_{1,\epsilon,b}(t)$ and integrate from t to b, respectively.

We obtain

$$X_{j,\epsilon,a,i} = \sum_{k=1}^{3} \int\limits_a^t A_{k,\epsilon,a} B_{j,\epsilon,a} X_{k,\epsilon,a,i} \mathrm{d}t + \int\limits_a^t A_{1,\epsilon,b} B_{1,\epsilon,a} X_{1,\epsilon,b,i} \mathrm{d}t$$

$$+ \int\limits_a^t B_{j,\epsilon,a} \tilde{\Omega}_{\epsilon,i} \mathrm{d}t,$$

$$X_{1,\epsilon,b,i} = \sum_{k=1}^{3} \int\limits_t^b A_{k,\epsilon,a} B_{1,\epsilon,b} X_{k,\epsilon,a,i} \mathrm{d}t + \int\limits_t^b A_{1,\epsilon,b} B_{1,\epsilon,b} X_{1,\epsilon,b,i} \mathrm{d}t$$

$$+ \int\limits_t^b B_{1,\epsilon,b} \tilde{\Omega}_{\epsilon,i} \mathrm{d}t, \;\; j = 1, 2, 3, \;\; i = \alpha_\epsilon, \beta_\epsilon.$$

Differentiating these equations and taking into consideration the definition of $X_{j,\epsilon,a}$, $X_{1,\epsilon,b}$ we obtain the boundary value problem for the system of linear differential equations

$$X'_{j,\epsilon,a,i} = \sum_{k=1}^{3} A_{k,\epsilon,a} B_{j,\epsilon,a} X_{k,\epsilon,a,i}$$

$$+ A_{1,\epsilon,b} B_{1,\epsilon,a} X_{k,\epsilon,b,i} + B_{j,\epsilon,a} \tilde{\Omega}_{\epsilon,i}, \tag{G.1.8}$$

$$X'_{1,\epsilon,b,i} = - \sum_{k=1}^{3} A_{k,\epsilon,a} B_{1,\epsilon,b} X_{k,\epsilon,a,i}$$

$$- A_{1,\epsilon,b} B_{1,\epsilon,b} X_{1,\epsilon,b,i} - B_{1,\epsilon,b} \tilde{\Omega}_{\epsilon,i}, \tag{G.1.9}$$

$$X_{j,\epsilon,a,i}(a) = 0, \;\; X_{1,\epsilon,b,i}(b) = 0, \tag{G.1.10}$$

$j = 1, 2, 3, i = \alpha_\epsilon, \beta_\epsilon$ or in the block matrix notation

$$X' = \begin{pmatrix} P_{1,\epsilon}(t) & P_{3,\epsilon}(t) \\ P_{2,\epsilon}(t) & P_{4,\epsilon}(t) \end{pmatrix} X + D_{\epsilon,i}(t)$$

where

$$X = \left(X_{1,\epsilon,a,i}(t), X_{2,\epsilon,a,i}(t), X_{3,\epsilon,a,i}(t), X_{1,\epsilon,b,i}(t) \right)^T,$$

$$P_{1,\epsilon}(t)$$

$$= \begin{pmatrix} A_{1,\epsilon,a}(t)B_{1,\epsilon,a}(t) & A_{2,\epsilon,a}(t)B_{1,\epsilon,a}(t) & A_{3,\epsilon,a}(t)B_{1,\epsilon,a}(t) \\ A_{1,\epsilon,a}(t)B_{2,\epsilon,a}(t) & A_{2,\epsilon,a}(t)B_{2,\epsilon,a}(t) & A_{3,\epsilon,a}(t)B_{2,\epsilon,a}(t) \\ A_{1,\epsilon,a}(t)B_{3,\epsilon,a}(t) & A_{2,\epsilon,a}(t)B_{3,\epsilon,a}(t) & A_{3,\epsilon,a}(t)B_{3,\epsilon,a}(t) \end{pmatrix},$$

$$P_{2,\epsilon}(t)$$

$$= -\begin{pmatrix} A_{1,\epsilon,a}(t)B_{1,\epsilon,b}(t) & A_{2,\epsilon,a}(t)B_{1,\epsilon,b}(t) & A_{3,\epsilon,a}(t)B_{1,\epsilon,b}(t) \end{pmatrix},$$

$$P_{3,\epsilon}(t) = \begin{pmatrix} A_{1,\epsilon,b}(t)B_{1,\epsilon,a}(t) \\ A_{1,\epsilon,b}(t)B_{2,\epsilon,a}(t) \\ A_{1,\epsilon,b}(t)B_{3,\epsilon,a}(t) \end{pmatrix},$$

$$P_{4,\epsilon}(t) = -\left(A_{1,\epsilon,b}(t)B_{1,\epsilon,b}(t)\right)$$

and

$$D_{\epsilon,i}(t) = \tilde{\Omega}_{\epsilon,i}(t)\left(B_{1,\epsilon,a}(t), B_{2,\epsilon,a}(t), B_{3,\epsilon,a}(t), -B_{1,\epsilon,b}(t)\right)^T,$$

$i = \alpha_\epsilon, \beta_\epsilon$. Thus,

$$r_{\epsilon,i}(t) = r_{\epsilon,i}\left(\tilde{z}_{\epsilon,i}(t)\right) = \sum_{k=1}^{3} A_{k,\epsilon,a}(t)X_{k,\epsilon,a,i}(t) + A_{1,\epsilon,b}(t)X_{1,\epsilon,b,i}(t) + \tilde{\Omega}_{\epsilon,i}(t) \quad \text{(G.1.11)}$$

where X is a solution of the linear boundary value problem (G.1.8), (G.1.9), (G.1.10).

The conditions (G.1.5), (G.1.6) we may write in the form

$$-\tilde{z}_{\epsilon,i}(t) \le r_{\epsilon,i}(t) \le 0, \ \ i = \alpha_\epsilon, \beta_\epsilon \quad \text{(G.1.12)}$$

or

$$0 \le \sum_{k=1}^{3} A_{k,\epsilon,a}(t)X_{k,\epsilon,a,i}(t) \quad \text{(G.1.13)}$$

$$+A_{1,\epsilon,b}(t)X_{1,\epsilon,b,i}(t) + \Omega_{\epsilon,i}(t) \leq \tilde{z}_{\epsilon,i}(t). \qquad \text{(G.1.14)}$$

Remark 6. The matrix

$$\begin{pmatrix} P_{1,\epsilon}(t) & P_{3,\epsilon}(t) \\ P_{2,\epsilon}(t) & P_{4,\epsilon}(t) \end{pmatrix}$$

of the system is periodic with period p tendings to 0 for $\epsilon \to 0^+$, $\epsilon \in \mathcal{M}$ and using the Floquet theory, then the solution of the linear homogeneous system

$$X' = \begin{pmatrix} P_{1,\epsilon}(t) & P_{3,\epsilon}(t) \\ P_{2,\epsilon}(t) & P_{4,\epsilon}(t) \end{pmatrix} X$$

can be written as $X_{hom,\epsilon}(t) = p_\epsilon(t)e^{\Theta_\epsilon t}$ where $p_\epsilon(t)$ is a periodic function and a matrix Θ_ϵ is time independent. This fact is instructive for the numerical description and the computer simulation of the system (G.1.8), (G.1.9).

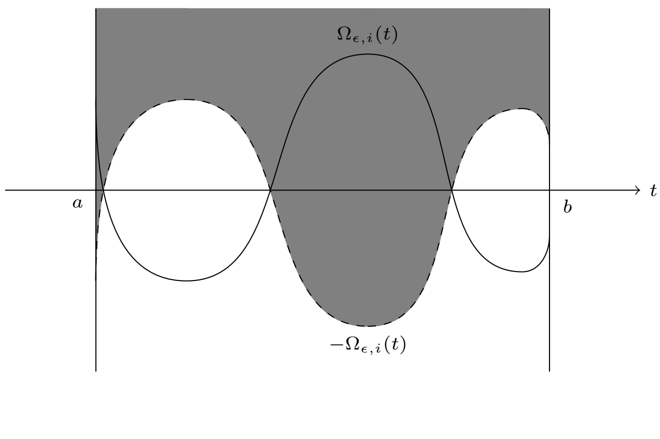

Fig. G.1.1 *The region for* $v_{c,\epsilon,i}(t)$

Remark 7. The condition (G.1.14) is the fundamental assumption for existence of the barrier functions α_ϵ, β_ϵ for proving Theorem 1.

Now let $v_{c,\epsilon,i}(t)$ is a solution of Neumann boundary value problem (G.1.2) for Diff. Eq.

$$\epsilon y'' + my = r_{\epsilon,i}(t),\ i = \alpha_\epsilon, \beta_\epsilon \tag{G.1.15}$$

i.e.

$$v_{c,\epsilon,i}(t) = \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right] \int_a^b \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right] \frac{r_{\epsilon,i}(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]}$$
$$+ \int_a^t \frac{\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right] \frac{r_{\epsilon,i}(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}}\mathrm{d}s = \mathcal{O}\left(r_{\epsilon,i}(t)\right), \epsilon \in \mathcal{M}.$$

As follows from (G.1.3), the functions $v_{c,\epsilon,i}(t)$ must appear in the region as illustrated in Figure ***G.1.1***. Now we may state the main result of this chapter.

G.2 Main result

Theorem 1. (see, [13])

($\mathcal{A}1$) *Let $\tilde{z}_{\epsilon,i}(t)$, $\epsilon \in (0, \epsilon_0] \cap \mathcal{M}$, $i = \alpha_\epsilon, \beta_\epsilon$ are the continuous functions such that (G.1.14) holds.*

($\mathcal{A}2$) *Let $f \in C^1(\mathcal{D}_\delta(u))$ satisfies the condition*

$$\left|\frac{\partial f(t,y)}{\partial y}\right| \leq w < k \text{ for every } (t,y) \in \mathcal{D}_\delta(u)$$

(nonhyperbolicity condition)

where

$$\delta \geq \max \{\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) + v_{c,\epsilon,i}(t);$$
$$i = \alpha_\epsilon, \beta_\epsilon,\ t \in \langle a, b\rangle,\ \epsilon \in (0, \epsilon_0] \cap \mathcal{M}\} .$$

Then the problem (G.1.1), (G.1.2) has for $\epsilon \in (0, \epsilon_0] \cap \mathcal{M}$ *a solution satisfying the inequality*

$$-\omega_{0,\epsilon}(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t) - v_{c,\epsilon,\alpha_\epsilon}(t) \leq y_\epsilon(t) - u(t)$$
$$\leq \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t)$$

on $\langle a, b\rangle$.

Proof. We define the lower solutions by

$$\alpha_\epsilon(t) = u(t) - (\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\alpha_\epsilon}(t))$$

and the upper solutions by

$$\beta_\epsilon(t) = u(t) + (\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t)) .$$

After simple algebraic manipulation we obtain

$$\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) + v_{c,\epsilon,i}(t) = z_{\epsilon,i}(t) \geq 0,\ i = \alpha_\epsilon, \beta_\epsilon$$

on $\langle a, b\rangle$. The functions $\alpha_\epsilon, \beta_\epsilon$ satisfy the boundary conditions prescribed for the lower and upper solutions of (G.1.1), (G.1.2) and $\alpha_\epsilon(t) \leq \beta_\epsilon(t)$ on $\langle a, b\rangle$.

Now we show that

$$\epsilon\alpha_\epsilon''(t) + k\alpha_\epsilon(t) \geq f(t, \alpha_\epsilon(t)) \tag{G.2.1}$$

and

$$\epsilon\beta_\epsilon''(t) + k\beta_\epsilon(t) \leq f(t, \beta_\epsilon(t)). \tag{G.2.2}$$

Denote $h(t, y) = f(t, y) - ky$. From the assumption $(\mathcal{A}2)$ on the function $f(t, y)$ we have

$$-m \le \frac{\partial h(t,y)}{\partial y} \le 2w - m < 0$$

in $\mathcal{D}_\delta(u)$. By the Taylor theorem we obtain

$$\begin{aligned}
\epsilon\alpha_\epsilon''(t) - h(t, \alpha_\epsilon(t)) &= \epsilon\alpha_\epsilon''(t) - [h(t, \alpha_\epsilon(t)) - h(t, u(t))] \\
&= \epsilon u''(t) - \epsilon\omega_{0,\epsilon}''(t) - \epsilon\omega_{1,\epsilon,\alpha_\epsilon}''(t) - \epsilon v_{c,\epsilon,\alpha_\epsilon}''(t) \\
&- \frac{\partial h\left(t, \theta_\epsilon(t)\right)}{\partial y}\left(-\omega_{0,\epsilon}(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t) - v_{c,\epsilon,\alpha_\epsilon}(t)\right) \\
&\ge \epsilon u''(t) - \epsilon\omega_{0,\epsilon}''(t) - \epsilon\omega_{1,\epsilon,\alpha_\epsilon}''(t) - \epsilon v_{c,\epsilon,\alpha_\epsilon}''(t) \\
&+ (-m)\left(\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\alpha_\epsilon}(t)\right) \\
&= -\epsilon v_{c,\epsilon,\alpha_\epsilon}''(t) - m v_{c,\epsilon,\alpha_\epsilon}(t) = -r_{\epsilon,\alpha_\epsilon}(t).
\end{aligned}$$

From the condition (G.1.6) is $-r_{\epsilon,\alpha_\epsilon}(t) \ge 0$ therefore $\epsilon\alpha_\epsilon''(t) - h(t, \alpha_\epsilon(t)) \ge 0$ on $\langle a, b\rangle$.

The inequality for $\beta_\epsilon(t)$:

$$\begin{aligned}
&h(t, \beta_\epsilon(t)) - \epsilon\beta_\epsilon''(t) \\
&\quad = \frac{\partial h\left(t, \tilde{\theta}_\epsilon(t)\right)}{\partial y}\left(\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t)\right) \\
&\quad - \epsilon u''(t) - \epsilon\omega_{0,\epsilon}''(t) - \epsilon\omega_{1,\epsilon,\beta_\epsilon}''(t) - \epsilon v_{c,\epsilon,\beta_\epsilon}''(t) \\
&\quad \ge (-m)\left(\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t)\right) \\
&\quad - \epsilon u''(t) - \epsilon\omega_{0,\epsilon}''(t) - \epsilon\omega_{1,\epsilon,\beta_\epsilon}''(t) - \epsilon v_{c,\epsilon,\beta_\epsilon}''(t) \\
&\quad = -\epsilon v_{c,\epsilon,\beta_\epsilon}''(t) - m v_{c,\epsilon,\beta_\epsilon}(t) = -r_{\epsilon,\beta_\epsilon}(t) \ge 0
\end{aligned}$$

where $(t, \theta_\epsilon(t))$ is a point between $(t, \alpha_\epsilon(t))$ and $(t, u(t))$, $(t, \theta_\epsilon(t)) \in \mathcal{D}_\delta(u)$. Analogously, $(t, \tilde{\theta}_\epsilon(t))$ is a point between $(t, u(t))$ and $(t, \beta_\epsilon(t))$, $(t, \tilde{\theta}_\epsilon(t)) \in \mathcal{D}_\delta(u)$ for $\epsilon \in (0, \epsilon_0] \cap \mathcal{M}$. The existence of a solution for (G.1.1), (G.1.2) satisfying the inequality above follows from Lemma 1.

Remark 8. We note, that if there exists the solution of G.1.3 such that $r_{\epsilon,i}(t) = \mathcal{O}(\epsilon^{\nu})$, $\nu > 0$ then for every sequence$\{\epsilon_n\}$, $\epsilon_n \in (0, \epsilon_0] \cap \mathcal{M}$, $\epsilon_n \in J_n$ we have

$$|y_{\epsilon_n}(t) - u(t)| \leq (|u'(a)| + |u'(b)|)\, \mathcal{O}\left(\sqrt{\epsilon_n}\right) + M_{u''}\mathcal{O}\left(\epsilon_n\right) + \mathcal{O}\left(\epsilon_n^{\nu}\right),$$

$M_{u''} = \max\left\{|u''(t)|, t \in \langle a, b\rangle\right\}$ on $\langle a, b\rangle$.

Remark 9. In the trivial case, when $u(t) = c =$const is

$$\omega_{0,\epsilon}(t) = \omega_{1,\epsilon,i}(t) \stackrel{\text{id}}{=} 0,\ r_{\epsilon,i}(t) \stackrel{\text{id}}{=} 0,\ i = \alpha_\epsilon, \beta_\epsilon$$

and

$$|y_\epsilon(t) - u(t)| \leq 0$$

i.e. $y_\epsilon(t) = u(t)$ on $\langle a, b\rangle$.

Example 2. Consider linear problem, i.e. $w = 0$,

$$\epsilon y'' + ky = e^t,\ t \in \langle a, b\rangle,\ k > 0,\ 0 < \epsilon << 1$$

$$y'(a) = 0,\ y'(b) = 0$$

and its solution

$$y_\epsilon(t) = \frac{-e^a \cos\left[\sqrt{\frac{k}{\epsilon}}(b-t)\right] + e^b \cos\left[\sqrt{\frac{k}{\epsilon}}(t-a)\right]}{\sqrt{\frac{k}{\epsilon}}(k+\epsilon)\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]} + \frac{e^t}{k+\epsilon}.$$

Hence, for every sequence $\{\epsilon_n\}$, $\epsilon_n \in J_n$ the solution of considered problem

$$y_{\epsilon_n}(t) = \frac{e^t}{k+\epsilon_n} + \mathcal{O}(\sqrt{\epsilon}_n)$$

converges uniformly for $n \to \infty$ to the solution $u(t) = \frac{e^t}{k}$ of the reduced problem on $\langle a, b\rangle$.

G.3 Generalization of the assumption $(\mathcal{A}1)$

The assumption of nonnegativity of $z_{\epsilon,i}(t)$ in (G.1.3) and the condition (G.1.12) may be generalized in the following sense.
Denote

$$I_{+,\epsilon,i} = \{t \in \langle a, b\rangle : z_{\epsilon,i}(t) \geq 0\}, \; i = \alpha_\epsilon, \beta_\epsilon$$

and

$$I_{-,\epsilon,i} = \{t \in \langle a, b\rangle : z_{\epsilon,i}(t) \leq 0\}, \; i = \alpha_\epsilon, \beta_\epsilon.$$

Let there exist the functions $\tilde{z}_{\epsilon,i}(t)$ such that

$$r_{\epsilon,i}(t) \leq 0 \text{ on } I_{+,\epsilon,i}, \; i = \alpha_\epsilon, \beta_\epsilon \tag{G.3.1}$$

and

$$r_{\epsilon,i}(t) \leq 2w z_{\epsilon,i}(t) \text{ on } I_{-,\epsilon,i}, \; i = \alpha_\epsilon, \beta_\epsilon \tag{G.3.2}$$

and

$$v_{c,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t) \geq -2\omega_{0,\epsilon}(t) \text{ on } I_{-,\epsilon,\alpha_\epsilon} \cup I_{-,\epsilon,\beta_\epsilon} \tag{G.3.3}$$

where $r_{\epsilon,i}(t)$ is from (G.1.11) and $z_{\epsilon,i}(t) = r_{\epsilon,i}(t) + \tilde{z}_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$.

Taking into consideration the fact that

$$\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) + v_{c,\epsilon,i}(t) = z_{\epsilon,i}(t) \leq 0 \text{ on } I_{-,\epsilon,i}, \; i = \alpha_\epsilon, \beta_\epsilon, \tag{G.3.4}$$

for the required inequality (G.2.1) for $\alpha_\epsilon(t)$ on the interval $I_{-,\epsilon,\alpha_\epsilon}$ (in the case of the inequality for $\beta_\epsilon(t)$ i.e. (G.2.2) on $I_{-,\epsilon,\beta_\epsilon}$, we proceed analogously) we obtain

$$\begin{aligned}\epsilon\alpha_\epsilon''(t) - h(t,\alpha_\epsilon(t)) &= \epsilon u''(t) - \epsilon\omega_{0,\epsilon}^{''}(t) - \epsilon\omega_{1,\epsilon,\alpha_\epsilon}^{''}(t) - \epsilon v_{c,\epsilon,\alpha_\epsilon}^{''}(t) \\ &- \frac{\partial h\left(t,\theta_\epsilon(t)\right)}{\partial y}\left(-\omega_{0,\epsilon}(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t) - v_{c,\epsilon,\alpha_\epsilon}(t)\right) \\ &\geq \epsilon u''(t) - \epsilon\omega_{0,\epsilon}^{''}(t) - \epsilon\omega_{1,\epsilon,\alpha_\epsilon}^{''}(t) - \epsilon v_{c,\epsilon,\alpha_\epsilon}^{''}(t) \\ &+ (-m+2w)\left(\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\alpha_\epsilon}(t)\right) \\ &= -r_{\epsilon,\alpha_\epsilon}(t) + 2w\left(\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\alpha_\epsilon}(t)\right).\end{aligned}$$

From (G.3.2) and (G.3.4),

$$-r_{\epsilon,\alpha_\epsilon}(t) + 2w\left(\omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\alpha_\epsilon}(t)\right) \geq 0$$

for $t \in I_{-,\epsilon,\alpha_\epsilon}$. The condition (G.3.3) guarantees that $\alpha_\epsilon(t) \leq \beta_\epsilon(t)$ on $\langle a,b\rangle$. Hence, Theorem 1 holds.

From (G.3.2), we get

$$(1-2w)r_{\epsilon,i}(t) \leq 2w\tilde{z}_{\epsilon,i}(t) \leq -2wr_{\epsilon,i}(t) \tag{G.3.5}$$

and we may generalize the assumption ($\mathcal{A}1$) as follows.

($\mathcal{A}1'$) Let $\tilde{z}_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$ are the continuous functions such that

$$\big[(G.1.12)\big] \vee \big[(G.3.5) \wedge \left(v_{c,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t) \geq -2\omega_{0,\epsilon}(t)\right)\big]$$

on $\langle a,b\rangle$, $\epsilon \in (0,\epsilon_0] \cap \mathcal{M}$ holds.

References

1. C. De Coster and P. Habets: *Two-Point Boundary Value Problems: Lower and Upper Solutions*, Volume 205 (Mathematics in Science and Engineering), Elsevier Science; 1 edition, 2006.

2. J. Grasman: *Asymptotic Methods for Relaxation Oscillations and Applications*, Springer- Verlag, New York, 1987.
3. Christopher K. R. T. Jones: *Geometric Singular Perturbation Theory*, C.I.M.E. Lectures, Montecatini Terme, June 1994, Lecture Notes in Mathematics 1609, Springer-Verlag, Heidelberg, 1995.
4. F.Gesztesy and K. A. Makarov: *(Modified) Fredholm Determinants for Operators with Matrix-Valued Semi-Separable Integral Kernels Revisited*, Integr. equ. oper. theory 48 (2004), 0378-620X/040561-42, DOI 10.1007/s00020-003-1279-z, pp. 561-–602.
5. M. Krupa and P. Szmolyan: *Relaxation oscillation and canard explosion*, J. Diff. Equations 174 (2001), pp. 312–368.
6. M. T. Nair, E. Schock and U. Tautenhahn: *Morozov's discrepancy principle under general source conditions*, J. Anal. Appl. 22 (2003), pp. 199-–214.
7. P. Mathé and S. V. Pereverzev: *Geometry of linear ill-posed problems in variable Hilbert scales*, Inverse Problems 19 (3) (2003), pp. 789-–803.
8. J. Mawhin: *Points fixes, points critiques et problemes aux limites*, Semin. Math. Sup. no. 92, Presses Univ. Montreal, 1985.
9. P. Szmolyan and M. Wechselberger: *Relaxation oscillations in R3*, J. Diff. Equations 200, (2004), pp. 69–104.
10. M. Thamban Nair and S. V. Pereverzev: *Regularized collocation method for Fredholm integral equations of the first kind*, Journal of Complexity (2006).
11. A. N. Tikhonov and V. Y. Arsenin: *Solutions of Ill-Posed Problems*, Winston, New York, 1977. ISBN 0470991240.
12. F. Verhulst: *Nonlinear Differential Equations and Dynamical Systems*, Springer-Verlag, New York, 2000.
13. R. Vrábeľ: *Singularly perturbed semilinear Neumann problem with non-normally hyperbolic critical manifold*, E. J. Qualitative Theory of Diff. Equ. - ISSN 1417–3875, No. 9. (2010), pp. 1–11.

H A NOTE ABOUT THE EQUIVALENCE FOR NEUMANN PROBLEM WITH NON-NORMALLY HYPERBOLIC MANIFOLD

H.1 Preliminaries

In [3], we have established the sufficient condition for existence and gave some remarks on asymptotic behavior (for $\epsilon \to 0^+$) of the solutions of Neumann problem

$$\epsilon y'' + ky = f(t, y),\ t \in \langle a, b\rangle,\ k > 0,\ 0 < \epsilon << 1 \tag{H.1.1}$$

$$y'(a) = 0,\ y'(b) = 0, \tag{H.1.2}$$

for

$$f \in C^1\left(\langle a, b\rangle \times [u(t) - z_{\epsilon,\alpha_\epsilon}(t), u(t) + z_{\epsilon,\beta_\epsilon}(t)]\right)$$

such that

$$\left|\frac{\partial f(t, y)}{\partial y}\right| \leq w < k \tag{H.1.3}$$

for

$$(t,y) \in \langle a,b\rangle \times [u(t) - z_{\epsilon,\alpha_\epsilon}(t), u(t) + z_{\epsilon,\beta_\epsilon}(t)]$$

where u is a solution of reduced problem

$$ku = f(t,u). \tag{H.1.4}$$

We will assume that $u \in C^2(\langle a,b\rangle)$ and for the functions $z_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$, see (H.2.1) below and α_ϵ, β_ϵ are the lower and upper solutions to (H.1.1), (H.1.2), for details see e.g. [1].

In this chapter we focus your attention on some equivalence for existence of the solutions for (H.1.1), (H.1.2). The problem is very difficult unless several special cases, but those ones may be instructive for future considerations.

Hereafter we follow the notation and definition of [3]. Let

$$J_n(\lambda) = \left\langle m\left(\frac{b-a}{(n+1)\pi - \lambda}\right)^2, m\left(\frac{b-a}{n\pi + \lambda}\right)^2\right\rangle, \quad n = 0,1,2,\dots$$

where $m = k + w$, $\lambda > 0$ is arbitrarily small, but fixed constant and let

$$\mathcal{M} = \left\{\bigcup J_n, n = 0,1,2,\dots\right\}.$$

Let

$$\begin{aligned} v_{1,\epsilon}(t) &= \quad |u'(a)| \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(b-t)\right]}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]} \\ v_{2,\epsilon}(t) &= -\,|u'(b)| \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]}. \end{aligned}$$

Denote

$$\omega_{0,\epsilon}(t) = v_{2,\epsilon}(t) - v_{1,\epsilon}(t)\left(= \mathcal{O}(\sqrt{\epsilon})\right), \epsilon \in \mathcal{M}.$$

Let

$$\omega_{1,\epsilon,i}(t) = \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right] \int\limits_a^b \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right](\pm u''(s))\,\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]} + \int\limits_a^t \frac{\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right](\pm u''(s))\,\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}}\mathrm{d}s\,(=\mathcal{O}(\epsilon))\,, \epsilon \in \mathcal{M},$$

i.e. $\omega_{1,\epsilon,i}(t)$ is a solution of Neumann problem (H.1.2) for linear differential equation

$$\epsilon y'' + my = \pm\epsilon u''(t),$$

where the sign $+$ and $-$ is considered for $i = \alpha_\epsilon$ and $i = \beta_\epsilon$, respectively.

Hence

$$\omega_{1,\epsilon,\alpha_\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) \equiv 0 \text{ on } \langle a, b\rangle. \tag{H.1.5}$$

Let $r_{\epsilon,i}(t)$ is a solution of Fredholm integral equation of second kind

$$\Gamma(\epsilon)\int\limits_a^b K_\epsilon(t,s)y(s)\mathrm{d}s + \tilde{\Omega}_{\epsilon,i}(t) = y(t),\ \ i = \alpha_\epsilon, \beta_\epsilon, \tag{H.1.6}$$

where $\tilde{\Omega}_{\epsilon,i}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) - \tilde{z}_{\epsilon,i}(t)$, $\tilde{z}_{\epsilon,i}(t)$ is an appropriate chosen continuous function (see the assumption $(\mathcal{A}1')$ below),

$$\Gamma(\epsilon) = \frac{1}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]} \cdot \frac{1}{\epsilon},\ \ \epsilon \in \mathcal{M}$$

and the kernel

$$K_\epsilon(t,s) = \begin{cases} K_{1,\epsilon}(t,s), & a \le s \le t \le b \\ K_{2,\epsilon}(t,s), & a \le t \le s \le b, \end{cases}$$

$$K_{1,\epsilon}(t,s) = \cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]\cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right] + \sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right]$$

$$K_{2,\epsilon}(t,s) = \cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]\cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right]$$

for $\epsilon \in \mathcal{M}$. Obviously, $r_{\epsilon,i}(t) = r_{\epsilon,i}\left(\tilde{z}_{\epsilon,i}(t)\right)$. Let

$$v_{c,\epsilon,i}(t) = \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]\int_a^b \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right]\frac{r_{\epsilon,i}(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]} + \int_a^t \frac{\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right]\frac{r_{\epsilon,i}(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}}\mathrm{d}s$$

$$(= \mathcal{O}(r_{\epsilon,i}(t)))\,,\ \epsilon \in \mathcal{M},\ i = \alpha_\epsilon, \beta_\epsilon,$$

i.e. $v_{c,\epsilon,i}(t)$ is a solution of Neumann problem (H.1.2) for linear differential equation

$$\epsilon y'' + my = r_{\epsilon,i}(t).$$

We will assume that

$(\mathcal{A}1')$ There exist the continuous functions $\tilde{z}_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$ such that for all $t \in \langle a, b\rangle$

$$\big[(H.1.7)\big] \vee \big[(H.1.8) \wedge (v_{c,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t) \geq -2\omega_{0,\epsilon}(t))\big]_{i=\alpha_\epsilon}$$

and

$$\big[(H.1.7)\big] \vee \big[(H.1.8) \wedge (v_{c,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t) \geq -2\omega_{0,\epsilon}(t))\big]_{i=\beta_\epsilon}$$

for $\epsilon \in (0, \epsilon_0] \cap \mathcal{M}$ where

$$-\tilde{z}_{\epsilon,i}(t) \leq r_{\epsilon,i}(t) \leq 0,\ i = \alpha_\epsilon, \beta_\epsilon \qquad \text{(H.1.7)}$$

$$(1-2w)r_{\epsilon,i}(t) \le 2w\tilde{z}_{\epsilon,i}(t) \le -2wr_{\epsilon,i}(t),\ i = \alpha_\epsilon, \beta_\epsilon. \tag{H.1.8}$$

H.2 Main results

Theorem 1. (see, [4]) *Let $w = 0$ and $u'(a) = u'(b) = 0$. Then the following three statement are equivalent*

(i) $(\mathcal{A}1')$

(ii) *There exist $\alpha_\epsilon, \beta_\epsilon \in C^2(\langle a,b\rangle)$ satisfying $\alpha_\epsilon''(t) + k\alpha_\epsilon(t) \ge f(t, \alpha_\epsilon(t))$ and $\alpha_\epsilon'(a) \ge 0$, $\alpha_\epsilon'(b) \le 0$, $\epsilon\beta_\epsilon''(t) + k\beta_\epsilon(t) \le f(t, \beta_\epsilon(t))$ and $\beta_\epsilon'(a) \le 0$, $\beta_\epsilon'(b) \ge 0$ and $\alpha_\epsilon(t) \le \beta_\epsilon(t)$ for all $t \in \langle a,b\rangle$.*

(iii) *There exists a solution $y = y_\epsilon(t)$ of (H.1.1), (H.1.2).*

Proof.
(i)⇒(ii) This result is proven in [3], of a more general form. We recall, that

$$\alpha_\epsilon(t) = u(t) - z_{\epsilon,\alpha_\epsilon}(t) \text{ and } \beta_\epsilon(t) = u(t) + z_{\epsilon,\beta_\epsilon}(t),$$

where

$$z_{\epsilon,i}(t) = r_{\epsilon,i}(t) + \tilde{z}_{\epsilon,i}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) + v_{c,\epsilon,i}(t), . \tag{H.2.1}$$

$i = \alpha_\epsilon, \beta_\epsilon$. The condition $\alpha_\epsilon(t) \le \beta_\epsilon(t)$ on $\langle a,b\rangle$ requires

$$z_{\epsilon,\alpha_\epsilon}(t) + z_{\epsilon,\beta_\epsilon}(t) \ge 0 \text{ for all } t \in \langle a,b\rangle. \tag{H.2.2}$$

(ii)⇒(iii) follows from the method of upper and lower solutions [1, 2].
(iii)⇒(i):

At first, the assumption $u'(a) = u'(b) = 0$ implies that $\omega_{0,\epsilon}(t) \equiv 0$ on $\langle a,b\rangle$.

Denote

$$I_{+,\epsilon,i} = \{t \in \langle a,b\rangle : z_{\epsilon,i}(t) \geq 0\}\,,\ i = \alpha_\epsilon, \beta_\epsilon$$

and

$$I_{-,\epsilon,i} = \{t \in \langle a,b\rangle : z_{\epsilon,i}(t) \leq 0\}\,,\ i = \alpha_\epsilon, \beta_\epsilon.$$

We set

$$z_{\epsilon,\alpha_\epsilon}(t) = u(t) - y_\epsilon(t)\,(= r_{\epsilon,\alpha_\epsilon}(t) + \tilde{z}_{\epsilon,\alpha_\epsilon}(t))$$

and

$$z_{\epsilon,\beta_\epsilon}(t) = y_\epsilon(t) - u(t)\,(= r_{\epsilon,\beta_\epsilon}(t) + \tilde{z}_{\epsilon,\beta_\epsilon}(t))$$

Obviously $I_{-,\epsilon,\alpha_\epsilon} \cup I_{-,\epsilon,\beta_\epsilon} = \langle a,b\rangle$. From the required conditions (H.1.7) and (H.1.8) we obtain for $\tilde{z}_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$ the equivalent inequalities with (H.1.7) and (H.1.8)

$$z_{\epsilon,\alpha_\epsilon}(t) \leq \tilde{z}_{\epsilon,\alpha_\epsilon}(t),\ z_{\epsilon,\beta_\epsilon}(t) \leq \tilde{z}_{\epsilon,\beta_\epsilon}(t) \text{ for } z_{\epsilon,i}(t) \geq 0 \qquad \text{(H.2.3)}$$

$$(1-2w)z_{\epsilon,\alpha_\epsilon}(t) \leq \tilde{z}_{\epsilon,\alpha_\epsilon}(t),\ (1-2w)z_{\epsilon,\beta_\epsilon}(t) \leq \tilde{z}_{\epsilon,\beta_\epsilon}(t) \qquad \text{(H.2.4)}$$

for $z_{\epsilon,i}(t) \leq 0$, i.e.

$$\tilde{z}_{\epsilon,\alpha_\epsilon}(t) \geq \begin{cases} u(t) - y_\epsilon(t) & \text{for } u(t) - y_\epsilon(t) \geq 0 \\ (1-2w)\,(u(t) - y_\epsilon(t)) & \text{for } u(t) - y_\epsilon(t) \leq 0 \end{cases}$$

$$\tilde{z}_{\epsilon,\beta_\epsilon}(t) \geq \begin{cases} y_\epsilon(t) - u(t) & \text{for } y_\epsilon(t) - u(t) \geq 0 \\ (1-2w)\,(y_\epsilon(t) - u(t)) & \text{for } y_\epsilon(t) - u(t) \leq 0 \end{cases}$$

on $\langle a,b\rangle$.

As illustrated in Figure ***H.2.1***, in the case $w = 0$ only we may choose $\tilde{z}_{\epsilon,\alpha_\epsilon}(t)$, $\tilde{z}_{\epsilon,\beta_\epsilon}(t)$ such that

$$\tilde{z}_{\epsilon,\alpha_\epsilon}(t) + \tilde{z}_{\epsilon,\beta_\epsilon}(t) \equiv 0 \text{ on } \langle a,b\rangle$$

and consequently

$$v_{c,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t) \equiv 0 \text{ on } \langle a, b\rangle.$$

More precisely,

$$v_{c,\epsilon,\alpha_\epsilon}(t) = v_{c,\epsilon,\beta_\epsilon}(t) \equiv 0 \text{ on } \langle a, b\rangle.$$

It follows from the fact that $v_{c,\epsilon,i}(t)$ is the solutions of Neumann problem for differential equation

$$\epsilon y'' + my = r_{\epsilon,i}(t)\,(= z_{\epsilon,i}(t) - \tilde{z}_{\epsilon,i}(t)) \;\; i = \alpha_\epsilon, \beta_\epsilon.$$

Thus, $(\mathcal{A}1')$ holds.

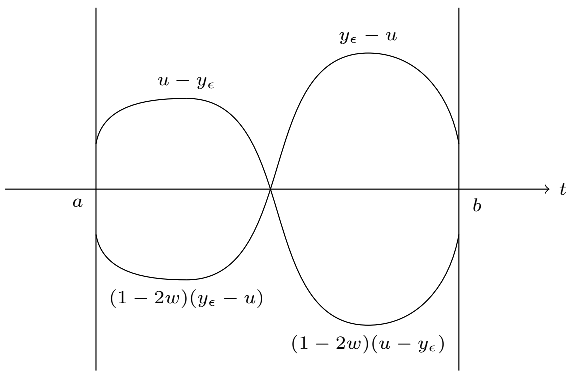

Fig. H.2.1 *The regions for* $\tilde{z}_{\epsilon,\alpha_\epsilon}(t)$ *and* $\tilde{z}_{\epsilon,\beta_\epsilon}(t)$, $w = 0$

Remark 10. We remark, that $r_{\epsilon,\alpha_\epsilon}(t) \equiv 0 \wedge r_{\epsilon,\beta_\epsilon}(t) \equiv 0$ on $\langle a, b\rangle$ in following two cases only

1. $w = 0$, $u'(a) = u'(b) = 0$

2. $w \neq 0$, $u'(a) = u'(b) = 0$, $u''(t) \equiv 0$ on $\langle a, b\rangle$.

In both cases $z_{\epsilon,i}(t) \equiv \tilde{z}_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$ and

1. $z_{\epsilon,i}(t) = \omega_{1,\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$

2. $z_{\epsilon,i}(t) = 0$, $i = \alpha_\epsilon, \beta_\epsilon$,

respectively. The second one is trivial, $y_\epsilon(t) = u(t) = const$ on $\langle a, b\rangle$.

In other cases (in all ones is $|u'(a)| + |u'(b)| \neq 0$), if $r_{\epsilon,\alpha_\epsilon}(t) \equiv 0 \wedge r_{\epsilon,\beta_\epsilon}(t) \equiv 0$ for $w = 0$ or $r_{\epsilon,\alpha_\epsilon}(t) \equiv 0 \vee r_{\epsilon,\beta_\epsilon}(t) \equiv 0$ for $w \neq 0$ then, as follows from the integral equation (H.1.6),

$$z_{\epsilon,\alpha_\epsilon}(t) = \tilde{z}_{\epsilon,\alpha_\epsilon}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t)$$

or

$$z_{\epsilon,\beta_\epsilon}(t) = \tilde{z}_{\epsilon,\beta_\epsilon}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t)$$

are not identically equal to zero on $\langle a, b\rangle$ and it leads to contradiction with (H.2.4) and consequently with $(\mathcal{A}1')$ (for $w \neq 0$) or (H.2.2) (for $w = 0$).

Example 3. Consider Neumann problem (H.1.2) for linear Diff. Eq.

$$\epsilon y'' + ky = f(t),\ t \in \langle a, b\rangle,\ k > 0,\ 0 < \epsilon << 1 \qquad \text{(H.2.5)}$$

where $f \in C^1(\langle a, b\rangle)$.

Assume that $f'(a) = f'(b) = 0$. This is case 1 from Remark 10. Therefore,

$$z_{\epsilon,i}(t) = \tilde{z}_{\epsilon,i}(t) = \omega_{1,\epsilon,i}(t),\ i = \alpha_\epsilon, \beta_\epsilon$$

and we may define the lower and upper solutions by

$$\alpha_\epsilon(t) = u(t) - z_{\epsilon,\alpha_\epsilon}(t) = u(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t) = \frac{f(t)}{k} - \omega_{1,\epsilon,\alpha_\epsilon}(t)$$

and

$$\beta_\epsilon(t) = u(t) + z_{\epsilon,\beta_\epsilon}(t) = u(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) = \frac{f(t)}{k} + \omega_{1,\epsilon,\beta_\epsilon}(t)$$

On the basis of Theorem 1 and the theory of method lower and upper solutions [1] there exists solution y_ϵ, $\epsilon \in \mathcal{M}$ of problem (H.2.5), (H.1.2) such that

$$\frac{f(t)}{k} - \omega_{1,\epsilon,\alpha_\epsilon}(t) \le y_\epsilon(t) \le \frac{f(t)}{k} + \omega_{1,\epsilon,\beta_\epsilon}(t)\ \epsilon \in \mathcal{M}$$

i.e.

$$y_\epsilon(t) = \frac{f(t)}{k} + \mathcal{O}(\epsilon),\ \ \epsilon \in \mathcal{M}.$$

In detail, we obtain that

$$\beta_\epsilon(t) - \alpha_\epsilon(t) = z_{\epsilon,\beta_\epsilon}(t) + z_{\epsilon,\alpha_\epsilon}(t) = \omega_{1,\epsilon,\beta_\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t) \equiv 0$$

on $\langle a, b\rangle$, therefore

$$y_\epsilon(t) = \alpha_\epsilon(t) = \beta_\epsilon(t),\ \ \epsilon \in \mathcal{M}.$$

On the other side, using the fact that a solution of (H.2.5), (H.1.2) we may write in the form

$$y_\epsilon(t) = \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right] \int\limits_a^b \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right] \frac{f(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]} + \int\limits_a^t \frac{\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right] \frac{f(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}}\mathrm{d}s$$

it is easy to verify, by integrating twice per-partes in $\omega_{1,\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$ that

$$y_\epsilon(t) = u(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t) = u(t) + \omega_{1,\epsilon,\beta_\epsilon}(t),$$

i.e. the choice of $\alpha_\epsilon, \beta_\epsilon$ in your example is optimal.

Now we introduce the notion of optimality of $v_{c,\epsilon,i}(t)$.

H.3 Optimality of $v_{c,\epsilon,i}(t)$

We say that $r^*_{\epsilon,i}(t) \leq 0$ on $\langle a, b\rangle$ is $v_{c,\epsilon,i}$ – optimal, if $v^*_{c,\epsilon,i}(t) \geq e_{c,\epsilon,i}(t)$ on $\langle a, b\rangle$ where

$$e_{c,\epsilon,\alpha}(t) = u(t) - y_\epsilon(t) - \omega_{0,\epsilon}(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t)$$
$$e_{c,\epsilon,\beta}(t) = y_\epsilon(t) - u(t) - \omega_{0,\epsilon}(t) - \omega_{1,\epsilon,\beta_\epsilon}(t),\ i = \alpha_\epsilon, \beta_\epsilon,$$

y_ϵ is a solution of (H.1.1), (H.1.2) and

1. The functions

$$\tilde{z}^*_{\epsilon,i}(t) = z^*_{\epsilon,i}(t) - r^*_{\epsilon,i}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) + v^*_{c,\epsilon,i}(t) - r^*_{\epsilon,i}(t)$$

satisfy $(\mathcal{A}1')$.

2. For every $v^{\triangle}_{c,\epsilon,i}(t) \geq e_{c,\epsilon,i}(t)$ on $\langle a, b\rangle$ such that

$$\tilde{z}^{\triangle}_{\epsilon,i}(t) = z^{\triangle}_{\epsilon,i}(t) - r^{\triangle}_{\epsilon,i}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) + v^{\triangle}_{c,\epsilon,i}(t) - r^{\triangle}_{\epsilon,i}(t)$$

satisfy $(\mathcal{A}1')$ is

$$\sup_{t\in\langle a,b\rangle} \left(v^{\triangle}_{c,\epsilon,i}(t) - e_{c,\epsilon,i}(t)\right) \geq \sup_{t\in\langle a,b\rangle} \left(v^*_{c,\epsilon,i}(t) - e_{c,\epsilon,i}(t)\right) \quad \text{(H.3.1)}$$

$i = \alpha_\epsilon, \beta_\epsilon$.

Remark 11. For problem (H.2.5), (H.1.2), may be computed the functions $e_{c,\epsilon,\alpha}(t)$, $e_{c,\epsilon,\beta}(t)$ exactly

$$e_{c,\epsilon,\alpha}(t) = \left(\frac{-f'(b) + |f'(b)|}{k}\right) \frac{\cos\left[\sqrt{\frac{k}{\epsilon}}(t-a)\right]}{\sqrt{\frac{k}{\epsilon}}\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]}$$
$$+ \left(\frac{f'(a) + |f'(a)|}{k}\right) \frac{\cos\left[\sqrt{\frac{k}{\epsilon}}(b-t)\right]}{\sqrt{\frac{k}{\epsilon}}\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]}$$

$$e_{c,\epsilon,\beta}(t) = \left(\frac{f'(b) + |f'(b)|}{k}\right) \frac{\cos\left[\sqrt{\frac{k}{\epsilon}}(t-a)\right]}{\sqrt{\frac{k}{\epsilon}}\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]}$$
$$+ \left(\frac{-f'(a) + |f'(a)|}{k}\right) \frac{\cos\left[\sqrt{\frac{k}{\epsilon}}(b-t)\right]}{\sqrt{\frac{k}{\epsilon}}\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]}.$$

These functions do not satisfy the boundary conditions (H.1.2) required for $v_{c,\epsilon,i}(t)$ if $|f'(a)| + |f'(b)| \neq 0$. Consequently, $v^*_{c,\epsilon,i}(t)$, is not identically equal to $e_{c,\epsilon,i}(t)$ on $\langle a, b\rangle$, therefore α_ϵ and β_ϵ are not identically equal to y_ϵ on $\langle a, b\rangle$.

For $|f'(a)| + |f'(b)| = 0$ is $v^*_{c,\epsilon,i}(t) = e_{c,\epsilon,i}(t) \equiv 0$ (i.e. $r^*_{\epsilon,i}(t) \equiv 0$), $i = \alpha_\epsilon, \beta_\epsilon$ which corresponds with Remark 10.

For $e_{c,\epsilon,\alpha}(t)$, $e_{c,\epsilon,\beta}(t)$ we obtain the following estimates

$$|e_{c,\epsilon,\alpha}(t)| \leq \left(\frac{-f'(b) + |f'(b)|}{k} + \frac{f'(a) + |f'(a)|}{k}\right) \frac{\sqrt{\epsilon}}{\sqrt{k}\sin\lambda}$$
$$|e_{c,\epsilon,\beta}(t)| \leq \left(\frac{f'(b) + |f'(b)|}{k} + \frac{-f'(a) + |f'(a)|}{k}\right) \frac{\sqrt{\epsilon}}{\sqrt{k}\sin\lambda}.$$

Thus, the functions $e_{c,\epsilon_n,i}$, $i = \alpha_\epsilon, \beta_\epsilon$ converge uniformly on $\langle a, b\rangle$ to zero for every sequence $\{\epsilon_n\}_{n=0}^{\infty}$, such that $\epsilon_n \in J_n$, i.e. $\epsilon_n \to 0^+$.

References

1. C. De Coster, P. Habets: *Two-Point Boundary Value Problems: Lower and Upper Solutions*, Volume 205 (Mathematics in Science and Engineering), Elsevier Science; 1 edition, 2006.

2. J. Mawhin: *Points fixes, points critiques et problemes aux limites*, Semin. Math. Sup. no. 92, Presses Univ. Montreal, 1985.

3. R. Vrábeľ: *Singularly perturbed semilinear Neumann problem with non-normally hyperbolic critical manifold*, E. J. Qualitative Theory of Diff. Equ. - ISSN: 1417–3875. - No. 9 (2010), pp. 1–11.

4. R. Vrábeľ: *Some remarks on an equivalence theorem for a singularly perturbed semilinear Neumann problem with non-normally hyperbolic critical manifold*, Int. Journal of Math. Analysis. - ISSN 1312-8876 - Vol. 5 (2011), no. 4, pp. 165–173.

Contents