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Continuous quantity and unit; their centrality to measurement

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Abstract – The Vocabulaire international de métrologie (3rd Edition) and the Système International d'Unités (8th Edition) between them elaborate three different measurement forms. Comparisons of these measurement forms leads, in some instances, to illogical conclusions regarding the relationships between number and unit that compromises the distinction between continuous quantity, ordered relationships and counts of entities.

To resolve these observed anomalies, we propose consideration is given to amending any subsequent VIM (4th Edition) to include restoration of definitions of measurement, unit and quantity that preserves the unit as the *standard of reference*, as elaborated by James Clerk Maxwell.

Keywords: Measurement forms, quantity, unit

1. MEASUREMENT FORMS

The three measurement forms promulgated by the Bureau International des Poids et Mesures are the:

> *classical form* [1], [2], wherein a standard of reference, called the unit [1 p 1], forms a *ratio* with the magnitude of a measurand sharing the same property of that unit [3 p 2].
>
> *ordinal quantity form* [3 p 15], which is a measurement procedure wherein an *ordering relationship* can be established, but no unit exists [3 p 15 Note 1].
>
> *counting of entities form* [4 pp 114-115] wherein a specified number of discrete entities compose the unit.

These differing measurement forms appear to be the result of a change to the permissible *standard of reference* adopted by the VIM [3]. Using the formalism of Quantity Calculus [5], where symbols in braces { } are numbers and symbols in square brackets [ ] are units, we distinguish between {N}[U], the Classical measurement form, and other references made permissible in the VIM [3]. To distinguish these alternate references from [U], we use angle brackets ⟨ ⟩, which maintains a similar symbolism but preserves the integrity of Quantity Calculus [5]. In so doing, first we demonstrate how number and quantity can be made interchangeable in "ordinal quantity" and lead, for instance, to the illogical $cm^{\wedge cm}$ or $cm^{\wedge 9}$ or $9^{\wedge cm}$. Second, we demonstrate that "ordinal quantity" is not falsifiable [6], a major tenet of science. Next, we compare measurement in the Classical form with the counting of entities. Berka [7] has distinguished between counting to determine an aggregate of entities as an ontological necessity, while "counting" to obtain the magnitude of a measurand is an epistemological auxiliary. Cooper and Humphry [9] have also argued the {N} of continuous quantity and the {N} of aggregates of entities are ontologically distinct. Where {N}[U] is a continuous quantity, {N} is a *real* number, and where for example {N}$\langle R_{\text{mole}}\rangle$, is a reference to an aggregate of discrete entities, then {N} is a *natural* number.

Therefore, we suggest counting is not measuring, employing "ordinal quantity" is not measuring and the term "measure" and its derivatives be preserved for continuous quantity as demonstrated by Maxwell [1].

We conclude by suggesting the subsequent VIM (4th Edition) give consideration to these issues to remove ambiguity and restore the distinctions between continuous quantity, ordered relationships and counts of entities.

2. THE CLASSICAL MEASUREMENT FORM

Michell [2 p 358] observes that, "…measurement is properly defined as *the estimation or discovery of the ratio of some magnitude of a quantitative attribute to a unit of the same attribute*. It is invariably along such lines that measurement is, and always has been, defined in the physical sciences…" (emphasis in original). This accords with the Maxwell definition [1 p 1], where every

> …expression of a Quantity consists of two factors or components. One of these is the name of a certain known quantity of the same kind as the quantity to be expressed, which is taken as a standard of reference. The other component is the number of times the standard is to be taken in order to make up the required quantity. The standard quantity is

technically called the Unit, and the number is called the Numerical Value of the quantity.

However, the VIM [3 p 2] defines quantity as a

> …property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference.

To this a qualification is provided in NOTE 2, where a

> …reference can be a **measurement unit,** a **measurement procedure,** a **reference material,** or a combination of such (emphasis in original).

In both the Maxwell and Michell definitions, only a unit of the same attribute can be the "standard of reference" but the VIM definition makes permissible three *different* "references":

(1) a unit, as classically defined.
(2) an ordered relationship.
(3) a count of discrete entities.

Removing the stricture of the unit as the only permissible expression accompanying the numerical component of a quantity has created the additional measurement forms, (2) and (3), above. We now elaborate the consequences of redefining quantity in this manner to permit non-unit alternatives in the expression of a quantity

3. QUANTITY CALCULUS AND THE EXPRESSION OF QUANTITY

Using the formalism of Quantity Calculus [5], numbers are symbolised in {N} and units in [U]. Therefore, the standard of reference, together with its numerical component, can be exemplified by {10}[K]. To preserve the integrity of Quantity Calculus we symbolise *ordered relationship* and a *count of entities* (2and 3 above) as the reference ⟨R⟩ with a subscript to ⟨R⟩ to *particularize* a reference. For example, subjective level of pain as an exemplification of an ordered relationship [3 p 15], would be particularized as $\{5\}\langle R_{\text{pain}}\rangle$ and a count of discrete entities may be particularized as $\{6\}\langle R_{\text{mole}}\rangle$.

This symbolism is now employed to make clear the consequence of re-defining quantity from *number* and a *unit* to *number* and a *reference*. This re-definition is exemplified in Figure 1 and Figure 2 by "Subjective level of abdominal pain on a scale from zero to (ten)" [3 p 15].

Let 5 metres be expressed as {5}[M] and 5 "pain" be expressed as $\{5\}\langle R_{\text{PAIN}}\rangle$, where ⟨R⟩ is the reference and the subscript $\langle_{\text{PAIN}}\rangle$ particularises the reference. The {5}[M] is the ratio {5}[M]/[M], which means {5} is a real number. By contrast, $\{5\}\langle R_{\text{PAIN}}\rangle$ is a subjective estimate based on categories of pain assumed to be order to which "numbers" have been assigned.

In this last instance, the {5} is neither a real number nor a natural number. It is an *ordering* of pain con-

Fig. 1 Ordered pain categories (after Bierie et al [6])

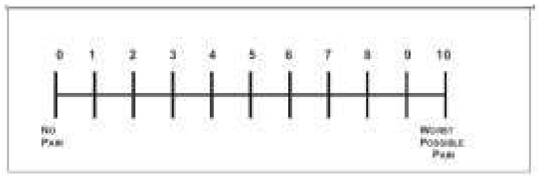

Fig. 2. Pain categories (after Downie et al [7])

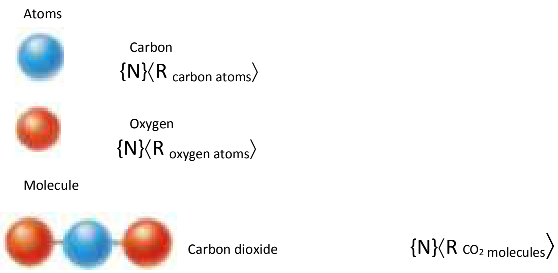

Fig. 3. The mole exemplified

structed from the pain descriptor categories and would best be described as 5^{th} $\langle R_{\text{PAIN}}\rangle$ or 5^{th} Label of Pain. Such numbers are therefore numerals or labels and carry no information regarding magnitude of a quantity. Further, such "ordinal quantity" *cannot* form an inverse relationship between its numerical component and its reference. This is only possible in the classical measurement form where, for example, {5}[M] can be re-expressed as {500}[cm] or {5000}[mm]. It is the decimal number and decimal unit that permit this relationship, one that cannot be sustained by any "ordinal quantity".

Finally, the VIM [3 p 6] defines"…measurement unit (as a) real scalar **quantity,** defined and adopted by convention, with which any other quantity of the same **kind** can be compared to express the ratio of the two quantities as a number". This definition makes clear that "ordinal quantity" cannot claim quantity status for its numerical component is neither "real" nor "scalar" and no ratio can be formed to express a number.

These logical inconsistencies in the VIM [3] require attention to ensure "…there is no fundamental difference in the basic principles of measurement in physics, chemistry, laboratory medicine, biology, or engineering."

4. S. S. STEVENS AND THE ASSIGNING OF NUMERALS - ORDINAL QUANTITY

The VIM includes a quantity named as "ordinal quantity" (3 p. 15), and defined as

> …..a conventional **measurement procedure,** for which a total ordering relation can be established, according to magnitude, with other quantities of the same **kind,** but for which no algebraic operations among those quantities exist" (emphasis in original).

Note 1 (p. 15) then qualifies the definition in the following manner:

> Ordinal quantities can enter into empirical relations only and have neither **measurement units** nor **quantity dimensions.** Differences and ratios of ordinal quantities have no physical meaning. (emphasis in original).

A logical inconsistency between the definition and its qualification are apparent. In the definition, the term "magnitude" is to be used to determine the order of quantities of the same kind. As detailed above, a quantity is formalised as {N} [U], distinct from other references {N} ⟨R⟩. From Note 1, it is made clear this form of "quantity" has no units. This means such a "quantity" can only be expressed as {N} ⟨ ⟩; that is, a number in the numeric component and a null in the reference component. This is a troublesome issue, for it means number and quantity are equivalent. To express an "ordinal quantity" as {9} ⟨ ⟩ would simply mean "9", and in turn mean number and quantity are the same thing. Clearly, numbers and quantities are *not* the same thing. For example, the power of a number is expressed as a base and an exponent. If numbers and quantities are the same thing, and therefore interchangeable, then $22^{\wedge[cm]}$, $[cm]^{\wedge[cm]}$ and $[cm]^{\wedge 9}$ would all have meaning. As these expressions are meaningless, numbers and quantities cannot be interchanged and are therefore *not* the same. Massey [10 p 3] made a similar observation when he noted that "…if x does have units, to write $e^{\wedge x}$ is meaningless."

Ordinal quantity also has all the attributes of the Stevens [11 p. 677] measurement definition, where "…we may say that measurement, in the broadest sense, is defined as the assignment of *numerals* to objects or events according to rules…" In VIM [3 p 15] this is exemplified as "Subjective level of abdominal pain on a scale from zero to five."

This assignment of numerals fails to discriminate between numerals and numbers, and fails to discriminate between objects or events and the *attributes* of those objects or events. First, numerals are labels, and can be found on mail boxes, street names and football jumpers, and like the letter labels for academic grades "A", "B" or "C", carry no information regarding magnitude. As elaborated by Michell [8 pp. 177-181], Stevens therefore needed to interpret numeral as number. However as demonstrated above, real number comes from the relationship between a quantity and its special reference quantity the unit, such that M / [M] = {M}. Number is *not assigned* as defined by Stevens or the VIM [3 p. 15). Second, objects or events are discrete entities and are best described as enumerated quantities [10]. The Stevens definition and the VIM promulgation of "ordinal quantity" fail to account for any of these issues.

Of logical concern, is the assigning of numerals and the classical measurement form are a comparator for truth. For example, the length of a jetty is discovered through measurement to be 7 metres. It is not the jetty that is related to the number 7. It is the *magnitude of the attribute length* of the jetty *relative to the metre unit* that is the number 7. That is, the number 7 is predicated on there being an attribute *length* that is reflected in the number of metre units that is 7 which describes the attribute length of my jetty (c.f. 8 p 16). This may be expressed as {7} [m]**:**{1} [m] or {7} [m]. If the same jetty was to be "assessed" from the point of view of "assigning numerals", or "ordinal quantity", the following may occur.

To descriptor categories for jetties that were qualitatively different and assumed ordered, numerals would be assigned to reflect the assumed order. This in itself is troublesome for such numerals are treated as numbers and the perceived qualitative difference then treated as a quantitative difference. However, say by using this procedure the jetty is assigned a "7". This produces a logical distinction between measurement as a discovery of ratios, {7}[m]/[m], and measurement as the assigning of a numeral, "7".

The discovery of ratios *is a commitment to truth***.** That is, "The jetty is {7} [m] long", *is a proposition that can be falsified*, a major tenet of science. By comparison, the assessed "7", based on descriptors of jetties to which each descriptor has numerals assigned, is not logically possible to falsify. The descriptor categories and the numerals by which the assignment was made may well be disputed, but this does not establish the truth or falsity of the statement; *the jetty has a quantitative measure and that measure is of a **magnitude*** {7} [m]. Measurement based on the ratio of two quantities, one of which is the standard of reference (the unit), is about truth and may be falsified. As the procedure of "ordinal quantity" assigns numerals (idiosyncratically), no "ordinal quantity" *can be falsified*. We therefore contend "ordinal quantity" is not scientific.

5. COUNTING OF ENTITIES

Berka [9 p 110] has observed there is a distinction between the counting of discrete objects and the counting of continuous units of quantity; the former is not expressive of measurement while the later is expressive of measurement. This arises because

> …counting determines the number of individual objects of a collection and, for this reason, refers primarily to discrete entities and individual objects and not, however-unlike measurement -to their continuous properties. If, with the help of the measurement unit, we decompose the measured magnitude into a series of individual elements of the same size, we might also secondarily apply counting to this specific sort of discrete objects and regard this operation as an auxiliary component of measurement procedures. However, one must not infer from this, as a general conclusion, that measurement is reducible to counting.

Here Berka has referred to units of continuous quantities as the "specific sort of discrete objects." This recognizes that a continuous quantity can only be *observed* as discrete; not that continuous quantity *is* discrete or is *made* discrete by the use of instrumentation to represent it. Therefore, his distinction between counting and measuring remains a distinction between counting of entities and the measuring of continuous quantity, wherein the former counting is an ontological requirement and in the later counting is an epistemological auxiliary.

The distinction between counting and measuring has been elaborated by Cooper & Humphry [12] in the boarder measurement context. In précis, Cooper & Humphry argue counting occurs with objects and events they call discrete entities and therefore deal with *natural* numbers. Measurement, in the classical sense defined above, is the formation of a *ratio* between two expressions of a quantity, one of which is called the standard of reference, the unit, and therefore deals with *real* numbers. Therefore, measurement does not occur by counting discrete entities. They conclude by observing that counting and measuring are *ontologically* distinct; counting involves *natural* numbers and measuring involves *real* numbers formed from the ratio of quantities.

In the SI [4], the counting of entities measurement form [pp 114-115] is a specified number of discrete entities exemplified by the unit mole, which is unit of "amount of substance". The mole is defined as proportional to the number of specified elementary entities in a sample, which is the same for all samples. The *unit* of amount of substance is called the mole, and is defined by specifying the mass of carbon 12 that constitutes one mole of carbon 12 atoms [4 pp 114–115]. Two notes are attached to its definition:

> 1. The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12; its symbol is "mol".
> 2. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles. It follows that the molar mass of carbon 12 is exactly 12 grams per mole, M(12C) = 12 g/mol.

By definition, the mole is a count of entities that must be specified (Fig. 3). Although the mole itself is divisible, the specified entities of its composition are not divisible, for they are entities. Unlike the classical measurement form where there is a continuous inverse relationship between the numeric and unit component, the composition of entities that form the mole will ultimately fail to be divisible. Therefore the mole is not a continuous quantity as the inverse relationship that characterises the classical measurement form does not hold. The mole will ultimately cease its inverse relationship between its numeric and unit component at one entity of its composition. The mole is therefore not a continuous unit and is best considered a count of discrete entities.

6..DISCUSSION

The differing expressions of quantity now exemplified in the international vocabulary of metrology, appear to have arisen from a change in the manner of defining quantity. Expressing quantity as {N}⟨Reference⟩, while not formal, has drawn attention to the formal requirement of {N}[U] where only a unit may be expressed in a quantity. There appears to be a sound reason to place a caveat on {N}[U]. Only {N}[U] can form the inverse relationship that makes permissible the formation of equivalences. No other ⟨Reference⟩ can achieve this with their respective {N}.

While ordering attributes and counting entities may be worthy means of establishing differences, they are not a means of establishing measurement. Measurement, the ratio of two quantities of the same attribute is to be preserved for quantification of continuous quantity and none other.

We are currently researching to document the history of measurement definitions of continuous quantity, ordering, and counting. This research will provide theoretical and empirical comparisons and contrasts of measurement based in these three contrasting forms. This research will be used to describe a framework within which the potential of ordering and counting to support inferences of continuous quantity may be experimentally evaluated.

7. CONCLUSION

The VIM [5] and SI [6] promulgation of three measurement forms belies the stated assumption there are no fundamental differences in the basic principles of measurement found in industry [5 p vii]. While the "…need to cover measurements in chemistry and laboratory medicine for the first time… and nominal properties" [5 p vii] may well explain why differing measurement forms are now defined in metrology, the consequence is a blurring of the distinction between continuous quantity, ordered relationships and counts of entities.

We suggest the VIM 4th Edition should explicitly define differing measurement forms and their limitations. The classical form, where in a ratio of two continuous quantities of the same type forms a unit [U], maintains the inverse relationship between the numerical component of a continuous quantity {N} and its unit [U]. No other measurement form does this and therefore cannot form systems of coherent units. Therefore, removing the relationship between the numerical component of a quantity and the "reference" ⟨R⟩, from the VIM [5] measurement definition, restores the unit [U] as the only permissible "standard of reference" [2 p 1] in a quantity.

Such a requirement may well be the litmus for distinguishing between measurement, ordering and counting and therefore differentiate measurement from these other forms of quantification.

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