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THE ROLE OF MATHEMATICAL MODELLING IN THE ANALYSIS AND DESIGN OF MEASUREMENT SYSTEMS

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Abstract − This paper briefly looks into the role and extent of mathematical modelling in the design and analysis of measurement systems, especially measurement sub-systems in the form of instruments and instrument elements. It also examines the role and use of mathematical modelling in the area of soft measurement (non-physical measurement). Based on a number of examples it demonstrates the use of modern modelling techniques in the design and analysis of sub-systems in measurement technology. In doing so, it will focus on the scope and importance of physical modelling at a sub-system level which ultimately contributes to modelling activities at a global systems level.

Keywords: mathematical models, measurement and instrumentation, design

1. INTRODUCTION

With the ever increasingly availability of computing technology significant progress is being made in the application of mathematical modelling techniques, especially numerical techniques for modelling, CAD, performance prediction and validation of measurement systems and sub-systems. Mathematical modelling is a key enabling tool and a means by which the functioning of systems and sub-systems can be predicted from a description of its physical principles, geometric features and material properties.

A model of a system is the description of the system in a formal language, such that relations between symbols in statements in the language imply and are implied by relations between the objects and attributes of the system and its components [1]. In other words, a model can be looked upon as the representation of a physical process and possesses the essential attributes of that physical process. Models are extensively used in design and by modelling it is meant the study of the mechanisms inside a system, and through using basic laws and relationships, a model is inferred. In terms of representation schemes, there could be linguistic, pictorial and mathematical models [2]. This paper focuses on mathematical models in which physical sub-systems are described as a set of mathematical relations (e.g. equations, discrete data, etc.) representing the physical processes, properties and behaviour of the sub-systems.

2. DEVELOPMENT OF INSTRUMENTATION AS MEASUREMENT SUB-SYSTEMS AND THEIR MATHEMATICAL MODELS

In modern instrumentation, information is generally carried by electrical signals. The analysis and design of these signals is generally performed by standard methods of signal theory. The signal and information processing components of modern instrumentation are generally standard components and are described by functional models. They are commonly implemented by standard information technology hardware and software and are analysed and designed by the general methods of information technology. However, the sensors and actuators of instrument systems are required to be analysed and designed in terms of their physical embodiment and function. Their analysis and design thus require special methods.

The ultimate objective of developing mathematical models and computer-aided methods of design of instrument sub-systems is the development of integrated computer environments in which the total design of these systems can be undertaken. Such an environment would be based on a modern model of the design process, based on the concepts of knowledge processing and problem solving. A model of the design process, based on a blackboard architecture, has been proposed and discussed in [3]. There continue to be developments reported in the literature of knowledge engineering, and artificial intelligence, problem solving and design. Models of the design process based on these advances have significant conceptual value for measurement science and education in the field. However, there is considerable gap between these models and practical application. With the availability of state of the art numerical modelling tools, both generic and application-specific, in all areas of measurement science and technology this gap between the reality and its mathematical representation is rapidly shrinking. However, significant challenges are still being posed by large and complex systems in both physical (e.g. biological) and non-physical (e.g. economic) areas.

3. EVLOLUTION OF MATHEMATICAL MODELLING OF INSTRUMENTS AS MEASUREMENT SUB-SYSTEMS

Design by computer modelling and simulation of measurement sub-systems is based on their appropriate representation by models, which can be handled by computers. The general concepts of instrument modelling have been considered in [2]. For example, in the case of sensors and actuators two kinds of models are used: power flow models which represent the functional relationship between physical inputs and physical outputs, and embodiment models that represent these relationships in terms of the geometry and material properties of the embodiment. Power flow models have seen substantial application. They are extensively used in the modelling and design of systems that consist of interacting components with diverse forms of energy. Mechatronics is an area in which such models are extensively and effectively used. In general instrumentation they provide a means of representing archetype models of sensors and actuators. They also are tools for modelling the interaction of sensors and the system being sensed, and that of actuators with the system upon which they act.

The main advances in these types of models have been in the development and application of computer software that automates model formulation and solution of system models. Significant advances have been made in languages and computer packages for power flow models. In particular they are bond graphs, Modelica, and the widely applied MATLAB [4-10]. The main requirement in the modelling of measurement sub-systems is for embodiment models. It is in this area where the principal advances have been made for sensors and actuators.

Qualitative, computer implemented, models have significant application potential in the description and in the design concept generation of complex instrumentation systems. Such models are making progress and they are beginning to provide useful insight find effective practical application [11].

3. MODELLING IN SOFT MEASUREMENT SYSTEMS

Soft measurement (or weakly-defined measurement) is a sub-set of widely-defined measurement which is not strongly defined It constitutes representation by symbols of properties of entities of the real world, based on an objective empirical process, but lacks some, or all, of the distinctive characteristics of strongly-defined measurement [12]. It lacks well-formed theories and involves predominantly non-physical sciences. Examples might be: psychology – intelligence, attitude, subjective perception of physical stimuli (colour, odour); sociology – class, status, segregation, poverty; economics, linguistics – measurement of phonological, lexical, grammatical and other attributes of natural language communication. In general, a soft system is any system for which there is not an adequately complete, empirically validated theory. This embraces much of the psychological, social and economic domains. Modelling and analysis by modelling of such soft systems pose significant challenges and it is made much more complicated by the fact that these systems involve human action, perception, feeling, decisions and the like. They can thus not be described by a system of invariant relations.

The main difficulties for modelling in soft measurement systems stem from the fact that soft measurement systems (a) are based on ill-defined concept of quality, (b) have significant uncertainty in the empirical relational system that it represents, (c) have symbolic relational system with limited relations defined on it, and (d) have no adequate theory relating the measurement to other measurements in the same domain.

In fact the whole area of widely-defined measurement, which is needed for the wide and diverse application of measurement, offers significant conceptual problems, compared with measurement in the physical sciences. These are in relation to: (i) experiments and observation – economic and biological systems, (ii) replicability –in relations with measurands: psychological and social sciences – humans, complex systems, (iii) utility – value judgement, quality and organisational performance measures, (iv) reliability, validity, generalizability – measurement in the social and psychological sciences, and (v) verifiability – economic and accountancy measurements, educational measurements.

An example could be measurement and natural language [12]. There is a strong relation between description by measurement, in the weakly defined sense, and description by natural language, which is in some of its functions a general form of symbolic representation. In measurement the meaning of a symbol is its reference. In natural language there are other views of meaning. Meaning may be related to an idea in the mind of the originator or receiver of an utterance or it may be considered as determined by conventional use. The function of measurement is informational but natural language has other functions such as aesthetic or phatic. A linguistic symbol, even in its informational function may, in addition to its denotation, convey other meanings, such as emotional colour. The essence of measurement is that it is an objective, empirical process. Description by natural language may be derived from empirical observation, though it is not necessarily so. It may be subjective, though it may have a high level of objectivity. Description by natural language has often a degree of ambiguity, and vagueness. Description by measurement has generally a high degree of precision. Finally description by measurement is generally more concise than description by natural language.

Despite these issues and difficulties there is now good progress in modelling soft measurement systems [13, 14].

4. MATHEMATICAL MODELLING FOR DESIGN AND ANALYSIS OF MEASUREMENT SUB-SYSTEMS – PHYSICAL MODELLING

Fig. 1 shows a simplified heuristic procedure for mathematical modelling and design of physical sub-systems in engineering and technology. It is a simple procedure yet it could be identified as the core activity in modelling and design of many sensors, actuators and devices, especially at a sub-system level.

The procedure comprises three main processes - the process of defining and setting up appropriate mathematical models (pre-processor), the process of solving the defining equations (solution-processor), and the process of calculating the necessary output parameters from the solution of these equations (post-processor). This procedure is repeated until a satisfactory design is obtained. In many areas of science and technology, sub-systems that are implemented using standard information technology, are modelled using the techniques of signals and information systems science. The relation between a sub-system's physical embodiment and its function can be represented by idealised lumped parameter models [15]. Such models are based on the relation between the input power flow to a sub-system, or system, and the output flow.

While idealised models like these are useful in the representation and analysis of concepts, detailed analysis and design require models, which relate the detailed geometric and material properties of the object modelled to its functional behaviour. Measurement sub-systems are characterized by complex geometries and distributed properties. The physical laws governing their behaviour are represented by partial differential equations, which are often non-linear and transcendental. There are a number ways of solving such realistic models – analytical, experimental and numerical. Analytical solutions are generally not feasible. They are normally applicable to problems with simple topology and linear materials. The experimental techniques, although applicable to many systems, are usually inaccurate, very time consuming and extremely expensive. In comparison, the numerical techniques based on, for example, the finite element (FEM) [16-18], boundary element (BEM) [19, 20] and hybrid finite element-boundary element (FEM-BEM) [21] methods can tackle a wide variety of electrical, mechanical, thermal, structural and coupled problems. With the availability of powerful and affordable desktop computers, these techniques have

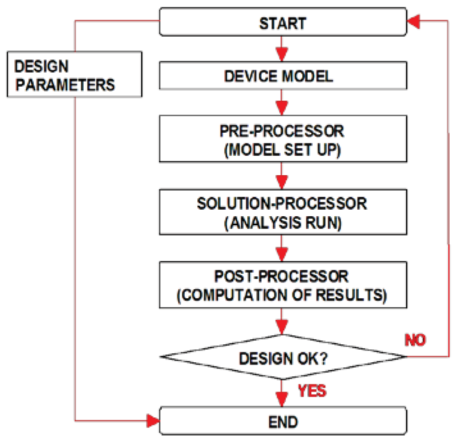

Fig. 1 Heuristic procedure for mathematical modelling and design of measurement sub-systems

revolutionised the formulation and solution of realistic models in the past few decades or so. This is especially true for the numerical finite element (FE) technique because of the relative ease of its computer implementation and the flexibility it provides in the definition of complex topology. FE models are fast, accurate and applicable to most physical systems. Finite element techniques have made possible the formulation and analysis of realistic models. The underlying principle of this method lies in the fact that the problem domain is divided ('discretised') into a number of triangular or rectangular elements of finite size ('finite elements') and the solution is sought at the vertices ('nodes') of these elements. The size, shape and the density of the 2D/3D 'mesh' thus obtained affect the accuracy of the numerical solution obtained by FEM. Today, significant progress has been made in this area. Various 2D/3D finite element models are being routinely used for computer aided design, investigation and performance modelling of instrument transducers and sensors.

5. PHYSICAL MODELLING OF MEASUREMEN SUB-SYSTEMS – GENERIC APPROACH

Here the examples are given for some of the most widely used electrical measurement sub-systems comprising capacitive sensors and electromagnetic (EM) actuators. They are used in a wide variety of diversified industrial applications ranging from measuring displacement to moving micro-mirrors in MEMS-based video projection systems. In general, they are based on the well-known capacitive technique in which the capacitance in a system of electrodes is changed owing to the redistribution of electric field caused by changes in the dielectric properties and/or geometric parameters in the system. In most cases, for modelling, design and performance evaluation of these sensor/actuator sub-systems, the core activities focus on the accurate computation and characterisation of 2D/3D electrostatic fields in complex geometry. This constitutes the main mathematical model of these subsystems, the solution of which involves the solution of the following Laplace's or Poisson's equation governing the field distribution in the 2D/3D problem domain Ω (x, y, z):

$$\nabla \cdot \varepsilon \nabla \Phi = 0 \tag{1}$$

$$\nabla \cdot \varepsilon \nabla \Phi = -\rho \tag{2}$$

where ρ is the charge density and $\varepsilon=\varepsilon(x, y, z)$ is the dielectric permittivity distribution in the problem domain.

Under appropriate boundary conditions, the solution of the above Laplace's (1) or Poisson's (2) equation gives the unknown electric potential distribution $\Phi=\Phi(x, y, z)$ in the problem domain Ω (x, y, z). In most cases it is assumed that the dielectric materials in Ω are linear, piece-wise homogeneous and isotropic. Following the solution of (1) or (2), the field intensity and flux density vectors $\boldsymbol{E}$ and $\boldsymbol{D}$, and other quantities like capacitance are calculated. The capacitance C is calculated either from the electric field energy E_e for a given potential difference V or from charge Q using the following relationships:

$$E_e = \frac{1}{2} C V^2 \tag{3}$$

$$C = \frac{Q}{V} \tag{4}$$

Here charge Q is calculated by integrating the flux density vector $\mathbf{D}$ over the appropriate electrode surfaces using the Gauss's law:

$$Q = \oint_s D_n ds = \oint_s D\cos\theta \, ds = \oint_s \mathbf{D} \cdot \mathbf{n} \, ds = \oint_s \mathbf{D} \cdot \mathbf{ds} \tag{5}$$

$$Q \approx \oint_s D \, ds \tag{6}$$

Although (1) and (2) are universal for sensors and actuators based on electrostatic principles, the specific formulation of their solution by FEM may vary depending on the material and geometric parameters, and the overall topology of the problem domain.

Electromagnetic (EM) actuators of wide variety of sizes, shapes, power outputs and technological realizations are used in many applications where discrete cyclic motions are required. Compared to other actuating mechanisms based on, for example, piezoelectric and hydraulic principles EM actuators are simpler, cheaper, easily repairable, robust, and easier to manufacture. EM actuators rarely operate in the steady state and various operational factors like start-stop duty, operating frequency, response time and damping have a significant influence on their design. The EM part of the system is represented by electric and magnetic circuits with self-inductance, resistance and reluctance which are subject to variations, in general, due to eddy currents, saturation conditions, motional electromotive force (e.m.f.), demagnetisation and hysteresis. The mechanical part is represented by friction, damping, elasticity and inertia as well as external forces. The nonlinear and transient EM, thermal, and motional problems that need to be solved in high speed actuators pose substantial challenges because of their high frequency of operation and the requirement, in many cases, for a continuous and fail-safe multibillion cycle operational regimes. In general, the mathematical model of an EM actuator can be adequately represented by the following four differential equations shown below: (7) an electrical circuit equation for the excitation coil and control circuitry, (8) a nonlinear magnetic field equation (Poisson's equation) for the flux, the change of which changes the EM energy storage in the system and produces the magnetic force, (9) a mechanical equation for this force, load (e.g. pneumatic force), friction, inertia, acceleration, speed and displacement, and (10) a nonlinear thermal diffusion equation for the conduction of heat produced by electrical power losses:

$$u(t) = iR + N\frac{d\Psi(i,z)}{dt} \tag{7}$$

$$\mathrm{curl}(\nu \,\mathrm{curl}\, \boldsymbol{A}) = \boldsymbol{J} - \sigma\frac{\partial \boldsymbol{A}}{\partial t} + \sigma \boldsymbol{V} \times (\mathrm{curl}\, \boldsymbol{A}) \tag{8}$$

$$F_m(i,z) = m\frac{d^2 z}{dt^2} + B\frac{dz}{dt} + Kz + F_e \tag{9}$$

$$\rho C\frac{\partial T}{\partial t} - \nabla \cdot \left[k(T)\nabla T\right] = q^B \tag{10}$$

In the above equations $u(t)$, i and $\Psi(i, z)$, and z are the applied voltage, coil current, flux linkage with the coil, and the displacement of moving part (plunger) respectively, R and N are the coil resistance and the number of turns in the coil, $\boldsymbol{J}$, $\boldsymbol{A}$, $\boldsymbol{V}$ are the coil current density, magnetic vector potential, and the plunger velocity; m, B, K, F_m and F_e are the mass of the plunger, viscous damping coefficient, spring constant, magnetic force and the load force respectively; and T, and q^B are the temperature and the internal rate of heat generated per unit volume respectively. The material parameters ν, σ, ρ, C and k denote the magnetic reluctivity (ν=1/μ, μ is the permeability), the electric conductivity, density, specific heat and the thermal conductivity respectively. In general, the above equations are nonlinear and inseparable. The current produced by (7) creates the magnetic field given by (8) and produces the magnetic force which causes the displacement, speed and acceleration of the actuator obtained from (9). The current also generates the heat and the resulting temperature distribution given by (10). There are two main approaches to the coupled solution of these equations: the direct coupled approach and the indirect coupled approach, neither of which alone is suitable to incorporate the whole array of factors which are expected to be encountered in the practical exploitation of high-speed EM actuators. The methodologies for modelling and design of EM actuators are normally based on modelling and computation of 2D/3D nonlinear magnetic field distribution using the numerical FE technique. This involves the steady-state and transient solutions of nonlinear Poisson's equation (8). The results are used for design optimisation and for investigating the effects of various geometric, material, EM and mechanical parameters on the output performance of actuators. As mentioned above the thermal modelling involves the development of 2D/3D thermal models and the FE solution of the heat transfer equation given by (10). The coupling of magnetic field and thermal equations may be realised either by indirect coupling or by direct coupling in which the equations are solved simultaneously. Following the solution of (8) above, the global quantities of interest such as inductance L and force F_m are calculated from the EM field energy E_m:

$$E_m = \frac{1}{2} L i^2 \Rightarrow L = \frac{2E_m}{i^2} \qquad (11)$$

$$F_m = \left. \frac{\partial E_m}{\partial x} \right|_{i=const.} \qquad (12)$$

Besides the above virtual work method of calculating force F_m, there exist two other methods for calculating the magnetic force – Maxwell stress tensor method and the magnetizing current method. This procedure is generic for all such EM instruments as measurement sub-systems.

6. CONCLUSIONS

Mathematical modelling has been playing and it will continue to play an important role in measurement theory and practice. This is particularly so for design, performance modelling and analyses of physical sub-systems in the form of instrumentation which measurement systems are often made up of. This has been the case for measurement systems since the inception by Helmholtz of the foundation of modern theory of measurement in 1887. Hence, today there has been significant development in the techniques and approaches to modelling in all areas of strongly-defined measurement (e.g. physical modelling). However, the same is not necessarily so for soft measurement systems where, as it is pointed out in Section 3 above significant challenges still remain despite considerable progress that has been made in recent years. There are objective empirical processes of assignment of numbers to properties of such systems so as to describe them. However, there are logical and philosophical problems underlying some of these processes. And, perhaps this indicates pathways for future development areas of measurement theory and practice.

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