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# Taking into account uncertainty of initial data when measuring quality on the basis of consensus relation

## ABSTRACT

In the paper an approach to introducing some number that defines maximal allowable uncertainty in assigning initial rankings and corresponding preference profile when measuring quality on the basis of consensus relation, that is median in form of linear order. The approach is borrowed from optimization theory and based on a concept of radius of stability. The main ideas of the paper are illustrated with individual problem examples. Direction of future investigations is outlined.

## 1. INTRODUCTION

Among different issues of quality measurement one is worth especially serious consideration. It is estimating its uncertainty. In fact, the initial rankings are shaped with errors. The question of how the errors influence on final consensus ranking is still open. It would be desirable to have some criterion which would set some restrictions onto feasible changes in initial rankings which would not lead to an incorrect solution. This would warrant a certain stability of the problem solution.

Let $A = \{a_1, a_2, ..., a_n\}$, $|A| = n$, be a set of objects. A group of $m$ experts ranks the objects in the order of preference. We have the relation set $\mathrm{A} = \{\alpha_1,...,\alpha_m\}$, where $|\mathrm{A}| = m$. Every *ranking* (preference relation $\succsim$) $\alpha = \{a_1 \succ a_2 \succ ... \sim a_s \sim a_t \succ ... \sim a_n\}$ includes $\succ$, a strict preference relation $\pi$, and $\sim$, an indifference relation $\nu$, so that $\alpha = \pi \cup \nu$. The relation $\pi$ is complete, transitive, irreflexive, and antisymmetric, and the relation $\nu$ is reflexive and symmetric. Such a relation $\alpha$ is generally called a *preorder*. The relation set A can be titled a *preference profile* for the given $m$ experts.

We can determine a single preference relation that would give an integrative characterization of the objects. The characterization can be called a *quality* of the objects. Let a space $\mathcal{D}_n$ be a set of all $n!$ strict (linear) order relations $\succ$ on $A$. Each linear order corresponds to one of permutations of first $n$ natural numbers $\mathbf{N_n}$. We consider one permutation $\beta \in \mathcal{D}_n$ of the objects $a_1, ..., a_n$ to represent the preference profile A, and we call it *consensus ranking*. It is desirable that $\beta$ would be nearest to any of rankings $\alpha_1, ..., \alpha_m$.

Finding the consensus ranking is possible by measuring a distance between pairs of rankings. This was first introduced by Kemeny [1] and discussed in many papers, see for instance [2, 3].

The ranking $\alpha$ can be represented by an $(n\times n)$ relation matrix $R = [r_{ij}]$ whose rows and columns are labeled by the states $a$ and

$$r_{ij} = \begin{cases} 1 & \text{if} \quad a_i \succ a_j \\ 0 & \text{if} \quad a_i \sim a_j \\ -1 & \text{if} \quad a_i \prec a_j \end{cases}.$$

The Kemeny distance function $d(\alpha_k,\alpha_l)$ between two rankings $\alpha_k$ and $\alpha_l$ is defined by formula

$$d(\alpha_k,\alpha_l)=\sum_{i<j}|r_{ij}^k-r_{ij}^l|. \qquad (1)$$

A distance between ranking α and profile A can then be defined as follows:

$$D(\alpha,\mathrm{A})=\sum_{k=1}^{m}d(\alpha,\alpha_k)=\sum_{i<j}\sum_{k=1}^{m}|r_{ij}^k-r_{ij}|=\sum_{i<j}\sum_{k=1}^{m}d_{ij}^k, \qquad (2)$$

where $d_{ij}^k=\begin{cases}0 & \text{if } a_i^k\succ a_j^k\\ 1 & \text{if } a_i^k\sim a_j^k\\ 2 & \text{if } a_i^k\prec a_j^k\end{cases}$.

We can now define an $(n\times n)$ *profile matrix* $P=[p_{ij}]$ which can represent the profile A in a compact form. In the profile matrix

$$p_{ij}=\sum_{k=1}^{m}d_{ij}^k, \quad i,j=1,\dots,n, \qquad (3)$$

In sense of the measure (2), the consensus linear ranking β is the closest relation (called also *median*) to the preference profile, i.e.

$$\beta=\arg\min_{\alpha}D(\alpha,\mathrm{A})=\arg\min_{\alpha}\sum_{i<j}p_{ij}. \qquad (4)$$

Every permutation of objects corresponds to transposition of the profile matrix rows and columns. Hence, the problem (4) means the determination of such a transposition of profile matrix $P$ rows and columns that the sum of elements of the upper triangle submatrix is minimal.

Thus, a solution of the problem (4) an optimal permutation β of n objects and corresponding minimal (optimal) total distance $D(\beta,\mathrm{A})$ from β to the profile A. It should be noticed that the problem may have more than one optimal solution.

An algorithm for determination of an optimal transposition of rows and columns of the profile matrix using the recursive branch and bound (B&B) technique has been described [4], and all our examples illustrating ideas of the paper are obtained with its help.

The following sections of the paper describes an attempt to take into account the fact that there always exists some uncertainty in matrix $P$ definition as object rankings may be erroneous by different reasons, both subjective and objective.

## 2. RADIUS OF STABILITY

This section is based on ideas from [5, 6]. Let $P\in\mathbf{R}_{\mathbf{n}^2}$ and in the space $\mathbf{R}_{\mathbf{n}^2}$ a norm is defined.

If $p_{11},\dots,p_{nn}$ are given with uncertainty not exceeding ε, and uncertainties in elements definition are independent, then by decision of the problem over $P$ we would like to believe that it is solved over any matrix $Q$ belonging to a sphere $S_\varepsilon(P)$ of a radius ε with center in $P$, that is

$$S_\varepsilon(P)=\{Q\mid Q\in\mathbf{R}_{\mathbf{n}^2},\ \|Q\|<\varepsilon\},$$

where $\|Q\|$ is the norm of $Q$.

This belief is based on an assumption that the solution is correct. However, in real situation

- $p_{ij}$ are always given under limited accuracy and
- in fact, there exists some particular $P'$, about which it is only known that $P'\in S_\varepsilon(P)$.

It should be noticed that $P'$ by no means always coincides with $P$. Indeed, for any $\varepsilon>0$ one can give an example (see section 3) of $P$ such that for some $P'\in S_\varepsilon(P)$ sets of optimal solutions on $P$

and on $P'$ are not intersected.

In this situation, having solved the problem on $P$, we will know nothing about the problem solution on really existing matrix $P'$. To resolve the challenge, it seems to be reasonable to have an algorithm that by $P$ gives out $\rho(P)$ (which will called *radius of stability*) such that the problem solution on $P$ is also a solution on any $P' \in S_\rho(P)$. Thus, $\rho(P)$ defines maximal admissible error in assigning numerical problem parameters. Then the problem can be considered to be correctly solved under the condition $\varepsilon < \rho(P)$, and, otherwise, the initial numerical data need to be defined more exactly. The problem solution without the revision would be meaningless.

Let $\mathcal{B}(P)$ is a set of indexes of optimal solutions of some problem on $P$. Denote the problem through $Z_P$. If $\mathcal{B}(P)$ includes all feasible solutions, then we suppose, by definition, that $\rho(P) = 0$. Otherwise, it can be easily shown that $\rho(P) > 0$.

Let as consider transition from $P$ to $P \oplus Q$, where $Q < \varepsilon$ and $\oplus$ is the operation of combining matrices $P$ and $Q$. If at any such transition no non-optimal on $P$ solution becomes optimal on $P \oplus Q$ (i.e. the gap between optimal and non-optimal solutions remains), then $P$ is $\varepsilon$-stable. This condition can be written as follows:

$$\mathcal{B}(P \oplus Q) \subseteq \mathcal{B}(P). \tag{5}$$

Now let $\rho(P) = \sup \varepsilon$ where supremum is taken by all $\varepsilon > 0$, for which $Z_P$ is stable.

Under $\rho(P) = 0$ the sphere $S_\rho(P)$ degenerates to a point. Otherwise, solution of all $Z_Q$ under $Q \in S_\rho(P)$ in view of (5) necessarily is among solutions of the problem $Z_P$.

## 3. EXAMPLES

In this section we illustrate the above statements with examples. All of the examples are produced for case $n = 4$. This value of $n$ allows to demonstrate meaningful instances still keeping satisfactory level of obviousness. The corresponding space of preorders is shown in Fig. 1. Every vertex in this diagram corresponds to one possible ordering (they are represented by indexes $i$ of objects $a_i$, and strict order symbols $\succ$ are omitted, i.e. $\{1234\} \equiv \{1\succ2\succ3\succ4\} \equiv \{a_1\succ a_2\succ a_3\succ a_4\}$ or $\{1\sim324\} \equiv \{1\sim3\succ2\succ4\} \equiv \{a_1\sim a_3\succ a_2\succ a_4\}$, and so on). Strict orderings in this space forms the solution space $\mathcal{D}_n$ of the problem (4). The space is closed but in order to have possibility to represent it on a plane we break some vertices into two ones with the same designation (a copy of each of the elements is white and elements of all corresponding pairs are connected with dashed line). Each edge has a number indicating the distance $d(a_k, a_l)$ between corresponding two elements. Central vertex of each hexagon in the space is connected to the element $\{1\sim2\sim3\sim4\}$ with the distance equal to 3.

*Example of stable solution.* Let a preference profile is given as follows: $\alpha_1$: 1342; $\alpha_2$: 3~421; $\alpha_3$: 4312; $\alpha_4$: 21~43; $\alpha_5$: 2413 (see Fig. 1). Profile matrix is

$$[p_{ij}^1] = \begin{bmatrix} 0 & 6 & 4 & 7 \\ 4 & 0 & 6 & 6 \\ 6 & 4 & 0 & 7 \\ 3 & 4 & 3 & 0 \end{bmatrix}.$$

The B&B algorithm gives the solution $\beta_1$: 4132, $D(\beta_1, A_1) = 24$.

Now we change objects order in the ranking $\alpha_4$. Let it be 12~43. In this case the profile matrix is

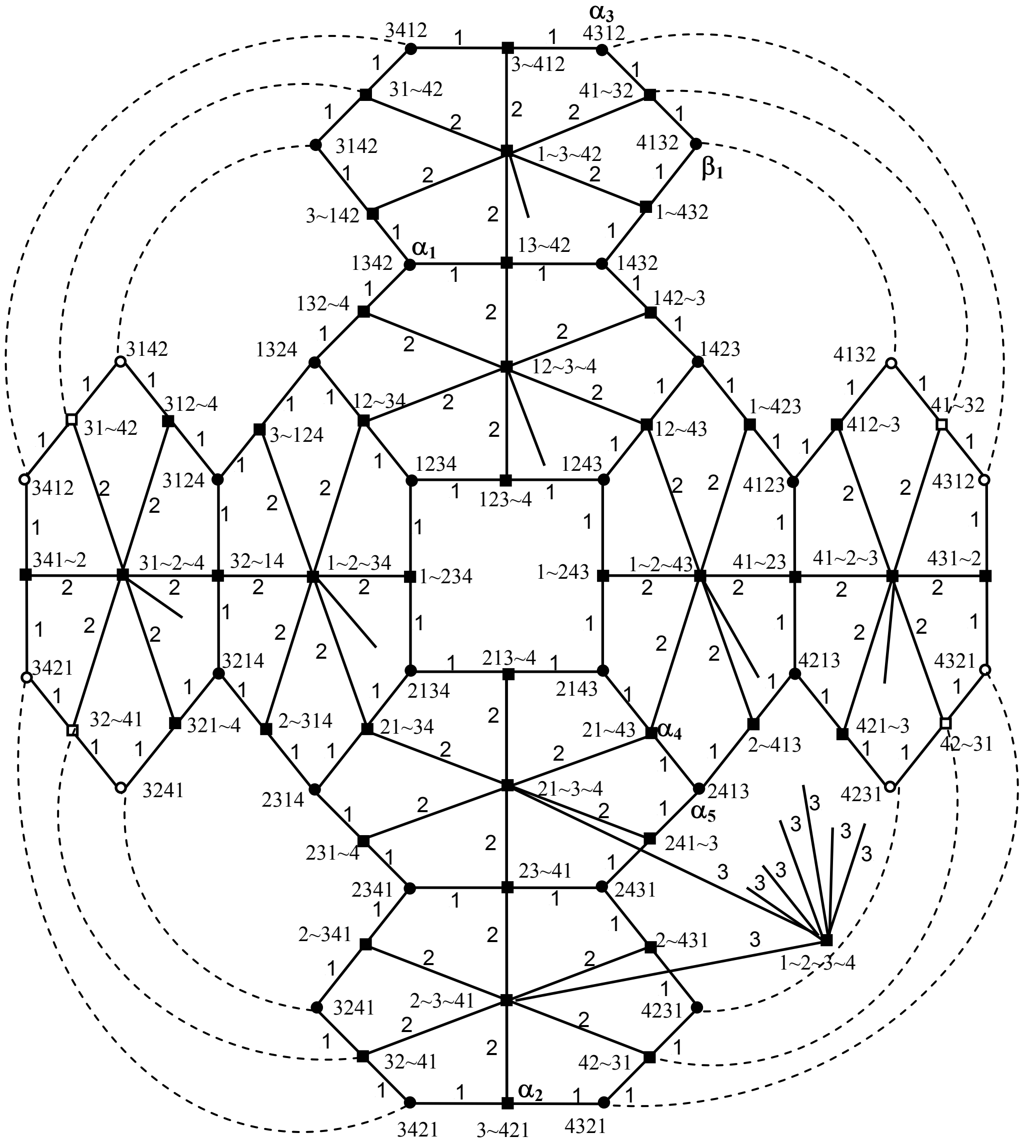


Fig. 1. Space of all possible preference relations for $n = 4$ including solution space of problem (4)

$$[p_{ij}^2] = \begin{bmatrix} 0 & 4 & 4 & 6 \\ 6 & 0 & 6 & 7 \\ 6 & 4 & 0 & 7 \\ 4 & 3 & 3 & 0 \end{bmatrix}.$$

The solution $\beta_2$ is the same: 4132, but $D(\beta_2, A_2) = 22$. It means that, in this case, the profile matrix allows to have some uncertainty in its elements.

*Examples of non-intersected solutions.* The solutions four individual problems are reduced in Table 1. Initial preference profile for all of them is $A_1$. An individual problem is obtained by small change in one of the five rankings; they are in the first column of Table 1.

It can be seen from the examples that sets of optimal solutions on $P$ and on $P_3, P_4, \ldots, P_7$ are not intersected.

Table 1. Examples of non-intersected solutions

| Ranking changed | $\alpha_1$: 3142 | $\alpha_2$: 1~324 | $\alpha_3$: 3412 | $\alpha_2$: 2~341 | $\alpha_1$: 3412 |
|---|---|---|---|---|---|
| Profile matrix | $\begin{bmatrix} 0 & 6 & 6 & 7 \\ 4 & 0 & 6 & 6 \\ 4 & 4 & 0 & 7 \\ 3 & 4 & 3 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 4 & 3 & 5 \\ 6 & 0 & 6 & 6 \\ 7 & 4 & 0 & 6 \\ 5 & 4 & 4 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 6 & 4 & 7 \\ 4 & 0 & 6 & 6 \\ 6 & 4 & 0 & 5 \\ 3 & 4 & 5 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 6 & 4 & 7 \\ 4 & 0 & 5 & 6 \\ 6 & 5 & 0 & 6 \\ 3 & 6 & 4 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 6 & 6 & 9 \\ 4 & 0 & 6 & 6 \\ 4 & 4 & 0 & 7 \\ 1 & 4 & 3 & 0 \end{bmatrix}$ |
| Optimal solution | $\beta_3$: 4321 $D(\beta_3,A_3)=22$ | $\beta_4$: 1432 $D(\beta_4,A_4)=24$ | $\beta_5$: 3421 $D(\beta_5,A_5)=26$ | $\beta_6$: 2413 $D(\beta_6,A_6)=24$ | $\beta_7$: 4312 $D(\beta_7,A_7)=22$ |

## 4. CONCLUSION

Let us fix A (and, consequently, $P$) and consider $X \in \mathbf{R}_{\mathbf{n}^2}$. Denote $D(\beta_i,A)$ through $D_i(P)$. For an arbitrary pair of solutions $\beta_i$ and $\beta_j$, $D_i(P) < D_j(P)$, one can state the following problem:

$$\|X\| \to \min, D_i(P \oplus X) \geq D_j(P \oplus X). \quad (6)$$

Consideration of problem (6) allows to obtain particular formula for radius of stability and specify an algorithm for its determination. This is the main purpose of future investigations of the authors on the question.

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