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Zero dynamics of time-varying linear systems

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Dedicated to the memory of Christopher I. Byrnes

Abstract

The Byrnes-Isidori form with respect to the relative degree is studied for time-varying linear multi-input, multi-output systems. It is clarified in which sense this form is a normal form. (A, B)-invariant time-varying subspaces are defined and the maximal (A, B)-invariant time-varying subspace included in the kernel of C is characterized. This is exploited to characterize the zero dynamics of the system. Finally, a high-gain derivative output feedback controller is introduced for the class of systems with higher relative degree and stable zero dynamics. All results are also new for time-invariant linear systems.

Keywords: Time-varying systems, linear systems, strict relative degree, zero dynamics, Byrnes-Isidori form, (A, B)-invariance, output feedback stablilization

1 Introduction

We study time-varying multi-input, multi-output linear systems of the form

$$\left.\begin{array}{rcl}\dot{x} &=& A(t)\,x + B(t)\,u(t)\\ y(t) &=& C(t)\,x(t),\end{array}\right\}\tag{1.1}$$

where $A \in \mathcal{C}^\ell(\mathcal{T}; \mathbb{R}^{n\times n})$, $B, C^\top \in \mathcal{C}^\ell(\mathcal{T}; \mathbb{R}^{n\times m})$ on some open set $\mathcal{T} \subseteq \mathbb{R}$ and $\ell \in \mathbb{N}$.
After we recall (see [IM07]) the definition of strict and uniform relative degree on $\mathcal{T}$ for systems (1.1) in Section 2, this concept is used to derive a Byrnes-Isidori form for the system. The latter is well-known; for time-invariant (nonlinear) systems see [Isi95, p. 137,220] and for time-varying systems see [IM07]. However, to the best of our knowledge, it has not yet been investigated in which sense the Byrnes-Isidori form is a normal form. This is clarified in Section 2. The Byrnes-Isidori form, certainly of theoretical interest in its own right, is also a main tool for the following sections: in Section 3, the vector space (or dynamical system) of zero dynamics (as defined in [IM07]) is characterized; the maximal (A, B)-invariant time-varying subspace included in the kernel of C (as defined in [Ilc89]) is characterized in Section 4; finally, in Section 5 a simple high-gain derivative output feedback controller is presented; this result generalizes well known results for time-invariant systems which have relative degree one (see e.g. [Ilc93]) and simplifies the controller [IM07, (4.2)] for time-varying systems which have higher relative degree. All main results of Sections 2-5 are also new for time-invariant linear systems.

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Nomenclature

$\mathbb{N}$, $\mathbb{N}_0$		the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{C}_+$, $\mathbb{C}_-$		the sets of complex numbers with positive and negative real parts, resp.
$\|x\|$	$=$	$\sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\|M\|$	$=$	$\max\{\|M\,x\| \mid x \in \mathbb{R}^m,\ \|x\| = 1\}$, induced matrix norm of $M \in \mathbb{R}^{n\times m}$
$\mathbf{Gl}_n(\mathbb{R})$		the general linear group of degree n, i.e. the set of invertible $n \times n$ matrices
$\mathcal{C}^\ell(\mathcal{T};\mathbb{R}^n)$		the set of ℓ-times continuously differentiable functions $f : \mathcal{T} \to \mathbb{R}^n$ on the open set $\mathcal{T} \subseteq \mathbb{R}$
$\mathcal{AC}(\mathcal{T};\mathbb{R}^n)$		the set of absolutely continuous functions $f : \mathcal{T} \to \mathbb{R}^n$ on the open set $\mathcal{T} \subseteq \mathbb{R}$, see [HP05, Def. A.3.12]
$\mathcal{PC}(\mathcal{T};\mathbb{R}^n)$		the set of piecewise continuous functions $f : \mathcal{T} \to \mathbb{R}^n$ on the open set $\mathcal{T} \subseteq \mathbb{R}$, i.e. f is left continuous and has only finitely many discontinuities on any compact subset of $\mathcal{T}$
$\mathcal{L}^\infty(\mathcal{T};\mathbb{R}^n)$		the set of essentially bounded functions $f : \mathcal{T} \to \mathbb{R}^n$ on the open set $\mathcal{T} \subseteq \mathbb{R}$

2 Relative degree and Byrnes-Isidori form

In this section we recall the concept of strict and uniform relative degree for time-varying linear systems. This allows to transform the system into Byrnes-Isidori form. The latter has been derived for time-varying linear systems in [IM07]. However, even for time-invariant systems, it was not clarified so far in which sense the Byrnes-Isidori form is a normal form. This will be done here. Moreover, we stress that the form allows to write the system as a decomposition into subsystems as depicted in Figure 1.

The following operator $\left(\frac{\mathrm{d}}{\mathrm{d}t} + A(t)_r\right)$, where the sub-script r in $A_r(C)$ indicates that A acts on C by multiplication from the right, has already been proved an advantageous notation for time-varying linear systems in [Fre71, IM07, Por69, Sil68].

Notation 2.1 (The operator $(\frac{\mathrm{d}}{\mathrm{d}t} + A(t)_r)^k$)**.**
Let $\ell \in \mathbb{N}_0$, $\mathcal{T} \subseteq \mathbb{R}$ an open set, $A \in \mathcal{C}^\ell(\mathcal{T};\mathbb{R}^{n\times n})$ and $C \in \mathcal{C}^\ell(\mathcal{T};\mathbb{R}^{m\times n})$. Set

$$\begin{aligned}
\forall\, t \in \mathcal{T} \ :\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A(t)_r\right)^0 \left(C(t)\right) &:= C(t)\,,\\
\forall\, t \in \mathcal{T} \ :\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A(t)_r\right) \left(C(t)\right) &:= \dot{C}(t) + C(t)A(t)\,,\\
\forall\, t \in \mathcal{T}\ \forall\, k \in \{1,\ldots,\ell\} \ :\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A(t)_r\right)^k \left(C(t)\right) &:= \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A(t)_r\right)\left(\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A(t)_r\right)^{k-1}\left(C(t)\right)\right).
\end{aligned}$$

The concept of relative degree is well known for time-invariant nonlinear SISO systems [Isi95, p. 137], time-invariant nonlinear MIMO systems [Isi95, p. 220], [LMS02], and for time-varying nonlinear MIMO systems [IM07, Def. 2.2]. When restricting the latter to time-varying linear systems, it is shown in [IM07, Thm. 2.7] that the definition of relative degree becomes as follows.

Definition 2.2 (Relative degree)**.**
Let $\rho, \ell \in \mathbb{N}$ with $\rho \le \ell$. Then the time-varying linear system (1.1) has *strict and uniform relative*

degree ρ (on $\mathcal{T}$) if, and only if,

$$\left.\begin{array}{rcl} \forall\, t\in\mathcal{T} \quad \forall\, k=0,\ldots,\rho-2 \;:\; \left(\frac{\mathrm{d}}{\mathrm{d}t}+A(t)_r\right)^k\left(C(t)\right)B(t) & = & 0_{m\times m} \\ \forall\; t\in\mathcal{T} \;:\; \left(\frac{\mathrm{d}}{\mathrm{d}t}+A(t)_r\right)^{\rho-1}\left(C(t)\right)B(t) & \in & \mathbf{Gl}_m(\mathbb{R})\,. \end{array}\right\} \tag{2.1}$$

Remark 2.3.

(i) If (1.1) is a time-invariant system, then it is straightforward to see that

$$\forall\; k\in\mathbb{N}_0 \;:\; \left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A(\cdot)_r\right)^k\left(C(\cdot)\right)B(\cdot)=CA^kB$$

and hence the conditions in (2.1) are equivalent to

$$CA^{\rho-1}B\in\mathbf{Gl}_m(\mathbb{R}) \qquad \text{and} \qquad \forall\; k=0,\ldots,\rho-2 \;:\; CA^kB=0\,.$$

(ii) The notion 'uniform' refers to the time set $\mathcal{T}$ in the sense that the conditions in (2.1) have to hold for all $t\in\mathcal{T}$.

(iii) The notion 'strict' is superfluous for single-input, single-output systems. However, for multivariable systems we may have $CA^kB=0$ for all $k=0,\ldots,\rho-2$ and $CA^{\rho-1}B\neq 0$ but $CA^{\rho-1}B\notin\mathbf{Gl}_m(\mathbb{R})$. In this case, one may introduce the concept of a vector relative degree: the vector $(\rho_1,\ldots,\rho_m)\in\mathbb{N}^m$ collects the smallest number of times ρ_j one has to differentiate $y_j(\cdot)$ so that the input occurs explicitly in $y_j^{(\rho_j)}(\cdot)$. This is not considered in the present note, for further details see [Isi95, Sec. 5.1] and [Mue09]. ⋄

As known for time-invariant systems, the relative degree ρ is the least number of times one has to differentiate the output $y(\cdot)$ so that the input occurs explicitly in $y^{(\rho)}(\cdot)$. That this also holds for time-varying systems is made explicit in the following proposition.

Proposition 2.4.
Let $\rho,\ell\in\mathbb{N}$ such that $\rho\leq\ell$ and consider a time-varying linear system (1.1) which has strict and uniform relative degree ρ. Then every solution $(x,u,y)\in\mathcal{AC}(\mathcal{T};\mathbb{R}^n)\times\mathcal{PC}(\mathcal{T};\mathbb{R}^m)\times\mathcal{AC}(\mathcal{T};\mathbb{R}^m)$ of (1.1) satisfies the following:

$$\forall\, j=0,\ldots,\rho-1 \;:\; y^{(j)} = \left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A_r\right)^j\left(C\right)x \qquad \text{a.e. on } \mathcal{T}\,, \tag{2.2}$$

$$y^{(\rho)} = \left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A_r\right)^\rho\left(C\right)x+\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A_r\right)^{\rho-1}\left(C\right)B\right]u \qquad \text{a.e. on } \mathcal{T}. \tag{2.3}$$

Proof: We show (2.2) by induction over $j=0,\ldots,\rho-1$. For $j=0$ the statement is clear. Suppose it holds for some $j\in\{0,\ldots,\rho-2\}$. Then, invoking Definition 2.2 we have, for almost all $t\in\mathcal{T}$,

$$\begin{aligned} y^{(j+1)}(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A(t)_r\right)^j\left(C(t)\right)x(t)\right] \\ &= \left[\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A(t)_r\right)^j\left(C(t)\right)\right]x(t)+\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A(t)_r\right)^j\left(C(t)\right)\left(A(t)x(t)+B(t)u(t)\right) \\ &= \left[\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A(t)_r\right)^j\left(C(t)\right)+\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A(t)_r\right)^j\left(C(t)\right)A(t)\right]x(t)+\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A(t)_r\right)^j\left(C(t)\right)B(t)u(t) \\ &\overset{(2.1)}{=} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A(t)_r\right)^{j+1}\left(C(t)\right)x(t)\,. \end{aligned}$$

Now we may derive that

$$\text{for almost all } t\in\mathcal{T} \;:\; y^{(\rho)}(t)=\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A(t)_r\right)^\rho\left(C(t)\right)x(t)+\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A(t)_r\right)^{\rho-1}\left(C(t)\right)B(t)u(t). \qquad \square$$

Note that we consider behaviours $(x, u, y) \in \mathcal{AC}(\mathcal{T}; \mathbb{R}^n) \times \mathcal{PC}(\mathcal{T}; \mathbb{R}^m) \times \mathcal{AC}(\mathcal{T}; \mathbb{R}^m)$. This is to some extend a matter of choice. We could choose the inputs more smooth, e.g. $u \in \mathcal{C}^k(\mathcal{T}; \mathbb{R}^m)$, then the solution x would become more smooth, $x \in \mathcal{C}^{k+1}(\mathcal{T}; \mathbb{R}^n)$.

The following theorem states a normal form for time-varying linear systems (1.1). The advantage of this form is that it expresses the dynamical properties of the system by allowing u only to effect the ρ^{th} derivative (ρ the relative degree) of the output and separating another part of the dynamics which is only influenced by y. See Figure 1.
The Byrnes-Isidori form has been derived for time-varying linear systems in [IM07, Th. 3.5]. Here we recall this form for later use and also clarify in which sense the form is a normal form. The latter is also new for time-invariant systems. We show that the entries $R_1, \ldots, R_\rho$, S and Γ in (2.5) are uniquely defined; whereas Q and P are unique modulo a coordinate transformation. For time-invariant systems, the Byrnes-Isidori form is implicitly contained in [Isi95, Sec. 5.1]. The form decouples the zero dynamics from the rest of the system, see Remark 5.2 and Figure 1.

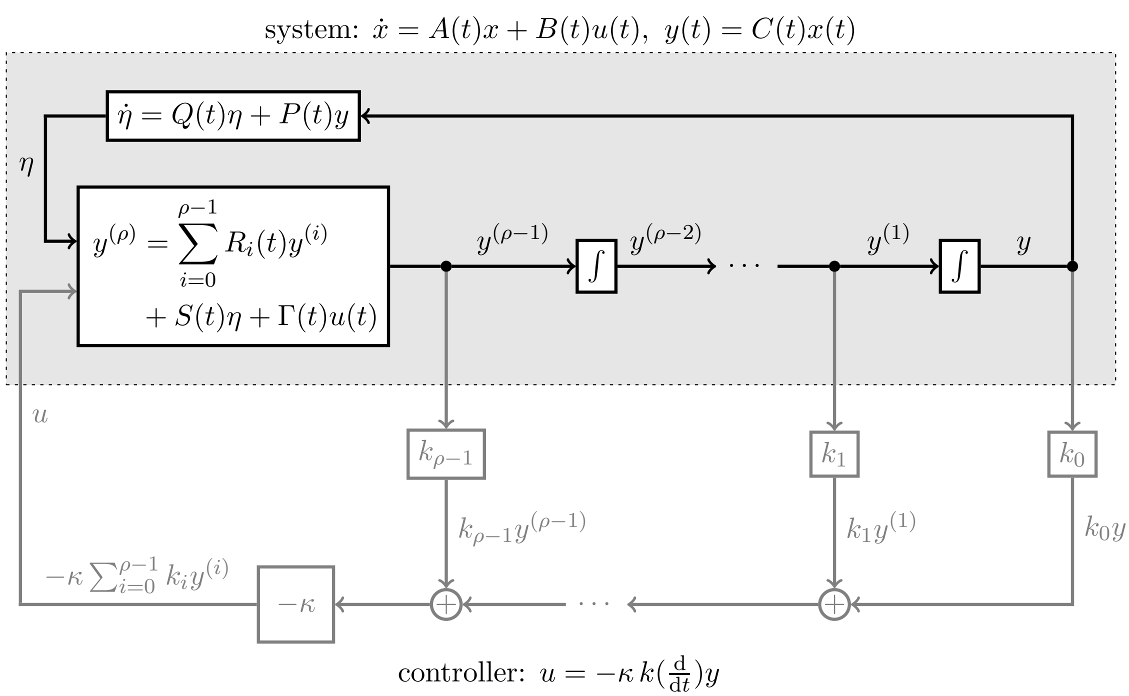

Figure 1: Byrnes-Isidori form and derivative output feedback controller

Theorem 2.5 (Byrnes-Isidori form)**.**
Let $\rho, \ell \in \mathbb{N}$ such that $\rho \leq \ell$ which has strict and uniform relative degree ρ. Then there exists a coordinate transformation $U \in \mathcal{C}^1(\mathcal{T}, \mathbf{Gl}_n(\mathbb{R}))$ such that

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} := \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_\rho(t) \\ \eta(t) \end{pmatrix} := \begin{pmatrix} y(t) \\ y^{(1)}(t) \\ \vdots \\ y^{(\rho-1)}(t) \\ \eta(t) \end{pmatrix} = U(t)\, x(t), \tag{2.4}$$

transforms (1.1) into *Byrnes-Isidori form*

$$\left.\begin{array}{rcl}\dot\xi(t) &=& \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m \\ R_1(t) & R_2(t) & \cdots & R_{\rho-1}(t) & R_\rho(t) \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ S(t) \end{bmatrix} \eta(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma(t) \end{bmatrix} u(t), \\ \dot\eta(t) &=& \begin{bmatrix} P(t) & 0 & \cdots & 0 & 0 \end{bmatrix} \xi(t) + Q(t)\,\eta(t) \\ y(t) &=& \begin{bmatrix} I_m & 0 & \cdots & 0 & | \; 0 \end{bmatrix} \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \end{array}\right\} \tag{2.5}$$

with initial condition

$$\begin{pmatrix} \xi(0) \\ \eta(0) \end{pmatrix} = \begin{pmatrix} \xi^0 \\ \eta^0 \end{pmatrix} = \begin{pmatrix} y(0) \\ \vdots \\ y^{(\rho-1)}(0) \\ \eta^0 \end{pmatrix} = U(0)\, x^0; \tag{2.6}$$

and

(i) $\Gamma = \left(\frac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho-1}(C)B \in \mathcal{C}^1(\mathcal{T}; \mathbf{Gl}_m(\mathbb{R}))$,

(ii) $[R_1, \ldots, R_\rho, S] = \left(\frac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho}(C)U^{-1} \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{m\times n})$,

(iii) $(P, Q) \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{(n-\rho m)\times m}) \times \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{(n-\rho m)\times(n-\rho m)})$ is unique up to $(\tilde Y^{-1}P, \tilde Y^{-1}Q\tilde Y)$ for some $\tilde Y \in \mathcal{C}^1(\mathcal{T}; \mathbf{Gl}_{n-\rho m}(\mathbb{R}))$.

(iv) A possible transformation is given by $U = \begin{bmatrix} \mathcal{C} \\ N \end{bmatrix}$, where

$$\begin{aligned} \mathcal{C} &:= \begin{bmatrix} C \\ \left(\frac{\mathrm{d}}{\mathrm{d}t} + A_r\right)(C) \\ \vdots \\ \left(\frac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho-1}(C) \end{bmatrix} \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{\rho m\times n}) \\ \mathcal{B} &:= \left[B, \left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\right)(B), \ldots, \left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\right)^{\rho-1}(B)\right] \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n\times\rho m}) \\ N &:= (V^\top V)^{-1}V^\top \left[I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right] \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{(n-\rho m)\times n}). \end{aligned}$$

and $V \in \mathcal{L}^\infty(\mathcal{T}; \mathbb{R}^{n\times(n-\rho m)}) \cap \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n\times(n-\rho m)})$ such that $(V^\top V)^{-1}V^\top \in \mathcal{L}^\infty(\mathcal{T}; \mathbb{R}^{(n-\rho m)\times n})$ and

$$\forall\, t \in \mathcal{T} \;:\; \operatorname{im} V(t) = \ker \mathcal{C}(t) \quad \text{and} \quad \operatorname{rk} V(t)^\top V(t) = n - \rho m.$$

Proof: The Byrnes-Isidori form (2.5) and the statements (i) and (ii) are proved in [IM07, Thm. 3.5]. The existence of V in statement (iv) is shown in [IM07, Rem. 3.4]. It is straightforward to show that $U^{-1} = \left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, V\right]$. So it remains to prove uniqueness in statement (iii). Note that Γ in (i) is unique, it depends on (A, B, C) only.
Let

$$(\hat A, \hat B, \hat C) := \left((UA + \dot U)U^{-1}, UB, CU^{-1}\right) \tag{2.7}$$

for U as in (iv). Then

$$\hat{A} = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix}, \quad \hat{C} = [I_m, 0, \dots, 0] \tag{2.8}$$

holds (see [IM07, Thm. 3.5]) for (P, Q) given by [IM07, (3.9)-(3.12)].
Consider next

$$(\tilde{A}\,, \tilde{B}\,, \tilde{C}) = \big((WA + \dot{W})W^{-1}, \, WB\,, \, CW^{-1}\big) \tag{2.9}$$

for any $W \in \mathcal{C}^1(\mathcal{T}; \mathbf{Gl}_n(\mathbb{R}))$ such that

$$\tilde{A} = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ \tilde{R}_1 & \tilde{R}_2 & \cdots & \tilde{R}_{\rho-1} & \tilde{R}_\rho & \tilde{S} \\ \tilde{P} & 0 & \cdots & 0 & 0 & \tilde{Q} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix}, \quad \tilde{C} = [I_m, 0, \dots, 0]\,. \tag{2.10}$$

We show that

$$\left. \begin{array}{lll} \tilde{S} = S, & \tilde{R}_i = R_i, & \forall\, i = 1, \dots, \rho, \\ \tilde{P} = \tilde{Y} P, & \tilde{Q} = \tilde{Y} Q\, \tilde{Y}^{-1} & \text{for some } \tilde{Y} \in \mathcal{C}^1(\mathcal{T}; \mathbf{Gl}_{n-\rho m}(\mathbb{R})). \end{array} \right\} \tag{2.11}$$

Set

$$WU^{-1} =: Y = \begin{bmatrix} Y^1 \\ \vdots \\ Y^{\rho+1} \end{bmatrix} = [Y_1, \dots, Y_{\rho+1}] \tag{2.12}$$

for $Y^i, (Y_i)^\top \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{m\times n})$, $i = 1, \dots, \rho$, and $Y^{\rho+1}, (Y_{\rho+1})^\top \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{(n-\rho m)\times n})$. Then (2.7) and (2.9) together with $\frac{\mathrm{d}}{\mathrm{d}t}(U^{-1}) = -U^{-1}\dot{U}U^{-1}$ yield

$$\begin{aligned} (Y\hat{A} + \dot{Y})Y^{-1} &= \big(WAU^{-1} + WU^{-1}\dot{U}U^{-1} + \dot{W}U^{-1} + W\tfrac{\mathrm{d}}{\mathrm{d}t}(U^{-1})\big)UW^{-1} \\ &= WAW^{-1} + \dot{W}W^{-1} = \tilde{A}\,. \end{aligned}$$

Thus

$$(Y\hat{A} + \dot{Y})Y^{-1} = \tilde{A}, \tag{2.13}$$

$$Y\hat{B} = \tilde{B}, \tag{2.14}$$

$$\hat{C} = \tilde{C}Y. \tag{2.15}$$

This gives

$$Y^1 \overset{(2.15)}{=} [I_m, 0, \dots, 0] \qquad \text{and} \qquad Y_\rho \overset{(2.14)}{=} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_m \\ 0 \end{bmatrix}, \tag{2.16}$$

and we proceed

$$\begin{array}{rcllcl}
[0,I_m,0,\dots,0] & \overset{(2.8)}{=} & Y^1\hat{A}+\frac{\mathrm{d}}{\mathrm{d}t}Y^1 & \overset{(2.16)}{=} Y^1(Y\hat{A}+\dot{Y}) & \overset{(2.13)}{=} Y^1\tilde{A}Y & \underset{(2.10)}{\overset{(2.12)}{=}} Y^2\\
[0,0,I_m,0,\dots,0] & \overset{(2.8)}{=} & Y^2\hat{A}+\frac{\mathrm{d}}{\mathrm{d}t}Y^2 & = Y^2(Y\hat{A}+\dot{Y}) & \overset{(2.13)}{=} Y^2\tilde{A}Y & \underset{(2.10)}{\overset{(2.12)}{=}} Y^3\\
 & \vdots & & & & \\
[0,\dots,0,I_m,0] & \overset{(2.8)}{=} & Y^{\rho-1}\hat{A}+\frac{\mathrm{d}}{\mathrm{d}t}Y^{\rho-1} & = Y^{\rho-1}(Y\hat{A}+\dot{Y}) & \overset{(2.13)}{=} Y^{\rho-1}\tilde{A}Y & \underset{(2.10)}{\overset{(2.12)}{=}} Y^{\rho}.
\end{array}$$

Therefore, Y is of the form

$$Y=\begin{bmatrix} I_m & 0 & \dots & 0 & 0\\ 0 & I_m & & & 0\\ \vdots & \ddots & \ddots & & \vdots\\ 0 & \dots & 0 & I_m & 0\\ Y_{\rho+1,1} & \dots & Y_{\rho+1,\rho-1} & 0 & \tilde{Y}\end{bmatrix} \qquad \text{for some } \tilde{Y}\in\mathcal{C}^1(\mathcal{T};\mathbf{Gl}_{n-\rho m}(\mathbb{R})).$$

Now consider the last $n-\rho m$ rows in $Y\hat{A}+\dot{Y}=\tilde{A}Y$, which read

$$\begin{aligned}&[\tilde{Y}P+\tfrac{\mathrm{d}}{\mathrm{d}t}Y_{\rho+1,1},Y_{\rho+1,1}+\tfrac{\mathrm{d}}{\mathrm{d}t}Y_{\rho+1,2},\dots,Y_{\rho+1,\rho-2}+\tfrac{\mathrm{d}}{\mathrm{d}t}Y_{\rho+1,\rho-1},Y_{\rho+1,\rho-1},\tilde{Y}Q]\\ &\qquad\qquad = [\tilde{P}+\tilde{Q}Y_{\rho+1,1},\tilde{Q}Y_{\rho+1,2},\dots,\tilde{Q}Y_{\rho+1,\rho-1},0,\tilde{Q}\tilde{Y}],\end{aligned}$$

and comparing successively the ρ^{th} block, ..., 1^{st} block yields $Y_{\rho+1,\rho-1}=0,\dots,Y_{\rho+1,1}=0$. Finally, $Y=\operatorname{diag}\{I_m,\dots,I_m,\tilde{Y}\}$ applied to (2.13)-(2.15) gives (2.11). □

Remark 2.6.

(i) In the time-invariant case $(A,B,C)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{m\times n}$, all matrices in Theorem 2.5(i)–(iv) are constant matrices over $\mathbb{R}$.

(ii) The uniqueness of (P,Q) in Theorem 2.5 up to $(\tilde{Y}^{-1}P,\tilde{Y}^{-1}Q\tilde{Y})$ for some $\tilde{Y}\in\mathcal{C}^1(\mathcal{T};\mathbf{Gl}_{n-\rho m}(\mathbb{R}))$ corresponds to the freedom in choosing V such that (iv) holds. If V is replaced by $V\tilde{Y}$ for arbitrary $\tilde{Y}\in\mathcal{C}^1(\mathcal{T};\mathbf{Gl}_{n-\rho m}(\mathbb{R}))$, then an easy calculation shows that N becomes $\tilde{Y}^{-1}N$ and therefore P and Q become $\tilde{Y}^{-1}P$ and $\tilde{Y}^{-1}Q\tilde{Y}$, resp.

(iii) A formula for (P,Q) is given in [IM07, (3.9)-(3.12)] but unfortunately with a typo in formula [IM07, (3.10)] for Q; the correct formula in terms of the notation from Theorem 2.5(iv) is

$$Q=-(V^\top V)^{-1}V^\top\left[(\tfrac{\mathrm{d}}{\mathrm{d}t}-A)V+B\Gamma^{-1}(\tfrac{\mathrm{d}}{\mathrm{d}t}+A_r)^{\rho}(C)V\right]. \tag{2.17}$$

For its proof see the proof of [IM07, Thm. 3.5].

(iv) As a technicality, quite useful in the following, we collect the fact that Theorem 2.5(iv) gives

$$\forall\, t\in\mathcal{T}:\ \mathcal{C}(t)U(t)^{-1}=\left[I_{\rho m},0_{\rho m\times(n-\rho m)}\right]\quad\text{and}\quad \ker(\mathcal{C}(t)U(t)^{-1})=\operatorname{im}\begin{bmatrix}0\\ \vdots\\ 0\\ I_{n-\rho m}\end{bmatrix}. \tag{2.18}$$

3 Zero dynamics

In this section we investigate the concept of zero dynamics. The zero dynamics of a system are, loosely speaking, those dynamics which are not visible at the output. We will use them to characterize (A, B)-invariant subspaces and to design an output feedback controller.

Definition 3.1 (Zero dynamics)**.**
Let $\ell \in \mathbb{N}_0$ and $\mathcal{T} \subseteq \mathbb{R}$ be an open set. The *zero dynamics* of system (1.1) on $\mathcal{T}$ is defined as the set of trajectories

$$\mathcal{ZD}(A,B,C) := \left\{ (x,u,y) \in \mathcal{AC}(\mathcal{T};\mathbb{R}^n) \times \mathcal{PC}(\mathcal{T};\mathbb{R}^m) \times \mathcal{AC}(\mathcal{T};\mathbb{R}^m) \;\middle|\; \begin{array}{l} (x,u,y) \text{ solves (1.1) on} \\ \mathcal{T} \text{ and } y \equiv 0 \end{array} \right\}.$$

◇

By linearity of (1.1), the set $\mathcal{ZD}(A,B,C)$ is a real vector space and, roughly speaking and made more precise in the following remark, it is also a dynamical system. We are indebted to our colleague Fabian Wirth (Würzburg) for pointing out this observation to us.

Remark 3.2 (Zero dynamics are a dynamical system)**.**
Let the *state transition map* of (1.1) be denoted by the unique solution

$$\varphi(\cdot\,; t_0, x^0, u(\cdot)) : \mathcal{T} \to \mathbb{R}^n$$

of the initial value problem (1.1), $x(t_0) = x^0$ for any $(t_0, x^0, u(\cdot)) \in \mathcal{T} \times \mathbb{R}^n \times \mathcal{PC}(\mathcal{T};\mathbb{R}^m)$, and the *output map* by

$$\eta(\cdot\,; t_0, x^0, u(\cdot)) : \mathcal{T} \to \mathbb{R}^m, \quad t \mapsto \eta(t; t_0, x^0, u(\cdot)) = C(t)\,\varphi(t; t_0, x^0, u(\cdot))\,.$$

Now it is readily verified that the axioms of a dynamical system as defined, e.g. in [HP05, Def. 2.1.1], are satisfied for the structure $(\mathcal{T}; \mathbb{R}^m, \mathcal{PC}(\mathcal{T};\mathbb{R}^m), \mathbb{R}^n, \mathbb{R}^m, \varphi, \eta)$ and the set

$$\mathcal{D}_\varphi := \left\{ \ (t, t_0, x^0, u(\cdot)) \in \mathcal{T}^2 \times \mathbb{R}^n \times \mathcal{PC}(\mathcal{T};\mathbb{R}^m) \ \middle|\ \forall\, t \in \mathcal{T} : \ \eta(t; t_0, x^0, u(\cdot)) = 0 \ \right\}.$$

Next we show that the vector space of orbits induced by $\mathcal{D}_\varphi$ fixed at $t_0 \in \mathcal{T}$ in the sense

$$\mathcal{D}_{\mathrm{orb},t_0} := \left\{ \ \varphi(\cdot\,; t_0, x^0, u(\cdot)) : \mathcal{T} \to \mathbb{R}^n \ \middle|\ (t_0, t_0, x^0, u(\cdot)) \in \mathcal{D}_\varphi \ \right\}$$

is isomorphic to $\mathcal{ZD}(A,B,C)$. This is a consequence of uniqueness and global existence of the solution of the initial value problem (1.1), $x(t_0) = x^0$ which gives that the map

$$\xi_{t_0} \colon \mathcal{D}_{\mathrm{orb},t_0} \to \mathcal{ZD}(A,B,C)\,, \quad \varphi(\cdot\,; t_0, x^0, u(\cdot)) \mapsto \big(\varphi(\cdot\,; t_0, x^0, u(\cdot)), u(\cdot), \eta(\cdot\,; t_0, x^0, u(\cdot))\big)$$

is a vector space isomorphism. ◇

The next corollary is an immediate consequence of Proposition 2.4.

Corollary 3.3 (Characterization of zero dynamics)**.**
Let $\rho, \ell \in \mathbb{N}$ such that $\rho \leq \ell$ and consider a time-varying linear system (1.1) which has strict and uniform relative degree ρ. Then $(x,u,y) \in \mathcal{ZD}(A,B,C)$ if, and only if, the following three conditions are satisfied on $\mathcal{T}$:

(i) $y(\cdot) = 0\,,$

(ii) $u(\cdot) = -\left[\left(\frac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho-1}(C)B\right]^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho}(C)\,x(\cdot)\,,$

(iii) $x(\cdot)$ solves $\dot{x} = \left[A - B\left[\left(\frac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho-1}(C)B\right]^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho}(C)\right]x\,.$ ◇

Remark 3.4 (Characterization of zero dynamics for time-invariant systems)**.**
If (1.1) is time-invariant, then Corollary 3.3(i)-(iii) reads

(i) $y(\cdot) \equiv 0$, (ii) $u(\cdot) = -(CA^{\rho-1}B)^{-1}\,CA^{\rho}\,x(\cdot)$, (iii) $x(\cdot)$ solves $\dot{x} = [A - B(CA^{\rho-1}B)^{-1}CA^{\rho}]\,x\,.$

◇

4 (A, B)-invariant subspaces

In this section we show that the function space of the zero dynamics of a system (1.1) which has some relative degree on $\mathcal{T}$ is isomorphic to the supremal (in fact maximal) (A, B)-invariant time-varying subspace included in $\ker C$ at some initial time. First we have to introduce some notations.

Let, for any open set $\mathcal{T} \subseteq \mathbb{R}$,

$$\mathcal{W}_n(\mathcal{T}) := \left\{ \mathcal{V} = \big(\mathcal{V}(t)\big)_{t\in\mathcal{T}} \;\middle|\; \exists\, k \in \mathbb{N}\ \exists\, V \in \mathcal{C}^1(\mathcal{T};\mathbb{R}^{n\times k})\ \forall\, t \in \mathcal{T}:\ \mathcal{V}(t) = \operatorname{im} V(t) \right\}$$

denote the set of all time-varying subspaces $\mathcal{V}$, generated by some continuously differentiable V.

Definition 4.1 ((A, B)-invariance)**.**
Let $\mathcal{T} \subseteq \mathbb{R}$ be an open set, $(A, B) \in \mathcal{C}^1(\mathcal{T};\mathbb{R}^{n\times n}) \times \mathcal{C}^1(\mathcal{T};\mathbb{R}^{n\times m})$ and $\mathcal{V} \in \mathcal{W}_n(\mathcal{T})$ be generated by $V \in \mathcal{C}^1(\mathcal{T};\mathbb{R}^{n\times k})$ for some $k \in \mathbb{N}$. Then $\mathcal{V}$ is called (A, B)*-invariant* if, and only if, there exists a discrete set $\mathcal{I} \subseteq \mathcal{T}$ and $N \in \mathcal{C}^1(\mathcal{T}\setminus\mathcal{I};\mathbb{R}^{k\times k})$, $M \in \mathcal{C}^1(\mathcal{T}\setminus\mathcal{I};\mathbb{R}^{m\times k})$, such that

$$\forall\, t \in \mathcal{T}\setminus\mathcal{I}:\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A(t)\right)(V(t)) = V(t)N(t) + B(t)M(t). \tag{4.1}$$

◇

The concept of (A, B)-invariance has been introduced by [BM69, WM70] and generalized in various directions, see the excellent textbook [Won85]. For time-varying linear systems see [Ilc89, Def. 4.1].

Remark 4.2.
If (A, B) has real analytic coefficients, then, as shown in [Ilc89, Rem. 4.4], Definition 4.1 is equivalent to

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A(t)\right)(V(t)) \subset \operatorname{im} V(t) + \operatorname{im} B(t) \qquad \text{for almost all } t \in \mathcal{T}.$$

For time-invariant (A, B), 'almost all' is redundant. The following example shows that 'almost all' is not redundant for time-varying systems. Set $\mathcal{T} = \mathbb{R}$ and, for all $t \in \mathbb{R}$,

$$V(t) = \begin{bmatrix}0\\t\end{bmatrix}, \qquad A = 0_{2\times 2}, \qquad B(t) = \begin{bmatrix}1\\0\end{bmatrix}.$$

Then

$$\forall\, t \in \mathbb{R}\setminus\{0\}:\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A(t)\right)(V(t)) = \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\t\end{bmatrix}\cdot t^{-1} + \begin{bmatrix}1\\0\end{bmatrix}\cdot 0\,.$$

◇

Obviously, the elements of $\mathcal{W}_n(\mathcal{T})$ are partially ordered by the relation $\subseteq^*$, defined by

$$\mathcal{V} \subseteq^* \hat{\mathcal{V}} \;:\Longleftrightarrow\; \forall\, t \in \mathcal{T}:\; \mathcal{V}(t) \subseteq \hat{\mathcal{V}}(t),$$

for any $\mathcal{V}, \hat{\mathcal{V}} \in \mathcal{W}_n(\mathcal{T})$. Hence, for any

$$(A, B, C) \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n\times n}) \times \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n\times m}) \times \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{m\times n}),$$

the existence of

$$\mathcal{V}^*(A, B; \ker C) := \sup\{\, \mathcal{V} \in \mathcal{W}_n(\mathcal{T}) \mid \mathcal{V} \text{ is } (A,B)\text{-invariant and } \mathcal{V}(t) \subseteq \ker C(t) \text{ for all } t \in \mathcal{T} \,\}$$

follows since the elements of $\mathcal{W}_n(\mathcal{T})$ are finite dimensional subspaces for all $t \in \mathcal{T}$, and the sum of (A, B)-invariant time-varying subspaces included in $\ker C$ is (A, B)-invariant and included in $\ker C$. Actually, the supremum in the definition of $\mathcal{V}^*(A, B; \ker C)$ is a maximum.

The following proposition shows that $\mathcal{V}^*(A, B; \ker C)$ has a simple representation if (1.1) has a strict and uniform relative degree. Note also that (4.2) becomes even simpler after the coordinate transformation (2.4) introduced in Theorem 2.5, see (4.5).

Proposition 4.3 (Representation of $\mathcal{V}^*(A, B; \ker C)$)**.**
Let $\rho, \ell \in \mathbb{N}$ with $\rho \le \ell$ and suppose system (1.1) has strict and uniform relative degree ρ on $\mathcal{T}$. Then, for $\mathcal{C}$ as in Theorem 2.5(iv),

$$\forall\, t \in \mathcal{T}\;:\; \mathcal{V}^*(A, B; \ker C)(t) = \ker \mathcal{C}(t). \tag{4.2}$$

Proof: Let U be as in Theorem 2.5(iv) and $(\hat{A}, \hat{B}, \hat{C})$ as in (2.7). At several occasions, we will make use of the fact that

$$\forall\, t \in \mathcal{T}:\; U(t)\, \ker \mathcal{C}(t) = \ker\big(\mathcal{C}(t) U(t)^{-1}\big). \tag{4.3}$$

Step 1: We first show

$$\forall\, t \in \mathcal{T}\;:\; \mathcal{V}^*(A, B; \ker C)(t) = U(t)^{-1}\, \mathcal{V}^*(\hat{A}, \hat{B}; \ker \hat{C})(t)$$

which is equivalent to

$$\forall\, t \in \mathcal{T}:\; U(t)\, \mathcal{V}^*(A, B; \ker C)(t) = \mathcal{V}^*(\hat{A}, \hat{B}; \ker \hat{C})(t). \tag{4.4}$$

Step 1a: Let $\mathcal{V} \in \mathcal{W}_n(\mathcal{T})$ be any (A, B)-invariant time-varying subspace included in $\ker C$ and generated by $V \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n\times k})$, $k \in \mathbb{N}$. Then (4.1) holds for some $N \in \mathcal{C}^1(\mathcal{T}\setminus\mathcal{I}; \mathbb{R}^{k\times k})$, $M \in \mathcal{C}^1(\mathcal{T}\setminus\mathcal{I}; \mathbb{R}^{m\times k})$, $\mathcal{I}$ a discrete set, and hence, for all $t \in \mathcal{T}\setminus\mathcal{I}$, we have

$$\big(\tfrac{\mathrm{d}}{\mathrm{d}t} - \hat{A}\big)(UV) = \dot{U}V + U\dot{V} - \hat{A}UV \overset{(2.7)}{=} \dot{U}V + U\dot{V} - (UA + \dot{U})V = U\big(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\big)(V) \underset{(2.7)}{\overset{(4.1)}{=}} (UV)N + \hat{B}M.$$

Therefore, $U\mathcal{V} = \big(U(t)\mathcal{V}(t)\big)_{t\in\mathcal{T}}$ is $(\hat{A}, \hat{B})$-invariant. Furthermore,

$$\forall\, t \in \mathcal{T}\;:\; \mathrm{im}(U(t)V(t)) = U(t)\,\mathrm{im}\, V(t) \subseteq U(t) \ker C(t) \overset{(4.3)}{=} \ker\big(C(t)U(t)^{-1}\big) \overset{(2.7)}{=} \ker \hat{C}(t)$$

and so "$\subseteq^*$" in (4.4) follows.
Step 1b: The proof of "$\supseteq^*$" in (4.4) is analogous and omitted.
Step 2: We show that

$$\forall\, t \in \mathcal{T}\;:\; U(t)^{-1}\, \mathcal{V}^*(\hat{A}, \hat{B}; \ker \hat{C})(t) = \ker \mathcal{C}(t)$$

which, in view of (4.3) and (2.18), is equivalent to

$$\mathcal{V}^*(\hat{A},\hat{B};\ker\hat{C}) = \mathcal{X}\,, \tag{4.5}$$

where

$$\mathcal{X} = \big(\mathcal{X}(t)\big)_{t\in\mathcal{T}} \in \mathcal{W}_n(\mathcal{T}) \text{ is generated by the constant matrix } \mathcal{X}(\cdot) := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix} \in \mathcal{C}^1(\mathcal{T};\mathbb{R}^{n\times(n-\rho m)})\,.$$

Step 2a: We show "$\supseteq^*$" in (4.5). The family $\mathcal{X}$ is $(\hat{A},\hat{B})$-invariant, since

$$\big(\tfrac{\mathrm{d}}{\mathrm{d}t} - \hat{A}\big)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix} \overset{(2.8)}{=} -\begin{bmatrix} 0 \\ \vdots \\ 0 \\ S \\ Q \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix}(-Q) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix}(-\Gamma^{-1}S) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix} N + \hat{B}M,$$

where $N := -Q \in \mathcal{C}^1(\mathcal{T};\mathbb{R}^{(n-\rho m)\times(n-\rho m)})$ and $M := -\Gamma^{-1}S \in \mathcal{C}^1(\mathcal{T};\mathbb{R}^{m\times(n-\rho m)})$; furthermore,

$$\forall\, t\in\mathcal{T}\ :\ \operatorname{im}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix} \overset{(2.18)}{=} \ker\big(\mathcal{C}(t)U(t)^{-1}\big) \overset{(2.7)}{=} \ker\begin{bmatrix} \hat{C}(t) \\ \big(\tfrac{\mathrm{d}}{\mathrm{d}t} + A(t)_r\big)\,\big(C(t)\big)U(t)^{-1} \\ \vdots \\ \big(\tfrac{\mathrm{d}}{\mathrm{d}t} + A(t)_r\big)^{\rho-1}\,\big(C(t)\big)U(t)^{-1} \end{bmatrix} \subseteq \ker\hat{C}(t)$$

and therefore $\mathcal{X} \subseteq^* \mathcal{V}^*(\hat{A},\hat{B};\ker\hat{C})$.
Step 2b: We show "$\subseteq^*$" in (4.5), i.e. that any $(\hat{A},\hat{B})$-invariant time-varying subspace $\hat{\mathcal{V}} \in \mathcal{W}_n(\mathcal{T})$ included in $\ker\hat{C}$ fulfills $\hat{\mathcal{V}} \subseteq^* \mathcal{X}$. Let $\hat{V} \in \mathcal{C}^1(\mathcal{T};\mathbb{R}^{n\times k})$, $k\in\mathbb{N}$, and $N \in \mathcal{C}^1(\mathcal{T}\setminus\mathcal{I};\mathbb{R}^{k\times k})$, $M \in \mathcal{C}^1(\mathcal{T}\setminus\mathcal{I};\mathbb{R}^{m\times k})$, $\mathcal{I}$ a discrete set, such that

$$\forall\, t\in\mathcal{T}:\ \operatorname{im}\hat{V}(t) \subseteq \ker\hat{C}(t) \quad\text{and}\quad \forall\, t\in\mathcal{T}\setminus\mathcal{I}:\ \big(\tfrac{\mathrm{d}}{\mathrm{d}t} - \hat{A}(t)\big)(\hat{V}(t)) = \hat{V}(t)N(t) + \hat{B}(t)M(t)\,. \tag{4.6}$$

It suffices to show that

$$\forall\, j = 1,...,\rho\ :\ S_j\hat{V} = 0\,, \tag{4.7}$$

where

$$S_j := \operatorname{diag}\{\underbrace{I_m,\ldots,I_m}_{j\text{-times}},0,\ldots,0\} \in \mathbb{R}^{n\times n}, \quad j = 1,...,\rho.$$

(4.7) is shown by induction. If $j = 1$, then

$$\forall\, t\in\mathcal{T}\ :\ S_1\hat{V}(t) \overset{(2.8)}{=} \hat{C}(t)^\top\hat{C}(t)\hat{V}(t) \overset{(4.6)}{=} 0.$$

Suppose $S_j\hat{V} = 0$ holds for some $j \in \{1,\ldots,\rho-1\}$ and set

$$\hat{V}_i = \operatorname{diag}\{\underbrace{0_{m\times m},\ldots,0_{m\times m}}_{(i-1)\text{-times}}, I_m, 0,\ldots,0\}\hat{V}\,, \quad i = 2,\ldots,j+1\,.$$

Then $\operatorname{im} \hat{V}(t) \subseteq \ker S_j$ for all $t \in \mathcal{T}$, and hence $\operatorname{im} \frac{\mathrm{d}}{\mathrm{d}t}\hat{V}(t) \subseteq \ker S_j$ for all $t \in \mathcal{T}$, thus $S_j \frac{\mathrm{d}}{\mathrm{d}t}\hat{V} = 0$ and

$$\begin{bmatrix} \hat{V}_2 \\ \vdots \\ \hat{V}_{j+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \underset{j\le\rho-1}{\overset{(2.8)}{=}} S_j \hat{A}\hat{V} = S_j\big(\tfrac{\mathrm{d}}{\mathrm{d}t} - \hat{A}\big)(\hat{V}) \underset{(4.6)}{\overset{\text{a.e.}}{=}} S_j \hat{V} N + S_j \hat{B} M \overset{(2.8)}{=} S_j \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix} M \overset{j\le\rho-1}{=} 0\,,$$

where "a.e." means "on $\mathcal{T} \setminus \mathcal{I}$" in this case. Hence we find $S_{j+1}\hat{V}(t) = 0$ for all $t \in \mathcal{T} \setminus \mathcal{I}$ and $\hat{V} \in \mathcal{C}^1(\mathcal{T};\mathbb{R}^{n\times k})$ gives $S_{j+1}\hat{V} = 0$. So the proof of Step 2 is complete, and the proof of the proposition follows from Step 1 and Step 2. □

We are now in a position to characterize the zero dynamics of systems which have strict and uniform relative degree. This result, motivated by some comments in [BLGS98] in the context of distributed parameter systems, seems, to the best of our knowledge, also new for time-invariant linear systems.

Proposition 4.4.
Let $\rho, \ell \in \mathbb{N}$ with $\rho \le \ell$ and suppose system (1.1) has strict and uniform relative degree ρ on $\mathcal{T}$. Let $(x,u,y) \in \mathcal{AC}(\mathcal{T};\mathbb{R}^n) \times \mathcal{PC}(\mathcal{T};\mathbb{R}^m) \times \mathcal{AC}(\mathcal{T};\mathbb{R}^m)$ be a solution of (1.1). Then

$$(x,u,y) \in \mathcal{ZD}(A,B,C) \quad \Longleftrightarrow \quad \Big[\forall\, t \in \mathcal{T} \;:\; x(t) \in \mathcal{V}^*(A,B;\ker\, C)(t)\Big]\,.$$

Proof: "⇒": Let $(x,u,y) \in \mathcal{ZD}(A,B,C)$. Applying the coordinate transformation (2.4), where U is as in Theorem 2.5(iv), it follows from $y = 0$ that $\xi = 0$ and therefore, for all $t \in \mathcal{T}$,

$$x(t) = U(t)^{-1}\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} = U(t)^{-1}\begin{pmatrix}0\\ \eta(t)\end{pmatrix} \in U(t)^{-1}\operatorname{im}\begin{bmatrix}0\\ \vdots\\ 0\\ I_{n-\rho m}\end{bmatrix} \overset{(2.18)}{=} \ker \mathcal{C}(t) \overset{(4.2)}{=} \mathcal{V}^*(A,B;\ker\, C)(t)\,.$$

"⇐": Since

$$\forall\, t \in \mathcal{T}:\; x(t) \in \mathcal{V}^*(A,B;\ker\, C)(t) \overset{(4.2)}{=} \ker \mathcal{C}(t) \overset{(2.18)}{=} U(t)^{-1}\operatorname{im}\begin{bmatrix}0\\ \vdots\\ 0\\ I_{n-\rho m}\end{bmatrix},$$

it follows that

$$\forall\, t \in \mathcal{T}:\; \begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} = U(t)x(t) \in \operatorname{im}\begin{bmatrix}0\\ \vdots\\ 0\\ I_{n-\rho m}\end{bmatrix},$$

and therefore (cf. (2.4)) $y = \xi_1 = 0$, whence $(x,u,y) \in \mathcal{ZD}(A,B,C)$. □

We now state the main result of this section: for any system (1.1) which has strict and uniform relative degree on $\mathcal{T}$, the zero dynamics is isomorphic to the maximal (A,B)-invariant time-varying subspace included in $\ker C$ at any initial time.

Theorem 4.5 (Vector space isomorphism)**.**
Let $\rho, \ell \in \mathbb{N}$ with $\rho \leq \ell$ and suppose (1.1) has strict and uniform relative degree ρ. Then, for every $t_0 \in \mathcal{T}$, the linear mapping

$$\begin{aligned} L_{t_0} : \mathcal{V}^*(A, B; \ker C)(t_0) &\to \mathcal{ZD}(A, B, C), \\ x^0 &\mapsto \left(x(\cdot),\ -\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho-1}(C)B\right]^{-1}\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho}(C)x(\cdot),\ Cx(\cdot)\right), \end{aligned}$$

where $x\colon \mathcal{T} \to \mathbb{R}^n$ solves

$$\dot{x} = \left(A - B\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho-1}(C)B\right]^{-1}\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho}(C)\right)x, \qquad x(t_0) = x^0,$$

is a vector space isomorphism. In particular, $t \mapsto \dim \mathcal{V}^*(A, B; \ker C)(t)$ is constant on $\mathcal{T}$.

Proof: *Step 1*: We show that L_{t_0} is well-defined, that means to show that for arbitrary $x^0 \in \mathcal{V}^*(A, B; \ker C)(t_0)$, the solution of

$$\dot{x} = \left(A - B\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho-1}(C)B\right]^{-1}\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho}(C)\right)x, \qquad x(t_0) = x^0 \tag{4.8}$$

on $\mathcal{T}$ satisfies

$$(x, u, y) := \left(x, -\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho-1}(C)B\right]^{-1}\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho}(C)x, Cx\right) \in \mathcal{ZD}(A, B, C). \tag{4.9}$$

It is an immediate consequence of (4.8) that (x, u, y) solves (1.1) on $\mathcal{T}$. In view of Corollary 3.3, it remains to show that $y = 0$.
Applying the coordinate transformation (2.4) and the notation as in (2.5) yields

$$\begin{aligned} \dot{\xi}(t) &= \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m \\ R_1(t) & R_2(t) & \cdots & R_{\rho-1}(t) & R_\rho(t) \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ S(t) \end{bmatrix} \eta(t) \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma(t) \end{bmatrix} \left(-\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A(t)_r\right)^{\rho-1}(C(t))B(t)\right]^{-1}\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A(t)_r\right)^{\rho}(C(t))U(t)^{-1}\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix}\right), \\ \dot{\eta}(t) &= P(t)\ \xi_1(t) + Q(t)\ \eta(t). \end{aligned}$$

It follows from (i) and (ii) of Theorem 2.5 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \left[\begin{array}{ccccc|c} 0 & I_m & & & & 0 \\ & \ddots & \ddots & & & \vdots \\ & & 0 & I_m & & 0 \\ \hline \multicolumn{6}{c}{[R_1, \ldots, R_\rho, S] - \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A_r\right)^{\rho}(C)U^{-1}} \\ \hline P, & 0, & \ldots, & 0 & & Q \end{array}\right]\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \left[\begin{array}{ccccc|c} 0 & I_m & & & & 0 \\ & \ddots & \ddots & & & \vdots \\ & & 0 & I_m & & 0 \\ \hline 0, & 0, & \ldots, & 0 & & 0 \\ \hline P, & 0, & \ldots, & 0 & & Q \end{array}\right]\begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

and the initial value satisfies

$$\begin{pmatrix}\xi(t_0)\\ \eta(t_0)\end{pmatrix} = U(t_0)x^0 \in U(t_0)\mathcal{V}^*(A,B;\ker C)(t_0) \overset{(4.2)}{=} U(t_0)\ker\mathcal{C}(t_0) \overset{(2.18)}{=} \operatorname{im}\begin{bmatrix} 0\\ \vdots \\ 0 \\ I_{n-\rho m}\end{bmatrix},$$

thus $y = \xi_1 = 0$.

Step 2: We show that L_{t_0} is injective. Let $x^1, x^2 \in \mathcal{V}^*(A,B;\ker C)(t_0)$ so that $L_{t_0}(x^1)(\cdot) = L_{t_0}(x^2)(\cdot)$. Then

$$(x^1,*,*) = L_{t_0}(x^1)(\cdot)\big|_{t=t_0} = L_{t_0}(x^2)(\cdot)\big|_{t=t_0} = (x^2,*,*).$$

Step 3: Surjectivity of L_{t_0} follows immediately from Proposition 4.4 and Corollary 3.3. $\square$

5 High-gain stabilization by output derivative feedback

In this section we design a simple high-gain output derivative feedback controller (see Figure 1) for systems (1.1), which have strict and uniform relative degree and uniformly asymptotically stable zero dynamics, so that the closed-loop system is uniformly exponentially stable. Throughout the section we set $\mathcal{T} = (0,\infty)$.

Recall (see e.g. [HP05, p. 257]) that a system $\dot x = A(t)x$, where $A \in \mathcal{PC}((0,\infty);\mathbb{R}^{n\times n})$, is called *uniformly exponentially stable* if, and only if, its transition matrix $\Phi_A(\cdot,\cdot)$ satisfies

$$\exists\, M,\lambda > 0\ \forall\, t \ge t_0 > 0\ :\ \|\Phi_A(t,t_0)\| \le M\,\mathrm{e}^{-\lambda(t-t_0)},$$

and this is equivalent (see e.g. [Rug96, Th. 6.7]) to

$$\exists\, M,\lambda > 0\ \forall \text{sln. } x(\cdot) \text{ of } \dot x = A(t)x\ \forall\, t \ge t_0 > 0\ :\ \|x(t)\| \le M\,\mathrm{e}^{-\lambda(t-t_0)}\|x(t_0)\|.$$

Stability of the zero dynamics is defined as follows:

Definition 5.1 (Stability of zero dynamics)**.**
The zero dynamics of system (1.1) is called

uniformly stable if, and only if,

$$\forall\,\varepsilon > 0\ \exists\,\delta > 0\ \forall\, t_0 > 0\ \forall\,(x,u,y) \in \mathcal{ZD}(A,B,C) \text{ s.t. } \|(x(t_0),u(t_0))\| < \delta \\ \forall\, t \ge t_0\ :\ \|(x(t),u(t))\| < \varepsilon\,;$$

attractive if, and only if,

$$\forall\,(x,u,y) \in \mathcal{ZD}(A,B,C) :\ \lim_{t\to\infty}\big(x(t),u(t)\big) = 0\,;$$

uniformly asymptotically stable or *exponentially stable* if, and only if, $\mathcal{ZD}(A,B,C)$ is uniformly stable and attractive. $\diamond$

The notions of uniformly stable and attractive zero dynamics introduced in Definition 5.1 are, precisely speaking, those of uniform stability and attractivity of the zero trajectory $(0,0,0) \in \mathcal{ZD}(A,B,C)$, resp. However, $\mathcal{ZD}(A,B,C)$ is a linear space and so the zero solution is uniformly stable (attractive) if, and only if, every solution $(x,u,y) \in \mathcal{ZD}(A,B,C)$ is uniformly stable (attractive), resp. Therefore, the abuse of terminology may be tolerated.

Remark 5.2 (Zero dynamics).
Let $\rho, \ell \in \mathbb{N}$ with $\rho \le \ell$ and suppose (1.1) has strict and uniform relative degree ρ on $\mathcal{T} = (0, \infty)$ and Byrnes-Isidori form (2.7), (2.8) as in Theorem 2.5. Then in [IM07, Prop. 4.2] it is shown that

$$\mathcal{ZD}(A, B, C) = \left\{ \ (V\eta, -\Gamma^{-1}S\eta, 0) \ \middle| \ \dot{\eta} = Q(t)\eta \ \right\}.$$

Furthermore, [IM07, Prop. 4.4] yields that the zero dynamics $\mathcal{ZD}(A, B, C)$ are uniformly asymptotically stable if, and only if, $\dot{\eta} = Q(t)\eta$ is a uniformly exponentially stable system. ⋄

We are now in a position to show that a system (1.1) which has some strict and uniform relative degree and uniformly asymptotically stable zero dynamics can be exponentially stabilized by a high-gain output feedback controller as depicted in Figure 1. We stress that the controller (5.1) is simpler than the controller [IM07, (4.2)]. Moreover, this result is also new for time-invariant systems.

Theorem 5.3 (High-gain derivative feedback stabilization).
Suppose

(i) system (1.1) has strict and uniform relative degree $\rho \in \mathbb{N}$ on $(0, \infty)$,

(ii) system (1.1) has uniformly asymptotically stable zero dynamics,

(iii) the matrix functions (A, B, C), $\mathcal{B}$, $\mathcal{C}$, $(\mathcal{C}\mathcal{B})^{-1}$ defined in Theorem 2.5(iv) are bounded on $(0, \infty)$,

(iv) the high-frequency gain matrix $\Gamma(t) := \left(\frac{\mathrm{d}}{\mathrm{d}t} + A(t)_r\right)^{\rho-1} \left(C(t)\right)B(t)$ is uniformly positive definite in the sense:
$$\exists\, \alpha > 0\ \forall\, t > 0 : \ \Gamma(t) + \Gamma(t)^\top \ge \alpha I_m\,.$$

Choose a Hurwitz polynomial $k(s) = \sum_{i=0}^{\rho-1} k_i s^i \in \mathbb{R}[s]$ such that $k_{\rho-1} > 0$. Then there exists $\kappa^* > 0$ such that, for all $\kappa \ge \kappa^*$, the derivative output feedback controller

$$u(t) = -\,\kappa\, k(\tfrac{\mathrm{d}}{\mathrm{d}t})(y(t)) = -\,\kappa \sum_{i=0}^{\rho-1} k_i\, y^{(i)}(t)\,, \tag{5.1}$$

applied to (1.1) yields a uniformly exponentially stable closed-loop system.

Remark 5.4 (Time-invariant systems).
For purpose of illustration, we discuss Theorem 5.3 for *time-invariant* systems (1.1). Suppose first that (1.1) has strict and uniform relative degree 1, i.e. $\det CB \neq 0$. Then the Byrnes-Isidori form of (1.1) is, in view of Remark 2.3(i), given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} R & S \\ P & Q \end{bmatrix} \begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} CB \\ 0 \end{bmatrix} u(t) \tag{5.2}$$

for some $R \in \mathbb{R}^{m\times m}$, $S, P^\top \in \mathbb{R}^{m\times(n-m)}$, $Q \in \mathbb{R}^{(n-m)\times(n-m)}$. Suppose further that the high-frequency gain matrix is positive in the sense $\sigma(CB) \subset \mathbb{C}_+$ and that (1.1) has asymptotically stable zero dynamics; the latter means, in view of Remark 5.2, $\sigma(Q) \subset \mathbb{C}_-$. Now apply, for $\kappa > 0$, the feedback

$$u(t) = -\,\kappa\, y(t)\,, \tag{5.3}$$

to (5.2). Then it is easy to see (see e.g. [Ilc93, Lem. 2.2.7]) that the closed-loop system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} R - \kappa CB & S \\ P & Q \end{bmatrix} \begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix}$$

is exponentially stable for all $\kappa \geq \kappa^*$ and some $\kappa^* > 0$ sufficiently large.
We would like to generalize this result to time-invariant systems with relative degree $\rho > 1$. The obstacle of the higher relative degree can be circumvented by introducing the compensator

$$u(t) = -\,\kappa\, k(\tfrac{\mathrm{d}}{\mathrm{d}t})\, v(t)\,, \tag{5.4}$$

for some polynomial $k(s) = \sum_{i=0}^{\rho-1} k_i s^i \in \mathbb{R}[s]$ such that $k_{\rho-1} > 0$. Then the transfer function of the series interconnection (1.1), (5.4) is given by

$$C(sI_n - A)^{-1} B\, k(s) = Ck(A)\,(sI_n - A)^{-1} B,$$

where equality follows from Remark 2.3(i), and the high-frequency gain matrix is $Ck(A)B = k_{\rho-1} CA^{\rho-1}B$. Therefore, the realization $(A, B, Ck(A))$ of the interconnection (1.1), (5.4) has strict relative degree 1. If system (1.1) has asymptotically stable zero dynamics and $k(\cdot)$ is Hurwitz, then $(A, B, Ck(A))$ has asymptotically stable zero dynamics, too; this follows from

$$\det \begin{bmatrix} sI - A & B \\ Ck(A) & 0 \end{bmatrix} = k(s) \det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix}$$

which is easy to see by invoking the Byrnes-Isidori form.
Finally, applying the feedback (5.1) to (1.1) is equivalent to applying

$$v(t) = -\,\kappa\, y(t)$$

to (1.1), (5.4), equivalently to $(A, B, Ck(A))$, and hence the findings from above, concerning relative degree one systems, yield that the closed-loop system interconnection (1.1), (5.1) is exponentially stable for all $\kappa \geq \kappa^*$ and some $\kappa^* > 0$ sufficiently large.
The analogous result for time-varying systems is the content of Theorem 5.3 but the proof is much more involved. ◇

Proof of Theorem 5.3: We proceed in several steps.
Step 1: Coordinate transformation of (1.1), (5.1).
By Theorem 2.5, the closed-loop system (1.1), (5.1) may be written, in terms of the coordinate transformation (2.4) and the transformed system matrices (2.7), (2.8) as

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} &= \hat{A}(t)\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} - \kappa\,\hat{B}(t)\sum_{i=0}^{\rho-1} k_i y(t)^{(i)} \\
&\overset{(2.2)}{=} \hat{A}(t)\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} - \kappa\,\hat{B}(t)\sum_{i=0}^{\rho-1} k_i \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + \hat{A}(t)_r\right)^i (\hat{C})\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} \\
&\overset{(2.8)}{=} \hat{A}(t)\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} - \kappa\,\hat{B}(t)\sum_{i=0}^{\rho-1} k_i \hat{C}\hat{A}(t)^i \begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} \\
&\overset{(2.8)}{=} \hat{A}(t)\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} - \kappa\,\hat{B}(t)\,\hat{C}\,k(\hat{A}(t))\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix}.
\end{aligned}$$

Clearly,

$$\hat{C}\,k(\hat{A}(t)) = [C_k, 0] \qquad \text{where} \qquad C_k := [k_0 I_m, \ldots, k_{\rho-1} I_m]\,.$$

Then the closed-loop system (1.1), (5.1) is equivalent to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} = \hat{A}(t)\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} + \hat{B}(t)\, v(t) \tag{5.5a}$$

$$\widetilde{y}(t) = [C_k, 0]\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} \tag{5.5b}$$

$$v(t) = -\kappa \widetilde{y}(t)\,. \tag{5.5c}$$

Furthermore, Assertion (iii) together with [IM07, Thm. 4.5] yields that the transformation matrix U, its inverse U^{-1} and all entries in (5.5) are bounded on $(0,\infty)$.

Step 2: Byrnes-Isidori form of (5.5a), (5.5b).
Note that

$$[C_k,0]\hat{B}(t) = k_{\rho-1}\Gamma(t), \quad t>0,$$

is the high-frequency gain matrix of the system (5.5a),(5.5b) and so it follows from Assertion (iv) that (5.5a), (5.5b) has strict and uniform relative degree one. Therefore, in view of

$$\operatorname{im}\tilde{V} = \ker[C_k,0], \qquad \text{where} \qquad \tilde{V} := \begin{bmatrix} 0 & k_{\rho-1}I_m & \cdots & k_2 I_m & k_1 I_m \\ 0 & 0 & & 0 & -k_0 I_m \\ \vdots & \vdots & \iddots & \iddots & 0 \\ 0 & 0 & \iddots & & \vdots \\ 0 & -k_0 I_m & & & \\ I_{n-\rho m} & 0 & \cdots & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times(n-m)}, \tag{5.6}$$

we may apply Theorem 2.5 for

$$\tilde{U} = \begin{bmatrix} [C_k,0] \\ \tilde{N} \end{bmatrix}, \qquad \tilde{N} = (\tilde{V}^\top\tilde{V})^{-1}\tilde{V}^\top\left(I - \hat{B}([C_k,0]\hat{B})^{-1}[C_k,0]\right)$$

to transform system (5.5a), (5.5b) into Byrnes-Isidori form

$$\left((\tilde{U}A + \tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{U})\tilde{U}^{-1}, \tilde{U}B, C\tilde{U}^{-1}\right) = \left(\begin{bmatrix} \tilde{R} & \tilde{S} \\ \tilde{P} & \tilde{Q} \end{bmatrix}, \begin{bmatrix} k_{\rho-1}\Gamma \\ 0_{(n-m)\times m} \end{bmatrix}, [I_m, 0_{(n-m)\times(n-m)}]\right) \tag{5.7}$$

or, equivalently,

$$\left.\begin{aligned} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{y}(t) \\ \dot{z}(t) \end{pmatrix} &= \begin{bmatrix} \tilde{R}(t) & \tilde{S}(t) \\ \tilde{P}(t) & \tilde{Q}(t) \end{bmatrix} \begin{pmatrix} \widetilde{y}(t) \\ z(t) \end{pmatrix} + \begin{bmatrix} k_{\rho-1}\Gamma(t) \\ 0_{(n-m)\times m} \end{bmatrix} v(t) \\ \widetilde{y}(t) &= [I_m, 0] \begin{pmatrix} \widetilde{y}(t) \\ z(t) \end{pmatrix}. \end{aligned}\right\} \tag{5.8}$$

In view of Assertion (iii), the transformation matrix $\tilde{U}$, $\tilde{U}^{-1}$ and all matrices in (5.7) are bounded on $(0,\infty)$.

Step 3: Zero dynamics of (5.5a), (5.5b).
By Remark 3.2, the zero dynamics of (5.8) are determined by $\dot{z} = \tilde{Q}(t)\,z$. We show that, for all $t>0$,

$$\begin{aligned} \tilde{Q}(t) = \begin{bmatrix} Q(t) & E(t) \\ 0 & H_k \end{bmatrix}, \quad \text{where} \quad E(t) &:= [k_{\rho-1}P(t), k_{\rho-2}P(t), \ldots, k_1 P(t)], \\ H_k &:= \begin{bmatrix} -\frac{k_{\rho-2}}{k_{\rho-1}}I_m & -\frac{k_{\rho-3}}{k_{\rho-1}}I_m & \cdots & -\frac{k_1}{k_{\rho-1}}I_m & -\frac{k_0}{k_{\rho-1}}I_m \\ I_m & 0 & \cdots & 0 & 0 \\ 0 & I_m & & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & I_m & 0 \end{bmatrix}. \end{aligned} \tag{5.9}$$

Formula (2.17) yields, for $\tilde{V}$ as in (5.6),

$$\begin{aligned} \tilde{Q} &= -(\tilde{V}^\top\tilde{V})^{-1}\tilde{V}^\top\left[-\hat{A}\tilde{V} + \hat{B}(k_{\rho-1}\Gamma)^{-1}(\tfrac{\mathrm{d}}{\mathrm{d}t} + \hat{A}_r)([C_k,0])\tilde{V}\right] \\ &= (\tilde{V}^\top\tilde{V})^{-1}\tilde{V}^\top\left[I_n - \frac{1}{k_{\rho-1}}\hat{B}\Gamma^{-1}[C_k,0]\right]\hat{A}\tilde{V}. \end{aligned} \tag{5.10}$$

Proceeding in steps, we calculate

$$
\begin{aligned}
\left[I_n - \frac{1}{k_{\rho-1}} \hat{B} \Gamma^{-1} [C_k, 0]\right] &= \begin{bmatrix} I_m & & & & \\ & \ddots & & & \\ & & I_m & & \\ -\frac{k_0}{k_{\rho-1}} I_m & \cdots & -\frac{k_{\rho-2}}{k_{\rho-1}} I_m & 0 & 0 \\ 0 & \cdots & 0 & 0 & I_m \end{bmatrix}, \\
\tilde{V}^\top \left[I_n - \frac{1}{k_{\rho-1}} \hat{B} \Gamma^{-1} [C_k, 0]\right] &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & I_m \\ \frac{k_0^2 + k_{\rho-1}^2}{k_{\rho-1}} I_m & \frac{k_0 k_1}{k_{\rho-1}} I_m & \cdots & \frac{k_0 k_{\rho-3}}{k_{\rho-1}} I_m & \frac{k_0 k_{\rho-2}}{k_{\rho-1}} I_m & 0 & 0 \\ k_{\rho-2} I_m & 0 & \cdots & 0 & -k_0 I_m & 0 & 0 \\ \vdots & \vdots & \iddots & \iddots & & \vdots & \vdots \\ k_2 I_m & 0 & \iddots & & & \vdots & \vdots \\ k_1 I_m & -k_0 I_m & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \\
\hat{A} \tilde{V} &= \begin{bmatrix} 0 & 0 & \cdots & 0 & -k_0 I_m \\ \vdots & \vdots & & \iddots & 0 \\ 0 & 0 & \iddots & \iddots & \vdots \\ 0 & -k_0 I_m & 0 & \cdots & 0 \\ S & k_{\rho-1} R_1 - k_0 R_\rho & k_{\rho-2} R_1 - k_0 R_{\rho-1} & \cdots & k_1 R_1 - k_0 R_2 \\ Q & k_{\rho-1} P & k_{\rho-2} P & \cdots & k_1 P \end{bmatrix}, \\
\tilde{V}^\top \left[I_n - \frac{1}{k_{\rho-1}} \hat{B} \Gamma^{-1} [C_k, 0]\right] \hat{A} \tilde{V} &= \begin{bmatrix} Q & k_{\rho-1} P & k_{\rho-2} P & \cdots & k_2 P & k_1 P \\ 0 & -\frac{k_0^2 k_{\rho-2}}{k_{\rho-1}} I_m & -\frac{k_0^2 k_{\rho-3}}{k_{\rho-1}} I_m & \cdots & -\frac{k_0^2 k_1}{k_{\rho-1}} I_m & -\frac{k_0^2 (k_{\rho-1}^2 + k_0^2)}{k_{\rho-1}} I_m \\ 0 & k_0^2 I_m & 0 & \cdots & 0 & -k_0 k_{\rho-2} I_m \\ 0 & 0 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & k_0^2 I_m & -k_0 k_1 I_m \end{bmatrix},
\end{aligned}
$$

$$
\tilde{V}^\top \tilde{V} = \begin{bmatrix} I_m & 0 & 0 & \cdots & 0 & \\ 0 & (k_{\rho-1}^2 + k_0^2) I_m & k_{\rho-1} k_{\rho-2} I_m & k_{\rho-1} k_{\rho-3} I_m & \cdots & k_{\rho-1} k_1 I_m \\ 0 & k_{\rho-1} k_{\rho-2} I_m & (k_{\rho-2}^2 + k_0^2) I_m & k_{\rho-2} k_{\rho-3} I_m & \cdots & k_{\rho-2} k_1 I_m \\ 0 & k_{\rho-1} k_{\rho-3} I_m & k_{\rho-2} k_{\rho-3} I_m & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & k_2 k_1 I_m \\ 0 & k_{\rho-1} k_1 I_m & k_{\rho-2} k_1 I_m & \cdots & k_2 k_1 I_m & (k_1^2 + k_0^2) I_m \end{bmatrix},
$$

and, for $\gamma := k_0^2 \sum_{j=0}^{\rho-1} k_j^2$,

$$\left(\tilde V^\top \tilde V\right)^{-1} = \gamma^{-1} \begin{bmatrix} \gamma I_m & 0 & 0 & \cdots & 0 & \\ 0 & \left(\sum_{\substack{j=0\\ j\neq\rho-1}}^{\rho-1} k_j^2\right) I_m & -k_{\rho-1}k_{\rho-2} I_m & -k_{\rho-1}k_{\rho-3} I_m & \cdots & -k_{\rho-1}k_1 I_m \\ 0 & -k_{\rho-1}k_{\rho-2} I_m & \left(\sum_{\substack{j=0\\ j\neq\rho-2}}^{\rho-1} k_j^2\right) I_m & -k_{\rho-2}k_{\rho-3} I_m & \cdots & -k_{\rho-2}k_1 I_m \\ 0 & -k_{\rho-1}k_{\rho-3} I_m & -k_{\rho-2}k_{\rho-3} I_m & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & -k_2k_1 I_m \\ 0 & -k_{\rho-1}k_1 I_m & -k_{\rho-2}k_1 I_m & \cdots & -k_2k_1 I_m & \left(\sum_{\substack{j=0\\ j\neq 1}}^{\rho-1} k_j^2\right) I_m \end{bmatrix}.$$

Inserting these findings into (5.10) yields (5.9). Note that E is bounded since P is bounded.

Step 4: Stability of the zero dynamics.
By Remark 3.2, the zero dynamics of (5.8) are uniformly asymptotically stable if, and only if, $\dot z = \tilde Q(t)\, z$ is uniformly exponentially stable.
Step 4a: We show that for H_k as defined in (5.9) *we have*

$$\exists\, M_1, \lambda_1 > 0\ \forall\, t > 0 : \ \|e^{H_k t}\| \le M_1\, e^{-\lambda_1 t}. \tag{5.11}$$

If we had shown that

$$\det(sI - H_k) = k_{\rho-1}^{-m}\ k(s)^m \ \in \mathbb{R}[s], \tag{5.12}$$

then (5.11) would be a consequence of the assumption that $k(\cdot)$ is Hurwitz.
To simplify the notation, set

$$\tilde k_i := \frac{k_i}{k_{\rho-1}} \qquad \text{for} \qquad i = 0, \ldots, \rho-2 \tag{5.13}$$

and

$$\underbrace{\begin{bmatrix} sI_m & & & \\ -I_m & sI_m & & 0 \\ & \ddots & \ddots & \\ 0 & & -I_m & sI_m \end{bmatrix}}_{=:F_{\rho-2}\in\mathbb{R}(s)^{(\rho-2)m\times(\rho-2)m}} \underbrace{\begin{bmatrix} s^{-1}I_m & & & \\ s^{-2}I_m & s^{-1}I_m & & 0 \\ \vdots & \ddots & \ddots & \\ s^{-(\rho-2)}I_m & \ldots & s^{-2}I_m & s^{-1}I_m \end{bmatrix}}_{=F_{\rho-2}^{-1}} = I_{(\rho-2)m}\,.$$

An application of the Schur complement formula (see e.g. [HP05, Lemma A.1.17]) yields

$$\det(sI - H_k) = \det \left[\begin{array}{c|cccc} (s+\tilde k_{\rho-2})I_m & \tilde k_{\rho-3} I_m & & \ldots & \tilde k_0 I_m \\ \hline -I_m & sI_m & & & \\ 0 & -I_m & sI_m & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & -I_m & sI_m \end{array}\right]$$

$$
\begin{aligned}
&= \det \left[\begin{array}{c|ccc} (s+\tilde{k}_{\rho-2})I_m & \tilde{k}_{\rho-3}I_m & \dots & \tilde{k}_0 I_m \\ \hline -I_m & & & \\ 0 & & & \\ \vdots & & F_{\rho-2} & \\ 0 & & & \end{array}\right] \\
&= \det(F_{\rho-2}) \cdot \det \left((s+\tilde{k}_{\rho-2})I_m - \left[\tilde{k}_{\rho-3}I_m \quad \dots \quad \tilde{k}_0 I_m\right] F_{\rho-2}^{-1} \begin{bmatrix} -I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \\
&= s^{(\rho-2)m} \cdot \det \left((s+\tilde{k}_{\rho-2}+\tilde{k}_{\rho-3}s^{-1}+\dots+\tilde{k}_0 s^{-(\rho-2)})I_m \right) \\
&= \left[s^{\rho-1}+\tilde{k}_{\rho-2}s^{\rho-2}+\dots+\tilde{k}_0 \right]^m \\
&\overset{(5.13)}{=} \left[\frac{1}{k_{\rho-1}} \, k(s) \right]^m .
\end{aligned}
$$

This completes the proof of (5.12).

Step 4b: We show that the transition matrix $\Phi_Q(\cdot\,,\cdot)$ of $\dot{x} = Q(t)x$ satisfies

$$
\exists\, M_2, \lambda_2 > 0 \; \forall\, t \geq t_0 > 0 \; : \; \|\Phi_Q(t,s)\| \leq M_2 \mathrm{e}^{-\lambda_2(t-t_0)} ; \tag{5.14}
$$

and we may assume that $\lambda_2 < \lambda_1$.
The zero dynamics of system (1.1) are uniformly asymptotically stable by Assertion (ii); they are determined by the system $\dot{\eta} = Q(t)\eta$ for Q as in (2.8). By Remark 5.2, system $\dot{\eta} = Q(t)\eta$ is uniformly exponentially stable, and so the transition matrix $\Phi_Q(\cdot\,,\cdot)$ of $\dot{\eta} = Q(t)\eta$ satisfies (5.14).

Step 4c: We show that $\dot{z} = \tilde{Q}(t)\, z$ is uniformly exponentially stable.
Consider the solution $(z_1(\cdot), z_2(\cdot))$ of the initial value problem

$$
\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \tilde{Q}(t) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} Q(t)z_1 + E(t)z_2 \\ H_k z_2 \end{pmatrix}, \qquad \begin{pmatrix} z_1(t_0) \\ z_2(t_0) \end{pmatrix} = \begin{pmatrix} z_1^0 \\ z_2^0 \end{pmatrix} \tag{5.15}
$$

for fixed but arbitrary $(z_1^0, z_2^0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ and $t_0 > 0$. By boundedness of E we have

$$
\exists\, c_1 > 0 \; \forall\, t > 0 \; : \; \|E(t)\| \leq c_1 . \tag{5.16}
$$

An application of the Variation of Constants formula to the first equation in (5.15) and taking norms yields, for all $t \geq t_0$,

$$
\begin{aligned}
\|z_1(t)\| &= \left\| \Phi_Q(t,t_0)z_1^0 + \int_{t_0}^t \Phi_Q(t,s)E(s)z_2(s) \,\mathrm{d}s \right\| \\
&\underset{(5.16)}{\overset{(5.14)}{\leq}} M_2\, \mathrm{e}^{-\lambda_2(t-t_0)} \|z_1^0\| + \int_{t_0}^t M_2\, \mathrm{e}^{-\lambda_2(t-s)}\, c_1 \, \|z_2(s)\| \,\mathrm{d}s \\
&\overset{(5.11)}{\leq} M_2\, \mathrm{e}^{-\lambda_2(t-t_0)} \|z_1^0\| + c_1 M_1 M_2 \|z_2^0\| \int_{t_0}^t \mathrm{e}^{-\lambda_2(t-s)}\, \mathrm{e}^{-\lambda_1(s-t_0)} \,\mathrm{d}s
\end{aligned}
$$

$$= M_2\,\mathrm{e}^{-\lambda_2(t-t_0)}\|z_1^0\| + \frac{c_1 M_1 M_2 \|z_2^0\|}{\lambda_1-\lambda_2}\left(\mathrm{e}^{-\lambda_2(t-t_0)} - \mathrm{e}^{-\lambda_1(t-t_0)}\right)$$

$$\leq M_2\,\mathrm{e}^{-\lambda_2(t-t_0)}\|z_1^0\| + \frac{c_1 M_1 M_2}{\lambda_1-\lambda_2}\,\mathrm{e}^{-\lambda_2(t-t_0)}\,\|z_2^0\|\,. \tag{5.17}$$

Now it is a straightforward calculation to see that (5.11) together with (5.17) yields

$$\exists M_3>0\ \forall t\geq t_0>0\ :\ \left\|\begin{pmatrix} z_1(t)\\ z_2(t)\end{pmatrix}\right\| \leq M_3\,\mathrm{e}^{-\frac{\lambda_2}{2}(t-t_0)}\ \left\|\begin{pmatrix} z_1^0\\ z_2^0\end{pmatrix}\right\|$$

and therefore $\dot z = \tilde Q(t)\,z$ is uniformly exponentially stable.

Step 5: We show that there exists $\kappa^*, M_4, \lambda_4>0$ such that, for all $\kappa\geq\kappa^*$, every solution $(\widetilde y(\cdot), z(\cdot))$ of the closed-loop system (5.8)*, $v=-\kappa\widetilde y$ satisfies*

$$\forall\, t\geq t_0>0\ :\ \left\|\begin{pmatrix}\widetilde y(t)\\ z(t)\end{pmatrix}\right\| \leq M_4\,\mathrm{e}^{-\lambda_4(t-t_0)}\left\|\begin{pmatrix}\widetilde y(t_0)\\ z(t_0)\end{pmatrix}\right\|.$$

Note that boundedness of E and Q yields boundedness of $\tilde Q$ and therefore we may apply [HP05, Th. 3.3.38] to conclude existence of a symmetric solution $P_{\tilde Q}\in\mathcal C^1((0,\infty),\mathbb R^{(n-m)\times(n-m)})$ to

$$\forall\, t>0:\ \tilde Q(t)^\top P_{\tilde Q}(t) + P_{\tilde Q}(t)\tilde Q(t) + \dot P_{\tilde Q}(t) = -I_{n-m} \tag{5.18}$$

which is bounded from above and below in the sense

$$\exists\,\beta_1,\beta_2>0\ \forall\, t>0:\ \beta_1 I_{n-m}\leq P_{\tilde Q}(t)\leq \beta_2 I_{n-m}\,. \tag{5.19}$$

Now differentiation of

$$V:\mathbb R\times\mathbb R^m\times\mathbb R^{n-m}\to\mathbb R,\quad (t,\widetilde y,z)\mapsto \widetilde y^\top\widetilde y + z^\top P_{\tilde Q}(t) z,$$

along any solution $(\widetilde y(\cdot), z(\cdot))$ of the closed-loop system (5.8), $v=-\kappa\widetilde y$ yields, for all $t>0$,

$$\begin{aligned}
\tfrac{\mathrm d}{\mathrm dt}V(t,\widetilde y(t),z(t)) &= 2\widetilde y(t)^\top\left(\tilde R(t)\widetilde y(t) + \tilde S(t)z(t) - \kappa k_{\rho-1}\Gamma(t)\widetilde y(t)\right)\\
&\quad +2z(t)^\top\left(P_{\tilde Q}(t)\tilde P(t)\widetilde y(t) + P_{\tilde Q}(t)\tilde Q(t)z(t)\right) + z(t)^\top\dot P_{\tilde Q}(t)z(t)\\
&\overset{(5.18)}{=} 2\widetilde y(t)^\top\left(\tilde R(t)\widetilde y(t) + \tilde S(t)z(t) - \kappa k_{\rho-1}\Gamma(t)\widetilde y(t)\right)\\
&\quad +2z(t)^\top P_{\tilde Q}(t)\tilde P(t)\widetilde y(t) - \|z(t)\|^2\\
&\overset{\text{Ass. (iv)}}{\leq} 2\widetilde y(t)^\top\left(\tilde R(t)\widetilde y(t) + \tilde S(t)z(t)\right) + 2z(t)^\top P_{\tilde Q}(t)\tilde P(t)\widetilde y(t)\\
&\quad -\|z(t)\|^2 - \kappa k_{\rho-1}\alpha\|\widetilde y(t)\|^2,
\end{aligned}$$

and by boundedness of $\tilde R,\tilde S,\tilde P$ (see Step 2) and $P_{\tilde Q}$ (see (5.19)) and making use of the inequality $ab\leq\frac12 a^2+\frac12 b^2$ for all $a,b\in\mathbb R$, it follows that there exists $M_4>0$ such that, for all $t>0$,

$$\begin{aligned}
\tfrac{\mathrm d}{\mathrm dt}V(t,\widetilde y(t),z(t)) &\leq (M_4-\kappa k_{\rho-1}\alpha)\,\|\widetilde y(t)\|^2 - \|z(t)\|^2 + M_4\|\widetilde y(t)\|\cdot\|z(t)\|\\
&\leq \left(M_4+\frac12 M_4^2-\kappa k_{\rho-1}\alpha\right)\|\widetilde y(t)\|^2 - \frac12\|z(t)\|^2\\
&\overset{(5.19)}{\leq} -\left(\kappa k_{\rho-1}\alpha - M_4 - \frac12 M_4^2\right)\|\widetilde y(t)\|^2 - \frac{1}{2\beta_2} z(t)^\top P_{\tilde Q}(t)z(t)\,.
\end{aligned}$$

Now set

$$\kappa^* \;:=\; \frac{1 + M_4 + M_4^2/2}{k_{\rho-1}\alpha} \quad \text{and} \quad M_5 := \min\left\{1, \frac{1}{2\beta_2}\right\};$$

then

$$\forall\, \kappa \geq \kappa^* \;\forall\, t > 0 \;:\; \tfrac{\mathrm{d}}{\mathrm{d}t} V(t, \widetilde{y}(t), z(t)) \leq -M_5 \; V(t, \widetilde{y}(t), z(t))$$

and separation of variables yields

$$\forall\, t \geq t_0 > 0 \;:\; V(t, \widetilde{y}(t), z(t)) \leq \mathrm{e}^{-M_5(t-t_0)} \; V(t_0, \widetilde{y}(t_0), z(t_0))\,. \tag{5.20}$$

Finally, the claim of Step 5 follows by an application of (5.19) to (5.20) and standard arguments, see e.g. the proof of [Rug96, Th. 7.4].

Step 6: Since

$$x = U^{-1}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = U^{-1}\tilde{U}^{-1}\begin{pmatrix}\widetilde{y}\\ z\end{pmatrix}$$

and U^{-1}, $\tilde{U}^{-1}$ are bounded (see Step 1 and Step 2) it follows that the closed-loop system (1.1), (5.1) is uniformly exponentially stable. This completes the proof of the theorem. □

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