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Long Cycles and Paths in Distance Graphs

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Abstract. For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the distance graph P_n^D has vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{ij \mid 0 \leq i, j \leq n-1, |j-i| \in D\}$. Note that the important and very well-studied circulant graphs coincide with the regular distance graphs.

A fundamental result concerning circulant graphs is that for these graphs, a simple greatest common divisor condition, their connectivity, and the existence of a Hamiltonian cycle are all equivalent. Our main result suitably extends this equivalence to distance graphs. We prove that for a set D, there is a constant c_D such that the greatest common divisor of the integers in D is 1 if and only if for every n, P_n^D has a component of order at least $n - c_D$ if and only if for every n, P_n^D has a cycle of order at least $n - c_D$. Furthermore, we discuss some consequences and variants of this result.

Keywords. Circulant graph; distance graph; connectivity; Hamiltonian cycle; Hamiltonian path

1 Introduction

Circulant graphs form an important and very well-studied class of graph [3, 18, 20, 21, 24, 28, 29]. They are Cayley graphs of cyclic groups and have been proposed for numerous network applications such as local area computer networks, large area communication networks, parallel processing architectures, distributed computing, and VLSI design. Their connectivity and diameter [3, 6, 20, 21, 30, 34], cycle and path structure [1, 2, 4, 5, 7, 11, 27], and further graph-theoretical properties have been studied in great detail. Polynomial time algorithms for isomorphism testing and recognition of circulant graphs have been long-standing open problems which were completely solved only recently [17, 26].

For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the *circulant graph* C_n^D has vertex set $[0, n-1] = \{0, 1, \ldots, n-1\}$ and the neighbourhood $N_{C_n^D}(i)$ of a vertex $i \in [0, n-1]$ in C_n^D is given by

$$N_{C_n^D}(i) = \{(i+d) \bmod n \mid d \in D\} \cup \{(i-d) \bmod n \mid d \in D\}.$$

Clearly, we may assume $\max(D) \leq \frac{n}{2}$ for every circulant graph C_n^D.

Our goal here is to extend some of the fundamental results concerning circulant graphs to the similarly defined yet more general class of distance graphs: For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the *distance graph* P_n^D has vertex set $[0, n-1]$ and

$$\begin{aligned} N_{P_n^D}(i) &= \{i+d \mid d \in D \text{ and } (i+d) \in [0, n-1]\} \\ &\quad \cup \{i-d \mid d \in D \text{ and } (i-d) \in [0, n-1]\} \end{aligned}$$

for all $i \in [0, n-1]$. Clearly, we may assume $\max(D) \leq n-1$ for every distance graph P_n^D.

Every distance graph P_n^D is an induced subgraph of the circulant graph $C_{n+\max(D)}^D$. More specifically, distance graphs are the subgraphs of sufficiently large circulant graphs induced by sets of consecutive vertices. Conversely, our following simple observation from [13] shows that every circulant graph is in fact a distance graph.

Proposition 1 ([13]) *A graph is a circulant graph if and only if it is a regular distance graph.*

Proof: Clearly, every circulant graph C_n^D is regular and isomorphic to the distance graph $P_n^{D'}$ for $D' = D \cup \{n - d \mid d \in D\}$.

Now let P_n^D be a regular distance graph. Let $D = \{d_1, d_2, \ldots, d_k\}$ with $d_1 < d_2 < \ldots < d_k \leq n-1$. Since the vertex 0 has exactly k neighbours D, P_n^D is k-regular.

Let $i \in [1, k]$. The vertex $d_i - 1$ has exactly $i-1$ neighbours j with $j < d_i - 1$. Hence $d_i - 1$ has exactly $k+1-i$ neighbours j with $j > d_i - 1$ which implies $(d_i - 1) + d_{k+1-i} \leq n-1$. The vertex d_i has exactly i neighbours j with $j < d_i$. Hence d_i has exactly $k-i$ neighbours j with $j > d_i$ which implies $d_i + d_{k+1-i} > n-1$.

We obtain $d_i + d_{k+1-i} = n$ for every $i \in [1, k]$ which immediately implies that P_n^D is isomorphic to the circulant graph $C_n^{D'}$ for $D' = \{d \in D \mid d \leq \frac{n}{2}\}$. $\Box$

Distance graphs lack the symmetry and algebraic interpretation of circulant graphs and the algebraic methods used in [17, 26] will not apply to them. In view of Proposition 1, the recognition of distance graphs will be at least as difficult as the recognition of circulant graphs.

Originally motivated by coloring problems for infinite distance graphs studied by Eggleton, Erdős, and Skilton [15, 16], most research on distance graphs focused on colorings [9, 12, 14, 22, 23, 31, 32].

One of the most fundamental results for circulant graphs is the following beautiful equivalence.

Theorem 2 (Boesch and Tindell [6], Burkard and Sandholzer [8], Garfinkel [19]) *For $n \in \mathbb{N}$ and a finite set $D \subseteq \mathbb{N}$, the following statements are equivalent.*

(i) C_n^D *is connected.*

(ii) The greatest common divisor $\gcd(\{n\} \cup D)$ *of the integers in* $\{n\} \cup D$ *equals* 1.

(iii) C_n^D *has a Hamiltonian cycle.*

In 1970 Lovász [10, 25, 33] asked whether every connected vertex-transitive graph has a Hamiltonian path. Since circulant graphs are clearly vertex-transitive, Theorem 2 is a positive example for this well-studied problem [10, 33].

In the present paper we suitably extend Theorem 2 to distance graphs. While connectivity and hamiltonicity of circulants are equivalent to a simple necessary gcd-condition, we prove that a similar condition for distance graphs is only equivalent to the existence of a large component and a long cycle. We also discuss consequences and variants of our result.

2 Cycles and Paths in Distance Graphs

We immediately proceed to our main result. The residue of an integer $n \in \mathbb{Z}$ modulo $d \in \mathbb{N}$ will be denoted by $n \bmod d$.

Theorem 3 *For a finite set $D \subseteq \mathbb{N}$, the following statements are equivalent.*

(i) There is a constant $c_1(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a component of order at least $n - c_1(D)$.

(ii) $\gcd(D) = 1$.

(iii) There is a constant $c_2(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a cycle of order at least $n - c_2(D)$.

Proof: (i) $\Rightarrow$ (ii): Let n be such that n is even and $n > 2c_1(D)$. By (i), more than half the vertices are in the same component of P_n^D. By the pigeonhole principle, there is some $i \in [0, n-2]$ such that the two vertices i and $i+1$ are in the same component of P_n^D. This implies that there is a path in P_n^D from i to $i+1$. Hence 1 is an integral linear combination of the elements in D. It is a well-known consequence of the Euclidean algorithm that this is equivalent to (ii).

(ii) $\Rightarrow$ (iii): The essential idea in order to obtain a cycle which contains almost all vertices of P_n^D is to use increasing and decreasing paths which only use edges uv such that $v - u$ is one fixed element d^* of D. Because the vertices on these paths always remain in the same residue class modulo d^*, such paths can be overlayed without intersecting. In order to connect these paths to a cycle, we use short paths which are close to 0 or $n-1$ and whose end vertices are in different residue classes modulo d^*. In this way the cycle can collect all vertices of P_n^D in some middle section and only misses vertices close to 0 or $n-1$ in terms of D (cf. Figure 1).

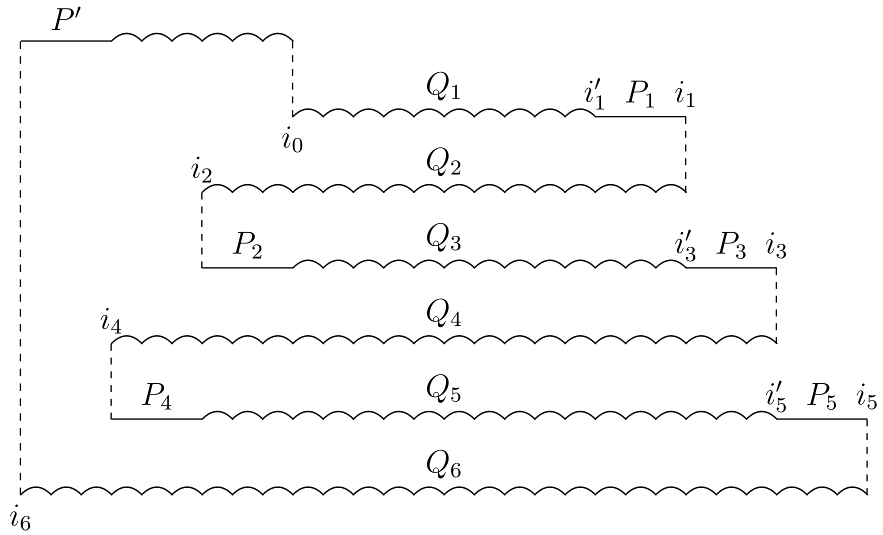

Figure 1

In view of the constant $c_2(D)$ in (iii), we may tacitly assume in the following that n is sufficiently large in terms of D.

Let $d_{\max} = \max(D)$ and $D^- = D \setminus \{d_{\max}\}$. Since $\gcd(D) = 1$, 1 is an integral linear combination of the elements of D. Hence there are integers n_d for $d \in D^-$ such that

$$1 = \left(\sum_{d \in D^-} n_d d\right) \bmod d_{\max} = \left(\sum_{d \in D^-} (n_d \bmod d_{\max})\, d\right) \bmod d_{\max}. \tag{1}$$

This implies the existence of a path

$$P : v_0 v_1 \ldots v_k$$

in P_n^D such that $v_i - v_{i-1} \in D^-$ for all $i \in [1, k]$ and

$$\{v_i \bmod d_{\max} \mid i \in [0, k]\} = [0, d_{\max} - 1],$$

i.e. P is a monotonously increasing path which only uses edges uv with $v - u \in D^-$ and contains a vertex from every residue class modulo $d_{\max}$.

We assume that P is chosen so as to be shortest possible. This implies that the residues modulo $d_{\max}$ of the vertices v_0 and v_k appear exactly once on P. Let

$$r_1, r_2, \ldots, r_{d_{\max}}$$

denote the residues modulo $d_{\max}$ in the order in which they appear for the first time when traversing P from v_0 to v_l. Clearly,

$$\begin{aligned} r_1 &= v_0 \bmod d_{\max}, \\ r_2 &= v_1 \bmod d_{\max}, \text{ and} \\ r_{d_{\max}} &= v_k \bmod d_{\max}. \end{aligned}$$

Furthermore, P is the concatenation of $(d_{\max}-1)$ edge-disjoint paths

$$P = P_1 P_2 \ldots P_{d_{\max}-1}$$

such that for $i \in [1, d_{\max}-1]$, the path P_i begins at the smallest vertex v_j on P with $r_i = v_j \bmod d_{\max}$ and ends at the smallest vertex $v_{j'}$ on P with $r_{i+1} = v_{j'} \bmod d_{\max}$; let $l_i = v_{j'} - v_j$ for these indices.

By the choice of P, for $i \in [1, d_{\max}-1]$, all internal vertices of P_i have residues modulo $d_{\max}$ in $\{r_j \mid 2 \leq j \leq i-1\}$.

Let

$$l = l_1 + l_2 + \ldots + l_{d_{\max}-1}.$$

Note that $l = v_k - v_0$.

We now describe a long cycle C in P_n^D. The general structure of C is illustrated in Figure 1. For simplicity we will first assume that $d_{\max}$ is even.

Let i_0 be the smallest integer at least

$$(d_{\max} + l_2) + (d_{\max} + l_4) + \ldots + (d_{\max} + l_{d_{\max}-2}) + (d_{\max} + l)$$

of residue r_1 modulo $d_{\max}$. Furthermore, let i'_1 be the largest integer at most

$$(n-1) - l_1 - (d_{\max} + l_3) - (d_{\max} + l_5) - \ldots - (d_{\max} + l_{d_{\max}-1})$$

of residue r_1 modulo $d_{\max}$.

We start C with an increasing path Q_1 only using edges uv with $v - u = d_{\max}$ which begins at i_0 and ends at i'_1. We continue C with P_1 shifted by a multiple of $d_{\max}$ such that it begins at i'_1 and ends at a vertex i_1.

For j from 1 up to $\frac{d_{\max}}{2} - 1$, we proceed as follows: We assume that we have already constructed C until the end of a shifted path P_{2j-1} which ends at a vertex i_{2j-1}. We continue C with a decreasing path Q_{2j} which only uses edges uv with $v - u = d_{\max}$ and ends at the largest integer i_{2j} at most $i_{2j-2} - l_{2j}$ with residue r_{2j} modulo $d_{\max}$. We continue C with P_{2j} shifted by a multiple of $d_{\max}$ such that it begins at i_{2j}. We continue C with an increasing path Q_{2j+1} which only uses edges uv with $v - u = d_{\max}$ and ends at the smallest integer i'_{2j+1} at least i_{2j-1} with residue r_{2j+1} modulo $d_{\max}$. We continue C with P_{2j+1} shifted by a multiple of $d_{\max}$ such that it begins at i'_{2j+1} and ends at a vertex i_{2j+1}. At this point, we increase j until it reaches $\frac{d_{\max}}{2} - 1$.

To complete C, we may assume now that we have already constructed C until the end of the shifted path $P_{d_{\max}-1}$ which ends at a vertex $i_{d_{\max}-1}$. We continue C with a decreasing path $Q_{d_{\max}}$ which only uses edges uv with $v - u = d_{\max}$ and ends at the largest integer $i_{d_{\max}}$ at most $i_{d_{\max}-2} - l$ with residue $r_{d_{\max}}$ modulo $d_{\max}$.

Let

$$P' : u_0 u_1 \dots u_{k'}$$

be a path in P_n^D such that $u_0 = i_{d_{\max}}$, $u_i - u_{i-1} \in D^-$ for all $i \in [1, k']$, $r_1 = u_{k'} \bmod d_{\max}$, and $l' = u_{k'} - u_0$ is minimum possible.

Clearly, $l' \leq l$. Furthermore, no internal vertex of P' has residue $r_{\max}$ modulo $d_{\max}$. We continue C with P'. Finally, we complete C with an increasing path which only uses edges uv with $v - u = d_{\max}$, begins at $u_{k'}$ and ends at i_0.

At this point we have completely described C as the concatenation of paths. Clearly, the choices of i_0 and i_1' imply that C never leaves $[0, n-1]$, i.e. C is in fact a closed walk within P_n^D. In order to show that C is a cycle, it remains to prove that it visits no vertex twice. This follows easily from the facts that

- the vertices on Q_i all have residue r_i modulo $d_{\max}$ for all $i \in [1, d_{\max}]$,
- the end vertices of the shifted paths P_i are the first vertices on C - traversed as constructed above - which have residue r_{i+1} modulo $d_{\max}$ for all $i \in [1, d_{\max} - 1]$,
- all internal vertices of P_i have residues modulo $d_{\max}$ in $\{r_j \mid 2 \leq j \leq i - 1\}$ for all $i \in [1, d_{\max} - 1]$, and
- no internal vertex of P' has residue $r_{\max}$ modulo $d_{\max}$.

Since C contains all vertices between i_0 and i_1', it misses at most

$$\begin{aligned} & 2d_{\max} + l_1 + (d_{\max} + l_2) + (d_{\max} + l_3) + \dots + (d_{\max} + l_{d_{\max}-1}) + (d_{\max} + l) \\ = \; & d_{\max}(d_{\max} + 1) + 2l \end{aligned}$$

many vertices of P_n^D. Since this expression is bounded in terms of D, the proof of (iii) in the case that $d_{\max}$ is even is complete.

Next, we consider the case that $d_{\max}$ is odd.

Let i_0 be the smallest integer at least

$$(d_{\max} + l_3) + (d_{\max} + l_5) + \dots + (d_{\max} + l_{d_{\max}-2}) + (d_{\max} + l)$$

of residue r_1 modulo $d_{\max}$. Furthermore, let i_2' be the largest integer at most

$$(n-1) - l_2 - (d_{\max} + l_4) - (d_{\max} + l_6) - \dots - (d_{\max} + l_{d_{\max}-1})$$

of residue r_2 modulo $d_{\max}$.

Let $d' = v_1 - v_0$. In the above construction, we replace the path Q_1 by the path $Q_{1,2}$ illustrated in Figure 2 with

$$\begin{aligned} Q_{1,2} \; : \; & i_0(i_0 + d')(i_0 + d_{\max} + d')(i_0 + d_{\max})(i_0 + 2d_{\max}) \\ & (i_0 + 2d_{\max} + d')(i_0 + 3d_{\max} + d')(i_0 + 3d_{\max})(i_0 + 4d_{\max}) \\ & \dots i_2' \end{aligned}$$

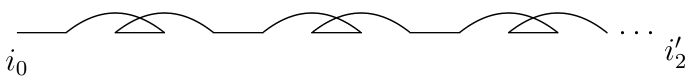

Figure 2

Note that $Q_{1,2}$ visits all but at most one of the vertices between i_0 and i'_1 of residues r_1 or r_2 modulo $d_{\max}$. The rest of the above construction is adapted accordingly as illustrated in Figure 3 replacing P_i and Q_i with P_{i+1} and Q_{i+1} for all $i \in [1, d_{\max} - 1]$.

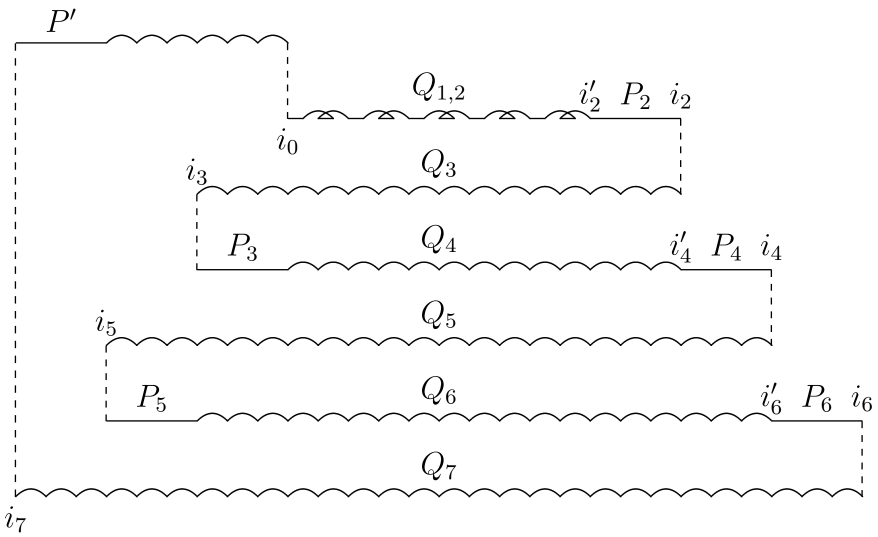

Figure 3

Again the same residue properties as before imply that the closed walk C is a cycle. Also as before, the number of vertices of P_n^D missed by C is at most $d_{\max}(d_{\max}+1)+2l$ which is bounded in terms of D. This completes the proof of the implication "(ii) $\Rightarrow$ (iii)".

(iii) $\Rightarrow$ (i): Since this implication is trivial, the proof is complete. $\Box$

We add some comments concerning Theorem 3.

It is easy to see that a distance graph P_n^D with $\gcd(D)=1$ and $n \geq 2\max(D)+1$ is actually connected. Hence (i) in Theorem 3 could be replaced by:

(i)' *There is a constant $c_3(D)$ such that for every $n \in \mathbb{N}$ with $n \geq c_3(D)$, the distance graph P_n^D is connected.*

For (iii) in Theorem 3, a similar change is not possible, i.e. no lower bound on the order n would imply that P_n^D has a Hamltonian cycle. If n as well as all elements of D are odd for instance, then P_n^D is bipartite and every cycle misses at least one vertex. In this sense, Theorem 3 is best-possible.

It follows easily from (1) that our proof yields the estimate

$$c_2(D) = O\left(\max(D)^2 + l\right) = O\left(\max(D)^3|D|\right).$$

For the case that $\gcd(D)$ is different from 1, Theorem 3 implies the following corollary.

Corollary 4 *For a finite set $D \subseteq \mathbb{N}$ and $d \in \mathbb{N}$, the following statements are equivalent.*

(i) There is a constant $c_4(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a component of order at least $\frac{n}{d} - c_4(D)$.

(ii) $\gcd(D) \leq d$.

(iii) There is a constant $c_5(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a cycle of order at least $\frac{n}{d} - c_5(D)$.

Theorem 3 trivially implies yet another condition which is equivalent to (i), (ii) and (iii) in Theorem 3:

(iv) *There is a constant $c_6(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a path of order at least $n - c_6(D)$.*

Clearly, such a path can be obtained from the cycle in (iii) by deleting one edge. It traverses $[0, n-1]$ several times back and forth just like the cycle does. We believe that there is also always a path containing almost all vertices of P_n^D which is *essentially monotonic*, i.e. it traverses $[0, n-1]$ once. The following conjecture makes this precise.

Conjecture 5 *For a finite set $D \subseteq \mathbb{N}$, the following statements are equivalent.*

(i) $\gcd(D) = 1$.

(ii) There are two constants $c_7(D)$ and $c_8(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a path $u_0 u_1 \dots u_l$ of order at least $n - c_7(D)$ such that $u_j > u_i$ for all $0 \leq i, j \leq l$ with $j - i \geq c_8(D)$.

A simple modification of the construction used in the proof of Theorem 3 implies the following weak version of Conjecture 5.

Theorem 6 *If $D \subseteq \mathbb{N}$ is a finite set with $\gcd(D) = 1$ and $\epsilon > 0$, then there are constants $c_9(D, \epsilon)$ and $c_{10}(D, \epsilon)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a path $u_0 u_1 \dots u_l$ of order at least*

$$(1 - \epsilon)n - c_9(D, \epsilon)$$

such that $u_j > u_i$ for all $0 \leq i, j \leq l$ with $j - i \geq c_{10}(D, \epsilon)$.

Proof: Since $\gcd(D) = 1$, D contains at least one odd element d_{odd}. Replacing the increasing and decreasing paths Q_i from the proof of Theorem 3 with increasing and decreasing paths R_i which only use edges uv with $v - u = d_{\text{odd}}$, and using the paths $P_1, P_2, \dots, P_{d_{\text{odd}}-1}$ and P' for parity changes as indicated in Figure 4, we obtain a path R from a vertex i_0 to a vertex $i_{d_{\text{odd}}} > i_0$ which both have residue r_1 modulo d_{odd}.

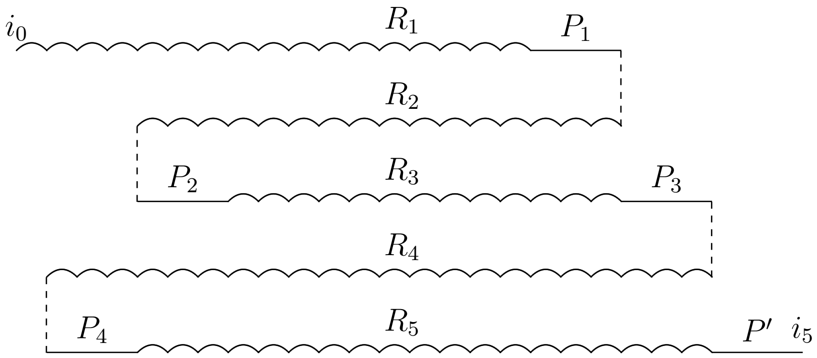

Figure 4

Note that R only visits vertices within $[i_0, i_{d_{\text{odd}}}]$. Furthermore, the number of vertices in $[i_0, i_{d_{\text{odd}}}]$ which R misses is bounded in terms of D. Therefore, increasing all paths $R_1, R_2, \ldots, R_{d_{\text{odd}}}$ by the same sufficiently large multiple of d_{odd}, we can ensure that R has order $(1-\epsilon)\,(i_{d_{\text{odd}}} - i_0 + 1)$. Concatenating shifted copies of R yields the desired path and completes the proof. □

Note that Conjecture 5 is trivial, if D contains only one element. If D contains exactly two elements, then Conjecture 5 easily follows from the following result.

Proposition 7 *If $d_1, d_2 \in \mathbb{N}$ are such that $d_1 > d_2$ and $\gcd(\{d_1, d_2\}) = 1$, then $P^{\{d_1,d_2\}}_{d_1+d_2+1}$ has a Hamiltonian path which begins at 0 and ends at $d_1 + d_2$.*

Proof: Consider the sequence $i_0, i_1, \ldots, i_{d_1+d_2}$ produced by Algorithm 1 below.

$i_0 := 0;$
$n := 0;$
$n_1 := 0;$
$n_2 := 0;$
while $n < d_1 + d_2$ **do**
  if $i_n \geq d_2$ *and* $n_2 < d_1 - 1$ **then**
    $i_{n+1} := i_n - d_2;$
    $n_2 := n_2 + 1;$
  else
    $i_{n+1} := i_n + d_1;$
    $n_1 := n_1 + 1;$
  end
  $n := n + 1;$
end

Algorithm 1

Clearly, $i_j \geq 0$ for $j \in [0, d_1 + d_2]$.

If $n_1 > d_2 + 1$ after the termination of the algorithm, then $n_2 = n - n_1 < d_1 - 1$ and hence

$$i_{d_1+d_2} = n_1 d_1 - n_2 d_2 \geq (d_2 + 2)d_1 - (d_1 - 2)d_2 = 2d_1 + 2d_2.$$

Let $j \in [1, d_1 + d_2 - 1]$ be maximum such that $i_{j+1} = i_j + d_1$. Clearly, $i_{j+1} \geq d_2$ and the algorithm would have set $i_{j+1} = i_j - d_2$ instead, which is a contradiction. Hence $n_1 \leq d_2 + 1$. Since $n_2 \leq d_1 - 1$, we obtain $n_1 = d_2 + 1$ and $n_2 = d_1 - 1$. This implies

$$i_{d_1+d_2} = (d_2 + 1)d_1 - (d_1 - 1)d_2 = d_1 + d_2.$$

If $i_j > d_1 + d_2$ for some $j \in [1, d_1 + d_2 - 1]$, then let j be largest with this property. Clearly, $i_{j-1} \geq d_2$ and at this moment of the execution of the algorithm $n_2 < d_1 - 1$. Therefore, the algorithm would have set $i_j = i_{j-1} - d_2$ instead, which is a contradiction. Hence $i_j \leq d_1 + d_2$ for all $j \in [0, d_1 + d_2]$,

If $i_r = i_s$ for some $r, s \in [0, d_1 + d_2]$ with $s > r$, then $i_s - i_r = a_1 d_1 - a_2 d_2 = 0$ for some $a_2 \in [1, d_1 - 1]$. This implies $a_1 d_1 = a_2 d_2$. Since $\gcd(\{d_1, d_2\}) = 1$, we obtain that a_2 must be a multiple of d_1, which is a contradiction. Hence all $d_1 + d_2 + 1$ integers $i_0, i_1, \ldots, i_{d_1+d_2}$ are distinct and define the desired Hamiltonian path of $P^{\{d_1,d_2\}}_{d_1+d_2+1}$. This completes the proof. $\Box$

It seems possible that Proposition 7 generalizes to sets D with more elements. For $D = \{6, 10, 15\}$ for instance, the pattern $0, 6, 12, 2, 8, 14, 4, 10, 16, 1, 7, 13, 3, 9, 15, 5, 11, 17$ yields a Hamiltonian path in P^D_{18} which begins at 0 and ends at 17.

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