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Abstract. For a graph $G$ let $\mu(G)$ denote the cyclomatic number and let $\nu(G)$ denote the maximum number of edge-disjoint cycles of $G$. We prove that for every $k \geq 0$ there is a finite set $P(k)$ such that every 2-connected graph $G$ for which $\mu(G) - \nu(G) = k$ arises by applying a simple extension rule to a graph in $P(k)$. Furthermore, we determine $P(k)$ for $k \leq 2$ exactly.

Keywords. graph; cycle; packing; cyclomatic number

1 Introduction

We consider finite and undirected graphs $G = (V_G, E_G)$ with vertex set $V_G$ and edge set $E_G$ which may contain multiple edges but no loops. We use standard terminology \cite{10} and only recall some basic notions. If an edge $e \in E_G$ has the two incident vertices $u$ and $v$ in $V_G$, then we write $e = uv$. The degree $d_G(u)$ in $G$ of a vertex $u \in V_G$ is the number of edges $e \in E_G$ incident with $u$. A path in $G$ of length $l \geq 0$ is a sequence $v_0e_1v_1e_2\ldots e_lv_l$ of distinct vertices $v_0, v_1, \ldots, v_l \in V_G$ and distinct edges $e_i = v_{i-1}v_i \in E_G$ for $1 \leq i \leq l$. A cycle in $G$ of length $l \geq 2$ is a sequence $v_1e_2v_2\ldots e_lv_lv_1$ such that $v_1e_2v_2\ldots e_lv_l$ is a path of length $(l - 1)$ and $e_l = v_lv_1 \in E_G$. The subgraph induced by some set $U \subseteq V_G$ is denoted by $G[U]$. An ear of $G$ is a path in $G$ of length at least 1 such that all internal vertices have degree 2 in $G$. An ear of $G$ is maximal, if it is not properly contained in another ear of $G$. If $P$ is an ear of $G$ and $I$ is the set of internal vertices of $P$, then we say that $G$ arises from $G' = (V_G \setminus I, E_G \setminus E_P)$ by adding the ear $P$ and that $G'$ arises from $G$ by removing the ear $P$. Whitney \cite{10,13} proved that a graph of order at least 2 is 2-connected if and only if it has an ear decomposition, i.e. it arises from a chordless cycle by iteratively adding ears. A graph is a cactus graph, if all of its cycles are edge-disjoint which is equivalent to the fact that all of its blocks are cycles or edges.

The cyclomatic number of a graph $G$ with $\kappa(G)$ components is

$$\mu(G) = |E_G| - |V_G| + \kappa(G).$$
A cycle packing \( C \) of order \( l \) is a set of \( l \) edge-disjoint cycles of \( G \). The maximum order of a cycle packing of \( G \) is denoted by

\[ \nu(G). \]

A cycle packing of maximum order is called optimal. For a cycle packing \( C \), the set of edges contained in some cycle in \( C \) is denoted by

\[ E_C. \]

Our research in the present paper is motivated by the well-known inequality

\[ \nu(G) \leq \mu(G) \]

which holds for every graph \( G \). As our main result, we prove that for every fixed \( k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) there is a finite set \( \mathcal{P}(k) \) of graphs such that every 2-connected graph \( G \) for which

\[ \mu(G) - \nu(G) = k \]

arises by applying a simple extension rule to one of the graphs in \( \mathcal{P}(k) \), i.e. there are essentially only finitely many configurations which cause \( \mu(G) \) and \( \nu(G) \) to deviate by \( k \). Furthermore, we determine \( \mathcal{P}(k) \) for \( k \leq 2 \) exactly.

The results which are most related to ours concern the minimum difference \( p(k) \) between the size \( |E_G| \) and the order \( |V_G| \) of a graph \( G \) which forces the existence of \( k \) edge-disjoint cycles, i.e.

\[ p(k) = \min \{p \mid \nu(G) \geq k \forall G = (V_G, E_G) \text{ with } |E_G| - |V_G| \geq p \}. \]

There are several classical results concerning this parameter

\[
p(k) = \begin{cases} 
0 & , k = 1 \\
4 & , k = 2 \quad [6] \\
10 & , k = 3 \quad [8] \\
18 & , k = 4 \quad [1, 14] \\
\Theta (k \log k) & \quad [6, 11, 12, 14].
\end{cases}
\]

Recently, algorithmic aspects of cycle packing problems have received considerable attention. While the problem to determine optimal cycle packings is APX-hard \([3, 4, 7, 9]\) and remains NP-hard even when restricted to Eulerian graphs of maximum degree 4 \([2]\), there are simple approximation algorithms \([3, 7]\).

In Section 2 we prove our main result about the finiteness of \( \mathcal{P}(k) \) and in Section 3 we determine \( \mathcal{P}(k) \) for \( k \leq 2 \) exactly.
2 Graphs $G$ with $\mu(G) - \nu(G) = k$

In this section we study the graphs $G$ for which $\mu(G)$ and $\nu(G)$ differ by some fixed $k$. It is well-known — and easy to see — that the graphs $G$ with $\mu(G) - \nu(G) = 0$ are exactly the cactus graphs, i.e. their blocks are either edges or arise by possibly subdividing the edges of a cycle of length 2.

For $k \in \mathbb{N}_0$ let

$$G(k)$$

denote the set of 2-connected graphs $G$ with $\mu(G) - \nu(G) = k$. In view of the above remark about cactus graphs, we obtain that $G \in G(0)$ if and only if $G$ is a cycle or an edge. The next lemma implies that in order to characterize the graphs $G$ with $\mu(G) - \nu(G) = k$, it suffices to characterize the 2-connected graphs with this property.

**Lemma 1** Let $k \in \mathbb{N}_0$. If $G$ is a graph with $\mu(G) - \nu(G) = k$ whose blocks $B_1, B_2, \ldots, B_l$ satisfy $B_i \in G(k_i)$ for $1 \leq i \leq l$, then $k = k_1 + k_2 + \cdots + k_l$.

**Proof:** This follows immediately from the fact that every cycle of $G$ is entirely contained in some block of $G$. \qed

In order to explain the simple extension rule mentioned in the introduction, we need some more notation.

An $l$-cycle-path is a cactus with at most 2 endblocks and exactly $l \in \mathbb{N}_0$ cycles.

An $l$-cycle-path-subgraph of a graph $G = (V_G, E_G)$ with attachment vertices $u$ and $v$ is an induced subgraph $H = (V_H, E_H)$ of $G$ which is an $l$-cycle-path such that $u$ and $v$ are two distinct vertices of $H$ for which $d_G(w) = d_H(w)$ for all $w \in V_H \setminus \{u, v\}$ and $H + uv = (V_H, E_H \cup \{uv\})$ is 2-connected, i.e. only the attachment vertices may have neighbours outside of $V_H$ and, if $H$ has more than one block, then the attachment vertices are two non-cutvertices from the two endblocks of $H$. Note that a 0-cycle-path-subgraph of $G$ with attachment vertices $u$ and $v$ is an ear of $G$ with endvertices $u$ and $v$.

A graph $H = (V_H, E_H)$ is said to arise from a graph $G = (V_G, E_G)$ by replacing the edge $e = uv \in E_G$ with an $l$-cycle-path, if $H$ has an $l$-cycle-path-subgraph $Q = (V_Q, E_Q)$ with attachment vertices $u$ and $v$ such that (cf. Figure 1)

$$V_G = V_H \setminus (V_Q \setminus \{u, v\})$$
$$E_G = (E_H \setminus E_Q) \cup \{e\}.$$
A graph $H$ is said to extend a graph $G$, if there is an optimal cycle packing $C$ of $G$ such that $H$ arises from $G$ by replacing every edge $e \in E_C$ with a 0-cycle-path and replacing every edge $e \in E_G \setminus E_C$ with an $l$-cycle-path for some $l \in \mathbb{N}_0$. A graph $H$ is said to be reduced, if there is no graph $G$ different from $H$ such that $H$ extends $G$.

For $k \in \mathbb{N}_0$ let
\[ P(k) \]
denote the set of reduced graphs in $G(k)$. Note that $P(0)$ contains exactly two elements, an edge and a cycle of length 2. It is instructive to verify that for $k \geq 1$ a graph in $P(k)$ contains neither vertices of degree at most 2 nor $l$-cycle-path-subgraphs for $l \geq 2$.

The next lemma summarizes some important properties of the above extension notion.

**Lemma 2** If $G_0 \in G(k)$, $G_1$ extends $G_0$, and $G_2$ extends $G_1$, then
- (i) $G_1 \in G(k)$,
- (ii) $G_2$ extends $G_0$, and
- (iii) every graph in $G(k)$ extends a graph in $P(k)$.

**Proof:** Let $C_0$ be an optimal cycle packing of $G_0$ such that $G_1$ arises from $G_0$ by replacing every edge $e \in E_{G_0}$ with an $l_e$-cycle-path $L_e$ with $l_e = 0$ for $e \in E_{C_0}$. Let $C_1'$ denote the set of the
\[ \sum_{e \in E_{G_0}} l_e \]
edge-disjoint cycles contained in the $l_e$-cycle-paths $L_e$ for $e \in E_{G_0}$.

Clearly,
\[ \mu(G_1) = \mu(G_0) + |C_1'|. \]

Since the set of cycles in $G_1$ which are subdivisions of the cycles in $C_0$ together with the cycles in $C_1'$ form a cycle packing of $G_1$, we obtain $\nu(G_1) \geq \nu(G_0) + |C_1'|$.

Let $C_1$ be an optimal cycle packing of $G_1$ such that $G_2$ arises from $G_1$ by replacing every edge $f \in E_{G_1}$ with an $h_f$-cycle-path $H_f$ with $h_f = 0$ for $f \in E_{C_1}$ and such that subject to this condition
\[ |C_1' \cap C_1| \]
is largest possible.

If $E_1'$ is an arbitrary set of edges which contains exactly one edge from each cycle in $C_1'$, then removing the $|C_1'|$ edges in $E_1'$ from $G_1$ can delete at most $|C_1'|$ cycles in $C_1$, which implies $\nu(G_0) \geq \nu(G_1) - |C_1'|$.

In view of the above, this implies that
\[ \nu(G_1) = \nu(G_0) + |C_1'| \] (1)
and hence (i).
Furthermore, this implies that every edge contained in a cycle in $C'_1$ belongs to $E_{C_1}$ and edges contained in different cycles in $C'_1$ are contained in different cycles in $C_1$. (Otherwise there would be a choice for $E'_1$ such that removing the edges in $E'_1$ would only delete at most $|C'_1| - 1$ cycles, which implies the contradiction $\nu(G_0) \geq \nu(G_1) - |C'_1| + 1$.)

If follows that, if $l_e \geq 2$ for some $e \in E_{G_0}$, then $C_1$ necessarily contains the $l_e$ edge-disjoint cycles contained in the $l_e$-cycle-path $L_e$.

Furthermore, if $l_e = 1$ for some $e \in E_{G_0}$ and $C_1$ does not contain the unique cycle $C_e$ contained in the 1-cycle-path $L_e$, then there are exactly two cycles $C'_e$ and $C''_e$ in $C_1$ which contain $E_{C_e}$. Since $(E_{C'_e} \cup E_{C''_e}) \setminus E_{C_e}$ contains the edge set of a cycle $C''''_e$,

$$\tilde{C}_1 = (C_1 \setminus \{C'_e, C''_e\}) \cup \{C_e, C''''_e\}$$

is an optimal cycle packing of $G_1$ such that $E_{\tilde{C}_1} \subseteq E_{C_1}$ and

$$|C'_1 \cap \tilde{C}_1| > |C'_1 \cap C_1|$$

which is a contradiction to the choice of $C_1$.

Hence $C'_1 \subseteq C_1$. By (1), the cycles in $C_1 \setminus C'_1$ are the subdivisions of the cycles in an optimal cycle packing $C'_0$ of $G_0$. Clearly, $l_e > 0$ implies $e \notin E_{C'_0}$. Since $h_f > 0$ for some $f \in E_{G_1} \setminus E_{C_1}$ implies that $f$ is a bridge of an $l_e$-cycle-path $L_e$ with $e \notin E_{C'_e}$, it follows that $G_2$ extends $G_0$, i.e. (ii) holds.

By definition, for every graph $H \in \mathcal{G}(k)$ there is a graph $G \in \mathcal{P}(k)$ such that $H$ arises from $G$ by a finite sequence of extensions. Applying (ii) in an inductive argument implies that $H$ extends $G$ and (iii) follows. This completes the proof. \(\square\)

We proceed to our main result.

**Theorem 3** The set $\mathcal{P}(k)$ is finite for every $k \in \mathbb{N}_0$.

**Proof:** We will prove the result by induction on $k$.

Since $|\mathcal{P}(0)| = 2$, we may assume that $k \geq 1$.

We will argue that every graph in $\mathcal{P}(k)$ arises from some graph in $\mathcal{P}(k-1)$ by applying a subset of a finite set of operations. Since, by induction, $\mathcal{P}(k-1)$ is finite, this clearly implies that $\mathcal{P}(k)$ is finite.

Let $H \in \mathcal{P}(k)$.

If a graph $H^-$ arises by removing an ear from $H$, then

$$\nu(H) - 1 \leq \nu(H^-) \leq \nu(H) \quad \text{and} \quad \mu(H^-) = \mu(H) - 1,$$

i.e. $H^- \in \mathcal{G}(k-1)$ or $H^- \in \mathcal{G}(k)$. Therefore, an ear decomposition of $H$ yields a sequence of 2-connected graphs

$$G_0, G_1, \ldots, G_l$$

such that

- $G_l = H$, 

5
\begin{itemize}
  \item $G_i$ arises by adding the ear $P_i$ to $G_{i-1}$ for $1 \leq i \leq l$,
  \item $\nu(G_0) = \nu(G_1)$ and
  \item $\nu(G_{i-1}) = \nu(G_i) - 1$ for $2 \leq i \leq l$.
\end{itemize}

We assume that the sequence is chosen to be shortest possible, i.e. $l$ is minimum.

Note that $G_0 \in \mathcal{G}(k-1)$ and $G_i \in \mathcal{G}(k)$ for $1 \leq i \leq l$.

By Lemma 2 (iii), $G_0$ extends some graph $G \in \mathcal{P}(k-1)$.

Let $C_l$ be an optimal cycle packing of $H = G_l$.

Since for $l \geq 2$ we have $\nu(G_{l-1}) = \nu(G_l) - 1$ and removing the ear $P_l$ from $G_l$ can only affect one cycle from $C_l$, the ear $P_l$ is contained in a unique cycle $C_l \in C_l$ and

\[ C_{l-1} := C_l \setminus \{C_l\} \]

is an optimal cycle packing of $G_{l-1}$. Iterating this argument, we obtain that for $i = l, (l-1), (l-2), \ldots, 2$, the ear $P_i$ is contained in a unique cycle $C_i \in C_i \subseteq C_l$

and that

\[ C_{i-1} := C_i \setminus \{C_i, C_{i+1}, \ldots, C_l\} \]

is an optimal cycle packing of $G_{i-1}$. Note that this argument does not apply to $i = 1$, because $\nu(G_0) = \nu(G_1)$.

Since each of the ears in

\[ \mathcal{E} = \{P_2, P_3, \ldots, P_l\} \]

is contained in a unique different cycle in $C_l$, no internal vertex of any $P_i$ is contained in any $P_j$ for $2 \leq i \leq l$ and $1 \leq j \leq l$ with $i \neq j$. Since $H$ is reduced and hence has no vertex of degree 2, this implies that the ears in $\mathcal{E}$ all have length 1, i.e. they are all edges.

Let $P = v_0e_1v_1e_2v_2\ldots erv_r$

be a maximal ear of $G_1$. Since $G_1$ is 2-connected and $k \geq 1$, the endvertices $v_0$ and $v_r$ of $P$ are of degree at least 3. Let

\[ I = \{v_1, v_2, \ldots, v_{r-1}\} \]

be the set of internal vertices of $P$. 
The next claim is obvious.

**Claim A** If an ear $P_i$ for $2 \leq i \leq l$ has exactly one endvertex in $I$, then $C_i$ contains either the edge $e_1$ or the edge $e_r$. Therefore, at most two ears in $\mathcal{E}$ have exactly one endvertex in $I$.

**Claim B** No ear $P_i$ for $2 \leq i \leq l$ has its two endvertices in $I$.

*Proof of Claim B:* For contradiction, we assume that the index $i$ with $2 \leq i \leq l$ is minimum such that $P_i$ has the endvertices $v_x, v_y \in I$ for $1 \leq x < y \leq r-1$. Since $\nu(G_{i-1}) = \nu(G_i) - 1$, the cycle $C_i$ is formed by $P_i$ and the subpath $P''$ of $P$ between $v_x$ and $v_y$. This implies that no internal vertex of $P'$ is an endvertex of an ear $P_j \in \mathcal{E} \setminus \{P_i\}$. Hence $P_i$ is an ear of $H$ and $C_i$ is a 1-cycle-path-subgraph of $H$.

Let $H'$ arise from $H$ by removing the ear $P_i$. If $\nu(H') = \nu(H)$, we may choose $\tilde{G}_0 = H'$, $\tilde{P}_1 = P_i$ and $\tilde{G}_1 = H$ contradicting the choice of the sequence $G_0, G_1, \ldots, G_l$ as shortest possible. Hence $\nu(H') = \nu(H) - 1$. This implies that $H'$ has an optimal cycle packing not using the edges of $P''$ and $H$ is not reduced, which is a contradiction. □

**Claim C** $G_1$ does not contain a 2-cycle-path-subgraph.

*Proof of Claim C:* For contradiction, we assume that $Q$ is a 2-cycle-path-subgraph of $G_1$ with attachment vertices $u$ and $v$. We may assume that $d_Q(u), d_Q(v) \geq 2$, i.e. that the 2 cycles $C'$ and $C''$ of $Q$ are the endblocks of $Q$.

Clearly, for every optimal cycle packing $\mathcal{C}'_1$ of $G_1$, we have $E_{C'} \cup E_{C''} \subseteq E_{\mathcal{C}_1}$. This implies that $E_{C'} \cup E_{C''} \subseteq E_{C_1}$ and, by Claims A and B, no ear in $\mathcal{E}$ has an endvertex in $V_Q \setminus \{u, v\}$. Hence $Q$ is also a 2-cycle-path-subgraph of $H$ and $H$ is not reduced, which is a contradiction. □

Since $G_1$ arises by adding the ear $P_1$ to $G_0$, Claim C implies that $G_0$ does not contain an $s$-cycle-path-subgraph for $s \geq 6$. Since every $s$-cycle-path-subgraph for $s \leq 5$ yields at most $2 \cdot 5 + 6 = 16$ maximal ears, this implies that the number of maximal ears of $G_0$ is at most $16|E_G|$ and hence the number of maximal ears of $G_1$ is at most $16|E_G| + 3$.

Since $H$ is reduced and hence has no vertex of degree 2, Claim A implies that no maximal ear of $G_1$ has more than 2 internal vertices. This implies that the order $|V_{G_1}|$ and size $|E_{G_1}|$ of $G_1$ is bounded in terms of the size $|E_G|$ of $G$.

Since all ears in $\mathcal{E}$ are edges between vertices of $G_1$, the number of ears in $\mathcal{E}$ with different endvertices is bounded in terms of $|V_{G_1}|$, i.e. it is bounded in terms of $|E_G|$.

Furthermore, since the ears in $\mathcal{E}$ all lie in different edge-disjoint cycles, the number of ears in $\mathcal{E}$ which have the same endvertices is bounded by the size $|E_{G_1}|$ of $G_1$, i.e. it is bounded in terms of $|E_G|$.

Altogether, $G_1$ arises from $G$ by applying a subset of a set of operations whose cardinality is bounded in terms of $|E_G|$, and $H$ arises from $G_1$ by applying a subset of a set of operations whose cardinality is also bounded in terms of $|E_G|$.
This completes the proof. □

The reader should note that the proof of Theorem 3 yields a — rather inefficient — algorithm which for \( k \geq 1 \) allows to derive \( \mathcal{P}(k) \) from \( \mathcal{P}(k-1) \) and has a running time which is bounded in terms of \(|\mathcal{P}(k-1)|\) and the maximum size of graphs in \( \mathcal{P}(k-1) \). Therefore, for every fixed \( k \), we can — in principle — determine \( \mathcal{P}(k) \) in finite time.

We finish this section with another algorithmic consequence of Theorem 3.

Let \( k \in \mathbb{N}_0 \) be fixed and let \( G \) be a fixed graph in \( \mathcal{P}(k) \).

For a given 2-connected graph \( H \) as input, we can decide in polynomial time whether \( H \) extends \( G \). The simplest argument implying this might be to consider all injective mappings of \( V_G \) to \( V_H \) and check whether the edges of \( G \) can be suitable replaced by cycle-paths in order to obtain \( H \). This can clearly be done in polynomial time.

Therefore, in view of Lemma 1 and Theorem 3, for a given graph \( H \) as input, we can decide in polynomial time whether \( \mu(H) - \nu(H) = k \). Furthermore, in view of the proof of Lemma 2, we can also efficiently construct an optimal cycle packing of \( H \) — even all of them — in this case.

### 3 \( \mathcal{P}(1) \) and \( \mathcal{P}(2) \)

In this section we illustrate Theorem 3 and determine \( \mathcal{P}(1) \) and \( \mathcal{P}(2) \) explicitly.

The following lemma captures a straightforward yet important observation which was essentially also used by the proof of Theorem 3.

**Lemma 4** Let \( k \geq 1 \).

(i) Every graph \( H \in \mathcal{P}(k) \) arises by adding an edge to a graph \( G \) such that either \( \nu(G) = \nu(H) \) and \( G \) extends a graph in \( \mathcal{P}(k-1) \), or \( \nu(G) = \nu(H) - 1 \) and \( G \) extends a graph in \( \mathcal{P}(k) \).

(ii) Let \( Q \subseteq \mathcal{P}(k) \).

If every graph \( H \in \mathcal{P}(k) \) which arises by adding an edge to a graph \( G \) such that either \( \nu(G) = \nu(H) \) and \( G \) extends a graph in \( \mathcal{P}(k-1) \), or \( \nu(G) = \nu(H) - 1 \) and \( G \) extends a graph in \( Q \), also belongs to \( Q \), then \( Q = \mathcal{P}(k) \).

**Proof:** (i) Let \( H \in \mathcal{P}(k) \) and let \( P \) be the last ear in some ear decomposition of \( H \).

Since \( H \) is reduced, \( P \) has length 1, i.e. it is an edge. Let \( G \) arise by removing \( P \) from \( H \).

Clearly, \( \mu(G) = \mu(H) - 1 \) while \( \nu(G) = \nu(H) \) or \( \nu(G) = \nu(H) - 1 \).

By the definition of \( \mathcal{P}(k) \), \( \nu(G) = \nu(H) \) implies that \( G \) extends a graph in \( \mathcal{P}(k-1) \) and \( \nu(G) = \nu(H) - 1 \) implies that \( G \) extends a graph in \( \mathcal{P}(k) \).

(ii) Let \( H \in \mathcal{P}(k) \).
Iteratively deleting edges as in (i) and reducing the constructed graphs, we obtain a sequence \( G_0, G_1, \ldots, G_l \) such that \( G_0 \in \mathcal{P}(k-1) \), \( G_i \in \mathcal{P}(k) \) for \( 1 \leq i \leq l \), \( G_i \) contains an edge \( e_i \) such that \( G_i - e_i \) extends \( G_{i-1} \) for \( 1 \leq i \leq l \) and \( G_l = H \).

Since \( G_{i-1} \) has less edges than \( G_i \) for \( 1 \leq i \leq l \), the sequence is finite.

Inductively applying the hypothesis, we obtain that \( G_i \in \mathcal{Q} \) for \( 1 \leq i \leq l \), i.e. \( H \in \mathcal{Q} \) which implies \( \mathcal{Q} = \mathcal{P}(k) \). □

Note that Lemma 4 (ii) yields a criterion to check whether some subset \( \mathcal{Q} \) of \( \mathcal{P}(k) \) already contains all of \( \mathcal{P}(k) \). Therefore, the proofs of the following two results reduce to tedious yet straightforward case analysis. The following result is in fact equivalent to a result in [5].

**Theorem 5** \( \mathcal{P}(1) = \{ K^3_2 \} \) where \( K^3_2 \) is the unique graph with two vertices and three parallel edges (cf. Figure 2).

**Proof:** It is easy to verify that \( K^3_2 \in \mathcal{P}(1) \).

Note that the only graphs extending graphs in \( \mathcal{P}(0) \) are cycle-paths. This easily implies that, if \( H \in \mathcal{P}(1) \) arises by adding an edge to a graph \( G \) with \( \nu(G) = \nu(H) \) such that \( G \) extends a graph in \( \mathcal{P}(0) \), then \( H = K^3_2 \).

Furthermore, if \( H \in \mathcal{P}(1) \) arises by adding an edge to a graph \( G \) with \( \nu(G) = \nu(H) - 1 \) and \( G \) extends \( K^3_2 \), then \( H \) extends \( K^3_2 \). Since \( H \) is reduced, we obtain \( H = K^3_2 \).

By Lemma 4 (ii), the proof is complete. □

![Figure 2](image)

**Figure 2** \( \mathcal{P}(1) = \{ K^3_2 \} \).

We say that the graphs which arise from one of the two graphs \( G_1 \) or \( G_2 \) in Figure 3 by contracting a subset of the edges indicated by dashed lines are *generated from* \( G_1 \) or \( G_2 \), respectively.

![Figure 3](image)

**Figure 3** The graphs \( G_1, G_2 \in \mathcal{P}(2) \).

**Theorem 6** \( \mathcal{P}(2) \) consists of \( K_4 \) and all graphs which are generated from \( G_1 \) or \( G_2 \).
Proof: It is easy to verify that $K_4$ and all graphs which are generated from $G_1$ or $G_2$ belong to $\mathcal{P}(2)$.

Let $H \in \mathcal{P}(2)$.

We consider different cases.

**Case 1** $H$ arises by adding an edge $uv$ to a graph $G$ with $\nu(G) = \nu(H) = 1$ such that $G$ extends $K_2^3$.

In this case $G$ is a subdivision of $K_2^3$.

Since $\nu(H) = 1$, the vertices $u$ and $v$ are not contained in a common maximal ear of $G$. This implies that $H = K_4$.

**Case 2** $H$ arises by adding an edge $uv$ to a graph $G$ with $\nu(G) = \nu(H) \geq 2$ such that $G$ extends $K_2^3$.

In this case $G$ has a unique optimal cycle packing $\mathcal{C}$.

- If $d_G(u) = d_G(v) = 2$ and $u$ and $v$ lie on a maximal ear contained in a cycle in $\mathcal{C}$, then $H = G_2$.
- If $d_G(u) = d_G(v) = 2$ and $u$ and $v$ lie in different maximal ears contained in one cycle in $\mathcal{C}$, then $H$ extends $K_4$. Since $H \neq K_4$, $H$ is not reduced which is a contradiction.
- If $d_G(u) = d_G(v) = 2$ and $u$ and $v$ lie in different cycles in $\mathcal{C}$, then $H$ is generated from $G_1$.
- If $d_G(u) \geq 3$, $d_G(v) = 2$ and $v$ lies in a cycle in $\mathcal{C}$, then $H$ extends $K_4$. Since $H \neq K_4$, $H$ is not reduced which is a contradiction.

In all remaining subcases, $H$ is generated from $G_2$.

**Case 3** $H$ arises by adding an edge $uv$ to a graph $G$ with $\nu(G) = \nu(H) - 1$ such that $G$ extends $K_4$.

Let $v_1, v_2, v_3, v_4$ denote the vertices of $K_4$. We may assume that $G$ arises by replacing the edges $v_i v_j$ with $l_{i,j}$-cycle-paths $Q_{i,j}$.

Since $H$ is reduced and $\nu(G) = \nu(H) - 1$, the vertices $u$ and $v$ are not both contained in one of the cycle-paths $Q_{i,j}$ and we obtain that $H$ is generated from $G_1$.

**Case 4** $H$ arises by adding an edge $uv$ to a graph $G$ with $\nu(G) = \nu(H) - 1$ such that $G$ extends a graph generated from $G_1$.

It is easy to verify that $\nu(G) = \nu(H) - 1$ implies that $H$ is generated from $G_1$.

**Case 5** $H$ arises by adding an edge $uv$ to a graph $G$ with $\nu(G) = \nu(H) - 1$ such that $G$ extends a graph generated from $G_2$.

It is easy to verify that $\nu(G) = \nu(H) - 1$ implies that $H$ is generated from $K_4$ or $G_2$.

By Lemma 4 (ii), the proof is complete. □
References


