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Spectral properties of singular Sturm-Liouville operators with indefinite weight $\operatorname{sgn} x$

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(MS received ;)

We consider a singular Sturm-Liouville expression with the indefinite weight $\operatorname{sgn} x$. To this expression there is naturally a self-adjoint operator in some Krein space associated. We characterize the local definitizability of this operator in a neighbourhood of ∞ . Moreover, in this situation, the point ∞ is a regular critical point. We construct an operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ with non-real spectrum accumulating to a real point. The obtained results are applied to several classes of Sturm-Liouville operators.

1. Introduction

We consider the singular Sturm-Liouville differential expression

$$a(y)(x) = (\operatorname{sgn} x)(-y''(x) + q(x)y(x)), \quad x \in \mathbb{R}, \quad (1.1)$$

with the signum function as indefinite weight and a real potential $q \in L^1_{loc}(\mathbb{R})$. We assume that (1.1) is in the limit point case at both $-\infty$ and $+\infty$. This differential expression is naturally connected with a self-adjoint operator A in the Krein space $(L^2(\mathbb{R}), [\cdot, \cdot])$ (see e.g. [12]), where the indefinite inner product $[\cdot, \cdot]$ is defined by

$$[f, g] = \int_{\mathbb{R}} f \bar{g} \operatorname{sgn} x \, dx, \quad f, g \in L^2(\mathbb{R}).$$

The operator $J : f(x) \mapsto (\operatorname{sgn} x)f(x)$ is a fundamental symmetry in the Krein space $(L^2(\mathbb{R}), [\cdot, \cdot])$. Let us define the operator $L := JA$. Then $L = -d^2/dx^2 + q$ is a self-adjoint Sturm-Liouville operator in the Hilbert space $L^2(\mathbb{R})$. It was shown in [12] that if L is a non-negative operator in the Hilbert space sense then A is a definitizable operator with ∞ as a regular critical point.

In general, the operator A may be not definitizable (in Section 3 we give a criterion). However, under certain assumptions, A is still locally definitizable over an appropriate subset of \mathbb{C} . It seems that the first result of such type was obtained in [5] for the operator $y \mapsto \frac{1}{w}[(py)'] + qy$ with w as indefinite weight function. Note that in [5] w may have many turning points, but rather strong assumptions on the spectra of certain associated self-adjoint operators are supposed.

As a main result we show the equivalence of the semi-boundedness from below of the operator L and the local definitizability of the operator A in a neighbourhood of ∞ . Moreover, we give a precise description of the domain of definitizability of A . If L is semi-bounded from below, we show the existence of a decomposition $A = \mathcal{A}_\infty \dot{+} \mathcal{A}_b$ such that the operator \mathcal{A}_∞ is similar to a self-adjoint operator in

the Hilbert space sense and \mathcal{A}_b is a bounded operator, that is, the point ∞ is a regular critical point. Hence, the non-real spectrum of A remains bounded. But, in contrast to the case of a non-negative operator L , now the non-real spectrum may accumulate to the real axis. We prove in Section 4 the existence of an even continuous potential q with a sequence of non-real eigenvalues of A accumulating to a real point. This potential q can be chosen in such a way that A is definitizable over $\mathbb{C} \setminus \{0\}$.

Finally, in Section 5, we discuss the spectrum and the sets of definitizability of A for various classes of potentials q .

Differential operators with indefinite weights appears in many areas of physics and applied mathematics (see [4, 21, 28, 43] and references therein). Under certain assumptions such operators are definitizable; this case was studied extensively (see [8, 12, 13, 14, 15, 18, 19, 20, 32, 35, 36, 42, 44, 47] and references therein). In [5, 6, 7, 29, 31, 33, 34] certain classes of differential operators that contain definitizable as well as not definitizable operators were considered.

Notation: Let T be a linear operator in a Hilbert space \mathfrak{H} . In what follows $\text{dom}(T)$, $\ker(T)$, $\text{ran}(T)$ are the domain, kernel, range of T , respectively. We denote the resolvent set by $\rho(T)$; $\sigma(T) := \mathbb{C} \setminus \rho(T)$ stands for the spectrum of T . By $\sigma_p(T)$ the set of eigenvalues of T is indicated. The discrete spectrum $\sigma_{disc}(T)$ is the set of isolated eigenvalues of finite algebraic multiplicity; the essential spectrum is $\sigma_{ess}(T) := \sigma(T) \setminus \sigma_{disc}(T)$. We denote the indicator function of a set S by $\chi_S(\cdot)$.

2. Sturm-Liouville operators with the indefinite weight $\text{sgn } x$

2.1. Differential operators

We consider the differential expression

$$\ell(y)(x) = -y''(x) + q(x)y(x), \quad x \in \mathbb{R} \quad (2.1)$$

with a real potential $q \in L^1_{loc}(\mathbb{R})$. Throughout this paper it is assumed that we have limit point case at both $-\infty$ and $+\infty$. We set

$$a(y)(x) = (\text{sgn } x)(-y''(x) + q(x)y(x)), \quad x \in \mathbb{R}.$$

Let \mathfrak{D} be the set of all $f \in L^2(\mathbb{R})$ such that f and f' are absolutely continuous with $\ell(f) \in L^2(\mathbb{R})$. On \mathfrak{D} we define the operators A and L as follows:

$$\text{dom}(A) = \text{dom}(L) = \mathfrak{D}, \quad Ay = a(y), \quad Ly = \ell(y).$$

We equip $L^2(\mathbb{R})$ with the indefinite inner product

$$[f, g] := \int_{\mathbb{R}} (\text{sgn } x) f(x) \overline{g(x)} dx, \quad f, g \in L^2(\mathbb{R}). \quad (2.2)$$

Then $(L^2(\mathbb{R}), [\cdot, \cdot])$ is a Krein space (for the definition of a Krein space and basic notions therein we refer to [2]). A fundamental symmetry J in $(L^2(\mathbb{R}), [\cdot, \cdot])$ is given by

$$(Jf)(x) = (\text{sgn } x)f(x), \quad f \in L^2(\mathbb{R}).$$

Obviously,

$$A = JL$$

holds.

Since the differential expressions $a(\cdot)$ and $\ell(\cdot)$ are in the limit point case both at $+\infty$ and $-\infty$, the operator L is self-adjoint in the Hilbert space $L^2(\mathbb{R})$. As $A = JL$, the operator A is self-adjoint in the Krein space $L^2(\mathbb{R}, [\cdot, \cdot])$.

Definition 2.1. *We shall say that A is the operator associated with the differential expression $a(\cdot)$.*

2.2. Titchmarsh-Weyl coefficients

In the following we denote by \mathbb{C}_\pm the set $\{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$. Let $c_\lambda(x)$ and $s_\lambda(x)$ denote the fundamental solutions of the equation

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in \mathbb{R}, \quad (2.3)$$

which satisfy the following conditions

$$c_\lambda(0) = s'_\lambda(0) = 1; \quad c'_\lambda(0) = s_\lambda(0) = 0.$$

Since the equation (2.3) is limit-point at $+\infty$, the Titchmarsh-Weyl theory (see, for example, [40]) states that there exists a unique holomorphic function $m_+(\lambda)$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$, such that the function $s_\lambda(\cdot) - m_+(\lambda)c_\lambda(\cdot)$ belongs to $L^2(\mathbb{R}_+)$. Similarly, the limit point case at $-\infty$ yields the fact that there exists a unique holomorphic function $m_-(\lambda)$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$, such that $s_\lambda(\cdot) + m_-(\lambda)c_\lambda(\cdot) \in L^2(\mathbb{R}_-)$. The function m_+ (m_-) is called *the Titchmarsh-Weyl m -coefficient for (2.3) on \mathbb{R}_+ (on \mathbb{R}_- , respectively)*.

We put

$$M_\pm(\lambda) := \pm m_\pm(\pm\lambda).$$

Definition 2.2. *The function $M_+(\cdot)$ ($M_-(\cdot)$) is said to be the Titchmarsh-Weyl coefficient of the differential expression $a(\cdot)$ on \mathbb{R}_+ (on \mathbb{R}_-).*

It is easy to see that for $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ the functions

$$\psi_\lambda^\pm(x) := \begin{cases} s_{\pm\lambda}(x) - M_\pm(\lambda)c_{\pm\lambda}(x), & x \in \mathbb{R}_\pm \\ 0, & x \in \mathbb{R}_\mp \end{cases} \quad (2.4)$$

belongs to $L^2(\mathbb{R})$. Moreover, the following formula (see [40]) for the norms of ψ_λ^\pm in $L^2(\mathbb{R})$ holds true

$$\|\psi_\lambda^\pm(x)\|^2 = \frac{\operatorname{Im} M_\pm(\lambda)}{\operatorname{Im} \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (2.5)$$

A holomorphic function $G : \mathbb{C}_+ \cup \mathbb{C}_- \rightarrow \mathbb{C}$ is called *Nevanlinna function* or *of class (R)*, see e.g. [27], if $G(\bar{\lambda}) = \overline{G(\lambda)}$ and $\operatorname{Im} \lambda \cdot \operatorname{Im} G(\lambda) \geq 0$ for $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$. It

follows easily from (2.5) that the functions M_+ and M_- (as well as m_{\pm}) belong to the class (R). Moreover, the functions M_{\pm} have the following asymptotic behavior

$$M_{\pm}(\lambda) = \pm \frac{i}{\sqrt{\pm\lambda}} + O\left(\frac{1}{|\lambda|}\right), \quad (\lambda \rightarrow \infty, 0 < \delta < \arg \lambda < \pi - \delta) \quad (2.6)$$

for $\delta \in (0, \frac{\pi}{2})$, see [17]. Here and below \sqrt{z} is the branch of the multifunction on the complex plane \mathbb{C} with the cut along \mathbb{R}_+ , singled out by the condition $\sqrt{-1} = i$.

2.3. The non-real spectrum of A

In the following we identify functions $f \in L^2(\mathbb{R})$ with elements $\begin{pmatrix} f_+ \\ f_- \end{pmatrix}$, where $f_{\pm} := f \upharpoonright_{\mathbb{R}_{\pm}} \in L^2(\mathbb{R}_{\pm})$. Similarly we write $q_{\pm} := q \upharpoonright_{\mathbb{R}_{\pm}} \in L^1_{\text{loc}}(\mathbb{R}_{\pm})$. Note that the differential expressions

$$-\frac{d^2}{dx^2} + q_+ \quad \text{and} \quad \frac{d^2}{dx^2} - q_-$$

in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$ are both regular at the endpoint 0 and in the limit point case at the singular endpoint $+\infty$ and $-\infty$, respectively. Therefore the operators

$$A_{\min}^+ f_+ = -f_+'' + q_+ f_+ \quad \text{and} \quad A_{\min}^- f_- = f_-'' - q_- f_-$$

defined on

$$\text{dom } A_{\min}^{\pm} = \{f_{\pm} \in \mathcal{D}_{\max}^{\pm} : f_{\pm}(0) = f_{\pm}'(0) = 0\},$$

with

$$\begin{aligned} \mathcal{D}_{\max}^+ &= \{f_+ \in L^2(\mathbb{R}_+) : f_+, f_+' \text{ absolutely continuous, } -f_+'' + q_+ f_+ \in L^2(\mathbb{R}_+)\}, \\ \mathcal{D}_{\max}^- &= \{f_- \in L^2(\mathbb{R}_-) : f_-, f_-' \text{ absolutely continuous, } f_-'' - q_- f_- \in L^2(\mathbb{R}_-)\}, \end{aligned}$$

are closed symmetric operators in the Hilbert spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, cf. [45, 46], with deficiency indices $(1, 1)$. The adjoint operators $(A_{\min}^{\pm})^*$ in the Hilbert space $L^2(\mathbb{R}_{\pm})$ are the usual maximal operators defined on \mathcal{D}_{\max}^{\pm} .

We introduce the operators

$$A_0^+ f_+ = -f_+'' + q_+ f_+ \quad \text{and} \quad A_0^- f_- = f_-'' - q_- f_-$$

defined on

$$\text{dom } A_0^{\pm} = \{f_{\pm} \in \mathcal{D}_{\max}^{\pm} : f_{\pm}'(0) = 0\},$$

Evidently, A_0^{\pm} are self-adjoint extensions of A_{\min}^{\pm} in the Hilbert spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, cf. [45, 46]. In the following we consider $\text{dom } A_{\min}^{\pm}$ as subsets of $L^2(\mathbb{R})$. Then above considerations imply the following lemma.

Lemma 2.3. *Let $\text{dom } A_{\min} := \text{dom } A_{\min}^+ \oplus \text{dom } A_{\min}^-$ and let the operator A_{\min} be defined on $\text{dom } A_{\min}$,*

$$A_{\min} := \begin{pmatrix} A_{\min}^+ & 0 \\ 0 & A_{\min}^- \end{pmatrix},$$

with respect to the decomposition $L^2(\mathbb{R}) = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$. Then A_{\min} is a closed symmetric operator in the Hilbert space $L^2(\mathbb{R})$ with deficiency indices $(2, 2)$. Moreover, we have

$$A_{\min} = A \upharpoonright_{\operatorname{dom} A_{\min}}, \quad A = A_{\min}^* \upharpoonright_{\mathfrak{D}},$$

where

$$\begin{aligned} \mathfrak{D} &= \operatorname{dom}(A) = \\ &= \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in \operatorname{dom}(A_{\min}^+) \oplus \operatorname{dom}(A_{\min}^-) : f_+(0) = f_-(0), f'_+(0) = f'_-(0) \right\}. \end{aligned}$$

In the following proposition we collect some spectral properties of A .

Proposition 2.4. *Let A be the operator associated with the differential expression $a(\cdot)$. Then:*

(i) $\{\lambda \in \mathbb{C} \setminus \mathbb{R} : M_+(\lambda) = M_-(\lambda)\} = \sigma_p(A) \setminus \mathbb{R};$

(ii) $\{\lambda \in \mathbb{C} \setminus \mathbb{R} : M_+(\lambda) \neq M_-(\lambda)\} = \rho(A) \setminus \mathbb{R};$

(iii) $\rho(A) \neq \emptyset.$

(iv) *The essential spectrum $\sigma_{ess}(A)$ of A is real and*

$$\sigma_{ess}(A) = \sigma_{ess}(A_0^+) \cup \sigma_{ess}(A_0^-).$$

The sets $\sigma_p(A) \cap \mathbb{C}_{\pm}$ are at most countable with possible limit points belonging to $\sigma_{ess}(A) \cup \{\infty\}$.

For a proof of Proposition 2.4 we refer to [34, Proposition 2.5] and [30, 31]. We mention only that the statements (iii) and (iv) follow from the first and second statement and (2.6).

3. Criteria for definitizability

3.1. Definitizable and locally definitizable operators

Let $(\mathcal{H}, [\cdot, \cdot])$ be a Krein space and let A be a closed operator in \mathcal{H} . We define the extended spectrum $\sigma_e(A)$ of A by $\sigma_e(A) := \sigma(A)$ if A is bounded and $\sigma_e(A) := \sigma(A) \cup \{\infty\}$ if A is unbounded. We set $\rho_e(A) := \overline{\mathbb{C}} \setminus \sigma_e(A)$. A point $\lambda_0 \in \mathbb{C}$ is said to belong to the *approximative point spectrum* $\sigma_{ap}(A)$ of A if there exists a sequence $(x_n) \subset \operatorname{dom}(A)$ with $\|x_n\| = 1$, $n = 1, 2, \dots$, and $\|(A - \lambda_0)x_n\| \rightarrow 0$ if $n \rightarrow \infty$. For a self-adjoint operator A in \mathcal{H} all real spectral points of A belong to $\sigma_{ap}(A)$ (see e.g. [9, Corollary VI.6.2]).

First we recall the notions of spectral points of positive and negative type.

The following definition was given in [37], [39] (for bounded self-adjoint operators).

Definition 3.1. *For a self-adjoint operator A in \mathcal{H} a point $\lambda_0 \in \sigma(A)$ is called a spectral point of positive (negative) type of A if $\lambda_0 \in \sigma_{ap}(A)$ and for every sequence $(x_n) \subset \operatorname{dom}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \rightarrow 0$ for $n \rightarrow \infty$, we have*

$$\liminf_{n \rightarrow \infty} [x_n, x_n] > 0 \quad (\text{resp. } \limsup_{n \rightarrow \infty} [x_n, x_n] < 0).$$

The point ∞ is said to be of positive (negative) type of A if A is unbounded and for every sequence $(x_n) \subset \text{dom}(A)$ with $\lim_{n \rightarrow \infty} \|x_n\| = 0$ and $\|Ax_n\| = 1$ we have

$$\liminf_{n \rightarrow \infty} [Ax_n, Ax_n] > 0 \quad (\text{resp. } \limsup_{n \rightarrow \infty} [Ax_n, Ax_n] < 0).$$

We denote the set of all points of $\sigma_e(A)$ of positive (negative) type by $\sigma_{++}(A)$ (resp. $\sigma_{--}(A)$). We shall say that an open subset δ of \mathbb{R} ($= \mathbb{R} \cup \infty$) is of positive type (negative type) with respect to A if

$$\delta \cap \sigma_e(A) \subset \sigma_{++}(A) \quad (\text{resp. } \delta \cap \sigma_e(A) \subset \sigma_{--}(A)).$$

An open set δ of $\overline{\mathbb{R}}$ is called of definite type if δ is of positive or negative type with respect to A .

The sets $\sigma_{++}(A)$ and $\sigma_{--}(A)$ are contained in $\overline{\mathbb{R}}$. The non-real spectrum of A cannot accumulate at a point belonging to an open set of definite type.

Recall, that a self-adjoint operator A in a Krein space $(\mathcal{H}, [\cdot, \cdot])$ is called definitizable if $\rho(A) \neq \emptyset$ and there exists a rational function $p \neq 0$ having poles only in $\rho(A)$ such that $[p(A)x, x] \geq 0$ for all $x \in \mathcal{H}$. Then the non-real part of the spectrum of A consists of no more than a finite number of points. Moreover, A has a spectral function E defined on the ring generated by all connected subsets of $\overline{\mathbb{R}}$ whose endpoints do not coincide with the points of some finite set which is contained in $\{t \in \mathbb{R} : p(t) = 0\} \cup \{\infty\}$ (see [38]).

A self-adjoint operator in a Krein space is definitizable if and only if it is definitizable over \mathbb{C} in the sense of the following definition (see e.g. [24, Definition 4.4]), which localizes the notion of definitizability.

Definition 3.2. Let Ω be a domain in $\overline{\mathbb{C}}$ such that

$$\Omega \text{ is symmetric with respect to } \mathbb{R}, \quad \Omega \cap \overline{\mathbb{R}} \neq \emptyset, \quad (3.1)$$

$$\text{and the domains } \Omega \cap \mathbb{C}^+, \Omega \cap \mathbb{C}^- \text{ are simply connected.} \quad (3.2)$$

Let A be a self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot, \cdot])$ such that $\sigma(A) \cap (\Omega \setminus \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A , and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the non-real spectrum $\sigma(A) \setminus \mathbb{R}$ of A . The operator A is called definitizable over Ω , if the following holds.

- (i) For every closed subset Δ of $\Omega \cap \overline{\mathbb{R}}$ there exist an open neighbourhood \mathcal{U} of Δ in $\overline{\mathbb{C}}$ and numbers $m \geq 1$, $M > 0$ such that

$$\|(A - \lambda)^{-1}\| \leq M(|\lambda| + 1)^{2m-2} |\text{Im } \lambda|^{-m} \quad (3.3)$$

for all $\lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}$.

- (ii) Every point $\lambda \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_λ in $\overline{\mathbb{R}}$ such that both components of $I_\lambda \setminus \{\lambda\}$ are of definite type (cf. Definition 3.1) with respect to A .

A self-adjoint operator definitizable over Ω where Ω is as in Definition 3.2 possesses a local spectral function E . For the construction and the properties of this

spectral function we refer to [24] (see also [23]). We mention only that $E(\Delta)$ is defined and is a self-adjoint projection in $(\mathcal{H}, [\cdot, \cdot])$ for every union Δ of a finite number of connected subsets Δ_i , $i = 1, \dots, n$, of $\Omega \cap \overline{\mathbb{R}}$, $\overline{\Delta}_i \subset \Omega \cap \overline{\mathbb{R}}$, such that the endpoints of Δ_i belong to intervals of definite type. A real point $\lambda \in \sigma(A) \cap \Omega$ belongs to $\sigma_{++}(A)$ if and only if there exists a bounded open interval $\Delta \subset \Omega$, $\lambda \in \Delta$, such that $E(\Delta)\mathcal{H}$ is a Hilbert space (cf. [3]). A point $t \in \mathbb{R} \cap \Omega$ is called a *critical point* of A if there is no open subset $\Delta \subset \Omega$ of definite type with $t \in \Delta$. The set of critical points of A is denoted by $c(A)$. A critical point t is called *regular* if there exists an open deleted neighbourhood $\delta_0 \subset \Omega$ of t such that the set of the projections $E(\delta)$ where δ runs through all intervals δ with $\delta \subset \delta_0$ is bounded. The set of regular critical points of A is denoted by $c_r(A)$. The elements of $c_s(A) := c(A) \setminus c_r(A)$ are called *singular* critical points.

We will make use of the following perturbation result, see [6].

Theorem 3.3. *Let T_1 and T_2 be self-adjoint operators in the Krein space \mathcal{H} , let $\rho(T_1) \cap \rho(T_2) \cap \Omega \neq \emptyset$ and assume that*

$$(T_1 - \lambda_0 I)^{-1} - (T_2 - \lambda_0 I)^{-1}$$

is a finite rank operator for some $\lambda_0 \in \rho(T_1) \cap \rho(T_2)$. Then T_1 is definitizable over Ω if and only if T_2 is definitizable over Ω .

Moreover, if T_1 is definitizable over Ω and $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is an open interval with end point $\eta \in \Omega \cap \overline{\mathbb{R}}$ and Δ is of positive type (negative type) with respect to T_1 , then there exist open interval Δ' , $\Delta' \subset \Delta$, with endpoint η such that Δ' is of positive type (resp. negative type) with respect to T_2 .

3.2. Definitizability of A

In this section we will give conditions which ensures the definitizability of the operator A from Definition 2.1. The following definition is needed below.

Definition 3.4. *We shall say that the sets S_1 and S_2 of real numbers are separated by a finite number of points if there exists a finite ordered set $\{\alpha_j\}_{j=1}^N$, $N \in \mathbb{N}$,*

$$-\infty = \alpha_0 < \alpha_1 \leq \dots \leq \alpha_N < \alpha_{N+1} = +\infty,$$

such that one of the sets S_j , $j = 1, 2$, is a subset of $\bigcup_{k \text{ is even}} [\alpha_k, \alpha_{k+1}]$ and another one is a subset of $\bigcup_{k \text{ is odd}} [\alpha_k, \alpha_{k+1}]$.

The operator $A_0^+ \oplus A_0^-$, where A_0^\pm are defined as in Section 2.3, is fundamentally reducible (cf. [22, Section 3]) in the Krein space $(L^2(\mathbb{R}), [\cdot, \cdot])$ (cf. (2.2)). Hence the following lemma is a easy consequence of Definitions 3.1 and 3.2.

Lemma 3.5. *Let $\lambda \in \mathbb{R}$. Then $\lambda \in \sigma_{++}(A_0^+ \oplus A_0^-)$ ($\lambda \in \sigma_{--}(A_0^+ \oplus A_0^-)$) if and only if $\lambda \in \sigma(A_0^+) \setminus \sigma(A_0^-)$ ($\lambda \in \sigma(A_0^-) \setminus \sigma(A_0^+)$, resp.). The operator $A_0^+ \oplus A_0^-$ is definitizable if and only if the sets $\sigma(A_0^+)$ and $\sigma(A_0^-)$ are separated by a finite number of points.*

It follows from Proposition 2.4 and $\sigma(A_0^+ \oplus A_0^-) \subset \mathbb{R}$ that $\rho(A) \cap \rho(A_0^+ \oplus A_0^-) \neq \emptyset$. Let $\lambda_0 \in \rho(A) \cap \rho(A_0^+ \oplus A_0^-)$. The operators $A_0^+ \oplus A_0^-$ and A are extensions of A_{\min}

and $\dim(\operatorname{dom}(A_0^+ \oplus A_0^-) / \operatorname{dom}(A_{\min})) = \dim(\operatorname{dom}(A) / \operatorname{dom}(A_{\min})) = 2$. This implies that

$$(A_0^+ \oplus A_0^- - \lambda_0 I)^{-1} - (A - \lambda_0 I)^{-1}$$

is an operator of rank 2. Then [25] and Lemma 3.5 imply the following theorem.

Theorem 3.6 ([30, 31]). *The operator A is definitizable if and only if the sets $\sigma(A_0^+)$ and $\sigma(A_0^-)$ are separated by a finite number of points.*

Example 3.7. *Let q be a constant potential, $q(x) \equiv c$, $c \in \mathbb{R}$. It is easy to calculate that $\sigma(A_0^+) = [c, +\infty)$ and $\sigma(A_0^-) = (-\infty, -c]$. Thus, Corollary 3.6 implies that the operator $(\operatorname{sgn} x)(-d^2/dx^2 + c)$ is definitizable in the Krein space $L^2(\mathbb{R}, \operatorname{sgn} x dx)$ if and only if $c \geq 0$.*

3.3. Local definitizability of A

In this subsection we consider Sturm-Liouville operators defined as in Section 2 and we prove that the operator A is a definitizable operator in a certain neighbourhood of ∞ (in the sense of the Krein space $(L^2(\mathbb{R}), [\cdot, \cdot])$) if and only if the operator L is semi-bounded from below (in the sense of the Hilbert space $L^2(\mathbb{R})$).

Remark 3.8. *Clearly, $L \geq \eta_0 > -\infty$ whenever $q(x) \geq \eta_0 > -\infty$, $x \in \mathbb{R}$.*

The operator $A_0^+ \oplus A_0^-$ is a self-adjoint operator both in the Hilbert space $L^2(\mathbb{R})$ and in the Krein space $(L^2(\mathbb{R}), [\cdot, \cdot])$, cf. (2.2).

Lemma 3.9. *The following statements are equivalent:*

- (i) *The operator L is semi-bounded from below.*
- (ii) *There exists $R > 0$ such that the operator $A_0^+ \oplus A_0^-$ is definitizable over the domain $\{\lambda \in \overline{\mathbb{C}} : |\lambda| > R\}$.*

Proof. (i) \Rightarrow (ii). Since $A_0^+ \oplus A_0^-$ is a self-adjoint operator in the Hilbert space $L^2(\mathbb{R})$, we see that

$$\sigma(A_0^+ \oplus A_0^-) \subset \mathbb{R} \quad \text{and (3.3) holds for all } \lambda \in \mathbb{C} \setminus \mathbb{R} \quad \text{with } m = 1. \quad (3.4)$$

Assume that $L \geq \eta_0$. The operator L is a self-adjoint extension of $A_{\min}^+ \oplus (-A_{\min}^-)$, hence the operator A_{\min}^+ is semi-bounded from below, $A_{\min}^+ \geq \eta_0$, and A_{\min}^- is semi-bounded from above, $A_{\min}^- \leq -\eta_0$. The operators A_0^\pm are self-adjoint extensions in $L^2(\mathbb{R}_\pm)$ of the symmetric operators A_{\min}^\pm with deficiency indices (1,1). Hence the spectrum of A_0^+ (A_0^-) lies, with the possible exception of at most one normal eigenvalue, in $[\eta_0, \infty)$ (in $(-\infty, -\eta_0]$, respectively), see e.g. [1, Section VII.85].

Choose $R := \eta_0$. Lemma 3.5 implies that the set $(R, +\infty)$, with the possible exception of at most one eigenvalue, is of positive type and the set $(-\infty, -R)$, with the possible exception of at most one eigenvalue, is of negative type with respect to $A_0^+ \oplus A_0^-$. Thus, the operator $A_0^+ \oplus A_0^-$ is definitizable over $\{\lambda \in \overline{\mathbb{C}} : |\lambda| > R\}$.

(i) \Leftarrow (ii) Obviously, the Sturm-Liouville operator A_0^+ (A_0^-) is not semi-bounded from above (below, resp.). That is,

$$\sup \sigma(A_0^+) = +\infty, \quad \inf \sigma(A_0^-) = -\infty. \quad (3.5)$$

Assume that L is not semi-bounded from below. Then A_{\min}^+ or $-A_{\min}^-$ is not semi-bounded from below. Thus, $\inf \sigma(A_0^+) = -\infty$ or $\sup \sigma(A_0^-) = +\infty$.

Consider the case

$$\inf \sigma(A_0^+) = -\infty. \quad (3.6)$$

It follows from (3.6), (3.5) and Lemma 3.5 that

$$(-\infty, -r) \cap \sigma_{++}(A_0^+ \oplus A_0^-) \neq \emptyset \quad \text{and} \quad (-\infty, -r) \cap \sigma_{--}(A_0^+ \oplus A_0^-) \neq \emptyset$$

for all $r > 0$. Thus, by definition, the operator $A_0^+ \oplus A_0^-$ is not definitizable over $\{\lambda \in \overline{\mathbb{C}} : |\lambda| > r\}$ for arbitrary $r > 0$. The case $\sup \sigma(A_0^-) = +\infty$ can be considered in the same way. \square

The following theorem is one of the main results.

Theorem 3.10. *The following assertions are equivalent:*

- (i) *The operator L is semi-bounded from below.*
- (ii) *There exists $R > 0$ such that the operator A is definitizable over the domain $\{\lambda \in \overline{\mathbb{C}} : |\lambda| > R\}$.*

Proof. It follows from Proposition 2.4 (iii) and $\sigma(A_0^+ \oplus A_0^-) \subset \mathbb{R}$ that $\rho(A) \cap \rho(A_0^+ \oplus A_0^-) \neq \emptyset$. Let $\lambda_0 \in \rho(A) \cap \rho(A_0^+ \oplus A_0^-)$. The operators $A_0^+ \oplus A_0^-$ and A are extensions of A_{\min} and $\dim(\operatorname{dom}(A_0^+ \oplus A_0^-) / \operatorname{dom}(A_{\min})) = \dim(\operatorname{dom}(A) / \operatorname{dom}(A_{\min})) = 2$. This implies that

$$(A_0^+ \oplus A_0^- - \lambda_0 I)^{-1} - (A - \lambda_0 I)^{-1} \quad (3.7)$$

is an operator of rank 2. Combining Lemma 3.9 and Theorem 3.3, Theorem 3.10 is proved. \square

By Theorem 3.10, the semi-boundedness of L implies the definitizability of A over some domain. Now we give a precise description of the domain of definitizability of A in terms of the spectra of A_0^+ and A_0^- .

Let T be an operator such that $\sigma(T) \subset \mathbb{R}$. Let us introduce the sets $\sigma^{\text{left}}(T)$ and $\sigma^{\text{right}}(T)$ by the following way: a point $\lambda \in \overline{\mathbb{R}} (= \mathbb{R} \cup \infty)$ is said to belong to $\sigma^{\text{left}}(T)$ ($\sigma^{\text{right}}(T)$) if there exists an increasing (resp. decreasing) sequence $\{\lambda_n\}_1^\infty \subset \sigma(T)$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda$.

Note that

$$\sigma^{\text{left}}(T) \cup \sigma^{\text{right}}(T) \subset \sigma_{\text{ess}}(T) \cup \{\infty\}. \quad (3.8)$$

For differential operators A_0^\pm , equality holds in (3.8) since every point of $\sigma_{\text{ess}}(A_0^\pm)$ is an accumulation point of $\sigma(A_0^\pm)$.

We put

$$\mathcal{S}_A := (\sigma^{\text{left}}(A_0^+) \cap \sigma^{\text{left}}(A_0^-)) \cup (\sigma^{\text{right}}(A_0^+) \cap \sigma^{\text{right}}(A_0^-)). \quad (3.9)$$

Theorem 3.11. *Let Ω be a domain in $\overline{\mathbb{C}}$ such that (3.1)-(3.2) are fulfilled. Then the operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ is definitizable over Ω if and only if $\Omega \subset \Omega_A$, where $\Omega_A := \overline{\mathbb{C}} \setminus \mathcal{S}_A$.*

Proof. Arguments from the proof of Theorem 3.10 show that it is enough to prove the theorem for the operator $A_0^+ \oplus A_0^-$.

Let $\lambda \in \mathcal{S}_A$ and let I_λ be an open connected neighbourhood of λ . Then (3.9) and Lemma 3.5 imply that one of the components of $I_\lambda \setminus \{\lambda\}$ is not of definite type. So if $A_0^+ \oplus A_0^-$ is definitizable over Ω , then $\lambda \notin \Omega$.

Conversely, if $\mathcal{S}_A \neq \overline{\mathbb{R}}$, then condition (ii) from Definition 3.2 is fulfilled for $\Omega_A = \mathbb{C} \setminus \mathcal{S}_A$. Taking (3.4) into account, we see that $A_0^+ \oplus A_0^-$ is definitizable over Ω_A . \square

Remark 3.12. Note that $\Omega_A \cap \overline{\mathbb{R}} = \emptyset$ is equivalent to $\sigma_{ess}(A_0^+) = \sigma_{ess}(A_0^-) = \mathbb{R}$. In the converse case, (3.1)-(3.2) are fulfilled for Ω_A and it is the greatest domain over which the operator A is definitizable.

The following statement is a simple consequence of Theorem 3.10, Theorem 3.11, and (3.8).

Corollary 3.13. Assume that L is semi-bounded from below. Then the operator A is definitizable over the set $\overline{\mathbb{C}} \setminus (\sigma_{ess}(A_0^+) \cap \sigma_{ess}(A_0^-))$.

3.4. Regularity of the critical point ∞

In the sequel we will use a result which follows easily from [12, Lemma 3.5 (iii)] and [12, Theorem 3.6 (i)].

Proposition 3.14. If the operator $\tilde{L} := -d^2/dx^2 + \tilde{q}(x)$, for some real $\tilde{q} \in L^1_{loc}(\mathbb{R})$, defined on \mathcal{D} is nonnegative in the Hilbert space $L^2(\mathbb{R})$, then the operator $\tilde{A} := (\text{sgn } x)\tilde{L}$ is definitizable and ∞ is a regular critical point of \tilde{A} .

The following theorem can be considered as the main result of this note.

Theorem 3.15. Assume that assertions (i), (ii) of Theorem 3.10 hold true. Then there exists a decomposition

$$A = \mathcal{A}_\infty \dot{+} \mathcal{A}_b \tag{3.10}$$

such that the operator \mathcal{A}_∞ is similar to a self-adjoint operator in the Hilbert space sense and \mathcal{A}_b is a bounded operator.

Remark 3.16. The conclusion of Theorem 3.15 is equivalent to the regularity of critical point ∞ of the operator A .

Proof of Theorem 3.15. Assume that A is an operator definitizable over $\{\lambda \in \overline{\mathbb{C}} : |\lambda| > R\}$, $R > 0$. By Theorem 3.10, this is equivalent to the fact that $L \geq \eta_0$ for certain $\eta_0 \in \mathbb{R}$.

Denote by E^A the spectral function of A . Choose $r > R$ such that $\sigma(A) \setminus \mathbb{R} \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$ and $E^A(\mathbb{R} \setminus (-r, r))$ is defined. Then A decomposes,

$$\begin{aligned} A &= \mathcal{A}_1 \dot{+} \mathcal{A}_0, & \mathcal{A}_1 &:= A \upharpoonright \text{dom}(A) \cap (E^A(\overline{\mathbb{R}} \setminus (-r, r))L^2(\mathbb{R})), \\ & & \mathcal{A}_0 &:= A \upharpoonright \text{dom}(A) \cap ((I - E^A(\overline{\mathbb{R}} \setminus (-r, r)))L^2(\mathbb{R})) \end{aligned}$$

and the following statements holds (cf. [22, Theorem 2.6]):

- \mathcal{A}_1 is a definitizable operator in the Krein space $(E^A(\overline{\mathbb{R}} \setminus (-r, r))L^2(\mathbb{R}), [\cdot, \cdot])$;
 \mathcal{A}_0 is a bounded operator and $\sigma(\mathcal{A}_0) \subset \{\lambda : |\lambda| \leq r\}$.

Let us show that ∞ is not a singular critical point of \mathcal{A}_1 .

Consider the operator \mathcal{A}_2 defined by $\mathcal{A}_2 = \mathcal{A}_1 \dot{+} 0$, where the direct sum is considered with respect to the decomposition

$$L^2(\mathbb{R}) = E^A(\overline{\mathbb{R}} \setminus (-r, r))L^2(\mathbb{R}) \dot{+} (I - E^A(\overline{\mathbb{R}} \setminus (-r, r)))L^2(\mathbb{R}),$$

and 0 is the zero operator in the subspace $\operatorname{ran}(I - E^A(\overline{\mathbb{R}} \setminus (-r, r)))$. Since \mathcal{A}_0 is a bounded operator, we have

$$\operatorname{dom}(\mathcal{A}_2) = \operatorname{dom} A.$$

It is easy to see that \mathcal{A}_2 is a definitizable operator in the Krein space $(L^2(\mathbb{R}), [\cdot, \cdot])$. Moreover, ∞ is not a singular critical point of \mathcal{A}_2 if and only if ∞ is not a singular critical point of A .

Now we prove that ∞ is not a singular critical point of \mathcal{A}_2 . Let $\eta_1 < \eta_0$. Since $L \geq \eta_0$, we see that $L - \eta_1 I$ is a uniformly positive operator in the Hilbert space $L^2(\mathbb{R})$ (i.e., $L - \eta_1 I \geq \delta > 0$). Therefore $\tilde{A} := J(L - \eta_1 I)$,

$$\tilde{A}y(x) = (\operatorname{sgn} x)(-y''(x) + q(x)y(x) - \eta_1 y(x)), \quad \operatorname{dom}(\tilde{A}) = \operatorname{dom}(A),$$

is a definitizable nonnegative operator in the Krein space $(L^2(\mathbb{R}), [\cdot, \cdot])$. By Proposition 3.14, ∞ is not a singular critical point of \tilde{A} . The Čurgus criterion of the regularity of critical point ∞ , see [11, Corollary 3.3], implies that ∞ is not a singular critical point of the operator \mathcal{A}_2 . So ∞ is not a singular critical point of \mathcal{A}_1 .

It follows from $L \geq \eta_0$ and Lemma 3.5 that for sufficiently large $r_1 > 0$ the set $(-\infty, -r_1]$ is of negative type and the set $[r_1, +\infty)$ is of positive type with respect to $A_0^+ \oplus A_0^-$. Combining this with Theorem 3.3, we obtain that there exists $r_2 \geq r_1$ such that $(-\infty, -r_2]$ is of negative type and the set $[r_2, +\infty)$ is of positive type with respect to the operator A . Evidently, we obtain the desired decomposition

$$A = \mathcal{A}_\infty \dot{+} \mathcal{A}_b, \quad \mathcal{A}_\infty := A \upharpoonright \operatorname{dom}(A) \cap (E^A(\overline{\mathbb{R}} \setminus (-r_2, r_2))L^2(\mathbb{R})), \\ \mathcal{A}_b := A \upharpoonright \operatorname{dom}(A) \cap ((I - E^A(\overline{\mathbb{R}} \setminus (-r_2, r_2)))L^2(\mathbb{R})),$$

where \mathcal{A}_b is a bounded operator and \mathcal{A}_∞ is similar to a self-adjoint operator in the Hilbert space sense. \square

4. Accumulation of non-real eigenvalues to a real point

By Proposition 2.4 (i), the non-real spectrum $\sigma(A) \setminus \mathbb{R}$ of A consists of eigenvalues.

Let \mathcal{S}_A be the set defined by (3.9). The following proposition is a consequence of Theorems 3.11 and 3.10.

Proposition 4.1. *If λ is an accumulation point of $\sigma(A) \setminus \mathbb{R}$, then $\lambda \in \mathcal{S}_A$. In particular, if the operator $L = -d^2/dx^2 + q(x)$ is semi-bounded from below, then non-real spectrum of A is a bounded set.*

The goal of this subsection is to show that there exists a potential q continuous in \mathbb{R} such that the set of non-real eigenvalues of the operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q(x))$ has a real accumulation point.

It is well known (e.g. [40]) that M_+ , the Titchmarsh-Weyl m -coefficient for (2.3) (see Subsection 2.2), admits the following integral representation

$$M_+(\lambda) = \int_{\mathbb{R}} \frac{d\Sigma_+(t)}{t - \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $\Sigma_+(\cdot)$ is a nondecreasing scalar functions such that $\int_{\mathbb{R}} (1 + |t|)^{-1} d\Sigma_+(t) < \infty$. The function Σ_+ is called a *spectral function* of the boundary value problem

$$-y''(x) + q_+(x)y(x) = \lambda y(x), \quad y'(0) = 0, \quad x \in [0, +\infty). \quad (4.1)$$

This means that the self-adjoint operator A_0^+ introduced in Subsection 2.3 is unitary equivalent to the operator of multiplication by the independent variable in the Hilbert space $L^2(\mathbb{R}, d\Sigma_+(t))$. This fact obviously implies

$$\sigma(A_0^+) = \operatorname{supp}(d\Sigma_+), \quad (4.2)$$

where $\operatorname{supp} d\tau$ denotes the *topological support* of a Borel measure $d\Sigma_+$ on \mathbb{R} (i.e., $\operatorname{supp} d\Sigma_+$ is the smallest closed set $\Omega \subset \mathbb{R}$ such that $d\Sigma_+(\mathbb{R} \setminus \Omega) = 0$).

Lemma 4.2. *Assume that q is an even potential, $q(x) = q(-x)$, $x \in \mathbb{R}$. If $\varepsilon > 0$, then $i\varepsilon \in \sigma_p(A)$ if and only if $\operatorname{Re} M_+(i\varepsilon) = 0$.*

Proof. Since q is even, we get $m_+(\lambda) = m_-(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. So $M_-(i\varepsilon) = -M_+(-i\varepsilon)$. Since M_+ is a Nevanlinna function, we see that $M_+(-i\varepsilon) = \overline{M_+(i\varepsilon)}$. Thus,

$$M_+(i\varepsilon) - M_-(i\varepsilon) = M_+(i\varepsilon) + \overline{M_+(i\varepsilon)} = 2 \operatorname{Re} M_+(i\varepsilon).$$

Proposition 2.4 completes the proof. \square

The following lemma follows easily from the Gelfand–Levitan theorem (see e.g. [41, Subsection 26.5]).

Lemma 4.3. *Let $\Sigma(t)$, $t \in \mathbb{R}$, be a nondecreasing function such that*

$$\int_{-\infty}^{T_1-0} d\Sigma(t) = 0 \quad \text{and} \quad (4.3)$$

$$\int_{-\infty}^{s-0} d\Sigma(t) = \int_0^s \frac{1}{\pi\sqrt{t}} dt \quad \left(= \frac{2}{\pi}\sqrt{s} \right) \quad \text{for all } s > T_2. \quad (4.4)$$

with certain constants $T_1, T_2 \in \mathbb{R}$, $T_1 < T_2$. Then there exists a potential q_+ continuous in $[0, +\infty)$ such that $\Sigma(t)$ is a spectral function of the boundary value problem

$$-y''(x) + q_+(x)y(x) = \lambda y(x), \quad y'(0) = 0, \quad x \in [0, +\infty).$$

Lemma 4.4. *There exist a nondecreasing function $\Sigma(t)$, $t \in \mathbb{R}$, with the following properties:*

(i) $\Sigma(t) = \Sigma_1(t) + \Sigma_2(t)$, where

$$\Sigma_1 \in AC_{loc}(\mathbb{R}), \quad \Sigma_1'(t) = \begin{cases} 0, & t \in (-\infty, 1), \\ \frac{1}{\pi\sqrt{t}}, & t \in (1, +\infty), \end{cases} \quad (4.5)$$

and the measure $d\Sigma_2$ has the form

$$d\Sigma_2(t) = \sum_{k=1}^{+\infty} h_k \delta(t - s_k),$$

$$h_k > 0, \quad s_k \in (-1, 1), \quad k \in \mathbb{N}; \quad \sum_{k=1}^{+\infty} h_k < \infty, \quad (4.6)$$

(here $\delta(t)$ is the Dirac delta-function).

(ii) Conditions (4.3)-(4.4) are valid for Σ with $T_1 = -1$ and $T_2 = 1$.

(iii) There exists a sequence $\varepsilon_k > 0$, $k \in \mathbb{N}$, such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ and $r(\varepsilon_k) = 0$, $k \in \mathbb{N}$, where the function $r(\varepsilon)$, $\varepsilon > 0$, is defined by

$$r(\varepsilon) := \operatorname{Re} \int_{\mathbb{R}} \frac{1}{t - i\varepsilon} d\Sigma(t) = \int_{\mathbb{R}} \frac{t}{t^2 + \varepsilon^2} d\Sigma(t).$$

Proof. Let $h_k = 2^{-k+1}/\pi$. Then

$$\sum_{k=1}^{\infty} h_k = 2/\pi. \quad (4.7)$$

Now, if $s_k \in (-1, 1)$ for all $k \in \mathbb{N}$, then Σ possesses property (ii). We should only choose $\{s_k\}_1^{\infty} \subset (-1, 1)$ such that statements (iii) holds true.

Consider for $\varepsilon \geq 0$ the functions

$$r_0(\varepsilon) = \int_1^{\infty} \frac{t}{t^2 + \varepsilon^2} d\Sigma_1(t)$$

and

$$r_n(\varepsilon) := \int_1^{\infty} \frac{t}{t^2 + \varepsilon^2} d\Sigma_1(t) + \sum_{k=1}^n \frac{s_k h_k}{s_k^2 + \varepsilon^2}, \quad n \in \mathbb{N}.$$

Let $s_k \neq 0$ for all $k \in \mathbb{N}$. Then r_n are well-defined and continuous on $[0, +\infty)$. Besides, $\lim_{n \rightarrow \infty} r_n(\varepsilon) = r(\varepsilon)$ for all $\varepsilon > 0$. It is easy to see that $\lim_{\varepsilon \rightarrow \infty} r_n(\varepsilon) = 0$, $n \in \mathbb{N}$. Since r_n are continuous on $[0, +\infty)$, we see that

$$\operatorname{SUP}_n := \sup_{\varepsilon \in [0, +\infty)} |r_n(\varepsilon)| < \infty, \quad n \in \mathbb{N}.$$

Now we give a procedure to choose $s_k \in (-1, 1) \setminus \{0\}$.

Let s_1 be an arbitrary number in $(-1, 0)$ such that

$$\frac{s_1 h_1}{s_1^2 + \varepsilon^2} \Big|_{\varepsilon=|s_1|} = \frac{1}{\pi} \frac{1}{2s_1} < -\text{SUP}_0 - 1,$$

$$\text{in other words, } -\frac{1}{2\pi(\text{SUP}_0 + 1)} < s_1 < 0.$$

Then

$$r_1(|s_1|) = r_0(|s_1|) + \frac{s_1 h_1}{s_1^2 + \varepsilon^2} \Big|_{\varepsilon=|s_1|} < r_0(|s_1|) - \sup_{\varepsilon \in [0, +\infty)} |r_0(\varepsilon)| - 1 < -1. \quad (4.8)$$

Let

$$\{s_k\}_2^\infty \in (-b_1, b_1) \setminus \{0\} \quad \text{with certain } b_1 \in (0, |s_1|/2). \quad (4.9)$$

Let us show that we may choose a number b_1 such that (4.9) implies

$$r(|s_1|) < 0. \quad (4.10)$$

Indeed, (4.8) and (4.7) yield

$$\begin{aligned} r(|s_1|) &= r_1(|s_1|) + \left[\sum_{k=2}^{\infty} \frac{s_k h_k}{s_k^2 + \varepsilon^2} \right]_{\varepsilon=|s_1|} < \\ &< -1 + \sum_{k=2}^{\infty} \frac{h_k |s_k|}{s_k^2 + s_1^2} < -1 + \frac{b_1}{s_1^2} \sum_{k=2}^{\infty} h_k < -1 + \frac{2b_1}{\pi s_1^2} \end{aligned}$$

and therefore (4.10) is valid whenever $0 < b_1 < \pi s_1^2/2$.

Similarly, there exist $s_2 \in (0, b_1)$ such that

$$\frac{s_2 h_2}{s_2^2 + \varepsilon^2} \Big|_{\varepsilon=s_2} = \frac{1}{2\pi} \frac{1}{2s_2} > \text{SUP}_1 + 1,$$

and therefore

$$r_2(s_2) > 1.$$

Further, there exist $b_2 \in (0, s_2/2)$ such that $\{s_k\}_3^\infty \subset (-b_2, b_2) \setminus \{0\}$ implies that $r(s_2) > 0$.

Continuing this process, we obtain a sequence $\{s_k\}_1^\infty \subset (-1, 1) \setminus \{0\}$ with the following properties:

$$\begin{aligned} s_k \in (-1, 0) \quad \text{if } k \text{ is odd,} \quad s_k \in (0, 1) \quad \text{if } k \text{ is even,} \\ |s_1| > \frac{|s_1|}{2} > |s_2| > \frac{|s_2|}{2} > |s_3| > \dots > |s_k| > \frac{|s_k|}{2} > |s_{k+1}| > \dots, \end{aligned} \quad (4.11)$$

$$r(|s_k|) < 0 \quad \text{if } k \text{ is odd,} \quad r(|s_k|) > 0 \quad \text{if } k \text{ is even.} \quad (4.12)$$

It is easy to show that r is continuous on $(0, +\infty)$. Combining this with (4.12), we see that there exists $\varepsilon_k \in (|s_{k-1}|, |s_k|)$ such that $r(\varepsilon_k) = 0$, $k \in \mathbb{N}$. Besides, (4.11) implies $\lim |s_k| = \lim \varepsilon_k = 0$. \square

Theorem 4.5. *There exist an even potential \widehat{q} continuous on \mathbb{R} and a sequence $\{\varepsilon_k\}_1^\infty \subset \mathbb{R}_+$ such that*

(i) *the operator \widehat{A} defined by the differential expression*

$$(\operatorname{sgn} x) \left(-\frac{d^2}{dx^2} + \widehat{q}(x) \right) \quad (4.13)$$

on the natural domain \mathfrak{D} (see Subsection 2.1) is a self-adjoint operator in the Krein space $L^2(\mathbb{R}, [\cdot, \cdot])$;

(ii) $\{i\varepsilon_k\}_1^\infty \subset \sigma_p(\widehat{A})$, *i.e., $i\varepsilon_k$, $k \in \mathbb{N}$, are non-real eigenvalues of \widehat{A} ;*

(iii) $\lim_{k \rightarrow \infty} \varepsilon_k = 0$;

(iv) *the operator \widehat{A} is definitizable over the domain $\overline{\mathbb{C}} \setminus \{0\}$.*

Proof. (i) Let Σ and $\{\varepsilon_k\}_1^\infty$ be from Lemma 4.4. Then, by Lemma 4.3, Σ is a spectral function of the boundary value problem (4.1) with a certain potential \widehat{q}_+ . Let us consider an even continuous potential $\widehat{q}(x) = \widehat{q}_+(|x|)$, $x \in \mathbb{R}$, and the corresponding operator $\widehat{A} = (\operatorname{sgn} x) \left(-\frac{d^2}{dx^2} + \widehat{q}(x) \right)$ defined as in Subsection 2.1.

It is well known that if equation (2.3) is in the limit-circle case at $+\infty$ then $M_+(\cdot)$ is a meromorphic function on \mathbb{C} and the spectral function Σ_+ is a step function with jumps at the poles of $M_+(\cdot)$ only (see e.g. [10, Theorem 9.4.1]). As $\Sigma_+(t) = \Sigma(t)$, $t > 0$, this condition does not hold for the function Σ since Σ satisfies (4.4). Indeed, (4.4) means that $\Sigma'(t) = \frac{1}{\pi\sqrt{t}}$ for $t > T_2 = 1$ and therefore Σ is not a step function. So (2.3) is limit-point at $+\infty$.

Since the potential \widehat{q} is even, the same is true for $-\infty$. Thus, \widehat{A} is a self-adjoint operator in the Krein space $L^2(\mathbb{R}, [\cdot, \cdot])$, see Subsection 2.1.

(ii) and (iii) follow from Lemma 4.2 and statement (iii) of Lemma 4.4.

(iv) Let \widehat{A}_0^\pm be the self-adjoint operators in the Hilbert spaces $L^2(\mathbb{R}_\pm)$ defined by the differential expression (4.13) in the same way as in Subsection 2.3 where q is replaced by \widehat{q} . By (4.2), $\sigma(\widehat{A}_0^+) = \{s_k\}_1^\infty \cup [1, +\infty)$. Since \widehat{q} is even, one gets $\sigma(\widehat{A}_0^-) = \{-s_k\}_1^\infty \cup (-\infty, -1]$. It follows from $\{s_k\}_1^\infty \subset (-1, 1)$ and $\lim_{k \rightarrow \infty} s_k = 0$ that

$$\min \sigma_{ess}(\widehat{A}_0^+) = \max \sigma_{ess}(\widehat{A}_0^-) = 0$$

and Theorem 3.13 concludes the proof. \square

5. Some classes of Sturm-Liouville operators

As an illustration of the results from the previous sections, we discuss in this section various potentials $q \in L^1_{loc}(\mathbb{R})$ such that the differential operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ is definitizable over specific subsets of $\overline{\mathbb{C}}$. As before it is supposed that the differential expression (2.1) is in limit point case at $+\infty$ and at $-\infty$ (for instance, the latter holds if $\liminf_{|x| \rightarrow \infty} \frac{q(x)}{x^2} > -\infty$, see e.g., [47, Example 7.4.1]).

5.1. The case $q(x) \rightarrow -\infty$

In this subsection we assume that for some $X > 0$ the potential q has the following properties on the interval $(X, +\infty)$:

$$q', q'' \text{ exist and are continuous on } (X, +\infty), \quad q(x) < 0, \quad q'(x) < 0, \quad (5.1)$$

$$q''(x) \text{ is of fixed sign, i.e., } q''(x_1)q''(x_2) \geq 0 \text{ for all } x_1, x_2 > X, \quad (5.2)$$

$$\lim_{x \rightarrow +\infty} q(x) = -\infty, \quad \int_X^{+\infty} |q(x)|^{-1/2} dx = \infty, \quad \text{and} \quad \limsup_{x \rightarrow +\infty} \frac{|q'(x)|}{|q(x)|^p} < \infty, \quad (5.3)$$

where $p \in (0, 3/2)$ is a constant.

Then the well-known result of Titchmarsh (see e.g. [40, Theorems 3.4.1 and 3.4.2]) states that (2.1) is in the limit point case at $+\infty$ and $\sigma(A_0^+) = \mathbb{R}$. Hence the set \mathcal{S}_A defined by (3.9) coincides with $\sigma_{ess}(A_0^-) \cup \infty$. By Theorem 3.11, there are two cases:

- (i) Let $\sigma_{ess}(A_0^-) \neq \mathbb{R}$. Then the greatest domain over which A is definitizable is $\Omega_A := \mathbb{C} \setminus \sigma_{ess}(A_0^-)$ (note that $\infty \notin \Omega_A$).
- (ii) Let $\sigma_{ess}(A_0^-) = \mathbb{R}$. Then $\Omega_A \cap \overline{\mathbb{R}} = \emptyset$ and there exists no domain Ω in $\overline{\mathbb{C}}$ such that A is definitizable over Ω . In particular, the latter holds if the analogues of assumptions (5.1)-(5.3) are fulfilled for $x \in (-\infty, 0]$.

Example 5.1. *Let us consider the operator $A = (\operatorname{sgn} x)(-d^2/dx^2 - x)$. By [45, Theorem 6.6] the differential expression $-d^2/dx^2 - x$ is in limit point case at $+\infty$ and $-\infty$. Assumptions (5.1)-(5.3) hold for $x \in (0, +\infty)$, hence $\sigma_{ess}(A_0^+) = \sigma(A) = \mathbb{R}$. On the other hand, $\sigma_{ess}(A_0^-) = \emptyset$ (see Subsection 5.2 and [40, Section 3.1]). Therefore the operator A is definitizable over \mathbb{C} and there exists no domain Ω in $\overline{\mathbb{C}}$ with $\infty \in \Omega$ such that A is definitizable over Ω . By Proposition 4.1, the only possible accumulation point for non-real spectrum of A is the point ∞ .*

5.2. The case $q(x) \rightarrow +\infty$

Let us assume that the following conditions holds with certain constants $X, c > 0$:

$$q(x) \geq c \text{ for } x > X, \quad \text{and for any } \omega > 0, \quad \lim_{x \rightarrow +\infty} \int_x^{x+\omega} q(t) dt = +\infty. \quad (5.4)$$

Molčanov proved (see e.g., [40, Lemma 3.1.2] and [41, Subsection 24.5]) that (5.4) yields $\sigma_{ess}(A_0^+) = \emptyset$, i.e., the spectrum of the operator A_0^+ is discrete. Besides, (5.4) implies that A_0^+ is semi-bounded from below. It follows from the results of Subsection 3.3 that *the operator A is definitizable over \mathbb{C}* . More precisely,

- (i) Let the operator A_0^- be semi-bounded from above. Then the operator A is definitizable, ∞ is a regular critical point of A (cf. [12]), and A admits decomposition (3.10).
- (ii) Let A_0^- be not semi-bounded from above. Then A is definitizable over \mathbb{C} and there exists no domain Ω in $\overline{\mathbb{C}}$ with $\infty \in \Omega$ such that A is definitizable over Ω . The only possible accumulation point for non-real spectrum of A is the point ∞ .

Note that A_0^- is not semi-bounded from above if $\lim_{x \rightarrow -\infty} q(x) = -\infty$.

5.3. Summable potentials

We denote by $q_{\text{neg}}(x) := \min\{q(x), 0\}$, $x \in \mathbb{R}$.

Assumption 5.2. $\int_t^{t+1} |q_{\text{neg}}(x)| dx \rightarrow 0$ as $|t| \rightarrow \infty$.

If Assumption 5.2 is fulfilled then the differential expression $-d^2/dx^2 + q$ is in limit point case at $+\infty$ and $-\infty$, cf. [46, Satz 14.21]. By [45, Theorem 15.1], A_0^+ is semi-bounded from below, A_0^- is semi-bounded from above with

$$\sigma_{\text{ess}}(A_0^+) \subset [0, +\infty) \quad \text{and} \quad \sigma_{\text{ess}}(A_0^-) \subset (-\infty, 0].$$

This implies that the negative spectrums of the operators A_0^+ and $-A_0^-$ consist of eigenvalues,

$$\sigma(\pm A_0^\pm) \cap (-\infty, 0) = \{\pm \lambda_n^\pm\}_1^{N^\pm} \subset \sigma_p(\pm A_0^\pm),$$

where $0 \leq N^\pm \leq \infty$. Besides, $\lim_{n \rightarrow \infty} \lambda_n^\pm = 0$ if $N^\pm = \infty$. Then, by Theorem 3.13, A is definitizable over $\overline{\mathbb{C}} \setminus \{0\}$. Theorems 3.11 and 3.15 imply easily the following statement.

Theorem 5.3. *Let Assumption 5.2 be fulfilled. Then the operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ admits the decomposition (3.10). Moreover,*

- (i) *If $\min \sigma_{\text{ess}}(A_0^+) > 0$ or $\max \sigma_{\text{ess}}(A_0^-) < 0$, then A is a definitizable operator and ∞ is a critical point of A .*
- (ii) *If $\min \sigma_{\text{ess}}(A_0^+) = \max \sigma_{\text{ess}}(A_0^-) = 0$ and $N^+ + N^- < \infty$, then A is a definitizable operator, 0 and ∞ are critical points of A .*
- (iii) *If $\min \sigma_{\text{ess}}(A_0^+) = \max \sigma_{\text{ess}}(A_0^-) = 0$ and $N^+ + N^- = \infty$, then the operator A is not definitizable. It is definitizable over $\overline{\mathbb{C}} \setminus \{0\}$. In particular, 0 is the only possible accumulation point of the non-real spectrum of A .*

We mention (cf. [5]) that Assumption 5.2, and therefore the statements of Theorem 5.3, hold true if $q \in L^1(\mathbb{R})$.

Remark 5.4. *By Theorem 3.15 (see also [12]) we have that if the operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ is definitizable, then ∞ is its regular critical point. In the case when A has a finite critical point, the question of the character of this critical point is difficult (see [13, 14, 18, 19, 33, 34, 32] and references therein). Let us mention one case. Assume that q is continuous in \mathbb{R} and $\int_{\mathbb{R}} (1+x^2)|q(x)| dx < \infty$, then $\min \sigma_{\text{ess}}(A_0^+) = \max \sigma_{\text{ess}}(A_0^-) = 0$ and $N^+ < \infty$ and $N^- < \infty$ (see [40]). Therefore Theorem 5.3 (as well as [12, Proposition 1.1]) implies that $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ is definitizable. It was shown (implicitly) in [18] that 0 is a regular critical point of A .*

In the following case, more detailed information may be obtained.

Corollary 5.5. *Suppose $\lim_{x \rightarrow \infty} q(|x|) = 0$. Then $\min \sigma_{\text{ess}}(A_0^+) = \max \sigma_{\text{ess}}(A_0^-) = 0$ and either the case (ii) or the case (iii) of Theorem 5.3 takes place. Moreover, the following holds.*

- (i) If $\liminf_{x \rightarrow \infty} x^2 q(|x|) > -1/4$, then A is a definitizable operator and 0 and ∞ are critical points of A .
- (ii) If $\limsup_{x \rightarrow \infty} x^2 q(|x|) < -1/4$, then the operator A is not definitizable. It is definitizable over $\overline{\mathbb{C}} \setminus \{0\}$.

Proof. The statement follows directly from [16, Corollary XIII.7.57], which was proved in [16] for infinitely differentiable q . Actually, this proof is valid for bounded potentials q . Finally, note that $\lim_{x \rightarrow \infty} q(|x|) = 0$ implies that q is bounded on $(-\infty, -X] \cup [X, +\infty)$ with X large enough. On the other hand, L^1 perturbations of potential q on any finite interval does not change $\sigma_{ess}(A_0^+)$, $\sigma_{ess}(A_0^-)$. Also such perturbations increase or decrease N^+ , N^- on finite numbers only due to Sturm Comparison Theorem (see e.g., [47, Theorem 2.6.3]). This completes the proof. \square

Example 5.6. Let $q(x) = -\frac{1}{1+|x|}$. Then Corollary 5.5 yields that the operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ is not definitizable. It is definitizable over $\overline{\mathbb{C}} \setminus \{0\}$.

It was shown above that under certain assumption on the potential q the operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ is not definitizable, but it is definitizable over the domain $\overline{\mathbb{C}} \setminus \{\lambda_0\}$, where $\lambda_0 \in \overline{\mathbb{R}}$ ($\lambda_0 = \infty$ in Example 5.1 and $\lambda_0 = 0$ in Example 5.6). In this case, unusual spectral behavior may appear near points of the set $c(A) \cup \{\lambda_0\}$ only ($c(A)$ is the set of critical points, see Subsection 3.1). Indeed, a bounded spectral projection $E^A(\Delta)$ exists for any connected set $\Delta \subset \overline{\mathbb{R}} \setminus \{\lambda_0\}$ such that the endpoints of Δ do not belong to $c(A) \cup \{\lambda_0\}$. Note also that $c(A)$ is at most countable and that λ_0 is the only possible accumulation point of the non-real spectrum of A .

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