Perturbation theory for self-adjoint operators in Krein spaces

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Perturbation Theory for Self-Adjoint Operators in Krein spaces

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Abstract. We show that the spectrum of negative type and the spectrum of positive type of self-adjoint operators in Krein spaces are stable under perturbations small in the gap metric. Moreover, we show how these notions can be used to decide whether a given operator has real spectrum only. We apply this to a $\mathcal{PT}$-symmetric multiplication operator.

1. Introduction

It is well-known that an isolated real eigenvalue of definite type of a self-adjoint operator \( A \) in a Krein space remain real under sufficiently small self-adjoint perturbations.

It is the aim of this paper to extend these results to points from the continuous spectrum and for non-isolated eigenvalues. In order to do this we recall the definition of different kind of spectra: The spectral points of positive and of negative type and the spectral points of type \( \pi_+ \) and of \( \pi_- \). A real point \( \lambda \) of the spectrum \( \sigma(A) \) is called a spectral point of positive (negative) type, if for every normed approximative eigensequence \( (x_n) \) corresponding to \( \lambda \) all accumulation points of the sequence \( ([x_n, x_n]) \) are positive (resp. negative). These spectral points were introduced by P. Lancaster, A. Markus and V. Matsaev in [19]. In [21] the existence of a local spectral function was proved for intervals containing only spectral points of positive (negative) type or points of the resolvent set \( \rho(A) \). Moreover it was shown that, if \( A \) is perturbed by a compact selfadjoint operator, a spectral point of positive type of \( A \) becomes either an inner point of the spectrum of the perturbed operator or it becomes an eigenvalue of type \( \pi_+ \). A point from the approximative point spectrum of \( A \) is of type \( \pi_+ \) if the abovementioned property of approximative eigensequences \( (x_n) \) holds only for sequences \( (x_n) \) belonging to some linear manifold of finite codimension (see Definition 4 below). Every spectral point of a selfadjoint operator in a Pontryagin space with finite index of negativity is of type \( \pi_+ \). For a detailed study of the properties of the spectrum of type \( \pi_+ \) we refer to [2].

In this paper we show how these notions can be used to decide whether a given operator has real spectrum only. Moreover, as the main result of this paper, we show that the spectrum of negative type and the spectrum of positive type of self-adjoint operators in Krein spaces are stable under perturbations small in the gap metric.

Sign type spectrum is used in the classification of eigenvalues, e.g. [6, 7, 9, 12, 18, 22] and it can be applied to \( PT \)-symmetric problems. We will give an example with a \( PT \)-symmetric multiplication operator in Section 3. Moreover, it is used in the theory of indefinite Sturm-Liouville operators, e.g. [3, 5, 10, 16], and in the mathematical system theory, see e.g. [14, 20].

2. Sign type spectrum of self-adjoint operators in Krein spaces

2.1. Self-adjoint operators in Krein spaces

Let \( (\mathcal{H}, [\cdot, \cdot]) \) be a Krein space. We briefly recall that a complex linear space \( \mathcal{H} \) with a hermitian nondegenerate sesquilinear form \( [\cdot, \cdot] \) is called a Krein space if there exists a so-called fundamental decomposition

\[
\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-
\]

with subspaces \( \mathcal{H}_\pm \) being orthogonal to each other with respect to \( [\cdot, \cdot] \) such that \( (\mathcal{H}_\pm, \pm [\cdot, \cdot]) \) are Hilbert spaces. In the following all topological notions are understood
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with respect to some Hilbert space norm \( \| \cdot \| \) on \( \mathcal{H} \) such that \( [\cdot, \cdot] \) is \( \| \cdot \| \)-continuous. Any two such norms are equivalent. An element \( x \in \mathcal{H} \) is called positive (negative, neutral, respectively) if \( [x, x] > 0 \) (\( [x, x] < 0 \), \( [x, x] = 0 \), respectively). For the basic theory of Krein space and operators acting therein we refer to [8], [1] and, in the context of \( \mathcal{PT} \) symmetry, we refer to [22].

Let \( A \) be a closed operator in \( \mathcal{H} \). We define the extended spectrum \( \sigma_e(A) \) of \( A \) by

\[
\sigma_e(A) := \sigma(A) \quad \text{if} \quad A \quad \text{is bounded} \quad \text{and} \quad \sigma_e(A) := \sigma(A) \cup \{ \infty \} \quad \text{if} \quad A \quad \text{is unbounded}.
\]

The resolvent set of \( A \) is denoted by \( \rho(A) \). The operator \( A \) is called Fredholm if the dimension of the kernel of \( A \) and the codimension of the range of \( A \) are finite. The set

\[
\sigma_{ess}(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is not Fredholm} \}
\]

is called the essential spectrum of \( A \). We say that \( \lambda \in \mathbb{C} \) belongs to the approximate point spectrum of \( A \), denoted by \( \sigma_{ap}(A) \), if there exists a sequence \( (x_n) \subset \text{dom}(A) \) with \( \|x_n\| = 1 \), \( n = 1, 2, \ldots \), such that

\[
\|x_n\| = 1 \quad \text{and} \quad \lim_{n \to \infty} \|Ax_n - \lambda x_n\| = 0
\]

(see e.g. [11, 25]). Obviously, the continuous and the point spectrum of a closed operator are subsets of the approximate point spectrum. Moreover, we have the following.

**Remark 1** The boundary points of \( \sigma(A) \) in \( \mathbb{C} \) belong to \( \sigma_{ap}(A) \).

Let \( A \) be a self-adjoint operator in the Krein space \( (\mathcal{H}, [\cdot, \cdot]) \), i.e., \( A \) coincides with its adjoint \( A^+ \) with respect to the indefinite inner product \( [\cdot, \cdot] \). Then all real spectral points of \( A \) belong to \( \sigma_{ap}(A) \) (see e.g. Corollary VI.6.2 in [8]).

### 2.2. Spectral points of positive and negative type of \( A \)

The indefiniteness of the scalar product \( [\cdot, \cdot] \) on \( \mathcal{H} \) induces a natural classification of isolated real eigenvalues: A real isolated eigenvalue \( \lambda_0 \) of \( A \) is called of positive (negative) type if all the corresponding eigenelements (i.e. all elements of all Jordan chains corresponding to \( \lambda_0 \)) are positive (negative, respectively). Observe that there is no Jordan chain of length greater than one which corresponds to a eigenvalue of \( A \) of positive type (or of negative type). This classification of real isolated eigenvalues is used frequently, we mention here only [6, 7, 9, 12, 18, 22].

There is a corresponding notion for points from the approximate point spectrum. The following definition was given in [19] and [21] for bounded self-adjoint operators.

**Definition 2** For a self-adjoint operator \( A \) in the Krein space \( (\mathcal{H}, [\cdot, \cdot]) \) a point \( \lambda_0 \in \sigma(A) \) is called a spectral point of positive (negative) type of \( A \) if \( \lambda_0 \in \sigma_{ap}(A) \) and for every sequence \( (x_n) \subset \text{dom}(A) \) with \( \|x_n\| = 1 \) and \( \|(A - \lambda_0 I)x_n\| \to 0 \) as \( n \to \infty \) we have

\[
\liminf_{n \to \infty} [x_n, x_n] > 0 \quad (\text{resp.} \quad \limsup_{n \to \infty} [x_n, x_n] < 0).
\]
We denote the set of all points of \( \sigma_e(A) \) of positive (negative) type by \( \sigma_{++}(A) \) (resp. \( \sigma_{--}(A) \)).

The sets \( \sigma_{++}(A) \) and \( \sigma_{--}(A) \) are contained in \( \mathbb{R} \). Indeed, for \( \lambda \in \sigma_{++}(A) \setminus \{ \infty \} \) and \( (x_n) \) as in the first part of Definition 2 we have \(- (\text{Im } \lambda)[x_n, x_n] = \text{Im } [(A - \lambda)x_n, x_n] \to 0 \) for \( n \to \infty \) which implies \( \text{Im } \lambda = 0 \). Here \( \mathbb{R} \) denotes the set \( \mathbb{R} \cup \{ \infty \} \) and \( \mathbb{C} \) the set \( \mathbb{C} \cup \{ \infty \} \), each equipped with the usual topology. In the following proposition we collect some properties. For a proof we refer to [2].

**Proposition 3** Let \( \lambda_0 \) be a point of \( \sigma_{++}(A) \) (\( \sigma_{--}(A) \)), respectively. Then there exists an open neighbourhood \( U \) in \( \mathbb{C} \) of \( \lambda_0 \) such that the following holds.

(i) We have

\[
U \setminus \mathbb{R} \subset \rho(A),
\]

this is, the non-real spectrum of \( A \) cannot accumulate to \( \sigma_{++}(A) \cup \sigma_{--}(A) \).

(ii) \( U \cap \sigma_e(A) \cap \mathbb{R} \subset \sigma_{++}(A) \) (resp. \( U \cap \sigma_e(A) \cap \mathbb{R} \subset \sigma_{--}(A) \)).

(iii) There exists a number \( M > 0 \) such that

\[
\|(A - \lambda)^{-1}\| \leq \frac{M}{|\text{Im } \lambda|} \text{ for all } \lambda \in U \setminus \mathbb{R}.
\]

2.3. **Spectral points of type \( \pi_+ \) and of type \( \pi_- \) of \( A \)**

In a similar way as above we define subsets \( \sigma_{\pi_+}(A) \) and \( \sigma_{\pi_-}(A) \) of \( \sigma_e(A) \) containing \( \sigma_{++}(A) \) and \( \sigma_{--}(A) \), respectively (cf. Definition 5 in [2]). They will play an important role in the following.

**Definition 4** For a self-adjoint operator \( A \) in \( \mathcal{H} \) a point \( \lambda_0 \in \sigma(A) \) is called a spectral point of type \( \pi_+ \) (type \( \pi_- \)) of \( A \) if \( \lambda_0 \in \sigma_{ap}(A) \) and if there exists a linear submanifold \( \mathcal{H}_0 \subset \mathcal{H} \) with \( \text{codim } \mathcal{H}_0 < \infty \) such that for every sequence \( (x_n) \subset \mathcal{H}_0 \cap \text{dom}(A) \) with \( \|x_n\| = 1 \) and \( \|(A - \lambda_0 I)x_n\| \to 0 \) as \( n \to \infty \) we have

\[
\lim_{n \to \infty} [x_n, x_n] > 0 \quad (\text{resp. } \lim_{n \to \infty} [x_n, x_n] < 0).
\]

The point \( \infty \) is said to be of type \( \pi_+ \) (type \( \pi_- \)) of \( A \) if \( A \) is unbounded and if there exists a linear submanifold \( \mathcal{H}_0 \subset \mathcal{H} \) with \( \text{codim } \mathcal{H}_0 < \infty \) such that for every sequence \( (x_n) \subset \mathcal{H}_0 \cap \text{dom}(A) \) with \( \lim_{n \to \infty} \|x_n\| = 0 \) and \( \|Ax_n\| = 1 \) we have

\[
\lim_{n \to \infty} [Ax_n, Ax_n] > 0 \quad (\text{resp. } \lim_{n \to \infty} [Ax_n, Ax_n] < 0).
\]

We denote the set of all points of \( \sigma_e(A) \) of type \( \pi_+ \) (type \( \pi_- \)) of \( A \) by \( \sigma_{\pi_+}(A) \) (resp. \( \sigma_{\pi_-}(A) \)).
Spectral point of type $\pi_+$ and type $\pi_-$ of $A$ have properties comparable to those mentioned in Proposition 3. We will collect them in the following proposition (for a proof see [2] and [4]).

**Proposition 5** Let $\lambda_0$ be a point of $\sigma_{\pi_+}(A)$ ($\sigma_{\pi_-}(A)$, respectively). Then there exists an open neighbourhood $\mathcal{U}$ in $\mathbb{C}$ of $\lambda_0$ such that the following holds.

(i) We have
$$\mathcal{U} \setminus \mathbb{R} \subset \sigma_p(A).$$

Moreover, if $\lambda_0 \in \mathbb{C}$ is non-real then the operator $A - \lambda_0$ has a closed range and $\text{dim} \ker(A - \lambda_0) < \infty$.

(ii) $\mathcal{U} \cap \sigma_{ap}(A) \subset \sigma_{\pi_+}(A)$ (resp. $\mathcal{U} \cap \sigma_{ap}(A) \subset \sigma_{\pi_-}(A)$).

(iii) If $\lambda_0 = \infty$ then $\infty \in \sigma_{++}(A)$. If $\infty \in \sigma_{\pi_-}(A)$ then $\infty \in \sigma_{--}(A)$.

(iv) Assume, in addition, that $\lambda_0 \in \mathbb{R}$ and that there is $[a, b] \subset \mathcal{U}$, $\lambda_0 \in [a, b]$, such that each point of $[a, b]$ is an accumulation point of $\rho(A)$. Then there exists an open neighbourhood $\mathcal{V}$ in $\mathbb{C}$ of $[a, b]$ such that $\mathcal{V} \setminus \mathbb{R} \subset \rho(A)$ and either $\mathcal{V} \cap \sigma(A) \cap \mathbb{R} \subset \sigma_{++}(A)$ or there exists a finite number of points $\lambda_1, \ldots, \lambda_n \in \sigma_{\pi_+}(A) \cap \sigma_p(A)$ such that
$$\mathcal{V} \setminus \sigma(A) \cap \mathbb{R} \setminus \{\lambda_1, \ldots, \lambda_n\} \subset \sigma_{++}(A).$$

Moreover, in this case there exist numbers $m \geq 1$ and $M > 0$ such that
$$\|(A - \lambda)^{-1}\| \leq \frac{M}{|\text{Im} \lambda|^m} \text{ for all } \lambda \in \mathcal{V} \setminus \mathbb{R}.$$
Observe, that in the case of an unbounded operator $A$ condition (2) implies also $\infty \in \sigma_{++}(A)$. There are comparable results for the spectrum of type $\pi_+$. 

**Theorem 7** Let $A$ be a self-adjoint operator in $(\mathcal{H}, [\cdot, \cdot])$ with $\rho(A) \neq \emptyset$ satisfying 

$$
\sigma_{\text{ess}}(A) \subset \mathbb{R} \quad \text{and} \quad \sigma_e(A) = \sigma_{\pi_+}(A) \quad \text{resp.} \quad \sigma_e(A) = \sigma_{\pi_-}(A). 
$$

(3) Then $(\mathcal{H}, [\cdot, \cdot])$ is a Pontryagin space and the space $\mathcal{H}_-$ in the fundamental decomposition (1) is of finite dimension. Moreover, the set $\sigma(A) \setminus \mathbb{R}$ consists of at most finitely many eigenvalues with finite dimensional root subspaces, i.e.

$$
\sigma(A) \setminus \mathbb{R} \subset \sigma_p(A) \setminus \sigma_{\text{ess}}(A)
$$

If $A$ with $\rho(A) \neq \emptyset$ satisfies instead of (3) the following condition

$$
\sigma_{\text{ess}}(A) \subset \mathbb{R} \quad \text{and} \quad \sigma_e(A) = \sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A).
$$

(4) Then the non-real spectrum of $A$ consists of at most finitely many points which belong to $\sigma_p(A) \setminus \sigma_{\text{ess}}(A)$.

### 3.2. Stability properties of sign type spectrum under compact perturbations and under perturbations small in gap

Let $A$ be a self-adjoint operator in a Krein space $(\mathcal{H}, [\cdot, \cdot])$. We assume that $A$ is fundamentally reducible, that is, the operator $A$ admits a matrix representation

$$
A = 
\begin{pmatrix}
A_+ & 0 \\
0 & A_-
\end{pmatrix}
$$

(5) with respect to a fundamental decomposition (1) of $(\mathcal{H}, [\cdot, \cdot])$ such that $A_+$ and $A_-$ are self-adjoint operators in the Hilbert spaces $(\mathcal{H}_+, [\cdot, \cdot])$ and $(\mathcal{H}_-, [\cdot, \cdot])$, respectively. In the case of a bounded $A$ and a perturbed operator $B$ of the form

$$
B = 
\begin{pmatrix}
A_+ & C \\
-C^* & A_-
\end{pmatrix}
$$

with some bounded operator $C$ acting from $\mathcal{H}_-$ to $\mathcal{H}_+$, it was shown in [21] that

$$
\text{dist} (\lambda, \sigma(A_-)) > \|C\| \implies \lambda \in \rho(B) \cup \sigma_{++}(B),
$$

$$
\text{dist} (\lambda, \sigma(A_+)) > \|C\| \implies \lambda \in \rho(B) \cup \sigma_{--}(B).
$$

Theorem 8 below can be viewed as a generalization of this result.

Recall that the gap between two subspaces $M$ and $N$ of a Hilbert space is defined by

$$
\hat{\delta}(M, N) := \max \left\{ \sup_{u \in M, \|u\|=1} \text{dist} (u, N), \sup_{v \in N, \|v\|=1} \text{dist} (v, M) \right\}
$$

(cf. [17]). If $P_M$ and $P_N$ denote the orthogonal projections on $M$ and $N$, respectively, it follows

$$
\hat{\delta}(M, N) = \|P_M - P_N\|.$$
Theorem 8 Let $A$ and $B$ be self-adjoint operators in $(\mathcal{H}, [\cdot, \cdot])$. Let $A$ be a fundamentally reducible operator. If $A_+$ and $A_-$ are given by the matrix representation (5) and if there exists a real $\gamma > 0$ such that for $\lambda \in \mathbb{R}$

$$\hat{\delta}(\text{graph } (A - \lambda), \text{graph } (B - \lambda)) < \gamma \quad \text{and} \quad \gamma^2 \left(1 + \text{dist } (\lambda, \sigma(A_-))^{-2}\right) < \frac{1}{4},$$

then

$$\lambda \in \rho(B) \cup \sigma_{++}(B).$$

If there exists a real $\gamma > 0$ such that for $\lambda \in \mathbb{R}$

$$\hat{\delta}(\text{graph } (A - \lambda), \text{graph } (B - \lambda)) < \gamma \quad \text{and} \quad \gamma^2 \left(1 + \text{dist } (\lambda, \sigma(A_+))^{-2}\right) < \frac{1}{4},$$

then

$$\lambda \in \rho(B) \cup \sigma_{--}(B).$$

Theorem 8 can be considered as a generalization of Corollary 3.4 in [23].

Finally, we mention a perturbation result for spectral points of type $\pi_+$ (type $\pi_-$) which was already proved in [2].

Theorem 9 Let $A$ and $B$ be self-adjoint operators in the Krein space $(\mathcal{H}, [\cdot, \cdot])$. Assume that $\rho(A) \cap \rho(B) \neq \emptyset$ and that for some (and hence for all) $\mu \in \rho(A) \cap \rho(B)$

$$(A - \mu)^{-1} - (B - \mu)^{-1} \quad \text{is a compact operator.} \tag{6}$$

Then

$$(\sigma_{\pi_+}(A) \cup \rho(A)) \cap \mathbb{R} = (\sigma_{\pi_+}(B) \cup \rho(B)) \cap \mathbb{R}, \tag{7}$$

$$(\sigma_{\pi_-}(A) \cup \rho(A)) \cap \mathbb{R} = (\sigma_{\pi_-}(B) \cup \rho(B)) \cap \mathbb{R}. \tag{8}$$

Moreover, $\infty \in \sigma_{++}(A) \ (\infty \in \sigma_{--}(A))$ if and only if $\infty \in \sigma_{++}(B) \ (\text{resp. } \infty \in \sigma_{--}(B))$.

We mention that a similar statement as in Theorem 9 for spectral points of positive or negative type is in general not true.

3.3. Example

Denote by $I$ the closed interval $[-1, 1]$. Suppose $V$ and $W$ are functions from $L^\infty(I)$, that is, $V$ and $W$ are essentially bounded. Moreover, we assume that $V$ is real-valued and even, that is

$$V(-x) = V(x) = \overline{V(x)}, \quad x \in I,$$

and that $W$ is $\mathcal{PT}$-symmetric, that is

$$W(-x) = \overline{W(x)}, \quad x \in I.$$

Let $\mathcal{D}$ be the set of all $f \in L^2(I)$ such that $f$ and $f'$ are absolutely continuous, $f(-1) = f(1) = 0$ with $f'' \in L^2(I)$. In the Hilbert space $L^2(I)$, equipped with the usual inner product

$$(f, g) = \int_I f(x) \overline{g(x)} dx, \quad f, g \in L^2(I),$$
we consider the operators $A$ and $B$ defined on $\mathcal{D}$,

$$A := -\frac{d^2}{dx^2} + V \quad \text{and} \quad B := -\frac{d^2}{dx^2} + W, \quad \text{dom} \, A = \text{dom} \, B = \mathcal{D}. $$

It is easily seen that $A$ is a self-adjoint operator in the Hilbert space $(L^2(I), (\cdot, \cdot))$. In general the potential $W$ is not real-valued and the operator $B$ is not a self-adjoint operator in the Hilbert space $(L^2(I), (\cdot, \cdot))$. Therefore, we consider the inner product

$$[f, g] = \int_I f(x)\overline{g(-x)}dx, \quad f, g \in L^2(I).$$

Then $(L^2(I), [\cdot, \cdot])$ is a Krein space and $W$ is a self-adjoint operator in the Krein space $(L^2(I), [\cdot, \cdot])$, see [22]. The operator $A$ is also self-adjoint in the Krein space $(L^2(I), [\cdot, \cdot])$ and $A$ is fundamental reducible. If $\|V\|_{L^\infty} < \frac{3\pi^2}{8}$, then the spectrum consists of eigenvalues which are alternating between positive and negative type (see Theorem 4.1 in [22]), that is

$$\sigma(A) = \sigma_{++}(A) \cup \sigma_{--}(A). $$

Moreover, we have $\infty \notin \sigma_{++}(A) \cup \sigma_{--}(A)$.

Observe that

$$\delta(\text{graph} \,(A - \lambda), \text{graph} \,(B - \lambda)) \leq \|V - W\|_{L^\infty}.$$ 

Now Theorem 8 implies the following.

**Theorem 10** Let $\lambda \in \mathbb{R}$. If

$$\|V - W\|_{L^\infty}^2 \left(1 + \left(\text{dist} \,(\lambda, \sigma_{--}(A))\right)^{-2}\right) < \frac{1}{4},$$

then

$$\lambda \in \rho(B) \cup \sigma_{++}(B).$$

If

$$\|V - W\|_{L^\infty}^2 \left(1 + \left(\text{dist} \,(\lambda, \sigma_{++}(A))\right)^{-2}\right) < \frac{1}{4},$$

then

$$\lambda \in \rho(B) \cup \sigma_{--}(B).$$

4. Proofs

In this section we will prove Theorems 6-8 from Section 3.
4.1. Proof of Theorems 6 and 7

Let \( A \) be a self-adjoint operator in the Krein space \((\mathcal{H}, [\cdot, \cdot])\) with \( \rho(A) \neq \emptyset \) satisfying (4). The resolvent set of a self-adjoint operator in a Krein space is symmetric with respect to the real axis (cf. [8]), hence there are points from \( \rho(A) \) in the upper and in the lower half-plane. This and \( \sigma_{\text{ess}}(A) \subset \mathbb{R} \) imply that \( \sigma(A) \setminus \mathbb{R} \) consists only of isolated eigenvalues with finite algebraic multiplicity (see §5.6 in [17]). In particular, each point in \( \mathbb{R} \) is an accumulation point of \( \rho(A) \) and Proposition 5 implies that the spectrum of \( A \) cannot accumulate to a real point. Moreover, from (4) and Proposition 5 (iii) we conclude \( \infty \in \sigma_{++}(A) \cup \sigma_{--}(A) \). Therefore the non-real spectrum of \( A \) is bounded and the second part of Theorem 7 is proved. In order to show the first part of Theorem 7 we assume without loss of generality \( \sigma_{\text{ess}}(A) \subset \mathbb{R} \) and \( \sigma_{e}(A) = \sigma_{\pi+}(A) \).

It remains to show that \((\mathcal{H}, [\cdot, \cdot])\) is a Pontryagin space. Relation (9), Theorem 23 in [2] and Theorem 4.7 in [15] imply that \( A \) is a definitizable operator. Recall that a self-adjoint operator \( A \) in a Krein space \((\mathcal{H}, [\cdot, \cdot])\) is called definitizable if \( \rho(A) \neq \emptyset \) and if there exists a rational function \( p \neq 0 \) having poles only in \( \rho(A) \) such that \([p(A)x, x] \geq 0 \) for all \( x \in \mathcal{H} \). Then the spectrum of \( A \) is real or its non-real part consists of a finite number of points. Moreover, \( A \) has a spectral function \( E(\cdot) \) defined on the ring generated by all connected subsets of \( \mathbb{R} \) whose endpoints do not belong to some finite set which is contained in \( \{ t \in \mathbb{R} : p(t) = 0 \} \cup \{ \infty \} \) (see [24]). Now Corollary 28 and Theorem 26 of [2] show that Theorem 7 holds true.

Theorem 6 is now a consequence of Theorem 7: Assume without loss of generality

\[
\sigma_{e}(A) = \sigma_{++}(A).
\]

Then Theorem 7 implies that \((\mathcal{H}, [\cdot, \cdot])\) is a Pontryagin space and the space \( \mathcal{H}_- \) in the fundamental decomposition (1) is of finite dimension. If \( \mathcal{H}_- \neq 0 \), then there exists at least one non-positive eigenvector of \( A \) (see §12 in [13]) for some eigenvalue \( \lambda_0 \). This implies \( \lambda_0 \notin \sigma_{++}(A) \), hence \( \mathcal{H}_- = 0 \) and \((\mathcal{H}, [\cdot, \cdot])\) is a Hilbert space. The second part of Theorem 6 follows from Proposition 25 and Corollary 28 in [2].

4.2. Proof of Theorem 8

We will only prove the first part of Theorem 8. The second one follows then by a similar reasoning. Let \( \lambda \) be a real number in \( \sigma(B) \) and assume that there exists a \( \gamma > 0 \) such that

\[
\tilde{\delta}(\text{graph } (A - \lambda), \text{graph } (B - \lambda)) < \gamma \quad \text{and} \quad \gamma^2 \left( 1 + (\text{dist } (\lambda, \sigma(A_-)))^{-2} \right) < \frac{1}{4}.
\]

Then \( \lambda \in \sigma_{\text{ap}}(B) \). Let \( (x_n^+ + x_n^-) \in \text{dom } B, n = 1, 2 \ldots, x_n^+ \in \mathcal{H}_+, x_n^- \in \mathcal{H}_- \), be a sequence with

\[
\|x_n^+\|^2 + \|x_n^-\|^2 = 1 \quad \text{and} \quad \lim_{n \to \infty} \|(B - \lambda)(x_n^+ + x_n^-)\| = 0 \quad (10)
\]
We have
\[
\text{dist} \left( \left( x_n^+ + x_n^- \right), \text{graph} \left( A - \lambda \right) \right) < \gamma \left\| \left( \begin{array}{c} x_n^+ + x_n^- \\ (B - \lambda)(x_n^+ + x_n^-) \end{array} \right) \right\|.
\]
Hence, there exists \( y_n^+ \in \text{dom} \, A_+, \, y_n^- \in \text{dom} \, A_- \) with
\[
\|x_n^+ - y_n^+\|^2 + \|x_n^- - y_n^-\|^2 + \|(B - \lambda)(x_n^+ + x_n^-) - (A_+ - \lambda)y_n^+ - (A_- - \lambda)y_n^-\|^2
\]
is less than
\[
\gamma^2 \left\| \left( \begin{array}{c} x_n^+ + x_n^- \\ (B - \lambda)(x_n^+ + x_n^-) \end{array} \right) \right\|^2.
\]
In view of (10), we have
\[
\limsup_{n \to \infty} \|x_n^+ - y_n^+\|^2 + \|A_-y_n^- - \lambda y_n^-\|^2 < \gamma^2. \tag{11}
\]
With (11), (10) and \( \|A_-y_n^- - \lambda y_n^-\| \geq \text{dist} \,(\lambda, \sigma(A_-))\|y_n^-\| \) we obtain
\[
\liminf_{n \to \infty} \left[ x_n^+ + x_n^-, \, x_n^+ + x_n^- \right] =
= \liminf_{n \to \infty} \|x_n^+\|^2 - \|x_n^-\|^2 = \liminf_{n \to \infty} 1 - 2\|x_n^-\|^2
= 1 - 2\limsup_{n \to \infty} \|x_n^- - y_n^+ + y_n^-\|^2
\geq 1 - 2\limsup_{n \to \infty} \left(2\|x_n^- - y_n^-\|^2 + 2\|y_n^-\|^2\right)
\geq 1 - 4\limsup_{n \to \infty} \left(1 + \text{dist} \,(\lambda, \sigma(A_-))^{-2} \right) \left( \|x_n^- - y_n^-\|^2 + \|A_-y_n^- - \lambda y_n^-\|^2 \right)
\geq 1 - 4\gamma^2 \left(1 + \text{dist} \,(\lambda, \sigma(A_-))^{-2} \right) > 0
\]
and Theorem 8 is proved.

References

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