Convergence in Distribution of Random Closed Sets and Applications in Stability Theory of Stochastic Optimisation

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Contents

Introduction

In this work we investigate one-sided convergence in distribution of random closed sets. Set convergence methods, most notably the concept of epi-convergence, have proved to be useful tools in the investigation of parametric optimisation problems (cf. [3], [33]). They are often favoured above uniform convergence methods, as they allow for lower semicontinuous objective functions instead of continuous functions. One-sided convergence, in the sense of inner/outer set convergence, is needed for example in stability theory of parametric optimisation problems. Frequently the original problem is approximated by a sequence of (easier to solve or numerically obtained) surrogate problems. Since in general the solutions of the surrogate problems only approximate a subset of the solutions for the original problem, inner convergence is a fruitful concept. Stochastic optimisation problems bear a close resemblance to parametric optimisation problems and can for example have their origin in the estimation of the parameters or in simulations. To derive stochastic versions of stability results, the set valued methods have to be combined with probability theory (see [24] and for convergence in distribution [34]). Convergence in distribution methods are especially useful when the original problem is also random. They are asked for to obtain information about the distribution of optimal values and solutions for the original problem. In [27], [43] and [22] one-sided set convergence in distribution (predominantly inner convergence) has been investigated.

In this work our approach and contributions are organised as follows.

The first chapter contains a collection of properties of the topologies which describe one-sided set convergence and allow a sound topological foundation for the definition of one-sided convergence in distribution. The noted differences to the case of convergence in distribution in metric spaces hint on the expected difficulties and the need for workarounds. The basic notations for the different types of convergence are given.

The conditions in the definition of convergence in distribution are hard to verify for a given sequence of random closed sets or stochastic processes. We thus accompany them with useful sufficient conditions for one-sided convergence in distribution in the second chapter. Mostly we derive convergence in distribution in our set valued framework from other types of convergence (in distribution). These criteria can serve as a bridge between classical convergence for stochastic processes for important classes (e.g. $D[0,\infty)$) and set convergence in distribution. A central result are new convergence criteria for the epigraphs of random lower semicontinuous functions, which are in line with the frequently used finite dimensional approach to convergence in distribution for stochastic processes. They use the idea of stochastic equi lower semicontinuity. As a combination of the two one-sided criteria we obtain a corrected version of the convergence criterion for the full epi-convergence in distribution in [19]. Our sufficient conditions can help to make recent applications of epi-convergence to statistics (cf. [38],[23],[29]) accessible for stochastic processes. The second chapter is closed with a partial result for the convergence in distribution of vectors of random closed sets, which is important for stochastic optimisation problems with random restriction sets.

Starting with the third chapter we show, how one-sided set convergence in distribution can be applied in stability theory of stochastic optimisation with random constraints. We show generalisations to the case of ε- resp. ε_n-optimality for known results and provide the complementary outer convergence part to [43]. One-sided convergence in distribution does in general not yield the distribution of minima and argmins of the approximated problem. Instead we obtain one-sided bounds, which can for example be used to find approximate confidence regions for the argmin sets. It is shown, how results from [12] about argmax distributions in the non unique case can be obtained with the set convergence in distribution approach. The most important technique used in this chapter is to transfer results from parametric optimisation (found in [2]) to the setting of convergence in distribution with the help of the Continuous Mapping Theorem and its semicontinuous versions.

In the fourth and final chapter we derive 'in distribution' stability results for stochastic multiobjective optimisation problems. In these problems in addition to the solutions, the optimal values are usually set-valued and are thus tractable by set convergence methods. As in the third chapter we are able to transfer deterministic results (here [35] was a valueable source) to the case of convergence in distribution and to provide ε_n-optimality extensions.

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1. Convergence in Distribution of Random Closed Sets

Random sets occur in a variety of situations, for example as solutions of random equalities and inequalities and as sets of optimal points in stochastic optimisation problems. It is known (see [3],[33]) that set convergence of closed sets is useful for stability theory of parametric optimisation problems and that it is topologically very accessible. Random closed sets ([24],[37]) are of special interest, for example as epigraphs of lower semicontinuous objective functions and as restriction sets in stochastic optimisation problems.. In this chapter we deal with one-sided convergence in distribution for sequences of random closed sets and its topological foundations. First we consider inner-/outer convergence of closed sets in the deterministic case and investigate properties of the related topologies. Throughout this text we denote the space of all closed subsets of a given first countable topological space X by $\mathcal{F}(X)$.

1.1. Set Convergence and Associated Topologies

There are several concepts for convergence of sequences $(F_n)_n \subset \mathcal{F}(X)$. For applications in stochastic optimisation it has proved to be useful to choose the concept of Kuratowski–Painlevé convergence, which describes convergence of sequences of closed sets with the help of sequences of points and their limits and accumulation points.

Definition 1.1 Let (X, τ) be a first countable topological space. For a sequence $(F_n)_n \subset \mathcal{F}(X)$ the Kuratowski–Painlevé limit superior and the Kuratowski–Painlevé limit inferior are given by

$$\text{K--}\limsup_{n\to\infty} F_n := \left\{x : \exists\, (x_{n_k})_k\,,\ x_{n_k} \in F_{n_k},\ x_{n_k} \to x\right\}$$

and

$$\text{K--}\liminf_{n\to\infty} F_n := \left\{x : \exists\, (x_n)\,,\ x_n \in F_n,\ \text{for } n \geq n_0,\ x_n \to x\right\}.$$

Note that $K - \liminf_{n\to\infty} F_n \subset K - \limsup_{n\to\infty} F_n$. It is well known, that $K - \limsup_{n\to\infty} F_n$ and $K - \liminf_{n\to\infty} F_n$ are again closed subsets of X, if X is first countable. In the following we will write $\limsup$ and $\liminf$ instead of $K - \limsup$ and $K - \liminf$, as it will always be clear from the context, whether the limit inferior/superior is to be understood in the classical calculus sense, in the set–algebraic way or in the Kuratowski–Painlevé sense.

Definition 1.2 Let (X, τ) be a topological space. Let $(F_n)_n \subset \mathcal{F}(X)$, let $F \subset \mathcal{F}(X)$. F_n is said to inner-converge to F, if

$$\limsup_{n\to\infty} F_n \subset F.$$

F_n outer-converges to F, if

$$F \subset \liminf_{n\to\infty} F_n$$

and F_n converges to F (in the sense of Kuratowski–Painlevé), if

$$F = \liminf_{n\to\infty} F_n = \limsup_{n\to\infty} F_n.$$

Inner and outer limits are generally not determined uniquely. If F_n inner converges to F, then F_n inner converges to every $G \supset F$. If F_n outer converges to F, then F_n outer converges to every $G \subset F$. In particular every sequence $(F_n)_n$ of closed sets inner converges to X and outer converges to $\emptyset$. These convergence notions are well suited for dealing with deterministic sets, since they only require the calculation of limits and accumulation points for sequences of points from X. In the random setting, the definition of almost sure inner–/outer convergence is straightforward. On the other hand it is by no means clear how to define inner–/outer convergence in distribution. A look at the definition of convergence in distribution or at the Portmanteau Theorem (e.g. Theorem 2.1 in [5]) shows that the topology describing the considered convergence in the deterministic case plays an essential role, either by directly using the open sets of this topology or by investigating integrals of continuous functions. We therefore require topologies on the space $\mathcal{F}(X)$ such that convergence in these topologies is equivalent to inner–/outer convergence.

It is well known (see for example [3]), that Kuratowski–Painlevé convergence is equivalent to convergence in the Fell topology, if the underlying space X is first countable.

The Fell topology on $\mathcal{F}(X)$ is a so-called hit-and-miss topology, generated by the subbase which consists of all sets of the type

$$M(K) = \{F \in \mathcal{F}(X) : \; F \cap K = \emptyset\}$$

with compact $K \subset X$
and all

$$H(G) = \{F \in \mathcal{F}(X) : \; F \cap G \neq \emptyset\}$$

with open $G \subset X$.

Because of the two types of sets that make up the subbase of the Fell topology (the miss-sets $M(K)$ and hit-sets $H(G)$), it is natural to consider the following coarser topologies.

Definition 1.3 (a) On $\mathcal{F}(X)$ the topology τ_M is the topology generated by the subbase consisting of all sets of the form

$$M(K) = \{F \in \mathcal{F}(X) : \; F \cap K = \emptyset\}$$

with compact $K \subset X$.

(b) On $\mathcal{F}(X)$ the topology τ_H is the topology generated by the subbase consisting of all sets of the form

$$H(G) = \{F \in \mathcal{F}(X) : \; F \cap G \neq \emptyset\}$$

with open $G \subset X$.

These topologies have been investigated for example in [13]. In the following we will gather properties of τ_M and τ_H. They have been used as a rigorous topological foundation for convergence in distribution in [43] and [21].
Beer in [3] calls for admissibility as a minimal requirement for hyperspace topologies. A hyperspace topology on $\mathcal{F}(X)$ is called admissible, provided that the relative topology that X inherits from the hyperspace topology, under the identification $x \leftrightarrow \{x\}$, agrees with the initial topology on X.

Example 1.4 The topology τ_M is in general not admissible. Let $X = \mathbb{R}$ be equipped with the usual topology and let

$$x_n = \begin{cases} 0 & , \; n \text{ odd} \\ n & , \; n \text{ even} \end{cases} \quad , \; x = 0,$$

then $\{x_n\} \overset{\tau_M}{\to} \{x\}$, but $x_n \not\to x$.

Lemma 1.5 *The topology τ_H is always admissible.*

Proof. Let $i : X \to \mathcal{F}(X),\ i\,(x) = \{x\}$. It suffices to show that i and i^{-1} are continuous. To show continuity of i we show that $i^{-1}\,(S)$ is open for all S from the subbase of τ_H. Such S has the form $S = H\,(G)$ with $\tau-$open G. We obtain

$$\begin{aligned} i^{-1}(H(G)) &= \{x \in X : i(x) \in H(G)\} \\ &= \{x \in X : \{x\} \in H(G)\} \\ &= \{x \in X : x \in G\} = G \end{aligned}$$

To prove continuity of i^{-1} we have to show that $i(G)$ is open in $i(X)$ with respect to $\tau_H|_{i(X)}$ for all $\tau-$open $G \subset X$. Now

$$\begin{aligned} i(G) &= \{\{x\} : x \in G\} \\ &= H(G) \cap i(X) \end{aligned}$$

and $i(G)$ is $\tau_H|_{i(X)}-$open since $H(G)$ is open in $\mathcal{F}(X)$ with respect to τ_H. □

It is clear that each τ_M-open set $U \subset \mathcal{F}(X)$ can be written in the form

$$U = \bigcup_{i \in I} \bigcap_{j=1}^{m_i} M(K_j^i)$$

where I is an index set and all $K_j^i \subset X$ are compact.
From $\bigcap\limits_{j=1}^{m_i} M(K_j^i) = M\left(\bigcup\limits_{j=1}^{m_i} K_j^i\right)$ and the compactness of $K_i := \bigcup\limits_{j=1}^{m_i} K_j^i$ it follows that each τ_M-open U can be written as

$$U = \bigcup_{i \in I} M(K_i)$$

with compact $K_i \subset X$.
Each τ_H-open set $V \subset \mathcal{F}(X)$ can be written in the form

$$V = \bigcup_{i \in I} \bigcap_{j=1}^{m_i} H(G_j^i)$$

with an index set I and open $G_j^i \subset X$.

In the following we will show that under suitable conditions on the underlying space X, the topological spaces $(\mathcal{F}(X), \tau_M)$ and $(\mathcal{F}(X), \tau_H)$ are second countable. This could be done by using known results about second countability of the Fell topology. We will however give a detailed proof, which will not only show second countabililty, but also provide the corresponding countable bases. We will see, that the elements of these bases have the same hit–/miss structure as in the general case above. We will frequently make use of this in the following chapters.

Theorem 1.6 *Let (X, τ) be a regular, second countable topological space. Let there be a countable collection $\mathcal{K}$ of compact subsets of X with the following properties: $K_1, \ldots, K_n \in \mathcal{K}$ implies $\bigcup_{i=1}^{n} K_i \in \mathcal{K}$, $\bigcap_{i=1}^{n} K_i \in \mathcal{K}$ and each compact $K \subset X$ is the countable intersection of elements of $\mathcal{K}$. Then the topological space $(\mathcal{F}(X), \tau_M)$ is second countable. A countable base of open sets for τ_M is given by*

$$\mathcal{O} = \{M(A_n) : \quad A_n \in \mathcal{K} \}.$$

Proof. Since $\{M(K) : \ K \text{ compact}\}$ is a base of open sets for τ_M, it suffices to show that each $M(K)$ is the countable union of elements of $\mathcal{O}$ (see Lemma A.1). Because of the assumptions on (X, τ) we can find a sequence $(K_n)_n \subset \mathcal{K}$ with $K = \bigcap_{n=1}^{\infty} K_n$. Let $L_m = \bigcap_{n=1}^{m} K_n$, then $L_{m+1} \subset L_m,\ L_m \in \mathcal{K}$ and $K = \bigcap_{m=1}^{\infty} L_m$. We have (see Lemma A.2)

$$M(K) = M\left(\bigcap_{m=1}^{\infty} L_m\right) = \bigcup_{m=1}^{\infty} M(L_m).$$

□

Remark 1.7 We assume regularity of (X, τ), because the Hausdorff property ensures that every compact subset of X is closed and the T_3-property is needed to prove Lemma A.2 of the appendix. The conditions of the theorem are fulfilled for $\mathbb{R}^d$ equipped with the usual topology, this is the most important case for us. $\mathcal{K}$ can be chosen as the set consisting of all finite unions and finite intersections of axially parallel compact cuboids with corners in $\mathbb{Q}^d$. The conditions are also fulfilled for all compact, second countable regular spaces, which have a countable base of closed sets for the closed sets.

Theorem 1.8 *If (X, τ) is second countable with the countable base $\mathcal{U}$ of open sets for τ, then the topological space $(\mathcal{F}(X), \tau_H)$ is second countable. A countable base for τ_H is given by*

$$\mathcal{O} = \left\{ \bigcap_{n=1}^{m} H(W_n) : \ m \in \mathbb{N}, \ W_n = \bigcup_{l=1}^{s_n} B_l^n \text{ with } B_l^n \in \mathcal{U}, \ s_n \in \mathbb{N} \right\}.$$

Proof. From the definition of τ_H, a base for this topology is given by $\left\{ \bigcap_{k=1}^{m} H(G_k) : \ m \in \mathbb{N}, \ G_k \text{ open in } X \right\}$. It suffices to show that each $A = \bigcap_{n=1}^{m} H(G_n)$ is the countable union of elements of $\mathcal{O}$ (see Lemma A.1). Each $\tau-$open G_k is the countable union of elements of $\mathcal{U}$. It follows that we can find a sequence $(W_k^n)_k$ of $\tau-$open sets with $W_k^n = \bigcup_{l=1}^{s_k^n} B_l^{n,k}$ and $B_l^{n,k} \in \mathcal{U}$ such that $G_n = \bigcup_{k=1}^{\infty} W_k^n$. It can be assumed that $W_k^n \subset W_{k+1}^n$.
We obtain $A = \bigcap_{n=1}^{m} H\left(\bigcup_{k=1}^{\infty} W_k^n \right) = \bigcap_{n=1}^{m} \bigcup_{k=1}^{\infty} H(W_k^n)$ and claim that $A = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{m} H(W_k^n)$, which completes the proof.
First let $F \in A$, then for each n there is k_n such that $F \in H\left(W_{k_n}^n\right)$. It follows that $F \in H(W_k^n)$ for all $k \geq k_n$, because of $W_k^n \subset W_{k+1}^n$. We choose $k_0 = \max\{k_1, \ldots, k_m\}$, then $F \in H(W_k^n)$ for all $k \geq k_0$ and all n. This shows $F \in \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{m} H(W_k^n)$.
Now let $F \in \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{m} H(W_k^n)$, then there is k such that $F \in H(W_k^n)$ for $n = 1, \ldots, m$. Because of $W_k^n \subset G_n$ it follows that $F \in H(G_n)$, $n = 1, \ldots, m$, i.e. $F \in A$. □

Corollary 1.9 *(a) Every τ_M-open $U \subset \mathcal{F}\left(\mathbb{R}^d\right)$ can be written as*

$$U = \bigcup_{i=1}^{\infty} M(K_i)$$

with compact $K_i \subset \mathbb{R}^d$.

(b) Every τ_H-open $V \subset \mathcal{F}\left(\mathbb{R}^d\right)$ can be written as

$$V = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{m_i} H(G_j^i)$$

with open $G_j^i \subset \mathbb{R}^d$.

The second countability of $\left(\mathcal{F}\left(\mathbb{R}^d\right), \tau_M\right)$ and $\left(\mathcal{F}\left(\mathbb{R}^d\right), \tau_H\right)$ allows us to work with sequences instead of nets or filters, when we deal with convergence of closed sets and (semi–)continuity of functions on $\mathbb{R}^d$.
The following theorem, which is a restriction of Theorems 2.1 and 3.1 of [13] to the case of Hausdorff, second countable X, clarifies the relationship between convergence in the topologies τ_M and τ_H on one hand and inner-/outer convergence on the other hand.

Theorem 1.10 *Let (X, τ) be Hausdorff and second countable, let $(F_n)_n \subset \mathcal{F}(X)$, $F \in \mathcal{F}(X)$ then*

(i) $F_n \overset{\tau_M}{\to} F$ *if and only if* $\limsup\limits_{n\to\infty} F_n \subset F$.

(ii) if (X, τ) is additionally locally compact, then
$F_n \overset{\tau_H}{\to} F$ *if and only if* $F \subset \liminf\limits_{n\to\infty} F_n \subset F$.

As a consequence of this theorem we will from now on use the terms τ_M- and τ_H- convergence instead of inner- and outer convergence.

We continue to collect properties of the topologies which will be important for convergence in distribution with respect to τ_M and τ_H.

Theorem 1.11 *The topological space $(\mathcal{F}(X), \tau_H)$ has the following separation properties:*

(a) $\mathcal{F}(X)$ is a T_0-space.

(b) $\mathcal{F}(X)$ is not a T_1-space.

(c) $\mathcal{F}(X)$ is not a T_2-space.

(d) $\mathcal{F}(X)$ is not a T_3-space.

(e) $\mathcal{F}(X)$ is not a T_{3a}-space.

(f) $\mathcal{F}(X)$ is a T_4-space.

Proof. (a) and (b) are proven in [13], (c) immediately follows from (b). To show (d) let $V \subset X$ be open and $V \neq \emptyset$. Let $F = \mathrm{cl}(V)$, then $F \in \mathcal{F}(X)$. Since V is open in X, the set $M(V)$ is a τ_H-closed subset of $\mathcal{F}(X)$. We have $F \notin M(V)$. To show that $\mathcal{F}(X)$ is not T_3 it suffices to show that each open neighbourhood U of $M(V)$ contains F. We can assume that $U = \bigcup\limits_{i\in I} \bigcap\limits_{j=1}^{r_i} H\left(G_j^i\right)$. Let $B = \mathrm{bdy}(V)$ be the boundary of V with

respect to the topology on X, then B is closed in X, i.e. $B \in \mathcal{F}(X)$. Since V is open we have $B \cap V = \emptyset$ and thus $B \in M(V) \subset U$. This implies that there is $i \in I$ such that $B \cap G^i_j \neq \emptyset$, $j = 1, \ldots, r_i$. It follows that $F \cap G^i_j \neq \emptyset$, $j = 1, \ldots, r_i$, because the sets G^i_j are open. This shows $F \in U$.
(e) follows from (d). To show (f), let A and C be closed subsets of $\mathcal{F}(X)$. Since each closed $D \subset \mathcal{F}(X)$ has the form $D = \bigcap_{i \in I} \bigcup_{j=1}^{r_i} M\left(G^i_j\right)$ with open $G^i_j \subset X$, we see that each nonempty closed D contains $\emptyset \in \mathcal{F}(X)$. Thus for A and C to be disjoint it is necessary that without loss of generality A is empty. We can then choose $U = \emptyset \subset \mathcal{F}(X)$ and $V = \mathcal{F}(X)$ as disjoint open neighbourhoods of A resp. B. □

Theorem 1.12 *The topological space $(\mathcal{F}(X), \tau_M)$ has the following separation properties:*

(a) $\mathcal{F}(X)$ is a T_0-space.

(b) $\mathcal{F}(X)$ is not a T_1-space.

(c) $\mathcal{F}(X)$ is not a T_2-space.

(d) $\mathcal{F}(X)$ is not a T_3-space.

(e) $\mathcal{F}(X)$ is not a T_{3a}-space.

(f) $\mathcal{F}(X)$ is a T_4-space.

Proof. Again (a) and (b) are proven in [13] and (c) follows from (b). To show (d) let $K \subset X$ be compact and let $F \in \mathcal{F}(X)$ such that $F \cap K = \emptyset$. Then $H(K)$ is closed and $F \notin H(K) \neq \emptyset$. Let U and V be arbitrary τ_H-neighbourhoods of F and $H(K)$. Then $\emptyset \in U$ and $\emptyset \in V$ and thus $U \cap V \neq \emptyset$. (e) follows from (d). To prove (f) let A and B be closed subsets of $\mathcal{F}(X)$ with $A \cap B = \emptyset$. Since each τ_M-closed nonempty set C is of the form $C = \bigcap_{i \in I} H(K_i)$ with nonempty compact K_i it follows that $X \in C$. Thus if $A \cap B = \emptyset$ we can assume without loss of generality that $A = \emptyset$. Let $U = \emptyset$ and $V = \mathcal{F}(X)$, then U and V are open neighbourhoods of A, resp. B with $U \cap V = \emptyset$. □

As a consequence of (b) in Theorems 1.12 and 1.11 it follows that not every τ_M- resp. τ_H- compact subset of $\mathcal{F}(X)$ is τ_M- resp. τ_H-closed. Thus Borel measurability of compact sets is not guaranteed. The possible applications of the T_4-property of $(\mathcal{F}(X), \tau_M)$ and $(\mathcal{F}(X), \tau_H)$ seem to be very limited, as typical applications (e.g. the Urysohn

Lemma) call for disjoint, nonempty closed sets. We have seen in the proofs of (f) in the last theorems, that such sets do not exist for τ_M and τ_H.

Corollary 1.13 *The topological spaces* $(\mathcal{F}(X), \tau_H)$ *and* $(\mathcal{F}(X), \tau_M)$ *are not metrizable.*

Proof. This follows from part (c) of the previous theorems, because each metrizable topological space is a T_2-space. □

This corollary is the origin for many problems, when dealing with convergence in distribution with respect to the topologies τ_M and τ_H. Note that many proofs in [5] rely heavily on probability theory in metric spaces, e.g. the concept of inner regularity and the measurability of compact sets. We have to find workarounds, whenever possible. In [43] Vogel shows by employing set distance methods that convergence in $\left(\mathcal{F}\left(\mathbb{R}^d\right), \tau_M\right)$ can be described with the help of a quasi-pseudo metric. This is mainly used to prove a Portmanteau like theorem. In a straightforward way an analogous quasi-pseudo metric can be developed for the topological space $\left(\mathcal{F}\left(\mathbb{R}^d\right), \tau_H\right)$. We will not pursue this way here and stick to the purely topological description of convergence. Note that in this way we can obtain more general results, as we do not require a distance/metric on the underlying topological space X. Additionally the case of the closed set $\emptyset \in \mathcal{F}(X)$ is included in a natural way. If one is to work with set distance methods, the empty set is excluded, or a suitable definition for the distance to the empty set has to be given.
When metrizability fails, often uniform structures are used as a generalisation, we however have

Corollary 1.14 *The topological spaces* $(\mathcal{F}(X), \tau_H)$ *and* $(\mathcal{F}(X), \tau_M)$ *are not uniformizable.*

Proof. Since each uniformizable topological space is a T_{3a}-space (cf. Theorem 11.22 of [46]) the assertion follows with part (e) of the preceding theorems. □

We now turn to the investigation of continuous and semicontinuous functions on $(\mathcal{F}(X), \tau_M)$ and $(\mathcal{F}(X), \tau_H)$.

Lemma 1.15 *The only* τ_H-*open subset of* $\mathcal{F}(X)$*, which contains* $\emptyset$ *is the whole space* $\mathcal{F}(X)$.

Proof. Let $U \subset \mathcal{F}(X)$ be open with respect to τ_H. Then U can be written as

$$U = \bigcup_{i \in I} \bigcap_{j \in J_i} H(G_j^i)$$

with finite J_i and open $G_j^i \subset X$ for all $j \in J_i$. Since $\emptyset \notin H(G)$ for all open $G \subset X$ it can only be true that $\emptyset \in U$, if J_i is the empty index set for some i and this implies $U = \mathcal{F}(X)$. □

Theorem 1.16 *Let* $f : (\mathcal{F}(X), \tau_H) \to \mathbb{R}$.

(a) If f is lower semicontinuous, then f takes its minimum in $\emptyset$.

(b) If f is upper semicontinuous, then f takes its maximum in $\emptyset$.

(c) If f is continuous, then f is constant.

Proof. (a) f is lower semicontinuous in $\emptyset$. This means that for all $\varepsilon > 0$ there is an τ_H-open neighbourhood U of $\emptyset$, with $f(F) \geq f(\emptyset) - \varepsilon$ for all $F \in U$. The previous lemma shows that $U = \mathcal{F}(X)$ and since this holds independent of ε, by $\varepsilon \to 0$ we obtain $f(\emptyset) \leq f(F)$ for all $F \in \mathcal{F}(X)$. (b) is proven analogously. (c) If f is continuous, then f is lower semicontinuous and upper semicontinuous. By (a) and (b) this implies $f(F) \leq f(\emptyset) \leq f(F)$ for all $F \in \mathcal{F}(X)$, and this means $f(F) \equiv f(\emptyset)$. □

Lemma 1.17 *The only τ_M-open subset of $\mathcal{F}(X)$, which contains X is the whole space $\mathcal{F}(X)$.*

Proof. Let $U \subset \mathcal{F}(X)$ be τ_M-open, then $U = \bigcup_{i \in I} M(K_i)$ with compact K_i. Since $X \notin M(K)$ for $K \neq \emptyset$ it follows that $K_i = \emptyset$ for at least one $i \in I$, if $X \in U$. This implies $\mathcal{F}(X) = M(\emptyset) \subset U$ and thus $U = \mathcal{F}(X)$. □

Theorem 1.18 *Let* $f : (\mathcal{F}(X), \tau_M) \to \mathbb{R}$.

(a) If f is lower semicontinuous, then f takes its minimum in X.

(b) If f is upper semicontinuous, then f takes its maximum in X.

(c) If f is continuous, then f is constant.

Proof. This is proven analogously to the case of $(\mathcal{F}(X), \tau_H)$ in Theorem 1.16. □

Part (c) of Theorems 1.16 and 1.18 is mentioned in [18]. Together with the fact that each sequence in $(\mathcal{F}(X), \tau_H)$ resp. $(\mathcal{F}(X), \tau_M)$ converges we see that the coarse topologies τ_H and τ_M share properties with the coarsest possible topology, the indescrete topology.

Corollary 1.19 *Let $f : (\mathcal{F}(X), \tau_M) \to \mathbb{R}$ resp. $f : (\mathcal{F}(X), \tau_H) \to \mathbb{R}$. If f is continuous on a nonempty τ_M– resp. τ_H–closed set $A \subset \mathcal{F}(X)$, then f is constant on A.*

Proof. From the T_4–property of $(\mathcal{F}(X), \tau_M)$ resp. $(\mathcal{F}(X), \tau_H)$ it follows with the Tietze Extension Theorem (see Satz 7.7 of [46]) that f can be extended to a continuous mapping on $\mathcal{F}(X)$. It follows with 1.18 resp. 1.16 that this extension is constant on $\mathcal{F}(X)$. Hence f is constant on A. $\square$

Lemma 1.20 *If X is a Hausdorff space, then the topological spaces $(\mathcal{F}(X), \tau_H)$ and $(\mathcal{F}(X), \tau_M)$ are compact.*

Proof. This follows from the compactness of $(\mathcal{F}(X), \tau_{\text{Fell}})$, which is shown in [11], and from the continuity of the mappings $id_1 : (\mathcal{F}(X), \tau_{\text{Fell}}) \to (\mathcal{F}(X), \tau_H)$ and $id_2 : (\mathcal{F}(X), \tau_{\text{Fell}}) \to (\mathcal{F}(X), \tau_M)$, $id_1(F) = id_2(F) = F$ for all $F \in \mathcal{F}(X)$. $\square$

It follows that the topological spaces $(\mathcal{F}(X), \tau_M)$ and $(\mathcal{F}(X), \tau_H)$ are not almost metrizable, since a topological space is called almost metrizable, if each compact subset is metrizable.

Lemma 1.21 *Let X be a topological space which fulfills the assumptions of Theorem 1.6 resp. Theorem 1.8. In the topological spaces $(\mathcal{F}(X), \tau_M)$ resp. $(\mathcal{F}(X), \tau_H)$ the only subsets that are open and closed are $\mathcal{F}(X)$ and $\emptyset$.*

Proof. The whole space and the empty set are open and closed in all topological spaces. Let U be τ_M-open and let $U \neq \mathcal{F}(X)$, $U \neq \emptyset$. Then $U = \bigcup_{i=1}^{\infty} M(K_i)$ with compact K_i. We have $K_i \neq \emptyset$ for all i and $K_i \neq X$ for at least one i, because otherwise $U = \mathcal{F}(X)$ or $U = \emptyset$. Choose $A \in U$. It follows, that $A \neq X$. Let $F_n = A$, $F = X$, then $F_n \xrightarrow{\tau_M} F$, but since $F \notin U$ it is clear that U cannot be closed.
Now let U be τ_H-open and let $U \neq \mathcal{F}(X)$, $U \neq \emptyset$. Because of 1.15 we can find $A \in U$ with $A \neq \emptyset$. Let $F_n = A$ and $F = \emptyset$, then $F_n \xrightarrow{\tau_H} F$, but $F \notin U$, because of Lemma 1.15. It follows that U is not closed. $\square$

1.2. Convergence in Distribution

Following the deterministic preparations we will now deal with the case of random closed sets. Throughout this text we assume that the underlying probability space consists of a nonempty set Ω, a σ-field $\mathcal{A}$ on Ω and a probability measure P such that $\mathcal{A}$ is complete for P, i.e. $P(A) = 0$ implies $B \in \mathcal{A}$ for all $B \subset A$. When working with a topology $\tau \in \{\tau_{\text{Fell}}, \tau_M, \tau_H\}$ on $\mathcal{F}(X)$ we assume that a random closed set $F : \Omega \to \mathcal{F}(X)$ is Borel-measurable with respect to τ, i.e. $F^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}_\tau(\mathcal{F}(X))$.

For random variables Z_n, Z, that take their values in a metric space S, convergence in distribution of $(Z_n)_n$ to Z is usually defined in the following way:

$$Z_n \xrightarrow{D} Z$$

if and only if

$$\int f dP_{Z_n} \to \int f dP_Z$$

for all bounded, continuous $f : S \to \mathbb{R}$.

In the case of the nonmetrizable topological spaces $(\mathcal{F}(X), \tau_H)$ and $(\mathcal{F}(X), \tau_M)$ we have seen in Theorems 1.16 and 1.18 that each continuous real valued function is constant. If we were to adopt the above definition for convergence in distribution to the case of random closed sets F_n, F taking their values in $(\mathcal{F}(X), \tau_H)$ or in $(\mathcal{F}(X), \tau_M)$ we would immediately obtain, that each sequence $(F_n)_n$ converges in distribution to each limit F. In the metric space case the well known Portmanteau Theorem (cf. Theorem 2.1 of [5]) provides alternative characterisations of convergence in distribution. Each of these equivalent characterisations could be used as a definition of convergence in distribution. It is thus reasonable to take the following as a definition for convergence in distribution.

Definition 1.22 Let (T, τ) be a first countable topological space. Let $F_n, F : \Omega \to T$ be $\mathcal{B}_\tau$-measurable. We say that F_n converges in distribution to F with respect to τ, denoted by $F_n \xrightarrow{D_\tau} F$, if and only if

$$\liminf_{n\to\infty} P(F_n \in U) \geq P(F \in U)$$

for all $\tau-$open $U \subset T$.

We see that for $\tau = \tau_M$ this definition is in line with the definition of inner convergence in distribution in [43] and it is close to [21].

Lemma 1.23 *Let τ_1, τ_2 be topologies on T with $\tau_2 \subset \tau_1$.*
Let $F_n, F : \Omega \to T$ be $\mathcal{B}_{\tau_1}$-measurable. Then $F_n \xrightarrow{D_{\tau_1}} F$ implies $F_n \xrightarrow{D_{\tau_2}} F$.

Proof. Let $U \subset T$ be τ_2-open, then U is τ_1-open and it follows that

$$\liminf_{n\to\infty} P(F_n \in U) \geq P(F \in U). \qquad \square$$

In the deterministic case it is easy to see, that for $F_n,\ F \in \mathcal{F}\left(\mathbb{R}^d\right)$ we have $F_n \stackrel{\tau_{\text{Fell}}}{\to} F$, if and only if $F_n \stackrel{\tau_{\text{M}}}{\to} F$ and $F_n \stackrel{\tau_{\text{H}}}{\to} F$. We will now show with the help of Pflug's characterisation of epi–convergence in distribution (see [27]), that the same is true in the convergence in distribution setting.

Theorem 1.24 *Let $\left(F_n\right)_n,\ F$ be random closed sets in $\mathbb{R}^d$, measurable with respect to $\mathcal{B}_{\tau_{Fell}}$. Then $F_n \stackrel{D_{\tau_{Fell}}}{\to} F$ if and only if $F_n \stackrel{D_{\tau_M}}{\to} F$ and $F_n \stackrel{D_{\tau_H}}{\to} F$.*

Proof. First let $F_n \stackrel{D_{\tau_{\text{Fell}}}}{\to} F$. Because of $\tau_M \subset \tau_{\text{Fell}}$ and $\tau_H \subset \tau_{\text{Fell}}$ it follows immediately with Lemma 1.23 that $F_n \stackrel{D_{\tau_M}}{\to} F$ and $F_n \stackrel{D_{\tau_H}}{\to} F$.
Now let $F_n \stackrel{D_{\tau_M}}{\to} F$ and $F_n \stackrel{D_{\tau_H}}{\to} F$. In view of Theorem 2.3 in [27] it suffices to show that for all V, where V is a finite union of compact cuboids

$$\begin{aligned} P\left(F \in M\left(V\right)\right) &\leq \liminf_{n\to\infty} P\left(F_n \in M\left(V\right)\right) \\ &\leq \limsup_{n\to\infty} P\left(F_n \in M\left(\text{int}\left(V\right)\right)\right) \\ &\leq P\left(F \in M\left(\text{int}\left(V\right)\right)\right) \end{aligned}$$

Note that V is compact and that $\text{int}\left(V\right)$ is open. It follows that $M\left(V\right)$ is τ_M-open and that $M\left(\text{int}\left(V\right)\right)$ is τ_H-closed. We allways have $P\left(F_n \in M\left(V\right)\right) \leq P\left(F_n \in M\left(\text{int}\left(V\right)\right)\right)$ and so the first resp. last inequality follows from $F_n \stackrel{D_{\tau_M}}{\to} F$ resp. $F_n \stackrel{D_{\tau_H}}{\to} F$. $\square$

Similar to the Portmanteau Theorem we have

Theorem 1.25 *Let (T,τ) be a first countable topological space, let $F_n, F : \Omega \to T$ be $\mathcal{B}_\tau$-measurable. The following are equivalent*

(a) $F_n \stackrel{D_\tau}{\to} F$

(b) $\limsup_{n\to\infty} P\left(F_n \in C\right) \leq P\left(F \in C\right)$, *for all* $\tau-$*closed* $C \subset T$

(c) $\liminf_{n\to\infty} \int f dP_{F_n} \geq \int f dP_F$, *for all bounded lower semicontinuous* $f : T \to \mathbb{R}$

(d) $\limsup_{n\to\infty} \int f dP_{F_n} \leq \int f dP_F$, *for all bounded upper semicontinuous* $f : T \to \mathbb{R}$

each of the above implies

(e) $\lim_{n\to\infty} P\left(F_n \in A\right) = P\left(F \in A\right)$, *for all* A *with* $P\left(F \in \text{bdy}\left(A\right)\right) = 0$

(f) $\lim_{n\to\infty} \int f dP_{F_n} = \int f dP_F$, *for all bounded continuous* $f : T \to \mathbb{R}$

Proof. (a)$\Longleftrightarrow$(b) can be shown as in the proof of the Portmanteau Theorem. (c)$\Longleftrightarrow$(d) follows from the fact that f is lower semicontinuous if and only if $-f$ is upper semicontinuous. To establish equivalence of (a),(b),(c),(d) it suffices to show that (a)$\Longleftrightarrow$(c). First assume that (c) holds.
We show that for each open set U the function 1_U is lower semicontinuous in each $x \in T$. First let $x \in U$, then $1_U(x) = 1$ and since U is open, there is a neighbourhood V of x with $V \subset U$. It follows that $1_U(y) = 1 = 1_U(x)$ for all $y \in V$. Now let $x \notin U$, then $1_U(x) = 0$ and thus $1_U(y) \geq 0 = 1_U(x)$ for all $y \in T$.
With the help of (c) we obtain

$$\begin{aligned}\liminf_{n\to\infty} P\left(F_n \in U\right) &= \liminf_{n\to\infty} \int 1_U dP_{F_n} \\ &\geq \int 1_U dP_F \\ &= P\left(F \in U\right).\end{aligned}$$

Now we assume that (a) holds. Let f be lower semicontinuous and bounded. Without loss of generality we can assume that $0 \leq f \leq 1$. (Otherwise a linear transformation would yield $0 \leq f \leq 1$.) Note that

$$\begin{aligned}\int f dP_{F_n} &= \int_0^1 P_{F_n}\left(f > c\right)\, dc \\ &= \int_0^1 P\left(F_n \in \{x : f(x) > c\}\right)\, dc\end{aligned}$$

The set $\{x : f(x) > c\}$ is open for each $c \in \mathbb{R}$, since f is lower semicontinuous. With (a) and the Lemma of Fatou it follows that

$$\begin{aligned}\liminf_{n\to\infty} \int f dP_{F_n} &= \liminf_{n\to\infty} \int_0^1 P\left(F_n \in \{x : f(x) > c\}\right)\, dc \\ &\geq \int_0^1 \liminf_{n\to\infty} \left(P\left(F_n \in \{x : f(x) > c\}\right)\right)\, dc \\ &\geq \int_0^1 P\left(F \in \{x : f(x) > c\}\right)\, dc \\ &= \int f dP_F.\end{aligned}$$

Now assume that any of (a),(b),(c),(d) holds.
To show that (e) holds let $P(F \in \mathrm{bdy}(A)) = 0$, then $P(F \in \mathrm{int}(A)) = P(F \in \mathrm{cl}(A)) = P(F \in A)$. With (c) and (d) we obtain

$$\begin{aligned} P(F \in A) &= P(F \in \mathrm{int}(A)) \\ &\leq \liminf_{n\to\infty} P(F_n \in \mathrm{int}(A)) \\ &\leq \liminf_{n\to\infty} P(F_n \in A) \\ &\leq \limsup_{n\to\infty} P(F_n \in A) \\ &\leq \limsup_{n\to\infty} P(F_n \in \mathrm{cl}(A)) \\ &\leq P(F \in \mathrm{cl}(A)) \\ &= P(F \in A). \end{aligned}$$

It follows that we have equality everywhere in this chain of inequalities and thus

$$\lim_{n\to\infty} P(F_n \in A) = \liminf_{n\to\infty} P(F_n \in A) = P(F \in A).$$

To prove (f) let f be a continuous bounded function, then f is upper and lower semi-continuous. With (c) and (d) it follows that

$$\limsup_{n\to\infty} \int f dP_{F_n} \leq \int f dP_F \leq \liminf_{n\to\infty} \int f dP_{F_n}.$$

Together with $\limsup_{n\to\infty} \int f dP_{F_n} \geq \liminf_{n\to\infty} \int f dP_{F_n}$ we obtain the existence of $\lim_{n\to\infty} \int f dP_{F_n}$ and

$$\lim_{n\to\infty} \int f dP_{F_n} = \int f dP_F.$$

□

In view of the structure of the boundaries with respect to the topologies τ_M and τ_H (see Lemma A.11) it is not surprising that part (e) of the Theorem is not sufficient for convergence in distribution. Roughly spoken, there are in general not enough P_F continuity sets, i.e. sets A with $P(F \in \mathrm{bdy}(A)) = 0$ to form a convergence determining class in the sense of [5]. Indeed the following example for the topological space $(\mathcal{F}(\mathbb{R}), \tau_M)$ shows, that (e) in general does not imply (a). A similar example can be constructed with the topology τ_H in mind.

Example 1.26 Let $F_n = \mathbb{R}$, $P(F = \mathbb{R}) = P(F = [2,3]) = \frac{1}{2}$. Let V be a measurable subset of $\mathcal{F}(\mathbb{R})$. The set $\mathrm{bdy}\, V$ is always closed. We distinguish two cases. First let

$\operatorname{bdy} V \neq \emptyset$, then $\operatorname{bdy} V = \bigcap_{i=1}^{\infty} H(K_i)$ with nonempty, compact K_i (if $K_i = \emptyset$ for an i, then $\bigcap_{i=1}^{\infty} H(K_i) = \emptyset$). It follows that $\mathbb{R} \in H(K_i)$ for all i, which implies $P(F \in \operatorname{bdy} V) \geq \frac{1}{2}$.
In the second case let $\operatorname{bdy} V = \emptyset$. It is well known from topology, that this can only occur, if V is open and closed. From Lemma 1.21 it follows, that we either have $V = \mathcal{F}$, or $V = \emptyset$. In the first case we obtain $P(F_n \in \mathcal{F}) = 1 = P(F \in \mathcal{F})$. In the second case $P(F_n \in \emptyset) = 0 = P(F \in \emptyset)$.
This shows that we have

$$\lim_{n\to\infty} P(F_n \in V) = P(F \in V)$$

for all V with $P(F \in \operatorname{bdy} V) = 0$, i.e. (e) holds.
On the other hand for the τ_M-open set $M([0,1])$ we obtain

$$\liminf_{n\to\infty} P(F_n \in M([0,1])) = 0 < \frac{1}{2} = P(F \in M([0,1])),$$

which shows that (a) does not hold.

One of the most powerful tools for working with convergence in distribution is the so called Continuous Mapping Theorem, see e.g. (2.5) in [5]. For completeness we give a purely topological proof which does not rely on a metric.

Theorem 1.27 *Let (T_1, τ_1), (T_2, τ_2) be first countable topological spaces.
Let $Z_n, Z : \Omega \to T_1$ be $\mathcal{B}_{\tau_1}$-measurable, let $h : T_1 \to T_2$ be continuous.
If $Z_n \stackrel{D_{\tau_1}}{\to} Z$, then $h(Z_n) \stackrel{D_{\tau_2}}{\to} h(Z)$.*

Proof. Let $U \subset T_2$ be τ_2-open, then $h^{-1}(U)$ is an τ_1-open subset of T_1, because h is continuous. With $Z_n \stackrel{D_{\tau_1}}{\to} Z$ it follows that

$$\begin{aligned}\liminf_{n\to\infty} P(h(Z_n) \in U) &= \liminf_{n\to\infty} P\left(Z_n \in h^{-1}(U)\right) \\ &\geq P\left(Z \in h^{-1}(U)\right) \\ &= P(h(Z) \in U).\end{aligned}$$ □

For real valued functions there are corresponding tools for lower and upper semicontinuous functions. While they do not transfer convergence in distribution from a topological space to $\mathbb{R}$ they are useful for establishing one-sided bounds for probabilities. We will come back to this in the third chapter, when we deal with optimal values and asymptotic regions of confidence.

Theorem 1.28 *Let (T, τ) be a first countable topological space. Let $Z_n, Z : \Omega \to T$ be $\mathcal{B}_\tau$-measurable with $Z_n \xrightarrow{D_\tau} Z$. Let $h : T \to \overline{\mathbb{R}}$.*

(a) If h is lower semicontinuous, then $\liminf_{n\to\infty} P\left(h(Z_n) > c\right) \geq P\left(h(Z) > c\right)$ for all $c \in \mathbb{R}$.

(b) If h is upper semicontinuous, then $\liminf_{n\to\infty} P\left(h(Z_n) < c\right) \geq P\left(h(Z) < c\right)$ for all $c \in \mathbb{R}$.

Proof. (a) is shown in Theorem 3.2 of [43]. Let h be upper semicontinuous, let $c \in \mathbb{R}$. Then $\widetilde{h} := -h$ is lower semicontinuous and (a) yields

$$\begin{aligned}
\liminf_{n\to\infty} P\left(h\left(X_n\right) < c\right) &= \liminf_{n\to\infty} P\left(\widetilde{h}\left(X_n\right) > -c\right) \\
&\geq P\left(\widetilde{h}\left(X\right) > -c\right) \\
&= P\left(h(X) < c\right).
\end{aligned}$$

□

The following example shows, that the concept of robust functions, which is dealt with in detail in [14], does not allow a "Robust Mapping Theorem".

Example 1.29 A subset A of a topological space T is called robust, if $\operatorname{cl}(A) = \operatorname{cl}(\operatorname{int}(A))$. A mapping $f : T \to S$ is said to be robust, if $f^{-1}(U)$ is a robust set, for all open $U \subset S$. Let $\mathbb{R}$ be equipped with the usual topology, let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & , \ x \leq 0 \\ 1 & , \ x > 0 \end{cases} ,$$

then for open $U \subset \mathbb{R}$ we have

$$f^{-1}(U) = \begin{cases} \emptyset & , \ 0 \notin U, \ 1 \notin U \\ (-\infty, 0] & , \ 0 \in U, \ 1 \notin U \\ (0, +\infty) & , \ 0 \notin U, \ 1 \in U \\ (-\infty, \infty) & , \ 0 \in U, \ 1 \in U \end{cases} .$$

This shows that f is robust. Let $Z_n(\omega) = \frac{1}{n}$ and $Z(\omega) = 0$, for all $\omega \in \Omega$ then $Z_n \xrightarrow{D} Z$. But for the open set $U = \left(-\infty, \frac{1}{2}\right)$ we obtain

$$\liminf_{n\to\infty} P\left(f\left(Z_n\right) \in U\right) = \liminf_{n\to\infty} P\left(Z_n \in (-\infty, 0]\right) = 0$$

and

$$P\left(f\left(Z\right) \in U\right) = P\left(Z \in \left(-\infty, 0\right]\right) = 1.$$

This shows that $f\left(Z_n\right) \overset{D}{\not\to} f\left(Z\right).$

As to the other frequently used tools in convergence in distribution, we note for the topologies τ_M and τ_H, that the existence of a Skorohod Representation Theorem remains an open problem. In the case of metric spaces the Skorohod Representation Theorem (c.f. Theorem in [5]), which describes convergence in distribution with the help of almost sure convergence greatly facilitates many proofs, for example when convergence in distribution in product spaces is considered. Unfortunately there is no general topological version of the Skorohod Representation Theorem. Recent developments in the non metrizable case (see [1]) do not work for τ_M and τ_H as they call for almost metrizable spaces. We have already seen as a consequence of Lemma 1.20, that this condition is not fulfilled for τ_M and τ_H. On the other hand, the Prohorov Theorem on the existence of subsequences converging in distribution trivially holds for τ_M and τ_H. This follows from the fact that each sequence of random closed sets $(F_n)_n$ converges in distribution to X in the τ_M sense and it converges in distribution to $\emptyset$ in the τ_H sense.

1.3. Random Lower Semicontinuous Functions

Definition 1.30 A mapping $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is called lower semicontinuous in x, if

$$\liminf_{n\to\infty} f\left(x_n\right) \geq f(x)$$

for all sequences $\left(x_n\right)_n$ with $x_n \to x$. f is called lower semicontinuous, if f is lower semicontinuous in each x.

With the epigraph of f

$$\operatorname{epi} f = \{(x, y) : y \geq f(x)\}$$

and the level sets

$$\operatorname{lev}_{\leq a} f = \{x : f(x) \leq a\}, \; a \in \mathbb{R},$$

there are well known alternative characterisations of lower semicontinuity:

Lemma 1.31 (Theorem 1.6 of [33]) *Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$, then the following are equivalent:*

(i) f *is lower semicontinuous on* $\mathbb{R}^d$

(ii) $\operatorname{epi} f$ *is a closed subset of* $\mathbb{R}^d \times \mathbb{R}$

(iii) $\operatorname{lev}_{\leq a}(f)$ *is closed for each* $a \in \mathbb{R}$

Note that a function f is uniquely determined by its epigraph $\operatorname{epi} f$. Thus convergence of lower semicontinuous functions can be described by convergence of their epigraphs in the space $\mathrm{EPI}(\mathbb{R}^d)$ consisting of all closed subsets, which occur as epigraphs of lower semicontinuous functions $f : \mathbb{R}^d \to \overline{\mathbb{R}}$. Clearly $\mathrm{EPI}(\mathbb{R}^d)$ is a subset of $\mathcal{F}\left(\mathbb{R}^{d+1}\right)$ and can thus be equipped with each of the topologies τ_{Fell}, τ_M and τ_H.
In the following random lower semicontinuous functions will play an important role, for example as random objective functions in stochastic optimisation problems. A random lower semicontinuos function on $\mathbb{R}^d$ is a random variable $\omega \mapsto f(.,\omega)$, taking values in $LSC(\mathbb{R}^d)$. When we do not need to specifically address ω we will frequently use the abbreviated form $f(x)$ instead of the full form $f(x,\omega)$. Because of the one to one correspondence between a function and its epigraph, each random lower semicontinuous function f yields a random closed set $\omega \mapsto \operatorname{epi} f(.,\omega)$.
When $\operatorname{epi} f$ takes its values in $\mathcal{F}(\mathbb{R}^{d+1})$ equipped with a topology τ we will at least assume $\mathcal{B}_\tau$-measurability. According to Definition 14.27 and Theorem 4.4 in [33], a random lower semicontinuous function f is called a normal integrand, if the mapping $\omega \mapsto \operatorname{epi} f(.,\omega)$ is $\mathcal{B}_{\tau_{\text{Fell}}}$-measurable. With Corollary 14.34 of [33] the normal integrand property follows from the joint measurability of the mapping $(x,\omega) \mapsto f(x,\omega)$, if f is lower semicontinuous with respect to x and if, as in our general assumption, $(\Omega, \mathcal{A}, P)$ is a complete probability space. Even when we apply convergence in distribution with respect to τ_M or τ_H in stochastic optimisation problems (see Chapter 3), we will generally assume $\mathcal{B}_{\tau_{\text{Fell}}}$-measurability of $\omega \mapsto \operatorname{epi} f(.,\omega)$. This stronger measurability assumption is made, because it implies measurability of optimal values and solution sets. We will come back to this in Chapter 3.

It is sometimes necessary to restrict the convergence region. It may for example be sufficient, to investigate lower semicontinuous functions only on an open set $U \subset \mathbb{R}$. Lachout has developed two concepts to restrict/localise the region of convergence in

[21]. For our purpose we do not require these elaborate concepts. It will be sufficient to take $\mathrm{EPI}(U)$ with the subspace topology inherited from $\mathrm{EPI}(\mathbb{R}^d)$ and to apply the existing theory of τ_M/τ_H convergence in distribution.

2. Sufficient Conditions

In the first chapter we have dealt with the definition and the properties of convergence in distribution of random closed sets with respect to τ_M and τ_H. The structure of the open sets in these topologies makes it difficult to show for example $\operatorname{epi} f_n \overset{D_{\tau_M}}{\to} \operatorname{epi} f$ for a sequence of random lower semicontinuous functions by verifying that the conditon in Definition 1.22 is fulfilled. In this chapter we will establish sufficient conditions for convergence in distribution in the τ_M and in the τ_H setting. In the first section we will show that the relation between convergence in distribution and convergence in probability is exactly as in the case of random variables in metric spaces. We will then restrict ourselves to random closed sets, which are the epigraphs of random lower semicontinuous functions. We will show sufficient conditions that are in line with the finite dimensional approach to convergence in distribution for stochastic processes. Because of the one to one correspondence between functions and their epigraphs we obtain a convergence criterion for stochastic processes with lower semicontinuous trajectories. Then we show that convergence in distribution in the sense of Skorohod for the cadlag-modifications of lower semicontinuous functions is sufficient for convergence in distribution of their epigraphs with respect to τ_{Fell}, which makes set convergence in distribution methods accessible for the important class $D[0, \infty)$. This is followed by the investigaton of (semi-)continuous dependence on parameters. To complete this chapter we treat convergence in distribution in product spaces.

As mentioned in Section 1.3 we assume that for all occuring random lower semicontinuous functions f the mapping $\omega \mapsto \operatorname{epi} f(., \omega)$ is $\mathcal{B}_{\tau_{\text{Fell}}}$-measurable. This assumption is necessary even in the case that we derive convergence in distribution of the epigraphs from convergence in distribution of stochastic processes (e.g. Section 2.3), since the usual measurability for stochastic processes (i.e. $\omega \mapsto f(x, \omega)$ is measurable for all x) is not sufficient for measurability of $\omega \mapsto \operatorname{epi} f(., \omega)$

2.1. Convergence in Distribution from Convergence in Probability

From the theory of random variables in metric spaces it is well known, that convergence in probability implies convergence in distribution and that the converse holds, if the limit is almost surely deterministic. We will show, that this also holds true in the case of convergence in distribution and convergence in probability with respect to τ_M and τ_H. Salinetti and Wets show in [34] that τ_{Fell} convergence in probability implies τ_{Fell} convergence in distribution. Throughout this section we deal with random closed sets that take their values in $\mathcal{F} := \mathcal{F}(\mathbb{R}^p)$ for some $p \in \mathbb{N}$. We will specify the measurability of the random closed sets as needed. The following definition of convergence in probability is a slightly modified version of Vogel's Definition 2.5 in [39].

Definition 2.1 Let $(F_n)_n$, F be random elements of $\mathcal{F}$, measurable with respect to $\mathcal{B}_{\tau_M}$ for part (a), resp. measurable with respect to $\mathcal{B}_{\tau_H}$ for part (b).
(a) F_n converges to F in probability with respect to τ_M, denoted by $F_n \overset{P_{\tau_M}}{\to} F$, if

$$\lim_{n\to\infty} P\left(F_n \backslash U_\varepsilon\left(F\right) \cap K \neq \emptyset\right) = 0$$

for all $\varepsilon > 0$ and all compact $K \subset \mathbb{R}^p$.
(b) F_n converges to F in probability with respect to τ_H, denoted by $F_n \overset{P_{\tau_H}}{\to} F$, if

$$\lim_{n\to\infty} P\left(F \backslash U_\varepsilon\left(F_n\right) \cap K \neq \emptyset\right) = 0$$

for all $\varepsilon > 0$ and all compact $K \subset \mathbb{R}^p$.

The definition used by Vogel was obtained by decomposing Definition 1.18 of [34] for convergence in probability (i.e. convergence in probability with respect to τ_{Fell}) into a τ_M and a τ_H part. In contrast to Vogel we restrict ourselves to random closed sets. In another aspect, our definition is slightly more general, because we have weakened the measurabiltiy assumptions from $\mathcal{B}_{\tau_{\text{Fell}}}$ measurability to $\mathcal{B}_{\tau_M}$ resp. $\mathcal{B}_{\tau_H}$ measurability.
Since we have changed the measurability assumptions, we have to check that all events occuring in the definition are measurable.
First note that for closed sets $A, B \subset \mathbb{R}^p$ the set $U_\varepsilon\left(B\right)$, is open as the open ε-neighbourhood of B. This implies that $A \backslash U_\varepsilon\left(B\right) = A \cap \left(U_\varepsilon\left(B\right)\right)^C$ is closed.
Now in (a) measurablitity of $\{F_n \backslash U_\varepsilon\left(F\right) \cap K \neq \emptyset\} = \{F_n \backslash U_\varepsilon\left(F\right) \in H\left(K\right)\}$ follows, be-

cause $H(K)$ is a τ_M-closed set.
In (b) we have $\{F \backslash U_\varepsilon(F_n) \cap K \neq \emptyset\} = \{F \backslash U_\varepsilon(F_n) \in H(K)\}$. To show measurability of this event, we have to show, that $H(K) \in \mathcal{B}_{\tau_H}$. Recall, that this Borel $\sigma-$field is generated by sets of the form $H(G)$ for open G.

Lemma 2.2 *Let $K \subset \mathbb{R}^p$ be compact, then $H(K) \in \mathcal{B}_{\tau_H}$.*

Proof. Since the compact set $\emptyset$ is also open, there is nothing to show for $K = \emptyset$ and we can assume that $K \neq \emptyset$.
We have

$$H(K) = \bigcap_{n=1}^{\infty} H\left(U_{\frac{1}{n}}(K)\right).$$

First assume that $F \in H(K)$, i.e. $F \cap K \neq \emptyset$. From $K \subset U_{\frac{1}{n}}(K)$ for all $n \in \mathbb{N}$ it follows that $F \cap U_{\frac{1}{n}}(K) \neq \emptyset$ for all $n \in \mathbb{N}$ and thus $F \in \bigcap_{n=1}^{\infty} H\left(U_{\frac{1}{n}}(K)\right)$.
Now assume that $F \in \bigcap_{n=1}^{\infty} H\left(U_{\frac{1}{n}}(K)\right)$. If $F \notin H(K)$ then $\alpha := \mathrm{dist}(F, K) > 0$, since K is compact and F is closed. For $\frac{1}{n} < \alpha$ we have $F \cap U_{\frac{1}{n}}(K) = \emptyset$ in contradiction to the assumption.
For each $n \in \mathbb{N}$ the set $H\left(U_{\frac{1}{n}}(K)\right)$ is an τ_H-open set, which belongs to $\mathcal{B}_{\tau_H}$. Since the intersection of countably many elements of a $\sigma-$field belongs to the same $\sigma-$field it follows that $H(K) \in \mathcal{B}_{\tau_H}$. □

Before we continue the investigation of convergence in distribution, we have to show that in our case convergence in distribution does in general not coincide with convergence in probability. This is not immediately obvious and there are topological spaces where convergence in distribution is the same as convergence in probability, e.g. any nonempty set with the indiscrete topology. We have seen in Chapter 1 that in some aspects, e.g. seperation properties, the topologies τ_M and τ_H behave like the indiscrete topology.

Example 2.3 Let F, G be independent, random closed sets with
$P(F = [0,1]) = P(G = [0,1]) = \frac{1}{2}$, $P(F = [2,3]) = P(G = [2,3]) = \frac{1}{2}$. Set $F_n = G$ for all $n \in \mathbb{N}$.
(a) We have $F_n \stackrel{D_{\tau_M}}{\to} F$, since F_n and F have the same distribution. On the other hand for $K = [0,1]$ and $\varepsilon = \frac{1}{10}$ we have $P((F_n \backslash U_\varepsilon(F)) \cap K \neq \emptyset) = P(F_n = [0,1],\ F = [2,3]) = P(G = [0,1],\ F = [2,3]) = P(G = [0,1]) P(F = [2,3]) = \frac{1}{4} \not\to 0$, which shows that F_n does not converge to F in probability with respect to τ_M.

(b) Clearly $F_n \overset{D_{\tau_H}}{\to} F$, but for $K = [2,3]$ and $\varepsilon = \frac{1}{10}$ we obtain $P\left((F \backslash U_\varepsilon(F_n)) \cap K \neq \emptyset\right) = P\left(F = [2,3],\ F_n = [0,1]\right) = \frac{1}{4} \not\to 0$ and so F_n does not converge to F in probability with respect to τ_H.

The following main results of this section extend the well known properties from convergence of real-valued random variables to the case of random closed sets and τ_M/τ_H-convergence.

Theorem 2.4 *Let $(F_n)_n$, F be random elements of $\mathcal{F}$, measurable with respect to $\mathcal{B}_{\tau_M}$ for (a), resp. $\mathcal{B}_{\tau_H}$ for (b).*
(a) If $F_n \overset{P_{\tau_M}}{\to} F$, then $F_n \overset{D_{\tau_M}}{\to} F$.
(b) If $F_n \overset{P_{\tau_H}}{\to} F$, then $F_n \overset{D_{\tau_H}}{\to} F$.

Corollary 2.5 *Let $(F_n)_n$, F be random elements of $\mathcal{F}$, measurable with respect to $\mathcal{B}_{\tau_M}$ for (a), resp. $\mathcal{B}_{\tau_H}$ for (b).*
(a) If $F_n \overset{a.s.\tau_M}{\to} F$, then $F_n \overset{D_{\tau_M}}{\to} F$.
(b) If $F_n \overset{a.s.\tau_H}{\to} F$, then $F_n \overset{D_{\tau_H}}{\to} F$.

Proof. This follows immediately from the above theorem, because for both topologies τ_M and τ_H almost sure convergence implies convergence in probability, see Lemma 2.4 of [39]. $\square$

There is a converse to the last theorem if the limit is almost surely deterministic.

Theorem 2.6 *Let $(F_n)_n$, F be random elements of $\mathcal{F}$, measurable with respect to $\mathcal{B}_{\tau_M}$ for (a), resp. $\mathcal{B}_{\tau_H}$ for (b). Let $P\left(F = \widetilde{F}\right) = 1$ for a deterministic closed set $\widetilde{F}$.*
(a) If $F_n \overset{D_{\tau_M}}{\to} F$, then $F_n \overset{P_{\tau_M}}{\to} F$.
(b) If $F_n \overset{D_{\tau_H}}{\to} F$, then $F_n \overset{P_{\tau_H}}{\to} F$.

Before we prove the theorems, we show equivalent charcterisations of convergence in probability. These characterisations will greatly facilitate the proofs of the main results. They will furthermore be very helpful in the last section of this chapter, where we will deal with convergence in distribution in product spaces.

The following characterisation of convergence in probability with respect to τ_H can be seen as an 'in probability' version of the deterministic convergence criterion in Theorem 4.5 of [33].

Lemma 2.7 *Let $(F_n)_n$, be random elements of $\mathcal{F}(\mathbb{R}^p)$, measurable with respect to $\mathcal{B}_{\tau_H}$. Then the following are equivalent:*

(i)

$$F_n \overset{P_{\tau_H}}{\to} F$$

(ii) $\forall \varepsilon > 0 \ \forall r > 0 \ \forall G \subset \mathbb{R}^p$ *open*

$$\lim_{n\to\infty} P\left(\left(F \cap \overline{B}_r(0)\right) \backslash U_\varepsilon(F_n) \in H(G)\right) = 0$$

Proof. First assume that (ii) holds and that (i) is not true, then there is $\varepsilon > 0$ and a compact set $K \subset \mathbb{R}^p$ such that for some $a > 0$ and a subsequence $(F_{n_k})_k$

$$P\left((F\backslash U_\varepsilon(F_{n_k})) \in H(K)\right) \geq a$$

for all $k \in \mathbb{N}$.
Since the compact set K is bounded we can find $r > 0$ such that $K \subset B_r(0)$. Let $G = B_r(0)$, then G is open and we have

$$\begin{aligned}
&P\left(\left(F \cap \overline{B}_r(0)\right) \backslash U_\varepsilon(F_{n_k}) \in H(G)\right) \\
&\geq P\left(F\backslash U_\varepsilon(F_{n_k}) \in H(K)\right) \\
&\geq a
\end{aligned}$$

for all $k \in \mathbb{N}$ in contradiction to (ii).
Now assume that (i) holds, while (ii) does not hold. Then we can find $\varepsilon > 0$, $r > 0$ and an open set G such that

$$P\left(\left(F \cap \overline{B}_r(0)\right) \backslash U_\varepsilon(F_{n_k}) \in H(G)\right) \geq a$$

for an $a > 0$ and a subsequence $(F_{n_k})_k$.

Let $K = \overline{B}_r(0) \cap \mathrm{cl}(G)$, then K is compact and we obtain

$$\begin{aligned}
&P\left(F \backslash U_\varepsilon(F_{n_k}) \in H(K)\right) \\
&\geq P\left(\left(F \cap \overline{B}_r(0)\right) \backslash U_\varepsilon(F_{n_k}) \in H\left(\mathrm{cl}(G)\right)\right) \\
&\geq P\left(\left(F \cap \overline{B}_r(0)\right) \backslash U_\varepsilon(F_{n_k}) \in H\left(G\right)\right) \\
&\geq a
\end{aligned}$$

for all $k \in \mathbb{N}$, which yields a contradiction to (i). □

Note that up to this point, convergence in probability relies heavily on the metric space properties of $\mathbb{R}^p$. We have used balls and $\varepsilon-$neighbourhoods. The following two characterisations of convergence in probability only use the open sets of $\mathcal{F}$ and thus only the topological structure of $\mathcal{F}$. The possibility to express convergence in probability in a 'metric–free' way is not restricted to the spaces $(\mathcal{F}, \tau_H)$ and $(\mathcal{F}, \tau_M)$. In Lemma A.5 we show that for random variables $(X_n)_n$ and X that take their values in a metric space, we have $X_n \xrightarrow{P} X$ if and only if $\lim\limits_{n\to\infty} P\left(X_n \notin U, X \in U\right) = 0$ for all open U. It is thus reasonable to take (ii) in Lemma 2.9 as a definition for convergence in probability in the general case of non metrizable first countable topological spaces. It should be noted that in [22] (see Definition 4 therein) Lachout uses a very similar condition as the definition of convergence in probability.

Lemma 2.8 *Let $(F_n)_n, F$ be random elements of $\mathcal{F}\left(\mathbb{R}^p\right)$, measurable with respect to $\mathcal{B}_{\tau_H}$. Then the following are equivalent:*

(i)

$$F_n \xrightarrow{P_{\tau_H}} F$$

(ii)

$$\lim_{n\to\infty} P\left(F_n \notin H(G),\ F \in H(G)\right) = 0 \textit{ for all open } G \subset \mathbb{R}^p.$$

Proof. Assume that (ii) holds and that there are an open set $G \subset \mathbb{R}^p$ and $\varepsilon > 0,\ r > 0$ such that $P\left(\left(F \cap \overline{B}_r(0)\right) \backslash U_\varepsilon(F_n) \in H(G)\right)$ does not converge to 0. Then there are $(n_k)_k$ and $a > 0$ with $P\left(\left(F \cap \overline{B}_r(0)\right) \backslash U_\varepsilon(F_{n_k}) \in H(G)\right) > a$ for all $k \in \mathbb{N}$. Since the set $\mathrm{cl}(G) \cap \overline{B}_r(0)$ is compact we can find $x_1, \ldots, x_s \in \mathrm{cl}(G) \cap \overline{B}_r(0)$ such that $\mathrm{cl}(G) \cap \overline{B}_r(0) \subset \bigcup\limits_{i=1}^{s} B_{\frac{\varepsilon}{4}}(x_i)$.

If $\left(F\cap\overline{B}_r(0)\right)\backslash U_\varepsilon(F_{n_k})\in H(G)$, then there is at least one $i\in\{1,\ldots,s\}$ with $\left(F\in H\left(B_{\frac{\varepsilon}{4}}(x_i)\right),\ F_{n_k}\notin H\left(B_{\frac{\varepsilon}{4}}(x_i)\right)\right)$. From $\left(F\cap\overline{B}_r(0)\right)\in H(G)$ it is clear that there is $i\in\{1,\ldots,s\}$ with $F\in H\left(B_{\frac{\varepsilon}{4}}(x_i)\right)$. If we had $F_{n_k}\in H\left(B_{\frac{\varepsilon}{4}}(x_i)\right)$ for each such i, we would also have $B_{\frac{\varepsilon}{4}}(x_i)\subset U_\varepsilon\left(F_{n_k}\right)$, which would imply $\left(F\cap\overline{B}_r(0)\right)\backslash U_\varepsilon(F_{n_k})\notin H(G)$. Since $\{1,\ldots,s\}$ is finite, there is $i\in\{1,\ldots,s\}$ such that
$\Big(F\in H\left(B_{\frac{\varepsilon}{4}}(x_i)\right),\ F_{n_{k(l)}}\notin H\left(B_{\frac{\varepsilon}{4}}(x_i)\right)\Big)$ for all $l\in\mathbb{N}$.
It follows that

$$\begin{aligned}
&P\Big(\left(F\cap\overline{B}_r(0)\right)\backslash U_\varepsilon(F_{n_{k(l)}})\in H(G)\Big)\\
&\leq\sum_{i=1}^{s}P\Big(F\in H\left(B_{\frac{\varepsilon}{4}}(x_i)\right),\ F_{n_{k(l)}}\notin H\left(B_{\frac{\varepsilon}{4}}(x_i)\right)\Big)
\end{aligned}$$

and thus for at least one $i\in\{1,\ldots,s\}$:

$$P\Big(F\in H\left(B_{\frac{\varepsilon}{4}}(x_i)\right),\ F_{n_{k(l)}}\notin H\left(B_{\frac{\varepsilon}{4}}(x_i)\right)\Big)>\frac{a}{s},\ l\in\mathbb{N}$$

because otherwise $P\Big(\left(F\cap\overline{B}_r(0)\right)\backslash U_\varepsilon(F_{n_{k(l)}})\in H(G)\Big)\leq s\frac{a}{s}=a$ in contradiction to $P\left(\left(F\cap\overline{B}_r(0)\right)\backslash U_\varepsilon(F_{n_k})\in H(G)\right)>a$ for all $k\in\mathbb{N}$.
Now assume that (i) holds and that there are an open set $G\subset\mathbb{R}^p$ and $(n_k)_k$ such that

$$P\left(F\in H(G),\ F_{n_k}\notin H(G)\right)\geq a$$

for an $a>0$ and all $k\in\mathbb{N}$. It follows, that $G\neq\emptyset$.
Let $0<\delta<\frac{a}{4}$. Because of the continuity of the probability measure and Lemma A.4 we can find $r>0$ such that for all $k\in\mathbb{N}$:

$$\begin{aligned}
&P\left(\left(F\cap\overline{B}_r(0)\right)\in H(G),\ F_{n_k}\notin H(G)\right)\\
&\geq P\left(F\in H(G),\ F_{n_k}\notin H(G)\right)-\delta\\
&\geq a-\delta.
\end{aligned}$$

For the open set $G \cap B_r(0)$ we can find open $G_i,\ i \in \mathbb{N}$ with $G_i \subset G_{i+1},\ G \cap B_r(0) = \bigcup_{i=1}^{\infty} G_i$ and $\operatorname{bdy}(G_i) \cap \operatorname{bdy}(G \cap B_r(0)) = \emptyset$. There is $i_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&P\left(\left(F \cap \overline{B}_r(0)\right) \in H(G_{i_0}),\ F_{n_k} \notin H(G)\right) \\
&\geq P\left(\left(F \cap \overline{B}_r(0)\right) \in H(G),\ F_{n_k} \notin H(G)\right) - \delta \\
&\geq a - 2\delta.
\end{aligned}$$

Let $\varepsilon = \frac{1}{2} \operatorname{dist}\left(\operatorname{bdy}(G_{i_0}), \operatorname{bdy}\left(G \cap B_r(0)\right)\right)$. Then $\varepsilon > 0$, this follows from $\operatorname{bdy}(G_{i_0}) \cap \operatorname{bdy}\left(G \cap B_r(0)\right) = \emptyset$ and the fact that $\operatorname{bdy}(G_{i_0})$ is closed and that $\operatorname{bdy}\left(G \cap B_r(0)\right)$ is compact.
Note that $F_{n_k} \notin H(G)$ implies $U_\varepsilon\left(F_{n_k}\right) \cap G_{i_0} = \emptyset$, which in turn yields that $\left(\left(F \cap \overline{B}_r(0)\right) \in H(G_{i_0}),\ F_{n_k} \notin H(G)\right)$ implies $\left(\left(F \cap \overline{B}_r(0)\right) \setminus U_\varepsilon(F_{n_k}) \in H(G_{i_0})\right)$.
Thus we have

$$\begin{aligned}
&P\left(\left(F \cap \overline{B}_r(0)\right) \setminus U_\varepsilon(F_{n_k}) \in H(G_{i_0})\right) \\
&\geq P\left(F \cap \overline{B}_r(0) \in H(G_{i_0}),\ F_{n_k} \notin H(G)\right) \\
&\geq a - 2\delta \geq \frac{a}{2}
\end{aligned}$$

in contradiction to the assumption that (i) holds. □

Lemma 2.9 *Let $(F_n)_n, F$ be random elements of $\mathcal{F}(\mathbb{R}^p)$, measurable with respect to $\mathcal{B}_{\tau_H}$. Then the following are equivalent*

(i)

$$F_n \overset{P_{\tau_H}}{\to} F$$

(ii)

$$\lim_{n \to \infty} P\left(F_n \notin U,\ F \in U\right) = 0 \text{ for all open } \tau_H\text{-open } U.$$

Proof. Note that $H(G)$ is τ_H-open for each open G. It follows with the above lemma, that (ii) implies (i).
Let U be τ_H-open, then $U = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{k_i} H\left(G_j^i\right)$ with open $G_j^i \subset \mathbb{R}^p$. Because of the conti-

nuity of the probability measure, for each $\eta > 0$ there is m, such that

$$P\left(F \in \bigcup_{i=1}^{m} \bigcap_{j=1}^{k_i} H\left(G_j^i\right)\right) \geq P\left(F \in U\right) - \eta.$$

It follows that

$$\begin{aligned}
P\left(F_n \notin U,\ F \in U\right) &\leq P\left(F_n \notin U,\ F \in \bigcup_{i=1}^{m} \bigcap_{j=1}^{k_i} H\left(G_j^i\right)\right) + \eta \\
&= P\left(\bigcup_{i=1}^{m}\left(F_n \notin U,\ F \in \bigcap_{j=1}^{k_i} H\left(G_j^i\right)\right)\right) + \eta \\
&\leq \sum_{i=1}^{m} P\left(F_n \notin U,\ F \in \bigcap_{j=1}^{k_i} H\left(G_j^i\right)\right) + \eta \\
&\leq \sum_{i=1}^{m} P\left(F_n \notin \bigcap_{l=1}^{k_i} H\left(G_l^i\right),\ F \in \bigcap_{j=1}^{k_i} H\left(G_j^i\right)\right) + \eta \\
&= \sum_{i=1}^{m} P\left(\bigcup_{l=1}^{k_i}\left(F_n \notin H\left(G_l^i\right)\right),\ F \in \bigcap_{j=1}^{k_i} H\left(G_j^i\right)\right) + \eta \\
&\leq \sum_{i=1}^{m} \sum_{l=1}^{k_i} P\left(F_n \notin H\left(G_l^i\right),\ F \in \bigcap_{j=1}^{k_i} H\left(G_j^i\right)\right) + \eta \\
&\leq \sum_{i=1}^{m} \sum_{l=1}^{k_i} P\left(F_n \notin H\left(G_l^i\right),\ F \in H\left(G_l^i\right)\right) + \eta
\end{aligned}$$

Because of the previous lemma, the finite double sum converges to 0 and it follows, that

$$\lim_{n\to\infty} P\left(F_n \notin U,\ F \in U\right) \leq \eta.$$

It remains to let $\eta \to 0$. $\square$

Next we show, that analogous results hold in the case of the topology τ_M.

Lemma 2.10 *Let* $(F_n)_n, F$ *be random elemets of* $\mathcal{F}(\mathbb{R}^p)$, *measurable with respect to* $\mathcal{B}_{\tau_M}$. *Then the following are equivalent*

(i)

$$F_n \overset{P_{\tau_M}}{\to} F$$

(ii)

$$\lim_{n\to\infty} P\left(F_n \notin M\left(K\right), F \in M\left(K\right)\right) = 0 \textit{ for all compact } K \subset \mathbb{R}^p.$$

Proof. First assume that (i) holds, it follows, that for each $\varepsilon > 0$

$$\lim_{n\to\infty} P\left(\left(F_n \backslash U_\varepsilon(F)\right) \cap K \neq \emptyset,\ \overline{U}_\varepsilon\left(F\right) \cap K = \emptyset\right) = 0$$

and because of

$$\begin{aligned} & F_n \cap K \neq \emptyset,\ \overline{U}_\varepsilon\left(F\right) \cap K = \emptyset \\ & \Longrightarrow \left(F_n \backslash \overline{U}_\varepsilon\left(F\right)\right) \cap K \neq \emptyset,\ \overline{U}_\varepsilon\left(F\right) \cap K = \emptyset \\ & \Longrightarrow \left(F_n \backslash U_\varepsilon\left(F\right)\right) \cap K \neq \emptyset,\ \overline{U}_\varepsilon\left(F\right) \cap K = \emptyset \end{aligned}$$

we obtain

$$\begin{aligned} & P\left(F_n \notin M\left(K\right), \overline{U}_\varepsilon\left(F\right) \in M\left(K\right)\right) \\ & = P\left(F_n \cap K \neq \emptyset,\ \overline{U}_\varepsilon\left(F\right) \cap K = \emptyset\right) \\ & \leq P\left(\left(F_n \backslash U_\varepsilon\left(F\right)\right) \cap K \neq \emptyset,\ \overline{U}_\varepsilon\left(F\right) \cap K = \emptyset\right). \end{aligned}$$

Now with $\varepsilon = \frac{1}{k}$, we have

$$\begin{aligned} & \left(F_n \notin M\left(K\right),\ F \in M\left(K\right)\right) \\ & = \bigcup_{k=1}^{\infty} \left(F_n \notin M\left(K\right),\ \overline{U}_{\frac{1}{k}}\left(F\right) \in M\left(K\right)\right) \end{aligned}$$

and by using the continuity of the probability measure, for $\eta > 0$ we can find $k_0 \in \mathbb{N}$, such that

$$\begin{aligned}&P\left(F_n \notin M(K),\ \overline{U}_{\frac{1}{k_0}}(F) \in M(K)\right)\\&\geq P\left(F_n \notin M(K),\ F \in M(K)\right) - \eta.\end{aligned}$$

We obtain

$$\begin{aligned}&\lim_{n\to\infty} P\left(F_n \notin M(K),\ F \in M(K)\right)\\&\leq \lim_{n\to\infty} P\left(F_n \notin M(K),\ \overline{U}_{\frac{1}{k_0}}(F) \in M(K)\right) + \eta\\&\leq \lim_{n\to\infty} P\left((F_n \backslash U_\varepsilon(F)) \cap K \neq \emptyset,\ \overline{U}_\varepsilon(F) \cap K = \emptyset\right) + \eta\\&\leq \lim_{n\to\infty} P\left((F_n \backslash U_\varepsilon(F)) \cap K \neq \emptyset\right) + \eta\\&= \eta\end{aligned}$$

and it remains to let $\eta \to 0$.
Assume that $\lim\limits_{n\to\infty} P\left(F_n \in H(K), F \notin H(K)\right) = 0$, for all compact $K \subset \mathbb{R}^p$, but that there are $\varepsilon > 0$ and K compact such that $P\left(F_n \backslash U_\varepsilon(F) \in H(K)\right)$ does not tend to 0, then clearly $K \neq \emptyset$. There are $(n_k)_k$ and $a > 0$ such that

$$P\left(F_{n_k} \backslash U_\varepsilon(F) \in H(K)\right) > a,\ k \in \mathbb{N}.$$

Since K is compact we can find $x_1, \ldots, x_r \in K$ with $K \subset \bigcup\limits_{i=1}^{r} B_{\frac{\varepsilon}{4}}(x_i)$.
We show that $F_{n_k} \backslash U_\varepsilon(F) \in H(K)$ implies that there is $i \in \{1, \ldots, r\}$, such that $F_{n_k} \in H\left(\overline{B}_{\frac{\varepsilon}{4}}(x_i)\right),\ F \notin H\left(\overline{B}_{\frac{\varepsilon}{4}}(x_i)\right)$. From $F_{n_k} \backslash U_\varepsilon(F) \in H(K)$ it is clear that $F_{n_k} \in H\left(\overline{B}_{\frac{\varepsilon}{4}}(x_i)\right)$ for some $i \in \{1, \ldots, r\}$. Assume that for each such i we have $F \in H\left(\overline{B}_{\frac{\varepsilon}{4}}(x_i)\right)$. Then it follows that $B_{\frac{\varepsilon}{4}}(x_i) \subset U_\varepsilon(F)$ which yields the contradiction $F_{n_k} \backslash U_\varepsilon(F) \notin H(K)$.
Thus we have

$$\begin{aligned}P\left(F_{n_k} \backslash U_\varepsilon(F) \in H(K)\right) &\leq P\left(\bigcup_{i=1}^{r} \left(F_{n_k} \in H\left(\overline{B}_{\frac{\varepsilon}{4}}(x_i)\right),\ F \notin H\left(\overline{B}_{\frac{\varepsilon}{4}}(x_i)\right)\right)\right)\\&\leq \sum_{i=1}^{r} P\left(F_{n_k} \in H\left(\overline{B}_{\frac{\varepsilon}{4}}(x_i)\right),\ F \notin H\left(\overline{B}_{\frac{\varepsilon}{4}}(x_i)\right)\right).\end{aligned}$$

It follows that there is $i \in \{1, \ldots, r\}$ such that
$P\left(F_{n_k} \in H\left(\overline{B}_{\frac{\varepsilon}{4}}(x_i)\right),\ F \notin H\left(\overline{B}_{\frac{\varepsilon}{4}}(x_i)\right)\right) > \frac{a}{r}$. Because otherwise we would have

$$P\left(F_{n_k} \backslash U_\varepsilon(F) \in H(K)\right) \leq r\frac{a}{r} = a$$

in contradiction to $P\left(F_{n_k} \backslash U_\varepsilon(F) \in H(K)\right) > a$. Since $\{1, \ldots, r\}$ is finite there is at least one $i \in \{1, \ldots, r\}$ such that

$$P\left(F_{n_{k(l)}} \in H\left(\overline{B}_{\frac{\varepsilon}{4}}(x_i)\right),\ F \notin H\left(\overline{B}_{\frac{\varepsilon}{4}}(x_i)\right)\right) > \frac{a}{r}$$

for all $l \in \mathbb{N}$. Since $\overline{B}_{\frac{\varepsilon}{4}}(x_i)$ is compact, this contradicts the assumption
$\lim_{n\to\infty} P\left(F_n \in H(K), F \notin H(K)\right) = 0$, for all compact $K \subset \mathbb{R}^p$. □

Lemma 2.11 *Let $(F_n)_n, F$ be random elements of $\mathcal{F}(\mathbb{R}^p)$, measurable with respect to $\mathcal{B}_{\tau_M}$. Then the following are equivalent*

(i)

$$F_n \overset{P_{\tau_M}}{\rightarrow} F$$

(ii)

$$\lim_{n\to\infty} P\left(F_n \notin U,\ F \in U\right) = 0,\ \textit{for all } \tau_M - \textit{open } U.$$

Proof. Since for each compact $K \subset \mathbb{R}^p$, the set $M(K)$ is τ_M-open, it immediately follows with the above lemma, that (ii) implies (i).
Now let U be τ_M-open, then $U = \bigcup_{i=1}^{\infty} M(K_i)$ with compact $K_i \subset \mathbb{R}^p$. With the continuity of the probability measure, it follows, that for each $\eta > 0$ there is m such that

$$P\left(F \in \bigcup_{i=1}^{m} M(K_i)\right) \geq P(F \in U) - \eta.$$

It follows, that

$$\begin{aligned}
P(F_n \notin U,\ F \in U) &\leq P\left(F_n \notin U,\ F \in \bigcup_{i=1}^{m} M(K_i)\right) + \eta \\
&= P\left(\bigcup_{i=1}^{m} (F_n \notin U,\ F \in M(K_i))\right) + \eta \\
&\leq \sum_{i=1}^{m} P(F_n \notin U,\ F \in M(K_i)) + \eta \\
&\leq \sum_{i=1}^{m} P(F_n \notin M(K_i),\ F \in M(K_i)) + \eta.
\end{aligned}$$

The finite sum tends to 0, because of the above lemma. We obtain

$$\lim_{n\to\infty} P(F_n \notin U,\ F \in U) \leq \eta$$

and it remains to let $\eta \to 0$. □

We are now able to prove the main results. Note that Lemmas 2.11 and 2.9 allow us to use one proof for both topologies τ_M and τ_H.

Proof. (of Theorem 2.4) Fix τ as one of the topologies τ_M and τ_H. For each $\tau-$open U it is to show that

$$\liminf_{n\to\infty} P(F_n \in U) \geq P(F \in U).$$

We have

$$\begin{aligned}
P(F \in U) &= P(F_n \in U,\ F \in U) + P(F_n \notin U,\ F \in U) \\
&\leq P(F_n \in U) + P(F_n \notin U,\ F \in U)
\end{aligned}$$

and thus

$$\begin{aligned}
\liminf_{n\to\infty} P(F_n \in U) &\geq \liminf_{n\to\infty} \left(P(F \in U) - P(F_n \notin U,\ F \in U)\right) \\
&= P(F \in U) + \liminf_{n\to\infty} \left(-P(F_n \notin U,\ F \in U)\right) \\
&= P(F \in U),
\end{aligned}$$

where we have used that $F_n \overset{P_\tau}{\to} F$ implies $\lim_{n\to\infty} P(F_n \notin U,\ F \in U) = 0$, which was shown in Lemmas 2.11 resp. 2.9 . □

Proof. (of Theorem 2.6) In view of Lemmas 2.11 and 2.9 it suffices to show that

$$\lim_{n\to\infty} P\left(F_n \notin U,\ F \in U\right) = 0 \tag{2.1}$$

for (a) all τ_M−open, resp. (b) all τ_H−open U.
Since F is almost surely a deterministic constant the probability of the event $\{F \in U\}$ is either 0 or 1. In the first case (2.1) is obviously fulfilled. Now let $P\left(F \in U\right) = 1$. From $F_n \overset{D_{\tau_M}}{\to} F$, resp. $F_n \overset{D_{\tau_H}}{\to} F$ it follows that $\liminf_{n\to\infty} P\left(F_n \in U\right) \geq P\left(F \in U\right) = 1$, which implies that $\lim_{n\to\infty} P\left(F_n \in U\right) = 1$ and hence $\lim_{n\to\infty} P\left(F_n \notin U\right) = 0$. Thus it is clear, that (2.1) holds. □

2.2. Finite Dimensional Convergence

For this section we restrict ourselves to closed sets which are the epigraphs of lower semicontinuous functions. For the classes of stochastic processes with continuous trajectories ($C[0,\infty)$) and for those with cadlag trajectories, i.e. trajectories in $D[0,\infty)$ (which we will deal with in section 2.3) it has been shown in Example 2.5 of [5], that convergence in distribution of the finite dimensional sections, i.e. convergence in distribution of the random vectors $(f_n(x_1), \ldots, f_n(x_k))$ to $(f(x_1), \ldots, f(x_k))$ with $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in \mathbb{R}$ is not sufficient for the convergence in distribution of the stochastic processes as random elements of $C[0,\infty)$ or $D[0,\infty)$. An additional assumption on the continuity properties of the trajectories has to be made. This leads to the concept of stochastic equi continuity in the case of continuous trajectories (see [30]). In the case of cadlag processes, the modulus of continuity is used in [5]. Both concepts are used to limit (uniformly for the sequence $(f_n)_n$) the probability of a certain discontinuous behavior on small neighbourhoods. We will see, that an analogous idea can be used for stochastic processes with lower semicontinuous trajectories. If we identify a lower semicontinuous function with its epigraph, we obtain a criterion for convergence in distribution for stochastic processes with lower semicontinuous trajectories. Convergence criteria for this class of processes are for example required in [6] for the investigation of densities with jump discontinuities and Poisson limits.

The first example shows that convergence in distribution of the finite dimensional sections is not sufficient for $\operatorname{epi} f_n \overset{D_{\tau_M}}{\to} \operatorname{epi} f$.

Example 2.12 Let $(X_n)_n$ be an iid sequence with $X_n \sim U[0,1]$. Let

$$f_n(x) = \begin{cases} 0 & , \ x = X_n \\ 1 & , \ x \neq X_n \end{cases} \ , \ f(x) = 1.$$

First we have to show that

$$\liminf_{n\to\infty} P\left(\left(f_n\left(x_1\right), \ldots, f_n\left(x_k\right)\right) \in U\right) \\ \geq P\left(\left(f\left(x_1\right), \ldots, f\left(x_k\right)\right) \in U\right) \tag{2.2}$$

for all $k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathbb{R}$ and all open $U \subset \mathbb{R}^k$. We can assume that $(1, \ldots, 1) \in U$, because otherwise $P\left(\left(f\left(x_1\right), \ldots, f\left(x_k\right)\right) \in U\right) = 0$ and (2.2) is automatically fulfilled. We have

$$\begin{aligned} & P\left(\left(f_n\left(x_1\right), \ldots, f_n\left(x_k\right)\right) \in U\right) \\ & \geq P\left(\left(f_n\left(x_1\right), \ldots, f_n\left(x_k\right)\right) = (1, \ldots, 1)\right) \\ & = P\left(X_n \neq x_1, \ldots, X_n \neq x_k\right) \\ & = 1 = P\left(\left(f\left(x_1\right), \ldots, f\left(x_k\right)\right) \in U\right) \end{aligned}$$

for all n and hence (2.2) is fulfilled.
However, if we consider the τ_M-open set $M\left([0,1] \times [-\frac{1}{2}, \frac{1}{2}]\right)$, then

$$P\left(\operatorname{epi} f_n \in M\left([0,1] \times [-\frac{1}{2}, \frac{1}{2}]\right)\right) = 0,$$

while

$$P\left(\operatorname{epi} f \in M\left([0,1] \times [-\frac{1}{2}, \frac{1}{2}]\right)\right) = 1,$$

which shows that $\operatorname{epi} f_n \overset{D_{\tau_M}}{\not\to} \operatorname{epi} f$.

We have to find an additional condition that similarly to the concept of stochastic equi-continuity in the case of continuous trajectories ensures convergence in distribution of the stochastic processes.

In [7] equi-lower semicontinuity was defined as an analogue to equi-continuity.

Definition 2.13 A sequence of lower semicontinuous functions $(f_n)_n$ is called equi-lower semicontinuous, if to each x and each $\delta > 0$ there is a neighbourhood V of x such that

$$\inf_{y \in V} f_n(y) \geq \min\left(\frac{1}{\delta},\ f_n(x) - \delta\right), \text{ for all } n \in \mathbb{N}.$$

Salinetti and Wets have given an almost sure definition of equi-lower semicontinuity in [34], for random lower semicontinuous functions. We present a stochastic (in probability) version of equi-lower semicontinuity. This definition is inspired by the concept of stochastic equi-continuity in [30].

Definition 2.14 A sequence of random lower semicontinuous functions $(f_n)_n$ is called stochastically equi-lower semicontinuous, if for given $\varepsilon > 0$, $\delta > 0$, for each compact K there are finitely many $x_1, \ldots, x_k \in K$ and neighbourhoods $V(x_1), \ldots, V(x_k)$, such that

$$K \subset \bigcup_{i=1}^{k} V(x_i)$$

and

$$\limsup_{n\to\infty} P\left(\bigcup_{i=1}^{k}\left(\inf_{y \in V(x_i)} f_n(y) \leq \min\left(\frac{1}{\delta}, f_n(x_i) - \delta\right)\right)\right) \leq \varepsilon.$$

The measureability of the infimum over an uncountable index set is shown in [30].

We note, that Knight in [19] provides a very similar condition. A minor difference is the use of bounded instead of compact sets.

Inspired by Theorem 3 in Chapter V of [30] we develop sufficient conditions for convergence in distribution with respect to τ_M and τ_H. In [19] Knight follows a similar way to obtain conditions under which convergence in distribution of the finite dimensional sections and epi-convergence in distribution (in the sense of [27]) coincide. In the following counterexamples, we show that Knight's assumptions in his main theorems are not strong enough.

Counterexample 2.15 (to Theorem 2 of [19]) Let

$$f_n(x,\omega) = 1, \text{ for all } n \in \mathbb{N}$$
$$f(x,\omega) = \begin{cases} 1 & , \; x \neq X(\omega) \\ 0 & , \; x = X(\omega) \end{cases} ,$$

where $X \sim U[0,1]$.
The sequence $(f_n)_n$ is equi-lsc. We have $f_n \xrightarrow{D} f$, in the sense of convergence in distribution of the finite dimensional sections. This follows from the fact that
$P(f(x_1) = 1, \ldots, f(x_k) = 1) = 1$ for all $k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathbb{R}$.
But from $P\left(\inf_{y\in(0,1)} f_n(y) < \frac{1}{2}\right) = 0$ and $P\left(\inf_{y\in(0,1)} f(y) < \frac{1}{2}\right) = 1$ it follows that

$$\liminf_{n\to\infty} P\left(\inf_{y\in(0,1)} f_n(y) < \frac{1}{2}\right) < P\left(\inf_{y\in(0,1)} f(y) < \frac{1}{2}\right).$$

And because of Corollary 2.4 of [27] we have

$$\operatorname{epi} f_n \overset{D_{\tau_{\text{Fell}}}}{\not\to} \operatorname{epi} f.$$

Counterexample 2.16 (to Theorem 3 of [19]) For $X \sim U[0,1]$ let

$$g(x,\omega) = \begin{cases} 0 & , \; x \leq X(\omega) \\ 1 & , \; x > X(\omega) \end{cases} , \quad h(x,\omega) = \begin{cases} 1 & , \; x < X(\omega) \\ 0 & , \; x \geq X(\omega) \end{cases} .$$

Let Y be independent of X, with $P(Y=0) = P(Y=1) = \frac{1}{2}$.
Let

$$f(x,\omega) = Y(\omega)g(x,\omega) + (1 - Y(\omega))\, h(x,\omega).$$

Then $f(.,\omega)$ is lower semicontinuous for each ω.
For finite $m \in \mathbb{N}$ let $u_1, \ldots, u_m \in [0,1]$ with neighbourhoods $V(u_i)$, $i = 1, \ldots, m$, such that $[0,1] \subset \bigcup_{i=1}^{m} V(u_i)$. We set

$$L(u_i) = \{x \in V(u_i) : x < u_i\} \cap [0,1]$$
$$R(u_i) = \{x \in V(u_i) : x > u_i\} \cap [0,1].$$

Let λ denote the Lebesgue measure on $\mathcal{B}([0,1])$.

From $[0,1] = \left(\bigcup\limits_{i=1}^{m} L(u_i)\right) \cup \left(\bigcup\limits_{i=1}^{m} R(u_i)\right) \cup \left(\bigcup\limits_{i=1}^{m} \{u_i\}\right)$ and from $\lambda([0,1]) = 1$ it follows that we have
$\lambda\left(\bigcup\limits_{i=1}^{m} L(u_i)\right) \geq \frac{1}{2}$ or $\lambda\left(\bigcup\limits_{i=1}^{m} R(u_i)\right) \geq \frac{1}{2}$.
In the first case we obtain

$$\begin{aligned}
& P\left(\bigcup_{i=1}^{m}\left(\inf_{y\in V(u_i)} f(y) \leq f(u_i) - \frac{1}{2}\right)\right) \\
& \geq P\left(Y = 1,\ \bigcup_{i=1}^{m}(X \in L(u_i))\right) \\
& = P\left(Y = 1,\ X \in \bigcup_{i=1}^{m} L(u_i)\right) \\
& = P(Y = 1)\, P\left(X \in \bigcup_{i=1}^{m} L(u_i)\right) \\
& = \frac{1}{2}\lambda\left(\bigcup_{i=1}^{m} L(u_i)\right) \\
& \geq \frac{1}{4}.
\end{aligned}$$

In the same way for the second case we have

$$\begin{aligned}
& P\left(\bigcup_{i=1}^{m}\left(\inf_{y\in V(u_i)} f(y) \leq f(u_i) - \frac{1}{2}\right)\right) \\
& \geq P\left(Y = 0,\ \bigcup_{i=1}^{m}(X \in R(u_i))\right) \\
& \geq \frac{1}{4}.
\end{aligned}$$

Thus for $0 < \varepsilon < \frac{1}{4}$ and $\delta = \frac{1}{2}$ there is no choice of points $u_1, \ldots, u_m \in [0,1]$ and neighbourhoods $V(u_i)\,,\ i = 1, \ldots, m$ such that $[0,1] \subset \bigcup\limits_{i=1}^{m} V(u_i)$ and

$$P\left(\left(\bigcup_{i=1}^{m}\left(\inf_{y\in V(u_i)} f(y) \leq f(u_i) - \delta\right)\right)\right) \leq \epsilon.$$

If we choose $f_n(.,\omega) = f(.,\omega)$ for all $n \in \mathbb{N}$, then

$$\operatorname{epi} f_n \overset{D_{\text{Fell}}}{\to} \operatorname{epi} f$$

and

$$f_n \overset{D}{\to} f,$$

in the finite dimensional sense. This is clear since f_n, f are equal in distribution. We have shown that $(f_n)_n$ is not equi-lsc.

The central point in both counterexamples is, that only lower semicontinuity is assumed for f. In contrast to this we note that in Theorems 4.4 and 4.8 of [34], where stochastical equi-semicontinuity is expressed by the concepts of equi-outer regularity and equi-inner separability, the assumptions on the sequence $(f_n)_n$ are extended to f. We will follow this idea and will prove corrected versions of Knight's Theorems 2 and 3. In the following, we will deal with τ_M and τ_H seperately, since we are primarily interested in sufficient conditions for the associated modes of convergence in distribution. First we pay attention to τ_M.

The example $f_n(x) = 1,\ f(x) = 0,\ \forall x$, for which $\operatorname{epi} f_n \overset{D_{\tau_M}}{\to} \operatorname{epi} f$, shows that for convergence in distribution with respect to τ_M we do not need the full convergence in distribution of the finite dimensional sections . In view of this example one might assume, that only 'one half' of the finite dimensional convergence in distribution is a suitable condition.

The following example however shows, that

$$\liminf_{n\to\infty} P\left(f_n(x_1) > a_1, \ldots, f_n(x_k) > a_k\right) \geq P\left(f(x_1) > a_1, \ldots, f(x_k) > a_k\right)$$

for all $x_1, \ldots, x_k \in \mathbb{R},\ a_i \in \mathbb{R},\ i = 1, \ldots, k,\ k \in \mathbb{N}$ is not sufficient for

$$\operatorname{epi} f_n \overset{D_{\tau_M}}{\to} \operatorname{epi} f,$$

even in the case of equi-lsc $(f_n)_n$.

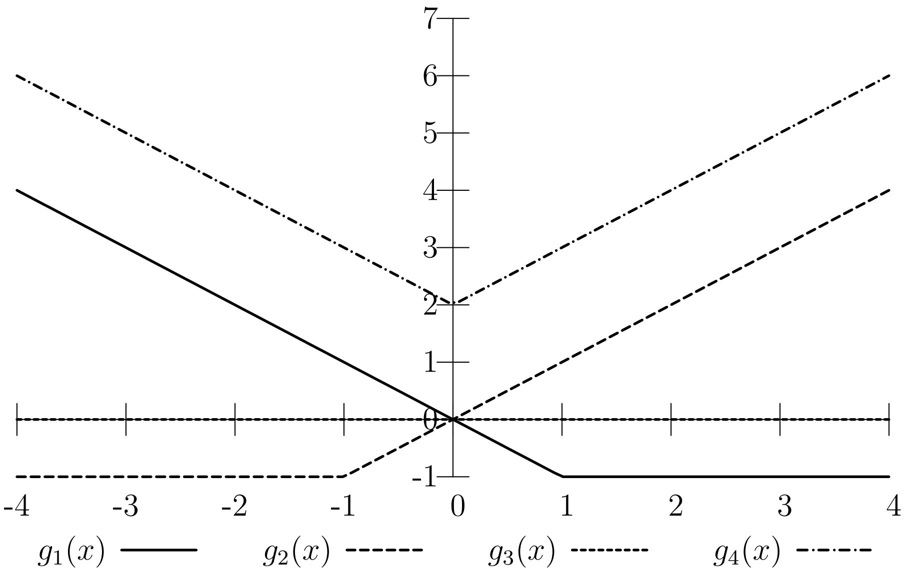
Example 2.17 Let

$$g_1(x) = \begin{cases} -x & , \ x \le 1 \\ -1 & , \ x > 1 \end{cases}$$

$$g_2(x) = \begin{cases} -1 & , \ x \le -1 \\ x & , \ x > -1 \end{cases}$$

$$g_3(x) = 0$$

$$g_4(x) = \begin{cases} 2-x & , \ x \le 0 \\ 2+x & , \ x > 0 \end{cases} .$$



Let $\omega \longmapsto \operatorname{epi} f_n(.,\omega)$ and $\omega \longmapsto \operatorname{epi} f(.,\omega)$ only take the values $\operatorname{epi} g_i, \ i = 1, \dots, 4$ with probabilities

$$P\left(\operatorname{epi} f_n = \operatorname{epi} g_i\right) = \frac{1}{4}, \ i = 1, \dots, 4$$

and

$$P\left(\operatorname{epi} f = \operatorname{epi} g_1\right) = P\left(\operatorname{epi} f = \operatorname{epi} g_2\right) = \frac{1}{2}.$$

First we show that

$$\liminf_{n\to\infty} P\left(\operatorname{epi} f_n \in M(K)\right) \ge P\left(\operatorname{epi} f \in M(K)\right)$$

for all compact $K \subset \mathbb{R}^2$.
There is nothing to show in the case $P\left(\operatorname{epi} f \in M(K)\right) = 0$. Now let $P\left(\operatorname{epi} f \in M(K)\right) = \frac{1}{2}$, then either $\operatorname{epi} g_1 \in M(K)$ or $\operatorname{epi} g_2 \in M(K)$ it follows that $\operatorname{epi} g_4 \in M(K)$ and we

obtain $P\left(\operatorname{epi} f_n \in M(K)\right) \geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
Finally let $P\left(\operatorname{epi} f \in M(K)\right) = 1$, then $\operatorname{epi} g_1 \in M(K)$ and $\operatorname{epi} g_2 \in M(K)$. It follows that $\operatorname{epi} g_3 \in M(K)$ and $\operatorname{epi} g_4 \in M(K)$ and thus $P\left(\operatorname{epi} f_n \in M(K)\right) = 1$.
Now we show that

$$\liminf_{n\to\infty} P\left(f_n(x_1) > a_1, \ldots, f_n(x_k) > a_k\right) \geq P\left(f(x_1) > a_1, \ldots, f(x_k) > a_k\right)$$

for all $x_1, \ldots, x_k \in \mathbb{R},\ a_i \in \mathbb{R},\ i = 1, \ldots, k,\ k \in \mathbb{N}$.
This follows from the above with the compact set $K = \{(x_1, a_1), \ldots, (x_k, a_k)\}$.
Note that the sequence $(f_n)_n$ is stochastically equi-lower semicontinuous. This can easily be obtained from the fact that for each ω the mapping $x \mapsto f_n(x, \omega)$ is Lipschitz continuous with Lipschitz constant 1.
We show that $\operatorname{epi} f_n \overset{D_{\tau_M}}{\not\to} \operatorname{epi} f$.
Let $K_1 = \{(-2, 0)\},\ K_2 = \{(2, 0)\}$, then $U = M(K_1) \cup M(K_2)$ is τ_M-open and we obtain

$$\begin{aligned}
&P\left(\operatorname{epi} f_n \in U\right)\\
&= P(\operatorname{epi} f_n \in M(K_1)) + P(\operatorname{epi} f_n \in M(K_2)) - P(\operatorname{epi} f_n \in M(K_1) \cap M(K_2))\\
&= P(\operatorname{epi} f_n \in M(K_1)) + P(\operatorname{epi} f_n \in M(K_2)) - P\left(\operatorname{epi} f_n \in M(K_1 \cup K_2)\right)\\
&= P(\operatorname{epi} f_n \in \{\operatorname{epi} g_1, \operatorname{epi} g_4\}) + P(\operatorname{epi} f_n \in \{\operatorname{epi} g_2, \operatorname{epi} g_4\}) - P\left(\operatorname{epi} f_n = \operatorname{epi} g_4\right)\\
&= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}
\end{aligned}$$

and

$$\begin{aligned}
&P\left(\operatorname{epi} f \in U\right)\\
&= P(\operatorname{epi} f \in M(K_1)) + P(\operatorname{epi} f \in M(K_2)) - P\left(\operatorname{epi} f \in M(K_1 \cup K_2)\right)\\
&= P\left(\operatorname{epi} f = \operatorname{epi} g_1\right) + P\left(\operatorname{epi} f = \operatorname{epi} g_2\right)\\
&= 1
\end{aligned}$$

and thus

$$\liminf_{n\to\infty} P\left(\operatorname{epi} f_n \in U\right) < P\left(\operatorname{epi} f \in U\right).$$

The following theorem contains a necessary condition for $\operatorname{epi} f_n \overset{D_{\tau_M}}{\to} \operatorname{epi} f$. We see that this condition is weaker than convergence in distribution of the finite dimensional sections, but stronger than the condition which was found to be insufficient in the above example.

Theorem 2.18 *If* $\operatorname{epi} f_n \overset{D_{\tau_M}}{\to} \operatorname{epi} f$, *then*

$$\begin{aligned}
&\liminf_{n\to\infty} P\left(\bigcup_{i=1}^{m} \big(f_n(x_1^i) > a_1^i, \ldots, f_n(x_{k_i}^i) > a_{k_i}^i\big)\right) \\
&\geq P\left(\bigcup_{i=1}^{m} \big(f(x_1^i) > a_1^i, \ldots, f(x_{k_i}^i) > a_{k_i}^i\big)\right)
\end{aligned}$$

for all $m \in \mathbb{N},\ x_1^i, \ldots, x_{k_i}^i \in \mathbb{R}^d,\ a_1^i, \ldots, a_{k_i}^i \in \mathbb{R},\ k_i \in \mathbb{N},\ i \leq m$.

Proof. Let $K_i = \left\{(\ x_1^i, a_1^i), \ldots, (\ x_{k_i}^i, a_{k_i}^i)\right\}$, then K_i is compact as the union of finitely many points. We have

$$\begin{aligned}
&\liminf_{n\to\infty} P\left(\bigcup_{i=1}^{m} \big(f_n(x_1^i) > a_1^i, \ldots, f_n(x_{k_i}^i) > a_{k_i}^i\big)\right) \\
&= \liminf_{n\to\infty} P\left(\bigcup_{i=1}^{m} (\operatorname{epi} f_n \in M(K_i))\right) \\
&= \liminf_{n\to\infty} P\left(\operatorname{epi} f_n \in \bigcup_{i=1}^{m} M(K_i)\right) \\
&\geq P\left(\operatorname{epi} f \in \bigcup_{i=1}^{m} M(K_i)\right) \\
&= P\left(\bigcup_{i=1}^{m} \big(f(x_1^i) > a_1^i, \ldots, f(x_{k_i}^i) > a_{k_i}^i\big)\right). \qquad \square
\end{aligned}$$

The next theorem shows, that this necessary condition, together with stochastic equi-lower semicontinuity is sufficient for convergence in distribution with respect to τ_M. Note that we only require lower semicontinuity of the limit function f.

Theorem 2.19 *Let $(f_n)_n$ be a sequence of stochastically equi-lower semicontinuous random lower semicontinuous functions, let f be a random lower semicontinuous function. If*

$$\liminf_{n\to\infty} P\left(\bigcup_{i=1}^{m}\left(f_n(x_1^i) > a_1^i, \dots, f_n(x_{k_i}^i) > a_{k_i}^i\right)\right)$$
$$\geq P\left(\bigcup_{i=1}^{m}\left(f(x_1^i) > a_1^i, \dots, f(x_{k_i}^i) > a_{k_i}^i\right)\right)$$

for all $m \in \mathbb{N},\ x_1^i, \dots, x_{k_i}^i \in \mathbb{R}^p,\ a_1^i, \dots, a_{k_i}^i \in \mathbb{R},\ k_i \in \mathbb{N},\ i = 1, \dots, m,$ then

$$\operatorname{epi} f_n \overset{D_{\tau_M}}{\to} \operatorname{epi} f.$$

Proof. We have to show that

$$\liminf_{n\to\infty} P\left(\operatorname{epi} f_n \in U\right) \geq P\left(\operatorname{epi} f \in U\right)$$

for all τ_M-open U. We can assume that $U = \bigcup\limits_{i=1}^{\infty} M(K_i)$ with compact $K_i \subset \mathbb{R}^{p+1}$. Because of the continuity of the probability measure for each $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that

$$P\left(\operatorname{epi} f \in \bigcup_{i=1}^{m} M(K_i)\right) \geq P\left(\operatorname{epi} f \in \bigcup_{i=1}^{\infty} M(K_i)\right) - \varepsilon.$$

For each compact set $K_i \subset \mathbb{R}^{p+1}$ there is a sequence $(Q_{i,l})_l$ with $K_i = \bigcap\limits_{l=1}^{\infty} Q_{i,l}$ and $Q_{i,l+1} \subset Q_{i,l}$, where each $Q_{i,l}$ is the union of finitely many axially parallel compact cuboids, i.e.

$$Q_{i,l} = \bigcup_{r=1}^{s_{i,l}} \left(A_r^{i,l} \times [c_r^{i,l}, d_r^{i,l}]\right),$$

where each $A_r^{i,l}$ is a axially parallel compact cuboid in $\mathbb{R}^p$. See for example Section 4.5 of [20]. From $M(Q_{i,l}) \subset M\left(Q_{i,l+1}\right)$ and $M(K_i) = \bigcup\limits_{i=1}^{\infty} M(Q_{i,l})$ it follows with the continuity of the probability measure that to each $\varepsilon > 0$ there is l_0 such that

$$P\left(\operatorname{epi} f \in \bigcup_{i=1}^{m} M(Q_{i,l})\right) \geq P\left(\operatorname{epi} f \in \bigcup_{i=1}^{m} M(K_i)\right) - \varepsilon$$

for all $l \geq l_0$.
Let $l \geq l_0$ be fixed. In the following we omit the index l.
For $\delta > 0$ let $Q_i^\delta = \bigcup\limits_{r=1}^{s_i} \left(A_r^i \times [c_r^i, d_r^i + \delta]\right)$. To $\varepsilon > 0$ we can find $\delta > 0$ such that

$$P\left(\operatorname{epi} f \in \bigcup_{i=1}^{m} M(Q_i^\delta)\right) \geq P\left(\operatorname{epi} f \in \bigcup_{i=1}^{m} M(Q_i)\right) - \varepsilon.$$

Without loss of generality we can assume that $\delta > 0$ is so small that $\frac{1}{\delta} > d_r^i$ for all i and r.
For each $x \in \bigcup\limits_{r=1}^{s_i} A_r^i$ let $d_i(x) = \max\left\{d_r^i : x \in A_r^i\right\}$, then $(\operatorname{epi} f_n \in M(Q_i))$ is equivalent to $\left(f_n(x) > d(x),\ \forall x \in \bigcup\limits_{r=1}^{s_i} A_r^i\right)$. Since A_r^i is compact and $(f_n)_n$ is equi-lower semicontinuous there are $x_r^i(1), \ldots, x_r^i(k_r^i)$ and neighbourhoods $V(x_r^i(1)), \ldots, V\left(x_r^i(k_r^i)\right)$ such that $A_r^i \subset \bigcup\limits_{j=1}^{k_r^i} V\left(x_r^i(j)\right)$ and

$$\limsup_{n\to\infty} P\left(\bigcup_{j=1}^{k_r^i}\left(\inf_{x \in V(x_r^i(j))} f_n(x) \leq \min\left(\frac{1}{\delta}, f_n(x_r^i(j)) - \delta\right)\right)\right) < \frac{\varepsilon}{m s_i}.$$

For the fixed $\delta > 0$ from above we have

$$\begin{aligned}
&P\left(\operatorname{epi} f_n \in \bigcup_{i=1}^{m} M(Q_i)\right)\\
&= P\left(\bigcup_{i=1}^{m} (\operatorname{epi} f_n \in M(Q_i))\right)\\
&= P\left(\bigcup_{i=1}^{m}\left(f_n(x) > d_i(x),\ \forall x \in \bigcup_{r=1}^{s_i} A_r^i\right)\right)\\
&\geq P\left(\bigcup_{i=1}^{m}\left(\begin{array}{c} f_n(x) > d_i(x),\ \forall x \in \bigcup\limits_{r=1}^{s_i} A_r^i, \\ f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1, \ldots, k_r^i,\ r = 1, \ldots, s_i \end{array}\right)\right).
\end{aligned}$$

Note that

$$\begin{aligned}
&\bigcup_{i=1}^{m}\big(f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1,\dots,k_r^i,\ r = 1,\dots,s_i\big)\\
&= \bigcup_{i=1}^{m}\left(\begin{array}{c}\left(\begin{array}{c} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1,\dots,k_r^i,\ r = 1,\dots,s_i,\\ f_n(x) > d_i(x),\ \forall x \in \bigcup_{r=1}^{s_i} A_r^i \end{array}\right)\\ \cup\left(\begin{array}{c} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1,\dots,k_r^i,\ r = 1,\dots,s_i,\\ \exists x \in \bigcup_{r=1}^{s_i} A_r^i:\ f_n(x) \le d_i(x) \end{array}\right)\end{array}\right)\\
&= \bigcup_{i=1}^{m}\left(\begin{array}{c} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1,\dots,k_r^i,\ r = 1,\dots,s_i,\\ f_n(x) > d_i(x),\ \forall x \in \bigcup_{r=1}^{s_i} A_r^i \end{array}\right)\\
&\cup \bigcup_{i=1}^{m}\left(\begin{array}{c} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1,\dots,k_r^i,\ r = 1,\dots,s_i,\\ \exists x \in \bigcup_{r=1}^{s_i} A_r^i:\ f_n(x) \le d_i(x) \end{array}\right).
\end{aligned}$$

Now

$$\begin{aligned}
&P\left(\bigcup_{i=1}^{m}\left(\begin{array}{c} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1,\dots,k_r^i,\ r = 1,\dots,s_i,\\ \exists x \in \bigcup_{r=1}^{s_i} A_r^i:\ f_n(x) \le d_i(x) \end{array}\right)\right)\\
&\le \sum_{i=1}^{m} P\left(\bigcup_{r=1}^{s_i}\left(\begin{array}{c} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1,\dots,k_r^i,\ r = 1,\dots,s_i,\\ \exists x \in A_r^i:\ f_n(x) \le d_i(x) \end{array}\right)\right)\\
&\le \sum_{i=1}^{m}\sum_{r=1}^{s_i} P\left(\begin{array}{c} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1,\dots,k_r^i,\ r = 1,\dots,s_i,\\ \exists x \in A_r^i:\ f_n(x) \le d_i(x) \end{array}\right)
\end{aligned}$$

We take a closer look at the term under the double sum. Let $x \in A_r^i$, then there is $x_r^i(j) \in A_r^i$ such that $x \in V(x_r^i(j))$ and $d_i(x) = d_i(x_r^i(j))$. We distinguish two cases. First let $\frac{1}{\delta} \le f_n(x_r^i(j)) - \delta$. Then $f_n(x) \le d_i(x)$ implies $f_n(x) < \frac{1}{\delta}$. This follows because $\delta > 0$ was chosen so small that $\frac{1}{\delta} > d_r^i$.
In the second case let $f_n(x_r^i(j)) - \delta < \frac{1}{\delta}$. With $d_i(x) = d_i(x_r^i(j))$ from $f_n(x) \le d_i(x)$ and

$f_n(x_r^i(j)) > d_i(x) + \delta$ that $f_n(x) \leq f_n(x_r^i(j)) - \delta$. Thus it follows that

$$\begin{aligned}
&P\begin{pmatrix} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1, \dots, k_r^i,\ r = 1, \dots, s_i, \\ \exists x \in A_r^i :\ f_n(x) \leq d_i(x) \end{pmatrix} \\
\leq &P\begin{pmatrix} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1, \dots, k_r^i,\ r = 1, \dots, s_i, \\ \bigcup_{j=1}^{k_r^i} \left(\inf_{x \in V(x_r^i(j))} f_n(x) \leq \min\left(\frac{1}{\delta}, f_n(x_r^i(j)) - \delta\right)\right) \end{pmatrix} \\
< &\frac{\varepsilon}{m s_i}
\end{aligned}$$

for all $n \geq n_0$.
It follows that

$$\begin{aligned}
&P\left(\bigcup_{i=1}^{m}\begin{pmatrix} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1, \dots, k_r^i,\ r = 1, \dots, s_i, \\ \exists x \in \bigcup_{r=1}^{s_i} A_r^i :\ f_n(x) \leq d_i(x) \end{pmatrix}\right) \\
&\leq \sum_{i=1}^{m}\sum_{r=1}^{s_i} P\begin{pmatrix} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1, \dots, k_r^i,\ r = 1, \dots, s_i, \\ \exists x \in A_r^i :\ f_n(x) \leq d_i(x) \end{pmatrix} \\
&\leq \varepsilon
\end{aligned}$$

We obtain

$$\begin{aligned}
&P\left(\bigcup_{i=1}^{m}\begin{pmatrix} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1, \dots, k_r^i,\ r = 1, \dots, s_i, \\ f_n(x) > d_i(x),\ \forall x \in \bigcup_{r=1}^{s_i} A_r^i \end{pmatrix}\right) \\
&\geq P\left(\bigcup_{i=1}^{m}\left(f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1, \dots, k_r^i,\ r = 1, \dots, s_i\right)\right) - \varepsilon
\end{aligned}$$

for all $n \geq n_0$ and consequently

$$\begin{aligned}
&\liminf_{n\to\infty} P\left(\operatorname{epi} f_n \in \bigcup_{i=1}^{\infty} M(K_i)\right)\\
&\geq \liminf_{n\to\infty} P\left(\operatorname{epi} f_n \in \bigcup_{i=1}^{m} M(K_i)\right)\\
&\geq \liminf_{n\to\infty} P\left(\operatorname{epi} f_n \in \bigcup_{i=1}^{m} M(Q_i)\right)\\
&\geq \liminf_{n\to\infty} P\left(\bigcup_{i=1}^{m}\left(\begin{array}{c} f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1, \ldots, k_r^i,\ r = 1, \ldots, s_i, \\ f_n(x) > d_i(x),\ \forall x \in \bigcup\limits_{r=1}^{s_i} A_r^i \end{array}\right)\right)\\
&\geq \liminf_{n\to\infty} P\left(\bigcup_{i=1}^{m}\big(f_n(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1, \ldots, k_r^i,\ r = 1, \ldots, s_i\big)\right) - \varepsilon\\
&\geq P\left(\bigcup_{i=1}^{m}\big(f(x_r^i(j)) > d_i(x_r^i(j)) + \delta,\ j = 1, \ldots, k_r^i,\ r = 1, \ldots, s_i\big)\right) - \varepsilon\\
&\geq P\left(\bigcup_{i=1}^{m}\left(f(x) > d_i(x) + \delta,\ \forall x \in \bigcup_{r=1}^{s_i} A_r^i\right)\right) - \varepsilon\\
&\geq P\left(\bigcup_{i=1}^{m}\big(\operatorname{epi} f \in M(Q_i^\delta)\big)\right) - \varepsilon\\
&\geq P\left(\bigcup_{i=1}^{m}(\operatorname{epi} f \in M\left(Q_i\right))\right) - 2\varepsilon.\\
&\geq P\left(\bigcup_{i=1}^{m}(\operatorname{epi} f \in M(K_i))\right) - 3\varepsilon\\
&\geq P\left(\bigcup_{i=1}^{\infty}(\operatorname{epi} f \in M(K_i))\right) - 4\varepsilon
\end{aligned}$$

$$= P\left(\operatorname{epi} f \in \bigcup_{i=1}^{\infty} M(K_i)\right) - 4\varepsilon.$$

Now let $\varepsilon \to 0$ to complete the proof. $\square$

We now turn to the τ_H case. The following example shows that

$$\limsup_{n\to\infty} P\left(f_n(x_1) \geq a_1, \ldots, f_n(x_k) \geq a_k\right) \leq P\left(f(x_1) \geq a_1, \ldots, f(x_k) \geq a_k\right)$$

for all $x_i,\ a_i,\ i = 1, \ldots, k,\ k \in \mathbb{N}$, is not sufficient for

$$\operatorname{epi} f_n \overset{D_{\tau_H}}{\to} \operatorname{epi} f.$$

Example 2.20 Let g_1, g_2, g_3, g_4 as in example 2.17 .
Let

$$P\left(\operatorname{epi} f_n = \operatorname{epi} g_i\right) = \frac{1}{4},\ i = 1, \ldots, 4,\ n \in \mathbb{N}$$

and

$$P\left(\operatorname{epi} f = \operatorname{epi} g_3\right) = P\left(\operatorname{epi} f = \operatorname{epi} g_4\right) = \frac{1}{2}.$$

First we show that $\limsup\limits_{n\to\infty} P\left(f_n(x_1) \geq a_1, \ldots, f_n(x_k) \geq a_k\right) \leq P\left(f(x_1) \geq a_1, \ldots, f(x_k) \geq a_k\right)$ holds for all $x_i,\ a_i,\ i = 1, \ldots, k,\ k \in \mathbb{N}$. Because of the definition of the random closed set $\operatorname{epi} f$, $P\left(f(x_1) \geq a_1, \ldots, f(x_k) \geq a_k\right)$ can only take the values 0, $\frac{1}{2}$ and 1. It suffices to consider only the first two cases. First let $P\left(f(x_1) \geq a_1, \ldots, f(x_k) \geq a_k\right) = 0$, then there is i such that $a_i > 2$. It follows that $f_n(x) < a_i$ for all x and all n and thus $P\left(f_n(x_1) \geq a_1, \ldots, f_n(x_k) \geq a_k\right) = 0$. In the second case let $P\left(f(x_1) \geq a_1, \ldots, f(x_k) \geq a_k\right) = \frac{1}{2}$, then there is i such that $a_i > 0 = f(x_i)$ this implies that for at least one $g \in \{g_1, g_2\}$ we have $g(x_i) < a_i$. It follows that $P\left(f_n(x_1) \geq a_1, \ldots, f_n(x_k) \geq a_k\right) \leq \frac{1}{2}$ for all n.
Now consider the τ_H-open set $U = H(B_{\frac{1}{4}}((-2, \frac{1}{2}))) \cap H(B_{\frac{1}{4}}((2, \frac{1}{2})))$.
Then $P\left(\operatorname{epi} f \in U\right) = P\left(\operatorname{epi} f = \operatorname{epi} g_4\right) = \frac{1}{2}$ and $P\left(\operatorname{epi} f_n \in U\right) = P\left(\operatorname{epi} f_n = \operatorname{epi} g_4\right) = \frac{1}{4}$ which shows that $\liminf\limits_{n\to\infty} P\left(\operatorname{epi} f_n \in U\right) < P\left(\operatorname{epi} f \in U\right)$ and thus $\operatorname{epi} f_n \overset{D_{\tau_H}}{\not\to} \operatorname{epi} f$.

The following definition extends the stochastic equi-lower semicontinuity condition to the designated limit f.

Definition 2.21 A random lower semicontinuous function f is called stochastically uniformly lower semicontinuous, if for each compact K, $\delta > 0$, $\varepsilon > 0$ there are $l \in \mathbb{N}$ and $x_1, \ldots, x_l \in A$ with neighbourhoods $V(x_i)$, $i = 1, \ldots, l$ such that $K \subset \bigcup_{i=1}^{l} V(x_i)$ and

$$P\left(\bigcup_{i=1}^{l}\left(\inf_{y \in V(x_i)} f(y) \leq \min\left(\frac{1}{\delta}, f(x_i) - \delta\right)\right)\right) \leq \varepsilon$$

Now we can provide a sufficient condition for convergence in distribution with respect to τ_H. Note that only lower semicontinuity of each f_n is needed. Especially we do not need stochastic equi-lower semicontinuity. However we assume stochastic uniform-lower semicontinuity for the limit f. A condition of this kind is missing in [19].

Theorem 2.22 *Let $(f_n)_n$ be a sequence of random lower semicontinuous functions. Let f be a random lower semicontinuous function which fulfills the stochastic uniform-lower semicontinuity property and let*

$$\begin{aligned}
&\limsup_{n\to\infty} P\left(\bigcup_{i=1}^{m}\left(f_n(x_1^i) \geq a_1^i, \ldots, f_n(x_{k_i}^i) \geq a_{k_i}^i\right)\right) \\
&\leq P\left(\bigcup_{i=1}^{m}\left(f(x_1^i) \geq a_1^i, \ldots, f(x_{k_i}^i) \geq a_{k_i}^i\right)\right)
\end{aligned}$$

for all $m \in \mathbb{N}$, $x_1^i, \ldots, x_{k_i}^i \in \mathbb{R}^p$, $a_1^i, \ldots, a_{k_i}^i \in \mathbb{R}$, $k_i \in \mathbb{N}$, $i = 1, \ldots, m$.
Then

$$\operatorname{epi} f_n \overset{D_{\tau_H}}{\to} \operatorname{epi} f.$$

Proof. We have to show that

$$\liminf_{n\to\infty} P(\operatorname{epi} f_n \in U) \geq P(\operatorname{epi} f \in U)$$

for each τ_H−open U. We can assume that $U = \bigcup_{i=1}^{\infty}\bigcap_{j=1}^{k_i} H(G_j^i)$ with open $G_j^i \subset \mathbb{R}^{p+1}$. It suffices to show, that

$$\limsup_{n\to\infty} P\left(\operatorname{epi} f_n \notin \bigcup_{i=1}^{\infty}\bigcap_{j=1}^{k_i} H(G_j^i)\right) \leq P\left(\operatorname{epi} f \notin \bigcup_{i=1}^{\infty}\bigcap_{j=1}^{k_i} H(G_j^i)\right).$$

Because of the continuity of the probability measure, to $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that

$$P\left(\operatorname{epi} f \notin \bigcup_{i=1}^{m} \bigcap_{j=1}^{k_i} H(G_j^i)\right) \leq P\left(\operatorname{epi} f \notin \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{k_i} H(G_j^i)\right) + \varepsilon.$$

Each of the open G_j^i can be approximated from within by a sequence $(Q_{j,l}^i)_l$ with $G_j^i = \bigcup_{l=1}^{\infty} Q_{j,l}^i$ and $Q_{j,l}^i \subset Q_{j,l+1}^i$, where each $Q_{j,l}^i$ is the union of finitely many axially parallel compact cuboids (see Section 4.5 of [20]), i.e.

$$Q_{j,l}^i = \bigcup_{r=1}^{s_{j,i,l}} \left(A_r^{j,i,l} \times [c_r^{j,i,l}, d_r^{j,i,l}]\right),$$

where each $A_r^{j,i,l}$ is a axially parallel compact cuboid in $\mathbb{R}^p$. Using the continuity of the probability measure we can find $l_0 \in \mathbb{N}$ with

$$P\left(\operatorname{epi} f \notin \bigcup_{i=1}^{m} \bigcap_{j=1}^{k_i} H\left(Q_{i,l}^j\right)\right) \leq P\left(\operatorname{epi} f \notin \bigcup_{i=1}^{m} \bigcap_{j=1}^{k_i} H(G_j^i)\right) + \varepsilon$$

for all $l \geq l_0$.
In the following we do not denote the dependece on l.
Let $\delta := \min\left\{\operatorname{dist}\left(Q_j^i, \operatorname{bdy}(G_j^i)\right) : i = 1, \ldots, m,\ j = 1, \ldots, k_i\right\}$. Then $\delta > 0$ since $\operatorname{bdy}(G_j^i)$ is closed and Q_j^i is compact for all i, j. Without loss of generality we can choose a smaller $\delta > 0$ such that $\frac{1}{\delta} > d_r^{j,i} + \delta$ for all r, j, i.
Let N be a positive number that only depends on m and $k_1, \ldots, k_m$. Each of the sets $A_r^{j,i}$ is compact and because of the stochastic uniform-lower semicontinuity property of f we can find $x_r^{j,i}(1), \ldots, x_r^{j,i}(u_r^{j,i}) \in A_r^{j,i}$ and neighbourhoods $V(x_r^{j,i}(1)), \ldots, V(x_r^{j,i}(u_r^{j,i}))$ such that $A_r^{j,i} \subset \bigcup_{q=1}^{u_r^{j,i}} V(x_r^{j,i}(q))$ and

$$P\left(\bigcup_{q=1}^{u_r^{j,i}} \left(\inf_{y \in V(x_r^{j,i}(q))} f(y) \leq \min\left(\frac{1}{\delta}, f(x_r^{j,i}(q)) - \delta\right)\right)\right) \leq \frac{\varepsilon}{N}.$$

For an open set $G \subset \mathbb{R}^{p+1}$ and for $x \in \mathbb{R}^p$ let $h_G(x) = \sup\{y \in \mathbb{R} : (x, y) \in G\}$.

We have

$$
\begin{aligned}
&\limsup_{n\to\infty} P\left(\operatorname{epi} f_n \notin \bigcup_{i=1}^{\infty}\bigcap_{j=1}^{k_i} H(G_j^i)\right)\\
&\leq \limsup_{n\to\infty} P\left(\operatorname{epi} f_n \notin \bigcup_{i=1}^{m}\bigcap_{j=1}^{k_i} H(G_j^i)\right)\\
&= \limsup_{n\to\infty} P\left(\bigcap_{i=1}^{m}\bigcup_{j=1}^{k_i} \operatorname{epi} f_n \notin H(G_j^i)\right)\\
&\leq \limsup_{n\to\infty} P\left(\bigcap_{i=1}^{m}\bigcup_{j=1}^{k_i}\left(f_n\left(x_r^{j,i}(q)\right) \geq h_{G_j^i}(x_r^{j,i}(q)),\ q=1,\dots,u_r^{j,i},\ r=1,\dots,s_{j,i}\right)\right)
\end{aligned}
$$

Note that the event in the last line can be written in the form

$$
\bigcup_{v=1}^{w}\left(f_n\left(z_1^v\right) \geq a_1^v, \dots, f_n\left(z_{t_v}^v\right) \geq a_{t_v}^v\right),
$$

with suitable $w,\ z_\cdot^\cdot,\ a_\cdot^\cdot$.
It then follows from the assumptions that

$$
\begin{aligned}
&\limsup_{n\to\infty} P\left(\bigcap_{i=1}^{m}\bigcup_{j=1}^{k_i}\left(f_n\left(x_r^{j,i}(q)\right) \geq h_{G_j^i}(x_r^{j,i}(q)),\ q=1,\dots,u_r^{j,i},\ r=1,\dots,s_{j,i}\right)\right)\\
&\leq P\left(\bigcap_{i=1}^{m}\bigcup_{j=1}^{k_i}\left(f\left(x_r^{j,i}(q)\right) \geq h_{G_j^i}(x_r^{j,i}(q)),\ q=1,\dots,u_r^{j,i},\ r=1,\dots,s_{j,i}\right)\right).
\end{aligned}
$$

We have

$$
\begin{aligned}
&P\left(\bigcap_{i=1}^{m}\bigcup_{j=1}^{k_i}\left(f\left(x_r^{j,i}\left(q\right)\right)\geq h_{G_j^i}(x_r^{j,i}\left(q\right)),\ q=1,\dots,u_r^{j,i},\ r=1,\dots,s_{j,i}\right)\right)\\
&=P\left(\bigcap_{i=1}^{m}\bigcup_{j=1}^{k_i}\left(\begin{array}{c}\left(\begin{array}{c}\left(f\left(x_r^{j,i}\left(q\right)\right)\geq h_{G_j^i}(x_r^{j,i}\left(q\right)),\ q=1,\dots,u_r^{j,i},\ r=1,\dots,s_{j,i}\right),\\ \operatorname{epi} f\notin H\left(Q_j^i\right)\end{array}\right)\\ \cup\left(\begin{array}{c}\left(f\left(x_r^{j,i}\left(q\right)\right)\geq h_{G_j^i}(x_r^{j,i}\left(q\right)),\ q=1,\dots,u_r^{j,i},\ r=1,\dots,s_{j,i}\right),\\ \operatorname{epi} f\in H\left(Q_j^i\right)\end{array}\right)\end{array}\right)\right)\\
&\leq P\left(\bigcap_{i=1}^{m}\bigcup_{j=1}^{k_i}\left(\begin{array}{c}\left(\operatorname{epi} f\notin H\left(Q_j^i\right)\right)\\ \cup\left(\begin{array}{c}\left(f\left(x_r^{j,i}\left(q\right)\right)\geq h_{G_j^i}(x_r^{j,i}\left(q\right)),\ q=1,\dots,u_r^{j,i},\ r=1,\dots,s_{j,i}\right),\\ \operatorname{epi} f\in H\left(Q_j^i\right)\end{array}\right)\end{array}\right)\right)
\end{aligned}
$$

The event in the last line can be written as

$$
\begin{aligned}
&\bigcap_{i=1}^{m}\left(\begin{array}{c}\left(\bigcup_{j=1}^{k_i}\operatorname{epi} f\notin H\left(Q_j^i\right)\right)\\ \cup\left(\bigcup_{j=1}^{k_i}\left(\begin{array}{c}\left(f\left(x_r^{j,i}\left(q\right)\right)\geq h_{G_j^i}(x_r^{j,i}\left(q\right)),\ q=1,\dots,u_r^{j,i},\ r=1,\dots,s_{j,i}\right),\\ \operatorname{epi} f\in H\left(Q_j^i\right)\end{array}\right)\right)\end{array}\right)\\
&=\left(\bigcap_{i=1}^{m}\bigcup_{j=1}^{k_i}\operatorname{epi} f\notin H\left(Q_j^i\right)\right)\cup R.
\end{aligned}
$$

Here R is a finite union and intersection of sets, where each of the sets is an intersection with a set of the form

$$
\left(\begin{array}{c}\left(f\left(x_r^{j,i}\left(q\right)\right)\geq h_{G_j^i}(x_r^{j,i}\left(q\right)),\ q=1,\dots,u_r^{j,i},\ r=1,\dots,s_{j,i}\right),\\ \operatorname{epi} f\in H\left(Q_j^i\right)\end{array}\right)
$$

for some i,j. The number N of sets that form R depends only on m and $k_1,\dots,k_m$. We now show that $P(R)\leq\varepsilon$. For fixed i and j let $d(x)=\max\{d_r^{j,i}: x\in A_r^{j,i}\}$. Assume that $\operatorname{epi} f\in H\left(Q_j^i\right)$, then there is $x\in A_r^{j,i}$ such that $f(x)\leq d(x)$. To x there is $x_r^{j,i}\left(q\right)$ such that $x\in V\left(x_r^{j,i}\left(q\right)\right)$ and $d_j^i(x_r^{j,i}\left(q\right))=d\left(x\right)$. We distinguish two cases. First let $\frac{1}{\delta}\leq f\left(x_r^{j,i}\left(q\right)\right)-\delta$. We have

$$
f(x)\leq d(x)<d(x_r^{j,i}\left(q\right))+\delta<\frac{1}{\delta}
$$

In the second case we have $f\left(x_r^{j,i}(q)\right) - \delta < \frac{1}{\delta}$. If $f(x) \le d(x)$ and $f\left(x_r^{j,i}(q)\right) \ge h_{G_j^i}(x_r^{j,i}(q))$, then $\operatorname{dist}\left(Q_j^i, \operatorname{bdy}(G_j^i)\right) = \delta$ implies

$$\begin{aligned} f(x) &\le d(x) \\ &= d_j^i(x_r^{j,i}(q)) \\ &\le h_{G_j^i}(x_r^{j,i}(q)) - \delta \\ &\le f\left(x_r^{j,i}(q)\right) - \delta. \end{aligned}$$

Together with (31) it follows from the two cases that

$$P\left(\begin{array}{c} \left(f\left(x_r^{j,i}(q)\right) \ge h_{G_j^i}(x_r^{j,i}(q)),\ q = 1, \ldots, u_r^{j,i},\ r = 1, \ldots, s_{j,i}\right), \\ \operatorname{epi} f \in H\left(Q_j^i\right) \end{array}\right) \le \frac{\varepsilon}{N}$$

and thus

$$P(R) \le N \frac{\varepsilon}{N} = \varepsilon.$$

This leads to

$$\begin{aligned} &\limsup_{n\to\infty} P\left(\operatorname{epi} f_n \notin \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{k_i} H(G_j^i)\right) \\ &\le P\left(\bigcap_{i=1}^{m} \bigcup_{j=1}^{k_i} \operatorname{epi} f \notin H\left(Q_j^i\right)\right) + P(R) \\ &\le P\left(\operatorname{epi} f \notin \bigcup_{i=1}^{m} \bigcap_{j=1}^{k_i} H\left(Q_j^i\right)\right) + \varepsilon \\ &\le P\left(\operatorname{epi} f \notin \bigcup_{i=1}^{m} \bigcap_{j=1}^{k_i} H\left(G_j^i\right)\right) + 2\varepsilon \\ &\le P\left(\operatorname{epi} f \notin \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{k_i} H\left(G_j^i\right)\right) + 3\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ completes the proof. $\square$

The next example shows that in contrast to the case τ_M, where we have seen in Theorem 2.18 that the finite dimensional part of the sufficient condition was also necessary, the condition

$$\begin{aligned}&\limsup_{n\to\infty} P\left(\bigcup_{i=1}^{m}\left(f_n(x_1^i)\geq a_1^i,\ldots,f_n(x_{k_i}^i)\geq a_{k_i}^i\right)\right)\\ \leq &P\left(\bigcup_{i=1}^{m}\left(f(x_1^i)\geq a_1^i,\ldots,f(x_{k_i}^i)\geq a_{k_i}^i\right)\right)\end{aligned}$$

for all $m\in\mathbb{N},\ x_j^i\in\mathbb{R}^p,\ a_j^i\in\mathbb{R},\ j=1,\ldots,k_i,\ i=1,\ldots,m$ from the last theorem is not necessary for $\operatorname{epi} f_n \stackrel{D_{\tau_H}}{\to} \operatorname{epi} f$.

Example 2.23 Let

$$f(x)=\begin{cases}-1 & ,\ x\leq 0\\ 1 & ,\ x>0\end{cases}$$

$$f_n(x)=\begin{cases}-1 & ,\ x\leq -\frac{1}{n}\\ nx & ,\ -\frac{1}{n}<x<\frac{1}{n}\\ 1 & ,\ x\geq \frac{1}{n}\end{cases}.$$

It is easy to see, that $\operatorname{epi} f\subset \liminf_{n\to\infty} \operatorname{epi} f_n$, which implies $\operatorname{epi} f_n \stackrel{D_{\tau_H}}{\to} \operatorname{epi} f$. On the other hand we have

$$\limsup_{n\to\infty} P\left(f_n(0)\geq 0\right)=1>P\left(f(0)\geq 0\right)=0.$$

The following theorem clarifies the relation between the finite dimensional convergence conditions in Theorems 2.19 and 2.22 and the convergence in distribution of the random vectors $(f_n(x_1),\ldots,f_n(x_k))$.

Theorem 2.24 *Let*

(i)

$$\begin{aligned}&\liminf_{n\to\infty} P\left(\bigcup_{i=1}^{m}\left(f_n(x_1^i)> a_1^i,\ldots,f_n(x_{k_i}^i)> a_{k_i}^i\right)\right)\\ \geq\ &P\left(\bigcup_{i=1}^{m}\left(f(x_1^i)> a_1^i,\ldots,f(x_{k_i}^i)> a_{k_i}^i\right)\right)\end{aligned}$$

for all $m\in\mathbb{N},\ x_1^i,\ldots,x_{k_i}^i\in\mathbb{R}^p,\ a_1^i,\ldots,a_{k_i}^i\in\mathbb{R},\ k_i\in\mathbb{N},\ i=1,\ldots,m.$

(ii)

$$\limsup_{n\to\infty} P\left(\bigcup_{i=1}^{m}\left(f_n(x_1^i)\geq a_1^i,\ldots,f_n(x_{k_i}^i)\geq a_{k_i}^i\right)\right)$$
$$\leq P\left(\bigcup_{i=1}^{m}\left(f(x_1^i)\geq a_1^i,\ldots,f(x_{k_i}^i)\geq a_{k_i}^i\right)\right)$$

for all $m\in\mathbb{N},\ x_1^i,\ldots,x_{k_i}^i\in\mathbb{R}^p,\ a_1^i,\ldots,a_{k_i}^i\in\mathbb{R},\ k_i\in\mathbb{N},\ i=1,\ldots,m.$

(iii)

$$(f_n(x_1),\ldots,f_n(x_k)) \xrightarrow{D} (f(x_1),\ldots,f(x_k))$$

for each $k\in\mathbb{N}$ *and all* $x_1,\ldots,x_k\in\mathbb{R}^p$.

Then

$$((i)\wedge(ii))\Leftrightarrow(iii).$$

Proof. First let (i) and (ii) be fulfilled. It suffices that (i) and (ii) hold with $m=1$. The system $\mathcal{A}:=\{[a_1,\infty)\times[a_2,\infty)\times\ldots\times[a_k,\infty):a_1,\ldots,a_k\in\mathbb{R}\}$ is a convergence determining class for convergence in distribution in $\mathbb{R}^k$ (see the considerations that follow Example 2.3. in [5]). It thus suffices to show that

$$\lim_{n\to\infty} P\left((f_n(x_1),\ldots,f_n(x_k))\in A\right)=P\left((f(x_1),\ldots,f(x_k))\in A\right)$$

for all $A\in\mathcal{A}$, with $P\left((f(x_1),\ldots,f(x_k))\in\mathrm{bdy}(A)\right)=0$. Note that for $A=[a_1,\infty)\times\ldots\times[a_k,\infty)\in\mathcal{A}$ we have $z=(z_1,\ldots,z_k)\in\mathrm{bdy}(A)$ if and only if $z_i=a_i$ for at least one $i\in\{1,\ldots,k\}$. Thus $P\left((f(x_1),\ldots,f(x_k))\in\mathrm{bdy}(A)\right)=0$ implies

$$P\left(f(x_1)\geq a_1,\ldots,f(x_k)\geq a_k\right)=P\left(f(x_1)>a_1,\ldots,f(x_k)>a_k\right).$$

With (i) and (ii) we obtain

$$\begin{aligned}
P\left((f(x_1),\ldots,f(x_k))\in A\right) &= P\left(f(x_1)>a_1,\ldots,f(x_k)>a_k\right)\\
&\le \liminf_{n\to\infty} P\left(f_n(x_1)>a_1,\ldots,f_n(x_k)>a_k\right)\\
&\le \liminf_{n\to\infty} P\left(f_n(x_1)\ge a_1,\ldots,f_n(x_k)\ge a_k\right)\\
&\le \limsup_{n\to\infty} P\left(f_n(x_1)\ge a_1,\ldots,f_n(x_k)\ge a_k\right)\\
&\le P\left(f(x_1)\ge a_1,\ldots,f(x_k)\ge a_k\right)\\
&= P\left((f(x_1),\ldots,f(x_k))\in A\right),
\end{aligned}$$

which implies that equality holds in the above chain of inequalities. It follows that

$$\begin{aligned}
&\lim_{n\to\infty} P\left((f_n(x_1),\ldots,f_n(x_k))\in A\right)\\
&=\liminf_{n\to\infty} P\left(f_n(x_1)\ge a_1,\ldots,f_n(x_k)\ge a_k\right)\\
&= P\left((f(x_1),\ldots,f(x_k))\in A\right).
\end{aligned}$$

Now assume that (iii) holds. Let

$$\begin{aligned}
U = &\left((a_1^1,\infty)\times\ldots\times(a_{k_1}^1,\infty)\right)\times\mathbb{R}^{k_2}\times\ldots\times\mathbb{R}^{k_m}\\
&\cup\left(\mathbb{R}^{k_1}\times\left((a_1^2,\infty)\times\ldots\times(a_{k_2}^2,\infty)\right)\times\mathbb{R}^{k_3}\times\ldots\times\mathbb{R}^{k_m}\right)\\
&\cup\ldots\cup\left(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}\times\ldots\times\mathbb{R}^{k_{m-1}}\times\left((a_1^m,\infty)\times\ldots\times(a_{k_m}^m,\infty)\right)\right).
\end{aligned}$$

Then U is an open subset of $\mathbb{R}^{k_1+\ldots+k_m}$ and we have

$$\begin{aligned}
&\left(f_n(x_1^1),\ldots,f_n(x_{k_1}^1),\ldots,f_n(x_1^m),\ldots,f_n(x_{k_m}^m)\right)\in U\\
&\Leftrightarrow \left(\left(f_n(x_1^1)>a_1^1,\ldots,f_n(x_{k_1}^1)>a_{k_1}^1\right)\vee\ldots\vee\left(f_n(x_1^m)>a_1^m,\ldots,f_n(x_{k_m}^m)>a_{k_m}^m\right)\right)
\end{aligned}$$

and thus with (iii) it follows that

$$\begin{aligned}
&\liminf_{n\to\infty} P\left(\bigcup_{i=1}^{m}\left(f_n(x_1^i) > a_1^i,\ldots,f_n(x_{k_i}^i) > a_{k_i}^i\right)\right)\\
&= \liminf_{n\to\infty} P\left(\left(f_n(x_1^1),\ldots,f_n(x_{k_1}^1),\ldots,f_n(x_1^m),\ldots,f_n(x_{k_m}^m)\right) \in U\right)\\
&\geq P\left(\left(f(x_1^1),\ldots,f(x_{k_1}^1),\ldots,f(x_1^m),\ldots,f(x_{k_m}^m)\right) \in U\right)\\
&= P\left(\bigcup_{i=1}^{m}\left(f(x_1^i) > a_1^i,\ldots,f(x_{k_i}^i) > a_{k_i}^i\right)\right),
\end{aligned}$$

i.e. (i) is true.
Let

$$\begin{aligned}
B = &\left([a_1^1,\infty)\times\ldots\times[a_{k_1}^1,\infty)\right)\times\mathbb{R}^{k_2}\times\ldots\times\mathbb{R}^{k_m}\\
&\cup\left(\mathbb{R}^{k_1}\times\left([a_1^2,\infty)\times\ldots\times[a_{k_2}^2,\infty)\right)\times\mathbb{R}^{k_3}\times\ldots\times\mathbb{R}^{k_m}\right)\\
&\cup\ldots\cup\left(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}\times\ldots\times\mathbb{R}^{k_{m-1}}\times\left([a_1^m,\infty)\times\ldots\times[a_{k_m}^m,\infty)\right)\right).
\end{aligned}$$

Then B is a closed subset of $\mathbb{R}^{k_1+\ldots+k_m}$ and we have

$$\begin{aligned}
&\left(f_n(x_1^1),\ldots,f_n(x_{k_1}^1),\ldots,f_n(x_1^m),\ldots,f_n(x_{k_m}^m)\right)\in B\\
&\Leftrightarrow \left(\left(f_n(x_1^1)\geq a_1^1,\ldots,f_n(x_{k_1}^1)\geq a_{k_1}^1\right)\vee\ldots\vee\left(f_n(x_1^m)\geq a_1^m,\ldots,f_n(x_{k_m}^m)\geq a_{k_m}^m\right)\right)
\end{aligned}$$

and with (iii) it follows that

$$\begin{aligned}
&\limsup_{n\to\infty} P\left(\bigcup_{i=1}^{m}\left(f_n(x_1^i) \geq a_1^i,\ldots,f_n(x_{k_i}^i) \geq a_{k_i}^i\right)\right)\\
&= \limsup_{n\to\infty} P\left(\left(f_n(x_1^1),\ldots,f_n(x_{k_1}^1),\ldots,f_n(x_1^m),\ldots,f_n(x_{k_m}^m)\right) \in B\right)\\
&\geq P\left(\left(f(x_1^1),\ldots,f(x_{k_1}^1),\ldots,f(x_1^m),\ldots,f(x_{k_m}^m)\right) \in B\right)\\
&= P\left(\bigcup_{i=1}^{m}\left(f(x_1^i) \geq a_1^i,\ldots,f(x_{k_i}^i) \geq a_{k_i}^i\right)\right),
\end{aligned}$$

i.e. (ii) holds. □

We are now able to proof a corrected version of Theorem 2 of [19].

Theorem 2.25 *Let $(f_n)_n$, f be random lsc-functions. Let $(f_n)_n$ be stochastically equi-lower semicontinuous. Let f be stochastically uniformly lower semicontinuous, then*

$$\operatorname{epi} f_n \stackrel{D_{\tau_{Fell}}}{\to} \operatorname{epi} f$$

if and only if

$$(f_n(x_1), \ldots, f_n(x_k)) \stackrel{D}{\to} (f(x_1), \ldots, f(x_k))$$

for all $k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathbb{R}^p$.

Proof. First let $\operatorname{epi} f_n \stackrel{D_{\tau_{\text{Fell}}}}{\to} \operatorname{epi} f$. To show convergence in distribution of the finite dimensional sections, we can adopt Knight's proof without modifications.
Now assume convergence in distribution of the finite dimensional sections. With Theorems 2.19 and 2.22 it follows that $\operatorname{epi} f_n \stackrel{D_{\tau_M}}{\to} \operatorname{epi} f$ and $\operatorname{epi} f_n \stackrel{D_{\tau_H}}{\to} \operatorname{epi} f$, thus by Theorem 1.24 it follows that $\operatorname{epi} f_n \stackrel{D_{\tau_{\text{Fell}}}}{\to} \operatorname{epi} f$. □

It remains to investigate the necessity of stochastic equi-lower semicontinuity. Let $(f_n)_n$ be a sequence of random lower semicontinuous functions, which is not stochastically equi-lower semicontinuous. We can assume that $f_n(x) \geq 0$ for all x and all $n \in \mathbb{N}$. Let $f(x) = 0$ for all x, then f is uniformly lower semicontinuous. From $\operatorname{epi} f_n \subset \operatorname{epi} f$ it follows that $\operatorname{epi} f_n \stackrel{D_{\tau_M}}{\to} \operatorname{epi} f$. It is easily verified, that the finite dimensional convergence property from Theorem 2.19 is fulfilled. This example shows that equi-lower semicontinuity is not necessary for τ_M convergence in distribution, in presence of our finite dimensional convergence condition. By assuming that $f_n(x) \leq 0$ for all x and all $n \in \mathbb{N}$ we obtain the analogous result for τ_H convergence in distribution. The situation is different in the case of convergence in distribution with respect to τ_{Fell}. To close the gap in Theorem 3 of [19], we additionally impose the stochastic uniform-lower semicontinuity condition on f and obtain

Theorem 2.26 *Let $(f_n)_n$ be a sequence of random lower semicontinuous functions. Let f be a random lower semicontinuous function, fulfilling the stochastic uniform-lower semicontinuity condition.*
If

$$(f_n(x_1), \ldots, f_n(x_k)) \xrightarrow{D} (f(x_1), \ldots, f_n(x_k))$$

and

$$\operatorname{epi} f_n \overset{D_{\tau_{Fell}}}{\to} \operatorname{epi} f,$$

then $(f_n)_n$ is stochastically equi-lower semicontinuous.

We can directly use the proof of Knight's Theorem 3. In this proof a stochastic uniform-lower semicontinuity property of f is taken for granted. This is not justified, since we have already encountered a stochastic process which is not stochastically uniformly-lower semicontinuous in Counterexample 2.16.

We will now show that the standard Brownian Motion on $[0, 1]$ is an example for a stochastic uniform-lower semicontinuous process. First we prove the following one-sided version of Lemma 7 from Chapter 5 of [30]. This lemma is helpful for investigating the behaviour of stochastic processes on an interval, when the behaviour in the endpoints is known.

Lemma 2.27 *Let $X(t),\ t \in [0, b]$ be a stochastic process with sample paths, which are left– or rightcontinuous in each t. Let $X(0) = 0$ and assume that there is a family of increasing sigma–fields $\mathcal{E}_t,\ t \in [0, b]$ such that $X(t)$ is $\mathcal{E}_t$–measurable.*
Let $\delta > 0$. If there is $\beta > 0$ depending only on δ such that

$$P\left(X(b) \leq \frac{1}{2} X(t) \middle| \mathcal{E}_t \right)(\omega) \geq \beta$$

for all $\omega \in \{X(t) < -\delta\}$,
then

$$P\left(\inf_{t \in [0,b]} X(t) < -\delta \right) \leq \frac{1}{\beta} P\left(X(b) < -\frac{1}{2}\delta \right).$$

Proof. The proof follows the ideas of Pollard's proof. Let S be a finite subset of $[0, b]$ with $0, b \in S$. If we let S approach a dense subset of $[0, b]$, we obtain $\min_{t \in S} X(t) \to \inf_{t \in [0,b]} X(t)$. This follows from the one-sided continuity assumptions on $X(.)$. Let τ denote the smallest point of S for which $X(\tau) < -\delta$ if such a point exists.

Otherwise let $\tau = \infty$. The event $\{\tau = t\}$ belongs to $\mathcal{E}_t$ and we have $\{\tau = t\} \cap \{\tau = s\} = \emptyset$ for $t \neq s$.
Now

$$
\begin{aligned}
P\left(\min_{t\in S} X(t) < -\delta\right) =& \sum_{t\in S} P\left(\tau = t\right) \\
=& \sum_{t\in S} \int 1_{(\tau=t)}(\omega) dP(\omega) \\
\leq& \frac{1}{\beta} \sum_{t\in S} \int 1_{(\tau=t)}(\omega) P\left(X(b) \leq \frac{1}{2} X(t) \middle| \mathcal{E}_t \right)(\omega) dP(\omega) \\
=& \frac{1}{\beta} \sum_{t\in S} P\left(\tau = t, X(b) \leq \frac{1}{2} X(t)\right) \\
\leq& \frac{1}{\beta} \sum_{t\in S} P\left(\tau = t, X(b) \leq -\frac{1}{2}\delta\right) \\
\leq& \frac{1}{\beta} P\left(X(b) \leq -\frac{\delta}{2}\right)
\end{aligned}
$$

□

Note that $\inf_{t\in S} X(t)$ can be replaced by $\min_{t\in S} X(t)$, if $X(.)$ has lower semicontinuous trajectories. We now show that the standard Brownian motion W on $[0, 1]$ fulfills the conditions of the above lemma. Let $\mathcal{E}_t = \sigma\{W_s : s \leq t\}$. For $\delta > 0$ let $\omega \in \{W(t) < -\delta\}$, then $-\frac{1}{2}W(t) > \frac{\delta}{2}$ for this ω. We obtain

$$
\begin{aligned}
P\left(W(b) \leq \frac{1}{2} W(t) \middle| \mathcal{E}_t \right)(\omega) =& P\left(W(b) - W(t) \leq -\frac{1}{2} W(t) \middle| \mathcal{E}_t \right)(\omega) \\
\geq& P\left(W(b) - W(t) \leq \frac{\delta}{2} \middle| \mathcal{E}_t \right)(\omega) \\
=& P\left(W(b) - W(t) \leq \frac{\delta}{2} \right)
\end{aligned}
$$

where we have used that the increment $W(b) - W(t)$ is independent of $\mathcal{E}_t$.
Since $W(b) - W(t)$ has a normal distribution with mean 0 and since $\delta > 0$ we have

$$
P\left(W(b) \leq \frac{1}{2} W(t) \,|\mathcal{E}_t \right)(\omega) \geq \frac{1}{2}
$$

and can thus choose $\beta = \frac{1}{2}$.

To make use of Lemma 2.27 we have to know $P\left(W(b) < -\frac{\delta}{2}\right)$. We will calculate this probability and choose a suitable $b > 0$. The choice of b will determine the number of neighbourhoods that are needed in Definition 2.21 to cover the interval $[0, 1]$. Let $\delta > 0$. Because of $W(b) \sim N(0, b)$ we have

$$\begin{aligned} P\left(W(b) < -\frac{\delta}{2}\right) &= P\left(\frac{W(b)}{\sqrt{b}} < -\frac{\delta}{2\sqrt{b}}\right) \\ &= P\left(\frac{W(b)}{\sqrt{b}} > \frac{\delta}{2\sqrt{b}}\right) \\ &\leq \frac{1}{\sqrt{2\pi}} \frac{2\sqrt{b}}{\delta} \exp\left(-\frac{\delta^2}{4\pi b}\right) \end{aligned}$$

For $\varepsilon > 0$ and large $k \in \mathbb{N}$ we want to bound this probability by $\frac{\varepsilon}{k^2}$ and choose

$$b = \frac{\pi\delta^2}{2k^2}$$

to obtain

$$P\left(W(b) < -\frac{\delta}{2}\right) \leq \frac{1}{k} \exp\left(-\frac{k^2}{2\pi^2}\right).$$

Because of the properties of the exponential function the second factor is less than $\frac{\varepsilon}{k}$ for all $k \geq k_0$. Thus we have

$$P\left(W(b) < -\frac{\delta}{2}\right) \leq \frac{\varepsilon}{k^2}.$$

Note that we require ck^2 intervals of length b to cover $[0, 1]$. Here c is a constant that only depends on δ.

Now we consider the interval $[0, 2b]$, then

$$\begin{aligned} &P\left(\min_{t\in[0,2b]} W(t) < W(b) - \delta\right) \\ \leq &P\left(\min_{t\in[0,b]} W(t) < W(b) - \delta\right) + P\left(\min_{t\in[b,2b]} W(t) < W(b) - \delta\right) \\ = &P\left(\min_{t\in[0,b]} W(t) - W(b) < -\delta\right) + P\left(\min_{t\in[b,2b]} W(t) - W(b) < -\delta\right). \end{aligned}$$

To investigate the second term we set $V(s) = W(b + s) - W(b)$, then V is a standard

Brownian motion on $[0,1]$. We apply Lemma 2.27 to V and receive

$$\begin{aligned} P\left(\min_{t\in[b,2b]} W(t)-W(b) < -\delta\right) =& P\left(\min_{s\in[0,b]} V(S) < -\delta\right) \\ \leq& \frac{1}{\beta} P\left(V(b) \leq -\frac{\delta}{2}\right) \\ \leq& \frac{\varepsilon}{\beta k^2} \end{aligned}$$

To deal with the first term, let $Z(s) = W(b-s) - W(b)$ to obtain a stochastic process which is equivalent to W. The application of Lemma 2.27 to Z yields

$$\begin{aligned} P\left(\min_{t\in[0,b]} W(t)-W(b) < -\delta\right) =& P\left(\min_{s\in[0,b]} Z(s) < -\delta\right) \\ \leq& \frac{1}{\beta} P\left(Z(b) \leq -\frac{\delta}{2}\right) \\ =& \frac{1}{\beta} P\left(W(0) - W(b) \leq -\frac{\delta}{2}\right) \end{aligned}$$

and because of $W(0) \overset{a.s.}{=} 0$ and since $-W(b)$ has a $N(0,b)$ distribution we obtain

$$P\left(\min_{t\in[0,b]} W(t)-W(b) < -\delta\right) \leq \frac{1}{\beta} P\left(W(b) \leq -\frac{\delta}{2}\right) \leq \frac{\varepsilon}{\beta k^2}.$$

Consequently we have

$$P\left(\min_{t\in[0,2b]} W(t) < W(b) - \delta\right) \leq \frac{2\varepsilon}{\beta k^2}$$

Now we cover the interval $[0,1]$ with at most ck^2 intervals of length $2b$. Let t_i be the middle point of the $i-$th interval I_i, then from the case $[0,2b]$ we receive by using the homogenity of the Brownian motion

$$
\begin{aligned}
&P\left(\bigcup_{i=1}^{ck^2}\left(\min_{t\in I_i} W(t) \le W(t_i)-\delta\right)\right)\\
\le&\sum_{i=1}^{ck^2} P\left(\min_{t\in I_i} W(t) \le W(t_i)-\delta\right)\\
\le&ck^2\frac{2\varepsilon}{\beta k^2}\\
=&\frac{2c}{\beta}\varepsilon.
\end{aligned}
$$

Since the constants c and β depend only on δ this suffices for stochastic uniform-lower semicontinuity of W.
The application of Lemma 2.27 is by no means restricted to the Browniam motion. The key was to calculate the probility in the assumptions of 2.27 and to calculate the number of sets needed to cover the intervall $[0,1]$. From the relation between these quantities we derived stochastic uniform-lower semicontinuity. If we can achieve this relation uniformly for a sequence of stochastic processes, we can use this method to show stochastic equi-lower semicontinuity. This can for example be done to show equi-lower semicontinuity of the sequence of normalized symmetrical random walks with lower semicontinuous paths on $[0,1]$. We are then able to combine the sufficient conditions from this section with the stability results from Chapter 3 to give new proofs for results about minima and argmins of Brownian Motion. We will not carry out the calculations for the random walks, since in the next section we achieve the same results in a more convenient way.

2.3. Skorohod Convergence in $D[0,\infty)$

When working with functions that, like lower semicontinuous functions, only have jump discontinuities, immediately the space of cadlag functions comes to mind. The acronym cadlag stands for continu à droite, limites à gauche, which means that the functions are continuous from the right and possess limits from the left in each point.
As we will do here, often the notation D is used for the space of cadlag functions. We will deal with the space $D[0,\infty)$ of all cadlag functions on $[0,\infty)$. For $x=0$ only right continuity is demanded.
Each $f \in D[0,\infty)$ has only jump discontinuities and thus allows a natural lower semicontinuous modification $\widetilde{f}$.

Let

$$\begin{aligned}\widetilde{f}(x) &= \min\left(f(x-), f(x+)\right)\\ &= \min\left(f(x-), f(x)\right)\end{aligned}$$

for all $x \in (0, \infty)$ and $\widetilde{f}(0) = f(0)$.

In this section we will investigate the relationship between convergence in distribution of a sequence $(f_n)_n \subset D[0,\infty)$ and convergence in distribution of $\left(\operatorname{epi}\widetilde{f}_n\right)_n$ with respect to τ_{Fell}. It should be noted, that parallel to this work Vogel in [44] has obtained similar results with other methods.

First we recall the deterministic versions of convergence in $D[0,\infty)$. It is shown in [30] that the topology of uniform convergence (on compact intervalls) is not very well suited for dealing with convergence in distribution of random $D[0,\infty)$ functions. Skorohod has developed a new metric on $D[0,\infty)$, under which $D[0,\infty)$ is a complete space, see [15] , [5] and [30] for details. We will denote the topology, which is generated by the Skorohod metric on $D[0,\infty)$, as τ_S.

The following criterion for τ_S-convergence will frequently be used in our proofs. It can for example be found in [30]. Let Λ_∞ be the space of all continuous, strictly increasing functions from $[0,\infty)$ onto $[0,\infty)$. Then $f_n \stackrel{\tau_S}{\to} f$ if and only if there is $(\lambda_n)_n \subset \Lambda_\infty$ such that for each $\varepsilon > 0$ and each $m \geq 0$ there is n_0 such that for all $n \geq n_0$

$$|\lambda_n(x) - x| \leq \varepsilon, \text{ for all } x \in [0,\infty)$$

and

$$|f_n(\lambda_n(x)) - f(x)| \leq \varepsilon, \text{ for all } x \in [0,m).$$

In the deterministic case we have

Theorem 2.28 *Let* $(f_n)_n \subset D[0,\infty),\ f \in D[0,\infty)$.
If

$$f_n \stackrel{\tau_S}{\to} f$$

then

$$\operatorname{epi}\widetilde{f}_n \stackrel{\tau_{Fell}}{\to} \operatorname{epi}\widetilde{f}.$$

We will prove this theorem by showing in the following lemmas that $\operatorname{epi}\widetilde{f}_n \stackrel{\tau_{\mathrm{M}}}{\to} \operatorname{epi}\widetilde{f}$ and $\operatorname{epi}\widetilde{f}_n \stackrel{\tau_{\mathrm{H}}}{\to} \operatorname{epi}\widetilde{f}$ follow from $f_n \stackrel{\tau_S}{\to} f$.

Lemma 2.29 *Let $f_n, f \in D[0,\infty)$, and let $f_n \overset{\tau_S}{\to} f$. Then* $\operatorname{epi} \widetilde{f}_n \overset{\tau_M}{\to} \operatorname{epi} \widetilde{f}$.

Proof. Assume that $\operatorname{epi} \widetilde{f}_n \overset{\tau_M}{\not\to} \operatorname{epi} \widetilde{f}$. Then there is a sequence $(x_n)_n \subset [0,\infty)$ converging to some $x \in [0,\infty)$ such that

$$\liminf_{n\to\infty} \widetilde{f}_n(x_n) < \widetilde{f}(x).$$

This implies that there is $\alpha > 0$ and a subsequence $\left(\widetilde{f}_{n_k}(x_{n_k})\right)_k$ with

$$\widetilde{f}_{n_k}(x_{n_k}) < \widetilde{f}(x) - 3\alpha$$

for all $k \in \mathbb{N}$.
Clearly x has to lie in $[0,m]$ for some $m > x$.
Let $(\lambda_n)_n \subset \Lambda_\infty$ be the sequence obtained from $f_n \overset{\tau_S}{\to} f$. Since the continuity points of each f_n are dense in $[0,m]$ and since $f_n(x_n-)$ and $f_n(x_n+)$ exist for each $x_n \in (0,m)$, we can find a sequence $(w_n)_n \subset (0,m)$ such that

$$\begin{aligned}
&|x_n - w_n| \to 0\\
&\left|f_n(w_n) - \widetilde{f}_n(x_n)\right| \le \alpha, \text{ for } n \ge n_0
\end{aligned}$$

and f_n is continuous in w_n.
In the case $x_n = 0$ we can choose $w_n = x_n$, since f_n is continuous in 0 and thus $\widetilde{f}_n(x_n) = f_n(x_n)$.
Let $z_{n_k} = \lambda_{n_k}^{-1}(w_{n_k})$, then

$$\begin{aligned}
f_{n_k}(\lambda_{n_k}(z_{n_k})) = f_{n_k}(w_{n_k}) = \widetilde{f}_{n_k}(w_{n_k}) \le \widetilde{f}_{n_k}(x_{n_k}) + \alpha < \widetilde{f}(x) - 2\alpha\\
\le \min\left(f(x-) - 2\alpha, f(x+) - 2\alpha\right).
\end{aligned}$$

From

$$\begin{aligned}
|x - z_{n_k}| &\le |x - \lambda_{n_k}(z_{n_k})| + |\lambda_{n_k}(z_{n_k}) - z_{n_k}|\\
&= |x - w_{n_k}| + |\lambda_{n_k}(z_{n_k}) - z_{n_k}|\\
&\le |x - x_{n_k}| + |x_{n_k} - w_{n_k}| + |\lambda_{n_k}(z_{n_k}) - z_{n_k}|
\end{aligned}$$

it follows that $z_{n_k} \to x$. Thus a subsequence of z_{n_k} converges to x from below or from above. In the first case we assume without loss of generality, that $z_{n_k} \le x$. Then

$f(z_{n_k}) \to f(x-)$, i.e. there is k_0 such that $f(z_{n_k}) > f(x-) - \alpha$ for all $k \geq k_0$. It follows that

$$f_{n_k}(\lambda_{n_k}(z_{n_k})) \leq f(x-) - 2\alpha < f(z_{n_k}) + \alpha - 2\alpha = f(z_{n_k}) - \alpha$$

and thus

$$f_{n_k}(\lambda_{n_k}(z_{n_k})) - f(z_{n_k}) < -\alpha$$

for all $k \in \mathbb{N}$.
In the second case we assume without loss of generality that $m > z_{n_k} \geq x$ for all $k \in \mathbb{N}$. Now $f(z_{n_k}) \to f(x+)$ and thus $f(z_{n_k}) > f(x+) - \alpha$ for all $k \geq k_0$. We obtain

$$f_{n_k}(\lambda_{n_k}(z_{n_k})) \leq f(x+) - 2\alpha < f(z_{n_k}) + \alpha - 2\alpha = f(z_{n_k}) - \alpha$$

which again yields

$$f_{n_k}(\lambda_{n_k}(z_{n_k})) - f(z_{n_k}) < -\alpha$$

for all $k \in \mathbb{N}$.
Now from $z_{n_k} \to x$ and from the uniform convergence of $(\lambda_n)_n$ to the identity mapping it follows that there is $m \in \mathbb{R}$ such that $x \in [0, m)$ and $z_{n_k},\ \lambda(z_{n_k}) \in [0, m)$ for all $k \geq k_0$. It follows that $f_{n_k}(\lambda_{n_k}(z_{n_k})) - f(z_{n_k}) < -\alpha$ for all $k \in \mathbb{N}$ yields a contradiction to $f_n(\lambda_n(z)) \to f(z)$ uniformly for all $z \in [0, m)$. $\square$

Lemma 2.30 *Let $f_n, f \in D[0,\infty)$, and let $f_n \stackrel{\tau_S}{\to} f$. Then* $\operatorname{epi} \widetilde{f_n} \stackrel{\tau_H}{\to} \operatorname{epi} \widetilde{f}$.

Proof. We have to show that for each $(x, y) \in \operatorname{epi} \widetilde{f}$ there is a sequence (x_n, y_n) with $(x_n, y_n) \to (x, y)$ and $(x_n, y_n) \in \operatorname{epi} \widetilde{f_n}$. First we deal with the case $(x, y) = (x, \widetilde{f}(x))$. We distinguish two subcases. In the first case let $f(x+) \leq f(x-)$, i.e. $\widetilde{f}(x) = f(x+) = f(x)$, because of right continuity. From $f_n \stackrel{\tau_S}{\to} f$ it follows that there is a sequence $(\lambda_n)_n \subset \Lambda_\infty$ with

$$\lambda_n(x) \to x$$

uniformly for all $x \in [0, \infty)$
and

$$f_n(\lambda_n(x)) \to f(x)$$

uniformly for all $x \in [0, m)$ and for all $m \in \mathbb{R}$.
We choose $(x_n, y_n) = (\lambda_n(x), f_n(\lambda_n(x)))$, then
$(x_n, y_n) \to (x, f(x)) = (x, \widetilde{f}(x))$ and because of

$$f_n(\lambda_n(x)) = f_n(\lambda_n(x)+) \geq \min(f(\lambda_n(x)-), f(\lambda_n(x)+)) = \widetilde{f}(\lambda_n(x))$$

we have $(x_n, y_n) \in \operatorname{epi} \widetilde{f}_n$.
In the second case let $f(x-) < f(x+)$, then $\widetilde{f}(x) = f(x-)$. We choose a sequence $(w_n)_n$ with $w_n \leq x$, $w_n \to x$. By existence of left limits we have $f(w_n) \to f(x-)$ and from $f_n \overset{\tau_S}{\to} f$ for each $\varepsilon > 0$ we obtain n_0 such that $|f_n(\lambda_n(w_n)) - f(w_n)| \leq \varepsilon$ for $n \geq n_0$. It follows that

$$\left| f_n(\lambda_n(w_n)) - \widetilde{f}(x) \right| \leq |f_n(\lambda_n(w_n)) - f(w_n)| + \left| f(w_n) - \widetilde{f}(x) \right| \leq 2\varepsilon$$

for $n \geq n_0$. Let $(x_n, y_n) = (\lambda_n(w_n), f_n(\lambda_n(w_n)))$, then $(x_n, y_n) \to \left(x, \widetilde{f}(x)\right)$ and from $f_n(\lambda_n(w_n)) = f_n(\lambda_n(w_n)+) \geq \min\left(f_n(\lambda_n(w_n)-), f_n(\lambda_n(w_n)+)\right) = \widetilde{f}_n\left(\lambda_n(w_n)\right)$ it follows that $(x_n, y_n) \in \operatorname{epi} \widetilde{f}_n$.
We still have to deal with the case $(x, y) \neq \left(x, \widetilde{f}\,(x)\right)$. In this case $y > \widetilde{f}(x)$, because of $(x, y) \in \operatorname{epi} \widetilde{f}$. Let $\delta = y - \widetilde{f}\,(x)$. With the above we can find $(x_n, \widetilde{y}_n) \in \operatorname{epi} \widetilde{f}_n$ such that $(x_n, \widetilde{y}_n) \to \left(x, \widetilde{f}\,(x)\right)$. Let $y_n = \widetilde{y}_n + \delta$, then $(x_n, y_n) \in \operatorname{epi} \widetilde{f}_n$ and
$(x_n, y_n) \to \left(x, \widetilde{f}\,(x) + \delta\right) = (x, y)$. □

Proof. (of Theorem 2.28) It only remains to observe that $\operatorname{epi} \widetilde{f}_n \overset{\tau_M}{\to} \operatorname{epi} \widetilde{f}$ together with $\operatorname{epi} \widetilde{f}_n \overset{\tau_H}{\to} \operatorname{epi} \widetilde{f}$ implies $\operatorname{epi} \widetilde{f}_n \overset{\tau_{\text{Fell}}}{\to} \operatorname{epi} \widetilde{f}$. □

The following example shows that even in the case of equi-lower semicontinuous $f_n, f \in D[0, \infty)$, convergence of the epigraphs in the Fell topology does in general not imply $f_n \overset{\tau_S}{\to} f$.

Example 2.31 Let

$$f_n(x) = \begin{cases} 1 & , \ 0 \leq x \leq 1 - \frac{1}{n} \\ nx + 2 - n & , \ 1 - \frac{1}{n} < x < 1 \\ 0 & , \ x \geq 1 \end{cases}$$

and

$$f(x) = \begin{cases} 1 & , \ 0 \leq \ x < 1 \\ 0 & , \ x \geq 1 \end{cases} .$$

Then f_n and f are lower semicontinuous and belong to $D[0, \infty)$. We have $\operatorname{epi} f_n \overset{\tau_{\text{Fell}}}{\to} \operatorname{epi} f$. Let $(x_{n_k}, y_{n_k}) \in \operatorname{epi} f_{n_k}$ and $(x_{n_k}, y_{n_k}) \to (x, y)$. If $x < 1$, then $x_{n_k} < 1$ for all $k \geq k_0$. It follows that $y_{n_k} \geq f_{n_k}(x_{n_k}) \geq 1$ and thus $y \geq 1$, which implies $(x, y) \in \operatorname{epi} f$. In the second case let $x \geq 1$. Since $y_{n_k} \geq 0$ for all k it follows that $y \geq 0$, which yields $(x, y) \in \operatorname{epi} f$.

Now let $(x, y) \in \operatorname{epi} f$. If $x \geq 1$ we can choose $(x_n, y_n) = (x, y) \in \operatorname{epi} f_n$. For $x < 1$ we have $y \geq 1$. Choose $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset (0,1)$. Then there is n_0 with $f_n(z) = 1$ for all $z \in (x-\varepsilon, x+\varepsilon)$ and all $n \geq n_0$. We can choose a sequence $(x_n)_n \subset (x-\varepsilon, x+\varepsilon)$ with $x_n \to x$. Let $(x_n, y_n) = (x_n, y) \in \operatorname{epi} f_n$.
On the other hand, f_n does not converge to f in the Skorohod topology: Let $(\lambda_n)_n \subset \Lambda_\infty$ be an arbitrary sequence with $\lambda_n(x) \to x$ uniformly for all $x \in [0, \infty)$.
Let $z_n = 1 - \frac{1}{2n}$, then $f_n(z_n) = \frac{3}{2}$. With $w_n = \lambda_n^{-1}(z_n)$ we have

$$|f_n(\lambda_n(w_n)) - f(w_n)| = |f_n(z_n) - f(w_n)| = \left|\frac{3}{2} - f(w_n)\right|.$$

Since $f(w_n)$ either takes the value 1 or the value 0 it follows that

$$|f_n(\lambda_n(w_n)) - f(w_n)| \geq \frac{1}{2} \tag{2.3}$$

for all n.
Because of $\lambda_n(x) \to x$ uniformly for all $x \in [0, \infty)$ and $z_n \in [0,1]$ for all n we have $w_n \in [0,2]$ for all $n \geq n_0$.
Now (2.3) shows that for given $0 < \varepsilon < \frac{1}{2}$ there is no n_0 such that

$$|f_n(\lambda_n(z)) - f(z)| \leq \varepsilon, \text{ for all } n \geq n_0 \text{ and all } z \in [0,2].$$

It follows that $f_n \overset{\tau_S}{\not\to} f$.
Note that the sequence $(f_n)_n$ is even equi-lower semicontinuous and that f is uniformly lower semicontinuous on $[0, \infty)$. Let $\delta > 0$. Let $x \in [0,1)$, then there is n_0 with $f_n(x) = 0$ for all $n \geq n_0$. For each $n < n_0$ there is $\eta_n > 0$ such that $f_n(z) \geq f_n(x) - \delta$ for all $z \in [0, \infty)$ with $|z - x| \leq \eta_n$. Now choose $\eta := \min\{\eta_n : n < n_0\}$ to obtain $f_n(z) \geq f_n(x) - \delta$ for all $n \in \mathbb{N}$ and all $z \in [0, \infty)$ with $|z - x| \leq \eta$.
In the case $x \in [1, \infty)$ we can choose $\eta = 1$. Then $f_n(z) \geq 0 \geq f_n(x) - \delta$ for all $n \in \mathbb{N}$ and all $z \in [0, \infty)$ with $|z - x| \leq \eta$.
To show uniform lower semicontinuity let $x_1 = 1$ and $V(x_1) = [0, \infty)$, then $f(x) > f(x_1) - \delta$ for all $x \in V(x_1)$.

As the main result of this section we show that for epi convergence in distribution of the lower semicontinuous modifications of random $D[0, \infty)$ functions we have convergence in distribution in the sense of Skorohod as a sufficient condition.

Theorem 2.32 *Let $(f_n)_n$ be a sequence of random functions in $D[0,\infty)$, let f be a random $D[0,\infty)$ function, then*

$$f_n \overset{D_{\tau_S}}{\to} f$$

implies

$$\operatorname{epi} \widetilde{f}_n \overset{D_{\tau_{Fell}}}{\to} \operatorname{epi} \widetilde{f}.$$

Proof. In Theorem 2.28 we have shown that the mapping

$$T : (D[0,\infty), \tau_S) \to (\mathrm{EPI}([0,\infty)), \tau_{\mathrm{Fell}})\,,\ T\,(f) = \operatorname{epi} \widetilde{f}$$

is continuous. It remains to apply the continuous mapping theorem. □

Remark 2.33 All of the above results can be transferred to the frequently investigated special case $D\,[0,1]$. In the proof of Theorem 2.28 special care has to be taken for the point 1.

As an application we obtain a new proof for the epi-convergence in distribution of normalized lower semicontinuous random walks to the standard Brownian Motion on $[0,1]$. Note that the Brownian Motion has almost surely continuous trajectories.
Let $\xi_1, \xi_2, \ldots$ be an iid sequence of random variables with mean μ and variance $\sigma^2 > 0$. Let $S_n = \xi_1 + \ldots + \xi_n, S_0 = 0$, we define the random walk with lower semicontinuos trajectories as

$$\begin{aligned}
\widetilde{X}_t^n &= \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor},\ \text{for } t \notin \left\{\frac{k}{n} : k = 1, \ldots, n\right\} \\
\widetilde{X}_t^n &= \min\left(\widetilde{X}_{t-}^n,\ \widetilde{X}_{t+}^n\right),\ \text{for } t \in \left\{\frac{k}{n} : k = 1, \ldots, n-1\right\} \\
\widetilde{X}_1^n &= \widetilde{X}_{1-}^n.
\end{aligned}$$

Theorem 2.34 *Let $\widetilde{X}_t^n$ be as defined above, let W denote the standard Brownian Motion on $[0,1]$, then*

$$\operatorname{epi} \widetilde{X}_{\cdot}^n \overset{D_{\tau_{Fell}}}{\to} \operatorname{epi} W.$$

Proof. We note that $\widetilde{X}_t^n$ is the lower semicontinuous modification of the random walk

$$X_t^n = \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor},\ t \in [0,1]$$

which has trajectories in $D[0,1]$. It is shown in Theorem 14.1 of [5], that $X^n_{\cdot} \overset{D_{\tau_S}}{\to} W$. The assertion now follows with Theorem 2.32 . $\square$

2.4. Dependence on Parameters

In this section we assume that there is a function $g : \mathbb{R}^p \times B \to \overline{\mathbb{R}}$, where B is a second countable topological space, such that for all $y \in B$, the mapping

$$x \mapsto g(x,y)$$

is lower semicontinuous. Let $(Y_n)_n, Y$ be random variables with values in B. Then

$$f_n(x) = g(x, Y_n), \ f(x) = g(x, Y)$$

are random lower semicontinuous functions.
We shall show in this section, under which conditions on g and $(Y_n)_n$, Y convergence in distribution of $(\operatorname{epi} f_n)_n$ to $\operatorname{epi} f$ follows. In [45] Lachout and Vogel dealt with the case of convergence in probability. We will transfer these results to the setting of convergence in distribution.
The case of estimated parameters, which is important for applications in statistics, is included in our general setting: let $y_0 \in B$ be the true parameter and let $(Y_n)_n$ be a sequence of estimators converging to y_0. We realise however, that convergence in distribution yields nothing new, since on one hand estimators are chosen to be at least weakly consistent, i.e. $Y_n \overset{P}{\to} y_0$ and on the other hand, even if at first we obtain $Y_n \overset{D}{\to} y_0$, then $Y_n \overset{P}{\to} y_0$, because convergence in distribution implies convergence in probabilitiy if the limit is deterministic (see Theorem 2.6).
We will now show, under which continuity assumptions on g, $Y_n \overset{D}{\to} Y$ implies $\operatorname{epi} f_n \overset{D}{\to} \operatorname{epi} f$ with respect to τ_M or τ_H or τ_{Fell}. First we investigate the deterministic case.

Lemma 2.35 *Let $g : \mathbb{R}^p \times B \to \overline{\mathbb{R}}$ be lower semicontinuous, then the mapping $y \mapsto \operatorname{epi} g(.,y)$ is continuous with respect to τ_M.*

Proof. We have to show that $y_n \to y$ implies $\limsup_{n\to\infty} (\operatorname{epi} g(.,y_n)) \subset \operatorname{epi} g(.,y)$. Let $(x_{n_k}, z_{n_k}) \in \operatorname{epi} g(.,y_{n_k})$ and $(x_{n_k}, z_{n_k}) \to (x,z)$. Since g is lower semicontinuous in (x,y) to each $\varepsilon > 0$ there is a neighbourhood U of (x,y) such that $(t,s) \in U$ implies

$g(t,s) \geq g(x,y) - \varepsilon$. From the convergence of x_{n_k} to x and of y_n to y it follows that there is $k_0 \in \mathbb{N}$ such that $(x_{n_k}, y_{n_k}) \in U$ for all $k > k_0$. We have

$$z_{n_k} \geq g(x_{n_k}, y_{n_k}) \geq g(x,y) - \varepsilon, \text{ for all } k > k_0.$$

And it follows that $z = \lim\limits_{k\to\infty} z_{n_k} \geq g(x,y) - \varepsilon$. Since this holds for all $\varepsilon > 0$ we obtain $z \geq g(x,y)$ and thus $(x,z) \in \operatorname{epi} g(.,y)$. □

For the following lemma we recall a definition from [31].

Definition 2.36 A function $g : \mathbb{R}^p \times B \to \overline{\mathbb{R}}$ is called epi-upper semicontinuous at x_0 as $y \to y_0$, if

$$\sup_{V\in\mathcal{U}(x_0)} \limsup_{y\to y_0} \inf_{x\in V} g(x,y) \leq g(x_0, y_0).$$

Lemma 2.37 *Let $g : \mathbb{R}^p \times B \to \overline{\mathbb{R}}$ be a mapping such that $x \mapsto g(x,y)$ is lower semicontinuous for each $y \in B$ and $y \mapsto g(x,y)$ is epi-upper semicontinuous for each $x \in \mathbb{R}^p$, then $y \mapsto \operatorname{epi} g(.,y)$ is continuous with respect to τ_H.*

Proof. Let $y_n \to y$ and let $(x,z) \in \operatorname{epi} g(.,y)$. We have to show that there is a sequence $((x_n,z_n))_n$, $n \geq n_0$ such that $(x_n,z_n) \in \operatorname{epi} g(.,y_n)$ and $(x_n,y_n) \to (x,y)$. Let $\overline{B}_{\frac{1}{k}}(x)$ be the closed ball with center x and radius $\frac{1}{k}$. Due to epi-upper semicontinuity we have

$$\limsup_{n\to\infty} \min_{u\in\overline{B}_{\frac{1}{k}}(x)} g(u,y_n) \leq g(x,y).$$

Thus there is n_k such that $\min\limits_{u\in B_{\frac{1}{k}}(x)} g(u,y_n) \leq g(x,y)$ for all $n \geq n_k$. This yields the existence of $(u_n^{(k)})_n \subset \overline{B}_{\frac{1}{k}}(x)$, $n \geq n_k$ with $g(u_n^{(k)}, y_n) \leq g(x,y)$. Now let $x_n = u_n^{(k)}$ for $n_k \leq n < n_{k+1}$, $k = 1,2,\ldots$. Then $x_n \to x$ and $g(x_n,y_n) \leq g(x,y) \leq z$. It follows that $(x_n,z) \in \operatorname{epi} g(.,y_n)$ and $(x_n,z) \to (x,z)$. □

Corollary 2.38 *Let $g : \mathbb{R}^p \times B \to \overline{\mathbb{R}}$ be lower semicontinuous, let $y \mapsto g(x,y)$ be epi-upper semicontinuous for each $x \in \mathbb{R}^p$, then the mapping $y \mapsto \operatorname{epi} g(.,y)$ is continuous with respect to τ_{Fell}.*

We now turn to the random setting. In the following let $(Y_n)_n$, Y be random variables with values in B.

Theorem 2.39 *Let $g : \mathbb{R}^p \times B \to \overline{\mathbb{R}}$ be lower semicontinuous.*
If $Y_n \xrightarrow{D} Y$, then
$$\operatorname{epi} g(., Y_n) \overset{D_{\tau_M}}{\to} \operatorname{epi} g(., Y).$$

Theorem 2.40 *Let $g : \mathbb{R}^p \times B \to \overline{\mathbb{R}}$ be a mapping such that $x \mapsto g(x, y)$ is lower semicontinuous for each $y \in B$ and $y \mapsto g(x, y)$ is epi-upper semicontinuous for each $x \in \mathbb{R}^p$. If $Y_n \xrightarrow{D} Y$, then*
$$\operatorname{epi} g(., Y_n) \overset{D_{\tau_H}}{\to} \operatorname{epi} g(., Y).$$

Theorem 2.41 *Let $g : \mathbb{R}^p \times B \to \mathbb{R}$ be lower semicontinuous and let $y \mapsto g(x, y)$ be epi-upper semicontinuous for each $x \in \mathbb{R}^p$. If $Y_n \xrightarrow{D} Y$, then*
$$\operatorname{epi} g(., Y_n) \overset{D_{\tau_{Fell}}}{\to} \operatorname{epi} g(., Y).$$

For the proofs of these theorems the Continuous Mapping Theorem in combination with the corresponding deterministic result (2.35, 2.37 or 2.38) is applied.
Possible applications of parameter dependency are not restricted to the convergence of epigraphs. We will now deal with sets of the form $\{x : h_i(x, y) \le 0,\ i = 1, \dots, d\}$ with suitable mappings $h_i : \mathbb{R}^p \times B \to \mathbb{R}$. These sets occur as restriction sets in optimisation problems or in general as solution sets of inequalities.

Lemma 2.42 *Let $h : \mathbb{R}^p \times B \to \mathbb{R}^d$, $h = (h_1, \dots, h_d)$. Let h_i be lower semicontinuous, $i = 1, \dots, d$, then the mapping $y \mapsto \{x : h_i(x, y) \le 0,\ i = 1, \dots, d\}$ is continuous with respect to τ_M.*

Proof. Let $y_n \to y$. We have to show that $\limsup\limits_{n\to\infty} \{x : h_i(x, y_n) \le 0,\ i = 1, \dots, d\} \subset \{x : h_i(x, y) \le 0,\ i = 1, \dots, d\}$. Let $z_{n_k} \in \{x : h(x, y_{n_k}) \le 0,\ i = 1, \dots, d\}$ and $z_{n_k} \to z$. Since h_i is lower semicontinuous in (z, y), to each $\varepsilon > 0$ there is a neighbourhood V_i of (z, y) such that $(s, t) \in V_i$ implies $h_i(s, t) \ge h_i(z, y) - \varepsilon$. Let $V = \bigcap\limits_{i=1}^{d} V_i$. Because of $z_{n_k} \to z$ and $y_n \to y$ we can find k_0 such that $(z_{n_k}, y_{n_k}) \in V$ for all $k \ge k_0$ and thus
$$h_i(z_{n_k}, y_{n_k}) \ge h_i(z, y) - \varepsilon, \text{ for all } k \ge k_0 \text{ and } i = 1, \dots, d.$$

From $h_i(z_{n_k}, y_{n_k}) \le 0$ we obtain $h_i(z, y) \le \varepsilon$. Now $z \in \{x : h_i(x, y) \le 0,\ i = 1, \dots, d\}$ follows with $\varepsilon \to 0$. □

Theorem 2.43 *Let $h : \mathbb{R}^p \times B \to \mathbb{R}$ be as above and let $Y_n \xrightarrow{D} Y$, then*

$$\{x : h(x, Y_n) \leq 0,\ i = 1, \ldots, d\} \overset{D_{\tau_M}}{\to} \{x : h(x, Y) \leq 0,\ i = 1, \ldots, d\}.$$

Proof. Apply the previous Lemma in combination with the Continuous Mapping Theorem. □

As an example for the application of dependence on parameters we deal with a risk process on $[0, \infty)$. Processes of this type, with downward jumps of random height, occuring a random time, can be used in insurance to model the flow of premiums and claims (see e.g. [9]). We will see here and in Chapter 3, that estimates for the time of ruin and the probability of ruin can be obtained with set convergence in distribution methods.
First we construct a counting process akin to the Poisson process.
Let $(Z_i)_i$ be a sequence of random variables with values in $(0, \infty)$, let $Z_0 = 0$. Z_i will be interpreted as the time between the $i-$th and the $i + 1-$th jump for $i \geq 1$.
For $t \in [0, \infty)$ we set

$$N(t) = N(t, Z_1, \ldots) = k, \text{ for } \sum_{i=0}^{k} Z_i \leq t < \sum_{i=0}^{k+1} Z_i.$$

Then the process N counts the number of jumps up to time t. Note that we obtain a Poisson process with parameter $\lambda > 0$, if the $(Z_i)_i$ form an iid sequence with $Z_i \sim \exp(\lambda)$.
Now let $(X_i)_i$ be a sequence of random variables with values in $[0, \infty)$.
For $t \in [0, \infty)$ let

$$g(t, Z_1, \ldots, X_1, \ldots) = -\sum_{i=0}^{N(t)} X_i, \tag{2.4}$$

then as desired $g(., ..)$ is a stochastic process which jumps downwards with random jump height X_k at random time $\sum_{i=0}^{k} Z_i$. For an application of our parameter dependency results we now consider the case that each Z_i, resp. X_i is the limit in distribution of a sequence of $(0, \infty)$ valued random variables $(Z_i^n)_n$ resp. $[0, \infty)$ valued random variables $(X_i^n)_n$.
Let $\tau_{|.|}$ denote the usual topology on $\mathbb{R}_+$. Let $\tau_<$, denote the topology on $\mathbb{R}_+$. with the open sets $[0, a)$, for $a > 0$. Then $x_n \xrightarrow{\tau_<} x$ if and only if for each $\varepsilon > 0$ there is n_0 such that $x_n < x + \varepsilon$ for all $n \geq 0$. Analogously the topology $\tau_>$ has the open sets (a, ∞) and $x_n \xrightarrow{\tau_>} x$ if and only if for each $\varepsilon > 0$ there is n_0 such that $x_n > x - \varepsilon$ for all $n \geq n_0$.
It is easy to see that the topological spaces $(\mathbb{R}_+, \tau_<)$ and $(\mathbb{R}_+, \tau_>)$ are second countable.
Countable bases for the open sets are given by $\{[0, a) : a \in \mathbb{Q}_+\}$ and $\{(a, \infty) : a \in \mathbb{Q}_+\}$.

Lemma 2.44 *The mapping* $g : \left(\mathbb{R}_+ \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty, \tau_{|.|} \times \tau_>^\infty \times \tau_<^\infty\right) \to \mathbb{R}$ *defined as in (2.4) is lower semicontinuous.*

Proof. Let $t_n \to t,\ z_i^n \xrightarrow{\tau_<} z_i,\ x_i^n \xrightarrow{\tau_>} x_i$ for all i as $n \to \infty$. There is k such that $\sum_{i=0}^{k} z_i \le t < \sum_{i=0}^{k+1} z_i$. For all sufficiently small $\alpha > 0$ we have $t < \sum_{i=0}^{k+1} z_i - \alpha$. We can find n_0 such that $\sum_{i=0}^{k+1} z_i^n > \sum_{i=0}^{k+1} z_i - \frac{\alpha}{3}$, $|t_n - t| \le \frac{\alpha}{3}$ and $-\sum_{i=0}^{k} x_i^n > -\sum_{i=0}^{k} x_i - \alpha$ for all $n \ge n_0$. It follows that $t_n < \sum_{i=0}^{k+1} z_i^n$ and thus

$g(t_n, z_1^n, \ldots, x_1^n, \ldots) \ge -\sum_{i=0}^{k} x_i^n > -\sum_{i=0}^{k} x_i - \alpha = g(t, z_1, \ldots, x_1, \ldots) - \alpha$ for all $n \ge n_0$. With $\alpha \to 0$ we obtain

$$\liminf_{n\to\infty} g(t_n, z_1^n, \ldots, x_1^n) \ge g(t, z_1, \ldots, x_1, \ldots). \qquad \square$$

Lemma 2.45 *The mapping* $g : \left(\mathbb{R}_+ \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty, \tau_{|.|} \times \tau_<^\infty \times \tau_>^\infty\right) \to \mathbb{R}$ *defined as in (2.4) is epi-upper semicontinuous in the first argument.*

Proof. It suffices to show that for each neighbourhood V of t and each $\eta > 0$ there is a neighbourhood W of $(z_1, \ldots, x_1, \ldots)$ such that for each $(w_1, \ldots, y_1, \ldots) \in W$ there is $s \in V$ with $g(s, w_1, \ldots, y_1, \ldots) \le g(t, z_1, \ldots, x_1, \ldots) + \eta$.
For $\varepsilon > 0$ let $V = B_\varepsilon(t)$. Let k be the largest integer such that $\sum_{i=0}^{k} z_i \le t$.
Let $\delta := \min\left(\frac{\varepsilon}{3k}, \frac{\eta}{k}\right)$, then with
$W := \{(w_1, \ldots, y_1, \ldots) \in \mathbb{R}^\infty : w_i < z_i + \delta,\ y_i > x_i - \delta,\ i = 1, \ldots, k\}$ we obtain
$\sum_{i=0}^{k} w_i < \sum_{i=0}^{k} z_i + \frac{\varepsilon}{3}$ for all $(w_1, \ldots, y_1, \ldots) \in W$. Let $s = t + \frac{2}{3}\varepsilon$, then $s \in V$ and because of $s > \sum_{i=0}^{k} w_i$ we have $g(s, w_1, \ldots, y_1, \ldots) \le -\sum_{i=0}^{k} y_i \le -\sum_{i=0}^{k} x_i + \eta = f(t, z_1, \ldots, x_1, \ldots) + \eta$. $\square$

Corollary 2.46 *The mapping* $g : \left(\mathbb{R}_+ \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty, \tau_{|.|} \times \tau_{|.|}^\infty \times \tau_{|.|}^\infty\right) \to \mathbb{R}$ *defined as in (2.4) is lower semicontinuous and it is epi-upper semicontinuous in the first argument.*

In the following we write

$$U_n(t) := g(t, Z_1^n, \ldots, X_1^n, \ldots)$$

and

$$U(t) := g(t, Z_1, \ldots, X_1, \ldots)$$

Theorem 2.47 *If* $(Z_1^n, \ldots, X_1^n, \ldots) \overset{D_{\tau_>^\infty \times \tau_<^\infty}}{\to} (Z_1, \ldots, X_1, \ldots)$, *then*

$$\operatorname{epi} U_n \overset{D_{\tau_M}}{\to} \operatorname{epi} U.$$

If $(Z_1^n, \ldots, X_1^n, \ldots) \overset{D_{\tau_<^\infty \times \tau_>^\infty}}{\to} (Z_1, \ldots, X_1, \ldots)$, *then*

$$\operatorname{epi} U_n \overset{D_{\tau_H}}{\to} \operatorname{epi} U.$$

If $(Z_1^n, \ldots, X_1^n, \ldots) \overset{D_{\tau_{|\cdot|}^\infty \times \tau_{|\cdot|}^\infty}}{\to} (Z_1, \ldots, X_1, \ldots)$, *then*

$$\operatorname{epi} U_n \overset{D_{\tau_{Fell}}}{\to} \operatorname{epi} U.$$

Proof. This follows immediately from 2.35, 2.37 and 2.38 in combination with 2.44, 2.45 and 2.46. □

To show that for given $(Z_i)_i$, $(X_i)_i$, $(Z_i^n)_i$, $(X_i^n)_i$ the assumptions are fulfilled, it suffices to show that $(Z_{i_1}^n, \ldots, Z_{i_k}^n, X_{j_1}^n, \ldots, X_{j_l}^n) \overset{D}{\to} (Z_{i_1}, \ldots, Z_{i_k}, X_{j_1}, \ldots, X_{j_l})$ for all $k, l \in \mathbb{N}$, $i_1, \ldots, i_k, j_1, \ldots, j_k \in \mathbb{N}$ (see Lemma A.6). It remains an open problem, whether for example under the assumption of independent components the convergence in distribution of the finite dimensional random vectors follows from convergence in distributions of its components when the nonmetrizable topologies $\tau_<$ or $\tau_>$ are involved. For a practical application consider initial capital $c > 0$ at time $t = 0$ and premium rate r. With claims of height X_k resp. X_k^n occuring at time $\sum_{i=1}^k Z_i$ resp. $\sum_{i=1}^k Z_i^n$, then the capital at time $t \geq 0$ is given by the lower semicontinuous functions

$$C(t) = c + rt + U(t), \text{ resp. } C_n = c + rt + U_n(t).$$

Under the assumption that $(Z_1^n, \ldots, X_1^n, \ldots) \overset{D_{\tau_>^\infty \times \tau_<^\infty}}{\to} (Z_1, \ldots, X_1, \ldots)$, from Theorem 2.47 we obtain

$$\operatorname{epi} C_n \overset{D_{\tau_M}}{\to} \operatorname{epi} C. \tag{2.5}$$

We are interested in the time of ruin, i.e. in

$$R(C) = \min\{t \geq 0 : \; C(t) \leq 0\}.$$

and investigate how $R(C_n)$ relates to $R(C)$.
First note that (2.5) implies $\{t: \ C_n(t) \leq 0\} \overset{D_{\tau_M}}{\to} \{t: \ C(t) \leq 0\}$. This follows from Theorem 3.2 on the convergence of sublevel sets. Now with Lemma A.3 and the Lower Semicontinuous Mapping Theorem 1.28 we obtain the following lower bound for the probability of ruin occuring up to time $a \geq 0$

$$\limsup_{n\to\infty} P(R(C_n) \leq a) \leq P(R(C) \leq a)$$

2.5. Pointwise Convergence of Convex Functions

In this section we investigate continuous, convex functions, defined on a convex open set $W \subset \mathbb{R}^d$, denoted by $C_c = C_c(W)$. We will show that for a subclass of C_c convergence in distribution of the finite dimensional sections in a pointwise sense is sufficient for convergence in distribution of the epigraphs with respect to τ_M. We note that it is possible to derive convergence in distribution with respect to τ_M from convergence in distribution in the sense of the topology of (upper-) pointwise convergence for the larger class of all lower semicontinuous functions. Due to the properties of the topology of pointwise convergence (this topology is not metrizable and convergence of sequences is not sufficient to describe convergence in the topological sense) we would not obtain useful sufficient conditons for convergence in distribution with respect to τ_M. Let

$$C_c^N = \{f \in C_c : |f(x)| \leq N, \ \forall x \in W\}$$

denote the class of all continuous, convex functions, with absolute value bounded by N. As our main result of this section we obtain the following sufficient condition:

Theorem 2.48 *Let $(f_n)_n$, f be random elements of C_c^N.*
If

$$\begin{aligned}\liminf_{n\to\infty} P&\left(\bigcup_{i=1}^{m}\left(f_n\left(x_1^i\right) > a_1^i, \ldots, f_n\left(x_{k_i}^i\right) > a_{k_i}^i\right)\right) \\ &\geq P\left(\bigcup_{i=1}^{m}\left(f\left(x_1^i\right) > a_1^i, \ldots, f\left(x_{k_i}^i\right) > a_{k_i}^i\right)\right)\end{aligned}$$

for all $m \in \mathbb{N}, \ k_i \in \mathbb{N}, \ x_1^i, \ldots, x_{k_i}^i \in W, \ a_1^i, \ldots, a_{k_i}^i \in \mathbb{R}$,
then

$$\operatorname{epi} f_n \overset{D_{\tau_M}}{\to} \operatorname{epi} f.$$

Consult Example 2.17 to see that in general the finite dimensional convergence condition cannot be weakened to $\liminf_{n\to\infty} P(f_n(x_1) > a_1, \ldots, f_n(x_k) > a_k) \geq P(f(x_1) > a_1, \ldots, f(x_k) > a_k)$ for all $x_1, \ldots, x_k \in W,\ a_1, \ldots, a_k \in \mathbb{R}$. To prepare the proof of the theorem, we investigate pointwise upper convergence and compact upper convergence.

Definition 2.49 Let $(f_n)_n,\ f$ be real-valued functions on W.

(i) We call $(f_n)_n$ upper pointwise convergent to f, if to each $\varepsilon > 0$ and to each $x \in W$ there is n_0 such that
$$f_n(x) > f(x) - \varepsilon$$
for all $n \geq n_0$.

(ii) We say that $(f_n)_n$ converges upper compact to f, if to each $\varepsilon > 0$ and to each compact $K \subset W$ there is n_0 such that
$$f_n(x) > f(x) - \varepsilon$$
for all $n \geq n_0$ and all $x \in K$.

A base of open neighbourhoods for the topology τ_{pu} of upper pointwise convergence on C is given by all sets of the type
$$V_{\varepsilon,x_1,\ldots,x_k}(g) = \{h \in C : h(x_i) > g(x_i) - \varepsilon,\ i = 1, \ldots, k\},$$
with $\varepsilon > 0$, $k \in \mathbb{N},\ x_1, \ldots, x_k \in W$ and $g \in C$.
The topology τ_{cu} of compact upper convergence is generated by the base of open neighbourhoods consisting of all sets of the form
$$U_{\varepsilon,K}(g) = \{h \in C : h(x) > g(x) - \varepsilon,\ \forall x \in K\},$$
where $\varepsilon > 0,\ K \subset W$ compact and $g \in C$.
By τ_c we denote the topology of uniform convergence on compact sets. With the Stone Theorem (Satz 9.11 of [46]) it follows that the topological space (C, τ_c) is separable. Since (C, τ_c) is also metrizable, it follows that (C, τ_c) is second countable. We require these properties of τ_c to show the following Lindelöf property of (C, τ_{cu}).

Lemma 2.50 *For each $A \subset C$, every $\tau_{cu}-$open cover of A has a countable subcover.*

Proof. Since (C, τ_c) is separable there is a countable dense subset $F \subset C$. By $\mathcal{U}$ we denote the set of all ρ_c-balls $B_r(g)$ with $g \in F$ and rational radius $r > 0$. The set $\mathcal{U}$ is countable.

Let $G_i,\ i \in I$ be an $\tau_{cu}-$open cover for $A \subset C$ and let $f \in A$, then there is $i \in I$ with $f \in G_i$. Since G_i is $\tau_{cu}-$open there is a $\tau_{cu}-$neighbourhood $U_{\varepsilon,K}(f)$ of f with $U_{\varepsilon,K}(f) \subset G_i$. Because of the separability of (C, τ_c) we can find $g \in F$ and a rational $\alpha > 0$ with $\alpha < \frac{\varepsilon}{2}$, such that $f \in B_\alpha(g)$. For each $h \in B_\alpha(g)$ we have $|h(x) - g(x)| < \alpha$ for all $x \in K$ and thus

$$h(x) > g(x) - \alpha > f(x) - 2\alpha > f(x) - \varepsilon$$

for all $x \in K$.

We have shown that

$$f \in B_\alpha(g) \subset U_{\varepsilon,K}(f) \subset G_i.$$

By numbering the balls that form the set $\mathcal{U}$ we can assume that $\mathcal{U} = \bigcup\limits_{k\in\mathbb{N}} B^k$. We have seen, that $A \subset \mathcal{U}$. For each $k \in \mathbb{N}$ we choose a set $G_{i_k},\ i_k \in I$ with $B^k \subset G_{i_k}$ if such a set exists and obtain $A \subset \bigcup\limits_{k\in\mathbb{N}} G_{i_k}$ which shows that the sets $(G_{i_k})_k$ form a countable subcover of A. $\square$

The space $\left(C_c^M, \tau_{cu}\right)$ has the Lindelöf property. This follows from the next lemma.

Lemma 2.51 *Let (T, τ) be a topological space with the Lindelöf property, then every subspace $S \subset T$ equipped with its subspace topology $\tau|_S$ has the Lindelöf property.*

Proof. Let $W \subset S$ and let $\widetilde{G}_i,\ i \in I$ be an open cover, i.e. $\widetilde{G}_i \subset S,\ \tau|_S$-open for all $i \in I$ and $W \subset \bigcup\limits_{i\in I} \widetilde{G}_i$. If $\widetilde{G}_i$ is $\tau|_S-$open, then there is $G_i \subset T$ such that G_i is $\tau-$open and $\widetilde{G}_i = G_i \cap S$. We obtain $W \subset \bigcup\limits_{i\in I} G_i$ and because of the weak Lindelöf property of (T, τ) there is $(i_k)_k \subset I$ such that $W \subset \bigcup\limits_{k\in\mathbb{N}} G_{i_k}$. It follows that $W \subset \bigcup\limits_{k\in\mathbb{N}} (G_{i_k} \cap S) = \bigcup\limits_{k\in\mathbb{N}} \widetilde{G}_{i_k}$, which shows that $\widetilde{G}_{i_k},\ k \in \mathbb{N}$ is a countable subcover. $\square$

Definition 2.52 Let $\widetilde{C} \subset C$ be a class of functions, $\widetilde{C}$ has the equi-Lipschitz property, if there is $\alpha \in [0, \infty)$ such that

$$|g(x) - g(y)| \leq \alpha \|x - y\|$$

for all $x, y \in W$ and all $g \in \widetilde{C}$.

An example for a class with the equi-Lipschitz property is given by
$\widetilde{C} = \{f \in C^1([a, b]) : |f'(x)| \leq \alpha, \ \forall x \in [a, b]\}$, where $\alpha \in (0, \infty)$,
see for example page 308 of [36].

Lemma 2.53 *For all $N \in \mathbb{R}$, the class C_c^N has the equi-Lipschitz property.*

Proof. This follows immediately from Theorem 1.6 of [32]. □

At this point we see, that we could obtain our main result as an immediate consequence of the results in Section 2.2. We can easily verify that the equi-Lipschitz property implies stochastic equi-lower semicontinuity. We will however give a different proof, which gives insight into the relation between pointwise convergence in distribution and τ_M convergence in distribution.

Theorem 2.54 *On every class $\widetilde{C} \subset C$ with the equi-Lipschitz property the topologies $\tau_{pu}|_{\widetilde{C}}$ and $\tau_{cu}|_{\widetilde{C}}$ coincide.*

Proof. It suffices to show that a set $U \subset \widetilde{C}$ is $\tau_{pu}|_{\widetilde{C}}-$open if and only if it is $\tau_{cu}|_{\widetilde{C}}-$open. Recall that a set U is $\tau-$open if and only if for each $f \in U$ there is a $\tau-$neighbourhood A of f with $A \subset U$.
First assume that U is $\tau_{pu}|_{\widetilde{C}}-$open and that $f \in U$. We can find a $\tau_{pu}|_{\widetilde{C}}-$neighbourhood

$$A = V_{\varepsilon, x_1, \ldots, x_k}(f) = \left\{ g \in \widetilde{C} : g(x_i) > f(x_i) - \varepsilon, \ i = 1, \ldots, k \right\}$$

with $A \subset U$. Let $K = \{x_1, \ldots, x_k\}$ then K is compact and because of
$A = \left\{ g \in \widetilde{C} : g(x) > f(x) - \varepsilon, \ x \in K \right\}$, the set A is also a $\tau_{cu}|_{\widetilde{C}}-$neighbourhood of f. It follows that U is $\tau_{cu}|_{\widetilde{C}}-$open.
Now assume that U is $\tau_{cu}|_{\widetilde{C}}-$open and $f \in U$, then to f there is a $\tau_{cu}|_{\widetilde{C}}-$neighbourhood

$$A = \left\{ g \in \widetilde{C} : g(x) > f(x) - \varepsilon, \ x \in K \right\},$$

with $A \subset U$. Here K is a compact set. Because of the equi-Lipschitz property of $\widetilde{C}$ we can find finitely many $x_1, \ldots, x_k \in K$ such that $K \subset \bigcup_{i=1}^{k} B_{\frac{\varepsilon}{3\alpha}}(x_i)$ and $|g(x) - g(x_i)| \leq \alpha \|x - x_i\| \leq \frac{\varepsilon}{3}$ for all $x \in B_{\frac{\varepsilon}{3\alpha}}(x_i)$. Let

$$\widetilde{A} = \left\{ g \in \widetilde{C} : g(x_i) > f(x_i) - \frac{\varepsilon}{3},\ i = 1, \ldots, k \right\}.$$

Then $\widetilde{A}$ is a $\tau_{pu}|_{\widetilde{C}}$ neighbourhood of f. We show that $\widetilde{A} \subset A$. Let $g \in \widetilde{A}$ and let $x \in K$, then $x \in B_{\frac{\varepsilon}{3\alpha}}(x_i)$ for some i and we obtain

$$g(x) \geq g(x_i) - \frac{\varepsilon}{3} > f(x_i) - \frac{2\varepsilon}{3} \geq f(x) - \varepsilon$$

and thus $g \in A$.
Now $\widetilde{A} \subset A \subset U$ yields the $\tau_{pu}|_{\widetilde{C}}$−openness of U. □

Corollary 2.55 *The topological space $\left(C_c^N, \tau_{pu}|_{C_c^N}\right)$ is metrizable and second countable. Covergence of sequences is sufficient to describe convergence with respect to $\tau_{pu}|_{C_c^N}$.*

Theorem 2.56 *Let $(f_n)_n$, f be random elements of C_c^N.*
If

$$f_n \overset{D_{\tau_{pu}|_{C_c^N}}}{\rightarrow} f$$

then

$$\operatorname{epi} f_n \overset{D_{\tau_M}}{\rightarrow} \operatorname{epi} f.$$

Proof. We show that the mapping $u : \left(C_c^N, \tau_{pu}|_{C_c^N}\right) \rightarrow (, \tau_M)$, $u(f) = \operatorname{epi} f$ is continuous. In view of Corollary 2.55 it suffices to show sequential continuity. Let $f_n \overset{D_{\tau_{pu}|_{C_c^N}}}{\rightarrow} f$, then $f_n \overset{D_{\tau_{cu}|_{C_c^N}}}{\rightarrow} f$. With compact $K_i \subset \mathbb{R}^{n+1}$ let $G = \bigcup_{i=1}^{\infty} M(K_i)$ be τ_M−open and let $\operatorname{epi} f \in G$. It is to show that there is n_0 such that $\operatorname{epi} f_n \in G$ for all $n \geq n_0$. We have $\operatorname{epi} f \in M(K_i)$ for some i. Since $\operatorname{epi} f$ is closed and K_i is compact, there is $\varepsilon > 0$ with $\operatorname{dist}(\operatorname{epi} f, K_i) = \varepsilon$. Let $R = \{x \in W : \exists y \in \mathbb{R} : (x, y) \in K_i\}$, then R is compact as the projection of the compact set K_i onto $W \subset \mathbb{R}$. We set $m(x) = \max\{y : (x, y) \in K_i\}$. From $\operatorname{dist}(\operatorname{epi} f, K_i) = \varepsilon > 0$ it follows that $f(x) > m(x) + \frac{\varepsilon}{2}$ for all $x \in R$. Let $U = \left\{g \in C_c^N : g(x) > f(x) - \frac{\varepsilon}{2},\ \forall x \in R\right\}$. The set U is $\tau_{cu}|_{C_c^N}$−open and from $f \in U$, $f_n \overset{D_{\tau_{cu}|_{C_c^N}}}{\rightarrow} f$ it follows that $f_n \in U$ for all $n \geq n_0$. This shows that we have $f_n(x) > f(x) - \frac{\varepsilon}{2} > m(x)$ for all $x \in R$ and all $n \geq n_0$ and it follows that

epi $f_n \in M(K_i) \subset G$ for all $n \geq n_0$.
The assertion of the theorem now follows with the Continuos Mapping Theorem. $\square$

Note that for the proof we did not directly use the equi-Lipschitz property, but only its consequence that convergence of sequences is sufficient for the description of convergence with respect to τ_{pu}.

It remains to establish a sufficient condition for convergence in distribution with respect to $\tau_{pu}|_{C_c^N}$, that relies on the finite dimensional sections of f_n and f.

Theorem 2.57 *Let $(f_n)_n$, f be random elements of C_c^N.*
If

$$\liminf_{n\to\infty} P\left(\bigcup_{i=1}^{m}\left(f_n\left(x_1^i\right) > a_1^i, \ldots, f_n\left(x_{k_i}^i\right) > a_{k_i}^i\right)\right)$$
$$\geq P\left(\bigcup_{i=1}^{m}\left(f\left(x_1^i\right) > a_1^i, \ldots, f\left(x_{k_i}^i\right) > a_{k_i}^i\right)\right)$$

for all $m \in \mathbb{N},\ k_i \in \mathbb{N},\ x_1^i, \ldots, x_{k_i}^i \in W,\ a_1^i, \ldots, a_{k_i}^i \in \mathbb{R}$,

then

$$\text{epi}\, f_n \overset{D_{\tau_{pu}|_{C_c^N}}}{\longrightarrow} \text{epi}\, f.$$

Proof. Let $U \subset C_c^N$ be $\tau_{pu}|_{C_c^N}$ open. From the Lindelöf property of $\left(C_c^N, \tau_{pu}|_{C_c^N}\right)$ it follows that

$$U = \bigcup_{i=1}^{\infty} V_{\varepsilon_i, x_1^i, \ldots, x_{k_i}^i}(g_i)$$

Because of the continuity of the probability measure, to $\eta > 0$ there is m such that $P\left(f \in \bigcup_{i=1}^{m} V_{\varepsilon_i, x_1^i, \ldots, x_{k_i}^i}(g_i)\right) \geq P(f \in U) - \eta$.

We obtain

$$\begin{aligned}
\liminf_{n\to\infty} P\left(f_n \in U\right) \geq & \liminf_{n\to\infty} P\left(f_n \in \bigcup_{i=1}^{m} V_{\varepsilon_i, x_1^i, \ldots, x_{k_i}^i}(g_i)\right) \\
= & \liminf_{n\to\infty} P\left(\bigcup_{i=1}^{m} \left(f_n(x_1^i) > g_i(x_1^i) - \varepsilon_i, \ldots, f_n(x_{k_i}^i) > g_i(x_{k_i}^i) - \varepsilon_i\right)\right) \\
\geq & P\left(\bigcup_{i=1}^{m} \left(f(x_1^i) > g_i(x_1^i) - \varepsilon_i, \ldots, f(x_{k_i}^i) > g_i(x_{k_i}^i) - \varepsilon_i\right)\right) \\
= & P\left(f \in \bigcup_{i=1}^{m} V_{\varepsilon_i, x_1^i, \ldots, x_{k_i}^i}(g_i)\right) \\
\geq & P\left(f \in U\right) - \eta,
\end{aligned}$$

and it remains to let $\eta \to 0$. □

Proof. (of Theorem 2.48) Theorem 2.48 now follows as a combination of Theorem 2.56 and Theorem 2.57. □

We now turn to τ_H-convergence.

Theorem 2.58 *Let $(f_n)_n$, f be random elements of C_c^N.*
If

$$\begin{aligned}
\limsup_{n\to\infty} P & \left(\bigcup_{i=1}^{\infty} \left(f_n(x_1) \geq a_1, \ldots, f_n(x_k) \geq a_k\right)\right) \\
& \leq P\left(\bigcup_{i=1}^{\infty} \left(f(x_1) \geq a_1, \ldots, f(x_k) \geq a_k\right)\right)
\end{aligned}$$

for all $k \in \mathbb{N}$, $x_1, \ldots, x_k \in W$, $a_1, \ldots, a_k \in \mathbb{R}$, then

$$\operatorname{epi} f_n \overset{D_{\tau_H}}{\to} \operatorname{epi} f.$$

For the proof the same preparations can be carried out as in the τ_M-case. Instead of pointwise/compact upper convergence, pointwise/compact lower convergence have to be used. We say that f_n is lower pointwise convergent to f, if for each $\varepsilon > 0$ and each $x \in W$ there is n_0 such that $f_n(x) \leq f(x) + \varepsilon$ for all $n \geq n_0$. We call f_n lower compactly convergent to f, if for each $\varepsilon > 0$ and each compact $K \subset W$ there is n_0 such that $f_n(x) \leq f(x) + \varepsilon$ for all $n \geq n_0$ and all $x \in K$. There are topologies τ_{pl} and τ_{cl} on C that describe pointwise lower and compact lower convergence. We can show with the same methods as in the τ_M case that 2.50–2.55 remain valid if we replace τ_{pu} and τ_{cu} by τ_{pl} and τ_{cl}.

Theorem 2.59 *Let $(f_n)_n$, f be random elements of C_c^N.*
If

$$f_n \overset{D_{\tau_{pl}|C_c^N}}{\rightarrow} f$$

then

$$\mathrm{epi}\, f_n \overset{D_{\tau_H}}{\rightarrow} \mathrm{epi}\, f.$$

Proof. We show that the mapping $u : \left(C_c^N, \tau_{pl}|_{C_c^N}\right) \to (, \tau_H)\,,\ u(f) = \mathrm{epi}\, f$ is continuous. It suffices to show sequential continuity. Let $f_n \overset{\tau_{pl}}{\rightarrow} f$, let U be τ_H-open, i.e. $U = \bigcup\limits_{i=1}^{\infty} \bigcap\limits_{j=1}^{m_i} H(G_j^i)$ with open G_j^i. If $\mathrm{epi}\, f \in U$, then there is i such that $\mathrm{epi}\, f \cap H(G_j^i \neq \emptyset,\ j = 1, \ldots m_i$. Thus there are $(x_j, y_j) \in G_j^i$ with $f(x_j) \leq y_j,\ j = 1, \ldots, m_i$. Since the G_j^i are open we can find $\varepsilon > 0$ such that $(x_j, y_j + \varepsilon) \in G_j^i,\ j = 1, \ldots, m_i$. From $f_n \overset{\tau_{pl}}{\rightarrow} f$ it follows that there is n_0 such that $f_n(x_j) < f(x_j) + \varepsilon$ for $j = 1, \ldots, m_i$ and all $n \geq n_0$. It follows that $f_n(x_j) \leq y_j + \varepsilon,\ j = 1, \ldots, m_i,\ n \geq n_0$ and thus $(x_j, y_j + \varepsilon) \in \mathrm{epi}\, f_n \cap G_j^i,\ j = 1, \ldots, m_i,\ n \geq n_0$, which shows that $f_n \in U$ for all $n \geq n_0$. It remains to apply the Continuous Mapping Theorem. $\square$

2.6. Convergence in Distribution in Product Spaces

It is well known that for random variables $X_n,\ Y_n,\ X,\ Y$, which take their values in separable metric spaces, convergence in distribution of (X_n, Y_n) to (X, Y) follows from $X_n \overset{D}{\rightarrow} X$ and $Y_n \overset{D}{\rightarrow} Y$, if X_n and Y_n are independent for all n and if X and Y are independent (see Example 3.2 of [5]).
Having the topologies τ_M and τ_H in mind, we deal with the case of nonmetrizable topologies in this section.
Convergence in distribution in product spaces will be of great importance for applications in stochastic optimisation in the following chapters. We will for example need convergence in distribution of the vectors consisting of the epigraph of the objective function and the restriction set. We provide conditions, that imply $(F_n, G_n) \overset{D_{\tau_1 \times \tau_2}}{\rightarrow} (F, G)$ if $F_n \overset{D_{\tau_1}}{\rightarrow} F$ and G_n converges to G in a suitable sense with respect to τ_2. First we have to ensure that all occuring random vectors are Borel measurable. In general Borel measurability of (F, G) does not follow from Borel measurability of F and G.

Lemma 2.60 *Let the topological spaces (T_1, τ_1) and (T_2, τ_2) be second countable. If the random variables F resp. G are measurable with respect to $\mathcal{B}_{\tau_1}$ resp. $\mathcal{B}_{\tau_2}$, then (F, G) is measurable with respect to $\mathcal{B}_{\tau_1 \times \tau_2}$.*

Proof. This follows from $\mathcal{B}_{\tau_1} \otimes \mathcal{B}_{\tau_2} = \mathcal{B}_{\tau_1 \times \tau_2}$, which is proven in [8] (Satz 5.10). □

In the following we will thus always assume, that (T_1, τ_1) and (T_2, τ_2) are second countable. Recall that $\left(\mathcal{F}\left(\mathbb{R}^d\right), \tau_M\right)$, $\left(\mathcal{F}\left(\mathbb{R}^d\right), \tau_H\right)$ and $\left(\mathcal{F}\left(\mathbb{R}^d\right), \tau_{\text{Fell}}\right)$ are second countable. The following condition will be helpful for establishing convergence in distribution in product spaces.

Condition 2.61 For G and $(G_n)_n$

$$\lim_{n \to \infty} P\left(G_n \notin W,\ G \in W\right) = 0,$$

holds for all $\tau-$open W.

Note that for metric spaces this condition is equivalent to convergence in probability (see Lemma A.5). We have already seen in Lemmas 2.11 and 2.9 that this is also true for the topological spaces $(\mathcal{F}(\mathbb{R}^p), \tau_M)$ and $(\mathcal{F}(\mathbb{R}^p), \tau_H)$.

Theorem 2.62 *Let $F_n \overset{D_{\tau_1}}{\to} F$, let Condition 2.61 be fulfilled for G and $(G_n)_n$. If F_n and G_n, F_n and G, F and G are independent, then*

$$(F_n, G_n) \overset{D_{\tau_1 \times \tau_2}}{\to} (F, G).$$

Proof. Let U be open with respect to $\tau_1 \times \tau_2$. With the assumed second countability it follows that $U = \bigcup_{i=1}^{\infty} V_i \times W_i$, where the V_i, resp. W_i are open with respect to τ_1, resp. τ_2. We have to show that

$$\liminf_{n \to \infty} P\left((F_n, G_n) \in U\right) \geq P\left((F, G) \in U\right).$$

With the continuity of the probability measure, it follows that for each $\eta > 0$ there is m such that

$$P\left((F, G) \in \bigcup_{i=1}^{m} V_i \times W_i\right) \geq P\left((F, G) \in U\right) - \eta.$$

The set Ω can be partitioned into 2^m disjoint sets R_j, where each set is of the form

$$R_j = \{G \in W_{u_1}, \ldots, G \in W_{u_k}, G \notin W_{v_1}, \ldots, G \notin W_{v_l}\}$$

with $u_1 < \ldots < u_k,\ v_1 < \ldots < v_l,\ k + l = m$ and $\{u_1, \ldots u_k\} \cap \{v_1, \ldots, v_l\} = \emptyset$.
To simplify notation, we have not denoted the dependence of $u_i,\ v_i$, k and l on j.
We obtain

$$\begin{aligned}
&\liminf_{n\to\infty} P\left((F_n, G_n) \in \bigcup_{i=1}^{m} V_i \times W_i\right) \\
&= \liminf_{n\to\infty} \sum_{j=1}^{2^m} P\left((F_n, G_n) \in \bigcup_{i=1}^{m} V_i \times W_i,\ R_j\right) \\
&\geq \sum_{j=1}^{2^m} \liminf_{n\to\infty} P\left((F_n, G_n) \in \bigcup_{i=1}^{m} V_i \times W_i,\ R_j\right).
\end{aligned}$$

For each fixed j we have

$$\begin{aligned}
&\liminf_{n\to\infty} P\left((F_n, G_n) \in \bigcup_{i=1}^{m} V_i \times W_i,\ R_j\right) \\
&= \liminf_{n\to\infty} P\left(\begin{array}{c} (F_n, G_n) \in \bigcup_{i=1}^{m} V_i \times W_i, \\ G \in W_{u_1}, \ldots, G \in W_{u_k}, G \notin W_{v_1}, \ldots, G \notin W_{v_l} \end{array}\right) \\
&\geq \liminf_{n\to\infty} P\left(\begin{array}{c} (F_n, G_n) \in \bigcup_{i=1}^{k} V_{u_i} \times W_{u_i}, \\ G \in W_{u_1}, \ldots, G \in W_{u_k}, G \notin W_{v_1}, \ldots, G \notin W_{v_l} \end{array}\right)
\end{aligned}$$

From Condition 2.61 we obtain that for each open W and each $\varepsilon > 0$ there is n_0 such that

$$P\left(G_n \in W,\ G \in W\right) \geq P\left(G \in W\right) - \varepsilon$$

for all $n \geq n_0$. By applying this to $W = W_{u_1} \cap \ldots \cap W_{u_k}$ we continue

$$\begin{aligned}
&\liminf_{n\to\infty} P\left((F_n, G_n) \in \bigcup_{i=1}^{m} V_i \times W_i,\ R_j \right) \\
&\geq \liminf_{n\to\infty} P\left(\begin{array}{c} F_n \in \bigcup_{i=1}^{k} V_{u_i}, \\ G \in W_{u_1}, \ldots, G \in W_{u_k}, G \notin W_{v_1}, \ldots, G \notin W_{v_l} \end{array} \right) - \varepsilon \\
&= \liminf_{n\to\infty} P\left(F_n \in \bigcup_{i=1}^{k} V_{u_i} \right) P\left(G \in W_{u_1}, \ldots, G \in W_{u_k}, G \notin W_{v_1}, \ldots, G \notin W_{v_l} \right) - \varepsilon \\
&\geq P\left(F \in \bigcup_{i=1}^{k} V_{u_i} \right) P\left(G \in W_{u_1}, \ldots, G \in W_{u_k}, G \notin W_{v_1}, \ldots, G \notin W_{v_l} \right) - \varepsilon \\
&= P\left(\begin{array}{c} F \in \bigcup_{i=1}^{k} V_{u_i}, \\ G \in W_{u_1}, \ldots, G \in W_{u_k}, G \notin W_{v_1}, \ldots, G \notin W_{v_l} \end{array} \right) - \varepsilon \\
&= P\left(\begin{array}{c} (F, G) \in \bigcup_{i=1}^{k} V_{u_i} \times W_{u_i}, \\ G \in W_{u_1}, \ldots, G \in W_{u_k}, G \notin W_{v_1}, \ldots, G \notin W_{v_l} \end{array} \right) - \varepsilon \\
&= P\left(\begin{array}{c} (F, G) \in \bigcup_{i=1}^{m} V_{i} \times W_{i}, \\ G \in W_{u_1}, \ldots, G \in W_{u_k}, G \notin W_{v_1}, \ldots, G \notin W_{v_l} \end{array} \right) - \varepsilon \\
&= P\left((F, G) \in \bigcup_{i=1}^{m} V_i \times W_i,\ R_j \right) - \varepsilon.
\end{aligned}$$

Let $\varepsilon \to 0$ to obtain

$$\begin{aligned}
&\liminf_{n\to\infty} P\left((A_n, B_n) \in U\right) \\
&\geq \liminf_{n\to\infty} P\left((A_n, B_n) \in \bigcup_{i=1}^{m} V_i \times W_i\right) \\
&\geq \sum_{j=1}^{2^m} \liminf_{n\to\infty} P\left((A_n, B_n) \in \bigcup_{i=1}^{m} V_i \times W_i,\ R_j\right) \\
&\geq \sum_{j=1}^{2^m} P\left((A, B) \in \bigcup_{i=1}^{m} V_i \times W_i,\ R_j\right) \\
&= P\left((A, B) \in \bigcup_{i=1}^{m} V_i \times W_i\right) \\
&\geq P\left((A, B) \in U\right) - \eta.
\end{aligned}$$

It remains to let $\eta \to 0$. □

Theorem 2.63 *Let* $\tau_1 \in \{\tau_M, \tau_H\}$, $\tau_2 \in \{\tau_M, \tau_H\}$.
Let $F_n \stackrel{D_{\tau_1}}{\to} F$, $G_n \stackrel{P_{\tau_2}}{\to} G$. *If* F_n *and* G_n, F_n *and* G, F *and* G *are independent, then*

$$(F_n, G_n) \stackrel{D_{\tau_1 \times \tau_2}}{\to} (F, G).$$

Proof. The topological spaces $\left(\mathcal{F}\left(\mathbb{R}^d\right), \tau_M\right)$ and $\left(\mathcal{F}\left(\mathbb{R}^d\right), \tau_H\right)$ are second countable. The proof follows from the previous theorem, since convergence in probability with respect to τ_M or τ_H implies that Condition 2.61 is fulfilled. This is contained in Lemmas 2.11 and 2.9. □

Corollary 2.64 *Let* $\tau_1 \in \{\tau_M, \tau_H\}$, $\tau_2 \in \{\tau_M, \tau_H\}$.
Let $F_n \stackrel{D_{\tau_1}}{\to} F$, $G_n \stackrel{a.s._{\tau_2}}{\to} G$. *If* F_n *and* G_n, F_n *and* G, F *and* G *are independent, then*

$$(F_n, G_n) \stackrel{D_{\tau_1 \times \tau_2}}{\to} (F, G).$$

Proof. This follows immediately from the above theorem, because almost sure convergence with respect to τ_M (resp. τ_H) implies convergence in probability with respect to τ_M (resp. τ_H). □

Since in the case $\tau_2 \in \{\tau_M, \tau_H\}$ Condition 2.61 is equivalent to convergence in probability we cannot expect that the assertion of Theorem 2.63 holds if we assume $G_n \stackrel{D_{\tau_2}}{\to} G$ instead of $G_n \stackrel{P_{\tau_2}}{\to} G$.
We now leave the case $\tau_2 \in \{\tau_M, \tau_H\}$. To obtain $(F_n, G_n)_n \stackrel{D_{\tau_1 \times \tau_2}}{\to} (F, G)$ from $F_n \stackrel{D_{\tau_1}}{\to} F$ and $G_n \stackrel{D_{\tau_2}}{\to} G$ we restrict the possible choices of τ_2. The most important case, which for example includes the Fell topology, is to allow only topologies that are generated by a metric.

Lemma 2.65 *If $F_n \stackrel{D_{\tau_1}}{\to} F$ and if F_n and G, F and G are independent, then*

$$(F_n, G) \stackrel{D_{\tau_1 \times \tau_2}}{\to} (F, G)\,.$$

Proof. From $P\left(G \notin U,\ G \in U\right) = 0$ it follows, that condition 2.61 is fulfilled. Thus the proof follows with Theorem 2.63 . □

The following approximation property for open sets will be helpful. We will later show that it is fulfilled for all metric spaces.

Condition 2.66 If $G_n \stackrel{D_{\tau_2}}{\to} G$, then for all open sets $W_1, \ldots, W_m$ there are sequences of Borel sets $\left(Z_k^j\right)_k$, $j = 1, .., m$ such that $W_i = \bigcup\limits_{k=1}^{\infty} Z_k^i$ and

$$\lim_{n\to\infty} P\left(G_n \in \left(\bigcup_{k=1}^{k_1} Z_k^{i_1}\right) \cap \ldots \cap \left(\bigcup_{k=1}^{k_s} Z_k^{i_s}\right)\right) = P\left(G \in \left(\bigcup_{k=1}^{k_1} Z_k^{i_1}\right) \cap \ldots \cap \left(\bigcup_{k=1}^{k_s} Z_k^{i_s}\right)\right)$$

for all $s \in \{1, \ldots, m\}$ and all $k_1, \ldots, k_s \in \mathbb{N}$.

Theorem 2.67 *Let (T_1, τ_1) and (T_2, τ_2) be second countable topological spaces. Let Condition 2.66 hold for (T_2, τ_2). If F_n and G_n, F_n and G, F and G are independent, then*

$$F_n \stackrel{D_{\tau_1}}{\to} F,\ G_n \stackrel{D_{\tau_2}}{\to} G$$

implies

$$(F_n, G_n) \stackrel{D_{\tau_1 \times \tau_2}}{\to} (F, G).$$

Proof. We have to show that

$$\liminf_{n\to\infty} P((F_n, G_n) \in U) \geq P((F, G) \in U)$$

for all open $U \subset T_1 \times T_2$. Since τ_1 and τ_2 are second countable we can write

$$U = \bigcup_{i=1}^{\infty} V_i \times W_i$$

with suitable $V_i \subset T_1$ and $W_i \subset T_2$ open with respect to τ_1 resp. τ_2. Because of the continuity of the probability measure to each $\varepsilon > 0$ there is an m such that with $U_m = \bigcup_{i=1}^{m} V_i \times W_i$ we have

$$P((F,G) \in U_m) \geq P\left((F,G) \in U\right) - \varepsilon.$$

Since Condition 2.66 is fulfilled, for each W_i, $i = 1, \ldots, m$ there is a sequence of Borel sets $(Z_k^i)_k$ with $W_i = \bigcup_{k=1}^{\infty} Z_k^i$. It follows that there are $k_1, \ldots, k_m$ such that

$$P\left((F,G) \in \bigcup_{i=1}^{m} \left(V_i \times \bigcup_{k=1}^{k_i} Z_k^i\right)\right) \geq P((F,G) \in U_m) - \varepsilon.$$

With the inclusion-exclusion formula ((2.7) in [4]) we obtain

$$
\begin{aligned}
& P\left((F_n, G_n) \in U\right) \\
& \geq P\left((F_n, G_n) \in \bigcup_{i=1}^{m} \left(V_i \times \bigcup_{k=1}^{k_i} Z_k^i\right)\right) \\
& = \sum_{i=1}^{m} P\left((F_n, G_n) \in \left(V_i \times \bigcup_{k=1}^{k_i} Z_k^i\right)\right) \\
& - \sum_{i_1<i_2} P\left((F_n, G_n) \in \left(V_{i_1} \times \bigcup_{k=1}^{k_{i_1}} Z_k^{i_1}\right) \cap \left(V_{i_2} \times \bigcup_{k=1}^{k_{i_2}} Z_k^{i_2}\right)\right) \\
& + - \ldots + (-1)^{m+1} P\left((F_n, G_n) \in \bigcap_{i=1}^{m} \left(V_i \times \bigcup_{k=1}^{k_i} Z_k^i\right)\right) \\
& = \sum_{i=1}^{m} P\left(F_n \in V_i, G_n \in \bigcup_{k=1}^{k_i} Z_k^i\right) \\
& - \sum_{i_1<i_2} P\left(F_n \in V_{i_1} \cap V_{i_2},\ G_n \in \left(\bigcup_{k=1}^{k_{i_1}} Z_k^{i_1}\right) \cap \left(\bigcup_{k=1}^{k_{i_2}} Z_k^{i_2}\right)\right) \\
& + - \ldots + (-1)^{m+1} P\left(F_n \in \bigcap_{i=1}^{m} V_i,\ G_n \in \bigcap_{i=1}^{m} \bigcup_{k=1}^{k_i} Z_k^i\right) \\
& = \sum_{i=1}^{m} P\left(F_n \in V_i\right) P\left(G_n \in \bigcup_{k=1}^{k_i} Z_k^i\right) \\
& - \sum_{i_1<i_2} P\left(F_n \in V_{i_1} \cap V_{i_2}\right) P\left(G_n \in \left(\bigcup_{k=1}^{k_{i_1}} Z_k^{i_1}\right) \cap \left(\bigcup_{k=1}^{k_{i_2}} Z_k^{i_2}\right)\right) \\
& + - \ldots + (-1)^{m+1} P\left(F_n \in \bigcap_{i=1}^{m} V_i\right) P\left(G_n \in \bigcap_{i=1}^{m} \bigcup_{k=1}^{k_i} Z_k^i\right)
\end{aligned}
$$

With Condition 2.66 it follows that there are $\alpha_n^i, \alpha_n^{i_1,i_2}, \alpha_n^{i_1,i_2,i_3}\ , \ldots, \alpha_n^{1,\ldots,m}$ all converging to 0, as $n \to \infty$, such that

$$\begin{aligned}
&P\left((F_n, G_n) \in U\right) \\
&\geq \sum_{i=1}^{m} P\left(F_n \in V_i\right) \left(P\left(G \in \bigcup_{k=1}^{k_i} Z_k^i\right) + \alpha_n^i\right) \\
&- \sum_{i_1 < i_2} P\left(F_n \in V_{i_1} \cap V_{i_2}\right) \left(P\left(G \in \left(\bigcup_{k=1}^{k_{i_1}} Z_k^{i_1}\right) \cap \left(\bigcup_{k=1}^{k_{i_2}} Z_k^{i_2}\right)\right) + \alpha_n^{i_1,i_2}\right) \\
&+ - \ldots + (-1)^{m+1} P\left(F_n \in \bigcap_{i=1}^{m} V_i\right) \left(P\left(G \in \bigcap_{i=1}^{m} \bigcup_{k=1}^{k_i} Z_k^i\right) + \alpha_n^{1,\ldots,m}\right) \\
&= \sum_{i=1}^{m} P\left(F_n \in V_i\right) P\left(G \in \bigcup_{k=1}^{k_i} Z_k^i\right) \\
&- \sum_{i_1 < i_2} P\left(F_n \in V_{i_1} \cap V_{i_2}\right) P\left(G \in \left(\bigcup_{k=1}^{k_{i_1}} Z_k^{i_1}\right) \cap \left(\bigcup_{k=1}^{k_{i_2}} Z_k^{i_2}\right)\right) \\
&+ - \ldots + (-1)^{m+1} P\left(F_n \in \bigcap_{i=1}^{m} V_i\right) P\left(G \in \bigcap_{i=1}^{m} \bigcup_{k=1}^{k_i} Z_k^i\right) \\
&+ R_n^m \\
&= \ldots = P\left((F_n, G) \in \bigcup_{i=1}^{m} \left(V_i \times \bigcup_{k=1}^{k_i} Z_k^i\right)\right) + R_n^m
\end{aligned}$$

where $R_n^m = \sum_{i=1}^{m} P\left(F_n \in V_i\right) \alpha_n^i - \sum_{i_1<i_2} P\left(F_n \in V_{i_1} \cap V_{i_2}\right) \alpha_n^{i_1,i_2} + -\ldots$, which implies that $R_n^m \to 0$ for $n \to 0$, because all $\alpha_n^{\cdots}$ converge to 0. With Lemma 2.65 we achieve

$$\begin{aligned}
&\liminf_{n\to\infty} P\left((F_n, G_n) \in U\right) \\
&\geq \liminf_{n\to\infty} P\left((F_n, G) \in \bigcup_{i=1}^{m} \left(V_i \times \bigcup_{k=1}^{k_i} Z_k^i\right)\right) + \liminf_{n\to\infty} R_n^m \\
&\geq P\left((F, G) \in \bigcup_{i=1}^{m} \left(V_i \times \bigcup_{k=1}^{k_i} Z_k^i\right)\right) \\
&\geq P((F,G) \in U_m) - \varepsilon \\
&\geq P((F,G) \in U) - 2\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ completes the proof. $\square$

Lemma 2.68 *For every metric space T with metric d, Condition 2.66 is fulfilled.*

Proof. Let $W \neq \emptyset$ be open, let $D = T \backslash W$, then D is closed. Consider the set $\{x : d(x, D) > \varepsilon\}$. Since D is closed, we have $d(x, D) = 0$ for all $x \in D$ and $d(x, D) > 0$ for all $x \notin D$. It follows that

$$W = \bigcup_{\varepsilon > 0} \{x : d(x, D) > \varepsilon\}.$$

For the boundaries we obtain

$$\operatorname{bdy}\{x : d(x, D) > \varepsilon\} \subset \{x : d(x, D) = \varepsilon\}.$$

Assume that $y \in \operatorname{bdy}\{x : d(x, D) > \varepsilon\}$, but $d(y, D) \neq \varepsilon$. In the first case let $d(y, D) < \varepsilon$, then $B_{\frac{\varepsilon}{2}}(y) \subset \{x : d(x, D) < \varepsilon\}$ and it follows that $y \notin \operatorname{bdy}\{x : d(x, D) > \varepsilon\}$. The second case $d(y, D) > \varepsilon$ is dealt with analogously.
It follows that $\operatorname{bdy}\{x : d(x, D) > \varepsilon_1\} \cap \operatorname{bdy}\{x : d(x, D) > \varepsilon_2\} = \emptyset$ for $\varepsilon_1 \neq \varepsilon_2$. Thus $P(G \in \operatorname{bdy}\{x : d(x, D) > \varepsilon\}) > 0$ can hold for only countably many ε. We can then find a sequence $(\varepsilon_k)_k$ such that with $Z_k = \{x : d(x, D) > \varepsilon_k\}$:

$$W = \bigcup_{k=1}^{\infty} Z_k$$

and $P(G \in \operatorname{bdy} Z_k) = 0$. From $G_n \xrightarrow{D_\tau} G$ we obtain $\lim_{n\to\infty} P(G_n \in Z_k) = P(G \in Z_k)$.
Now let $W_1, \ldots, W_m$ be open. In the above way, we find sequences $(Z_k^i)_k$ for each i. From the fact that $\operatorname{bdy}(A \cap B) \subset \operatorname{bdy}(A) \cup \operatorname{bdy}(B)$ and $\operatorname{bdy}(A \cup B) \subset \operatorname{bdy}(A) \cup \operatorname{bdy}(B)$ for arbitrary sets A and B we obtain

$$\begin{aligned}
&P\left(G \in \operatorname{bdy}\left(\left(\bigcup_{k=1}^{k_1} Z_k^{i_1}\right) \cap \ldots \cap \left(\bigcup_{k=1}^{k_s} Z_k^{i_s}\right)\right)\right) \\
&\leq P\left(G \in \bigcup_{j=1}^{s} \bigcup_{k=1}^{k_j} \operatorname{bdy}\left(Z_k^{i_j}\right)\right) \\
&\leq \sum_{j=1}^{s} \sum_{k=1}^{k_j} P\left(G \in \operatorname{bdy}\left(Z_k^{i_j}\right)\right) = 0
\end{aligned}$$

and it follows from $G_n \stackrel{D_\tau}{\to} G$, that

$$\lim_{n\to\infty} P\left(G_n \in \left(\left(\bigcup_{k=1}^{k_1} Z_k^{i_1}\right) \cap \ldots \cap \left(\bigcup_{k=1}^{k_s} Z_k^{i_s}\right)\right)\right)$$
$$= P\left(G \in \left(\left(\bigcup_{k=1}^{k_1} Z_k^{i_1}\right) \cap \ldots \cap \left(\bigcup_{k=1}^{k_s} Z_k^{i_s}\right)\right)\right). \qquad \square$$

Corollary 2.69 *If* $F_n \stackrel{D_{\tau_1}}{\to} F$ *, where* $\tau_1 \in \{\tau_M, \tau_H\}$ *,* $G_n \stackrel{D_{\tau_{Fell}}}{\to} G$ *and if* F_n *and* F*,* F_n *and* G*,* F *and* G *are independent, then*

$$(F_n, G_n) \stackrel{D_{\tau_1 \times \tau_{Fell}}}{\to} (F, G).$$

The following example shows, that Condition 2.66 is not fulfilled for the topology τ_M.

Example 2.70 Let $P(G = \{-3\}) = P(G = [1,4]) = \frac{1}{2}$,
let $P(G_n = \{-3\}) = P(G_n = \{3\}) = \frac{1}{2}$. Let U be τ_M−open, then $U = \bigcup_{i=1}^{\infty} M(K_i)$. Let $K \subset \mathbb{R}$ be compact. If $[1,4] \in M(K)$, then $\{3\} \in M(K)$ and thus $P(G_n \in U) \geq P(G \in U)$, which shows $G_n \stackrel{D_{\tau_M}}{\to} G$. To show that Condition 2.66 is not fulfilled, consider the open set $W = M([-4,2])$. Assume that there is a sequence $(Z_k)_k$ of measureable sets with $W = \bigcup_{k=1}^{\infty} Z_k$. From $\{3\} \in W$ it follows that there is k_0 such that $\{3\} \in Z_{k_0}$ and $Z_{k_0} \subset W$ implies that $\{-3\} \notin Z_{k_0}$. This yields $\lim_{n\to\infty} P(G_n \in Z_{k_0}) = \lim_{n\to\infty} P(G_n = \{3\}) = \frac{1}{2} \neq 0 = P(G \in Z_{k_0})$.

Similarly for the topology τ_H :

Example 2.71 Let $P(G = \{-3\}) = P(G = \{3\}) = \frac{1}{2}$,
$P(G_n = \{-3\}) = P(G_n = [1,4]) = \frac{1}{2}$. Then $G_n \stackrel{D_{\tau_H}}{\to} G$. Next consider the τ_H−open set $W = H((0,2))$. If there is a sequence $(Z_k)_k$ of measurable sets with $W = \bigcup_{k=1}^{\infty} Z_k$, then there is k_0 such that $[1,4] \in Z_{k_0}$. This follows from $[1,4] \in W$. From $\{-3\}, \{3\} \notin W$ it follows that $\{-3\}, \{3\} \notin Z_k$. Thus we obtain $\lim_{n\to\infty} P(G_n \in Z_{k_0}) = \lim_{n\to\infty} P(G_n = [1,4]) = \frac{1}{2} \neq 0 = P(G \in Z_{k_0})$.

These examples furthermore show, that for the topologies τ_M and τ_H there is no convergence determining class in the sense of Theorem 2.2. of [5].
The question, whether for example $F_n \stackrel{D_{\tau_M}}{\to} F$ and $G_n \stackrel{D_{\tau_M}}{\to} G$ imply that
$(F_n, G_n) \stackrel{D_{\tau_M \times \tau_M}}{\to} (F, G)$ under the independence assumptions of Theorem 2.67 , remains

open. We have seen in the above examples, that the method used for second countable metric spaces does not work for the topologies τ_M and τ_H. Nevertheless our results are useful, as for example in a stochastic optimisation problem it might be known that only the epigraphs of the objective functions converge distribution with respect to τ_M while the restriction sets converge in distribution in the stronger τ_{Fell} sense.
As an important application of convergence in distribution in product spaces we deal with sums of lower semicontinuous functions. Often the lower semicontinuous functions under investigation are the sum of lower semicontinuous functions or the sum of a continuous and a lower semicontinuous function; take for example the case of an objective function which is the sum of a continuous function and a lower semicontinuous penalty function. Recall that for lower semicontinuous f and g the function $f + g$, which is as usual defined by $(f+g)(x) = f(x) + g(x)$, is also lower semicontinuous, if f and g are proper lower semicontinuous functions. A lower semicontinuous function f is called proper, if $f(x) < \infty$ for at least one x and $f(x) > -\infty$ for all x.

Theorem 2.72 *Let $(f_n)_n, f, (g_n)_n, g$ be proper random lower semicontinuous functions. If*

$$(\operatorname{epi} f_n, \operatorname{epi} g_n) \overset{D_{\tau_M} \times D_{\tau_M}}{\to} (\operatorname{epi} f, \operatorname{epi} g),$$

then

$$\operatorname{epi}(f_n + g_n) \overset{D_{\tau_M}}{\to} \operatorname{epi}(f+g).$$

Proof. We show that the mapping $(\operatorname{epi} f, \operatorname{epi} g) \mapsto \operatorname{epi}(f+g)$ is $\tau_M \times \tau_M - \tau_M$ continuous. Let $(x_n, y_n) \in \operatorname{epi}(f_n + g_n)$ and $(x_n, y_n) \to (x, y)$. It is to show that $(x, y) \in \operatorname{epi}(f+g)$. Note that $(\operatorname{epi} f_n, \operatorname{epi} g_n) \overset{\tau_M \times \tau_M}{\to} (\operatorname{epi} f, \operatorname{epi} g)$ implies $\operatorname{epi} f_n \overset{\tau_M}{\to} \operatorname{epi} f$ and $\operatorname{epi} g_n \overset{\tau_M}{\to} \operatorname{epi} g$. It follows that $\liminf_{n\to\infty} f_n(x_n) \geq f(x)$ and $\liminf_{n\to\infty} g_n(x_n) \geq g(x)$.
We obtain

$$\begin{aligned} y =& \liminf_{n\to\infty} y_n \\ \geq& \liminf_{n\to\infty} (f_n(x_n) + g_n(x_n)) \\ \geq& \liminf_{n\to\infty} (f_n(x_n)) + \liminf_{n\to\infty} (g_n(x_n)) \\ \geq& f(x) + g(x), \end{aligned}$$

which shows that $(x, y) \in \operatorname{epi}(f+g)$. Applying the continuous mapping theorem completes the proof. □

The following (deterministic) example shows that the situation is different in the τ_H-case.
Let

$$f_n(x) = \begin{cases} 1 & , \; x \neq \frac{1}{n} \\ 0 & , \; x = \frac{1}{n} \end{cases} \; , \; f(x) = \begin{cases} 1 & , \; x \neq 0 \\ 0 & , \; x = 0 \end{cases}$$

and

$$g_n(x) = \begin{cases} 1 & , \; x \neq -\frac{1}{n} \\ 0 & , \; x = -\frac{1}{n} \end{cases} \; , \; g(x) = \begin{cases} 1 & , \; x \neq 0 \\ 0 & , \; x = 0 \end{cases}$$

It is easy to verify that $\operatorname{epi} f_n \stackrel{\tau_H}{\to} \operatorname{epi} f$ and $\operatorname{epi} g_n \stackrel{\tau_H}{\to} \operatorname{epi} g$. Note that we even have convergence with respect to τ_{Fell}. It follows that $(\operatorname{epi} f_n, \operatorname{epi} g_n) \stackrel{\tau_{\text{Fell}} \times \tau_{\text{Fell}}}{\longrightarrow} (\operatorname{epi} f, \operatorname{epi} g)$ and thus $(\operatorname{epi} f_n, \operatorname{epi} g_n) \stackrel{\tau_H \times \tau_H}{\longrightarrow} (\operatorname{epi} f, \operatorname{epi} g)$. But from

$$f_n(x) + g_n(x) = \begin{cases} 2 & , \; x \notin \left\{-\frac{1}{n}, \frac{1}{n}\right\} \\ 1 & , \; x \in \left\{-\frac{1}{n}, \frac{1}{n}\right\} \end{cases} \; , \; f(x) + g(x) = \begin{cases} 2 & , \; x \neq 0 \\ 0 & , \; x = 0 \end{cases}$$

it follows that $\operatorname{epi}(f_n + g_n) \stackrel{\tau_H}{\not\to} \operatorname{epi}(f + g)$, since there is no sequence $((x_n, y_n))_n$ with $(x_n, y_n) \in \operatorname{epi}(f_n + g_n)$ and $(x_n, y_n) \to (0, 0) \in \operatorname{epi}(f + g)$.

To obtain convergence in distribution of $\operatorname{epi}(f_n + g_n)$ with respect to τ_H we impose stronger conditions on $(g_n)_n$. Let τ_c denote the topology of uniform convergence on compact sets.

Theorem 2.73 *Let $(f_n)_n, f, (g_n)_n, g$ be proper random lower semicontinuous functions. If*

$$(\operatorname{epi} f_n, g_n) \stackrel{D_{\tau_H} \times D_{\tau_c}}{\longrightarrow} (\operatorname{epi} f, g),$$

then

$$\operatorname{epi}(f_n + g_n) \stackrel{D_{\tau_H}}{\to} \operatorname{epi}(f + g).$$

Proof. We show that the mapping $(\operatorname{epi} f, g) \mapsto \operatorname{epi}(f+g)$ is $\tau_H \times \tau_p - \tau_H$ continuous. Let $(x, y) \in \operatorname{epi}(f+g)$. We have to show that there is a sequence $((x_n, y_n))_n$ with $(x_n, y_n) \to (x, y)$ and $(x_n, y_n) \in \operatorname{epi}(f_n + g_n)$ for $n \geq n_0$. First assume that $(x, y) = (x, f(x) + g(x))$. From $(x, f(x)) \in \operatorname{epi} f$ and $\operatorname{epi} f_n \stackrel{\tau_H}{\to} \operatorname{epi} f$ it follows that there is $((x_n, \widetilde{y}_n))_n$ such that $(x_n, \widetilde{y}_n) \to (x, y)$ and $(x_n, \widetilde{y}_n) \in \operatorname{epi} f$ for $n \geq n_0$. Now let $y_n = \widetilde{y}_n + g_n(x)$, then $(x_n, y_n) \in \operatorname{epi}(f + g)$ for $n \geq n_0$ and with the pointwise convergence of $(g_n)_n$ to g it follows that $(x_n, y_n) \to (x, f(x) + g(x))$.
In the general case $(x, y) \in \operatorname{epi}(f + g)$ we have $y = f(x) + g(x) + \delta$ for some $\delta > 0$ and with $\widetilde{y}_n$ from the first case we choose $y_n = \widetilde{y}_n + g_n(x) + \delta$. Then $(x_n, y_n)_n \in \operatorname{epi}(f_n + g_n)$

for $n \geq n_0$ and $(x_n, y_n) \to (x, f(x) + g(x) + \delta) = (x, y)$.
It remains to apply the continuous mapping theorem. $\square$

Note that in the proof we have only used the pointwise convergence of $(g_n)_n$. We have assumed uniform convergence on compact sets because of the topological difficulties that occur when we work with pointwise convergence, e.g. lack of first countability.
The case of convergence in distribution with respect to τ_{Fell} is covered in Theorem 4 of [19].

3. Stochastic Optimisation

In this chapter we show how the theory of τ_M and τ_H convergence in distribution, which was investigated in the previous chapters, can be applied to stability theory of stochastic optimisation problems. We deal with stochastic optimisation problems of the form

$$\inf_{x \in C(\omega)} f(x, \omega)$$

with $C(\omega) \in \mathcal{F}(X)$, i.e. optimisation problems with a random lower semicontinuous objective function and a random closed restriction set.

Recall from Section 1.3, that we assume $\mathcal{B}_{\tau_{\text{Fell}}}$-measurability of the mapping $\omega \mapsto \operatorname{epi} f(.,\omega)$. If additionally $\omega \mapsto C(\omega)$ is $\mathcal{B}_{\tau_{\text{Fell}}}$-measurable, it can be shown as in Theorem 14.37 of [33], that the random optimal value function $\omega \mapsto \inf\limits_{x \in C(\omega)} f(x, \omega)$ is Borel measurable.

Often the original problem is approximated by a sequence of surrogate problems

$$\inf_{x \in C_n(\omega)} f_n(x, \omega), \;\; n \in \mathbb{N}.$$

The sequences of surrogate problems can be obtained in various ways. One frequently used method is to approximate the objective function by functions that are easier to deal with, either directly or numerically. For example continuous functions are approximated by polynomials. In the probabilistic setting we have seen that the Brownian Motion can be approximated by random walks, whose minima and argmin sets can easily be determined by a comparison of finitely many values. Other methods to create random surrogate problems are the estimation of parameters in the original problem and simulations.

Throughout this chapter we denote the optmial values by

$$\Phi\left(\omega\right) = \inf_{x \in C(\omega)} f(x,\omega)$$
$$\Phi_n\left(\omega\right) = \inf_{x \in C_n(\omega)} f_n(x,\omega).$$

If the functions $f_n,\ f$ are lower semicontinuous and if the restriction sets are compact, the infima in the above are actually minima and we can define the argmin sets as

$$\Psi\left(\omega\right) = \{x \in C(\omega) : f(x,\omega) = \Phi\left(\omega\right)\}$$
$$\Psi_n\left(\omega\right) = \{x \in C_n(\omega) : f_n(x,\omega) = \Phi_n\left(\omega\right)\}.$$

In the following we will not denote the dependence on ω of the objective functions, restriction sets, minima and argmin sets. It will always be clear whether we deal with the deterministic or with the random case.
When dealing with approximation in optimisation problems naturally the question arises, under which conditions do $f_n \to f$ and $C_n \to C$ imply that $\Phi_n \to \Phi$ and $\Psi_n \to \Psi$, and which types of convergence are suitable. Note that usually the argmin is not unique and thus set convergence methods can be applied. Besides convergence of the argmin sets and of sequences of argmins, the behaviour of the minimal values is of interest. Consider for example a ruin model (compare Section 2.4), where additionally to the time of ruin the probability of the event $\{\Phi \leq 0\}$ (i.e. the probability of ruin) can be investigated. The cases of almost sure convergence and of convergence in probability have been dealt with in [39], [41] . In [34], [43], [27], [28] convergence in distribution has been taken into consideration. Here we will present known results and develop extensions, e.g. one-sided convergence with respect to τ_H and consideration of points of ε-optimality. We will show, that recent results e.g. [12] can also be obtained with our methods. In the proofs we will frequently transfer results from stability theory in the deterministic case, which can be found in [2] and [43] or which are developed in this section, to the convergence in distribution setting with the help of the Continuous Mapping Theorem or its semicontinuous versions. Since for example the minimum Φ can depend lower semicontinuously on the objective function and on the restriction set, we often need convergence in distribution in product spaces, for which sufficient conditions have been established in Section 2.6 .

3.1. Restriction Sets

First we deal with convergence of the restriction sets. For the moment we assume that, with real valued functions g_n^j on a topological space, the restriction sets C_n take the form

$$C_n = \left\{ x : g_n^j(x) \leq 0,\ j = 1, \ldots, r \right\},$$

i.e. we deal with inequality constraints. Thus besides applications in stochastic optimisation, the results are of general interest for the solution sets of random inequalities, see [25]. Note that with the help of $\leq$-level sets, C_n can be written as

$$C_n = \bigcap_{j=1}^{r} \operatorname{lev}_{\leq 0}\left(g_n^j\right).$$

Since we are always interested in closed sets, the following lemma is helpful.

Lemma 3.1 *If the functions g_n^j, $j = 1, \ldots, r$ are lower semicontinuous, then C_n is closed.*

Proof. For a lower semicontinuous function g_n^j the sublevel set $\operatorname{lev}_{\leq a}\left(g_n^j\right)$ is closed. Thus C_n is closed as the intersection of finitely many closed sets. □

The following is the main result for τ_M convergence in distribution of the restriction sets. We assume that for each occuring random lower semicontinuous function g the mapping $\omega \mapsto \operatorname{epi} g(., \omega)$ is at least $\mathcal{B}_{\tau_M}$-measurable.

Theorem 3.2 *Let g_n^j, g^j, $j = 1, \ldots, r$ be random lower semicontinuous functions. If $\left(\operatorname{epi} g_n^1, \ldots, \operatorname{epi} g_n^r\right) \stackrel{D_\tau}{\to} \left(\operatorname{epi} g^1, \ldots, \operatorname{epi} g^r\right)$, where $\tau = \tau_M(EPI(p)) \times \ldots \times \tau_M(EPI(p))$, then*

$$C_n \stackrel{D_{\tau_M}}{\to} C.$$

Proof. In view of the Continuous Mapping Theorem it suffices to show that for deterministic lower semicontinuous functions h^j the mapping

$$S\left(\operatorname{epi} h^1, \ldots, \operatorname{epi} h^r\right) = \bigcap_{j=1}^{r} \operatorname{lev}_{\leq 0}\left(h^j\right)$$

is continuous with respect to the topologies τ and τ_M.
Let $\left(\operatorname{epi} h_n^1, \ldots, \operatorname{epi} h_n^r\right) \stackrel{\tau}{\to} \left(\operatorname{epi} h^1, \ldots, \operatorname{epi} h^r\right)$. We have to show that

$x_{n_k} \in S\left(\operatorname{epi} h^1_{n_k}, \ldots, \operatorname{epi} h^r_{n_k}\right)$ and $x_{n_k} \to x$ implies $x \in S\left(\operatorname{epi} h^1, \ldots, \operatorname{epi} h^r\right)$.
For each $j = 1, \ldots, r$ we have

$$h^j(x) \leq \liminf_{k \to \infty} h^j_{n_k}(x_{n_k}) \leq 0$$

and thus $x \in S\left(\operatorname{epi} h^1, \ldots, \operatorname{epi} h^r\right)$. □

The theorem can be extended to the sets of points that nearly solve the inequalities in the definition of the restriction sets.

Corollary 3.3 *Let the assumptions of Theorem 3.2 be fulfilled. Let $(\varepsilon_n)_n$ be a sequence of random variables with $\varepsilon_n \geq 0$ and $\varepsilon_n \xrightarrow{p} 0$, then*

$$\left\{x : g^j_n \leq \varepsilon_n,\ j = 1, \ldots, r\right\} \stackrel{D_{\tau_M}}{\to} \left\{x : g^j \leq 0,\ j = 1, \ldots, r\right\}.$$

Proof. From $\{x : g^j \leq 0,\ j = 1, \ldots, r\} = \bigcap_{\varepsilon > 0} \{x : g^j \leq \varepsilon,\ j = 1, \ldots, r\}$ it follows with the continuity of the probability measure, that for each $\eta > 0$ and each τ_M-open U we can find $\varepsilon > 0$ such that

$$P\left(\left\{x : g^j \leq \varepsilon,\ j = 1, \ldots, r\right\} \in U\right) \geq P\left(\left\{x : g^j \leq 0,\ j = 1, \ldots, r\right\} \in U\right) - \eta.$$

With the same method as in the proof of Theorem 3.2 we can show that

$$\left\{x : g^j_n \leq \varepsilon,\ j = 1, \ldots, r\right\} \stackrel{D_{\tau_M}}{\to} \left\{x : g^j \leq \varepsilon,\ j = 1, \ldots, r\right\}$$

for all $\varepsilon > 0$.
With $\varepsilon_n \xrightarrow{p} 0$ it follows that

$$\begin{aligned}
&\liminf_{n \to \infty} P\left(\left\{x : g^j_n \leq \varepsilon_n,\ j = 1, \ldots, r\right\} \in U\right) \\
\geq &\liminf_{n \to \infty} P\left(\left\{x : g^j_n \leq \varepsilon,\ j = 1, \ldots, r\right\} \in U\right) - \eta \\
\geq &P\left(\left\{x : g^j \leq 0,\ j = 1, \ldots, r\right\} \in U\right) - 2\eta
\end{aligned}$$

and it remains to let $\eta \to 0$. □

To obtain convergence in distribution of the restriction sets with respect to τ_H we impose different and stronger conditions on the functions g^j_n and g^j. The lower semicontinuity assumption is replaced by upper semicontinuity and moreover we demand that for one j (without loss of generality $j = 1$) the functions g_n^1, g^1 are continuous. Under these conditions it cannot be ensured, that the definition used so far for the restriction sets leads to closed sets. Thus we work with $\operatorname{cl}\{x : g^j(x) < 0,\ j = 1,\dots,r\}$ instead of $\{x : g^j(x) \le 0,\ j = 1,\dots,r\}$.

Theorem 3.4 *Let g_n^1, g^1 be continuous, let g_n^j, g^j , $j = 2,\dots,r$, $n \in \mathbb{N}$ be upper semicontinuous, then*

$$\left(\operatorname{graph} g_n^1, \operatorname{hypo} g_n^2, \dots, \operatorname{hypo} g_n^r\right) \overset{D_\tau}{\to} \left(\operatorname{graph} g^1, \operatorname{hypo} g^2, \dots, \operatorname{hypo} g^r\right),$$

with $\tau = \tau_{Fell}\left(GRA(p)\right) \times \tau_M\left(HYP(p)\right) \ldots \times \tau_M\left(HYP(p)\right)$ implies

$$\operatorname{cl}\left\{x : g_n^j(x) < 0,\ j = 1,\dots,r\right\} \overset{D_{\tau_H}}{\to} \operatorname{cl}\left\{x : g^j(x) < 0,\ j = 1,\dots,r\right\}.$$

Proof. We show that for deterministic continuous h^1 and upper semicontinuous h^j, $j = 1,\dots,r$ the mapping

$$S\left(\operatorname{graph} h^1, \operatorname{hypo} h^2, \dots, \operatorname{hypo} h^r\right) = \operatorname{cl}\left\{x : h^j < 0,\ j = 1,\dots,r\right\}$$

is continuous with respect to the topologies τ and τ_H. It will then remain to apply the Continuous Mapping Theorem.
Let $\left(\operatorname{graph} h_n^1, \operatorname{hypo} h_n^2, \dots, \operatorname{hypo} h_n^r\right) \xrightarrow{\tau} \left(\operatorname{graph} h^1, \operatorname{hypo} h^2, \dots, \operatorname{hypo} h^r\right)$.
If $\{x : h^j(x) < 0,\ j = 1,\dots,r\} \neq \emptyset$, then let $x_0 \in \{x : h^j(x) < 0,\ j = 1,\dots,r\}$. For each j there are $\varepsilon_j > 0$ and $n_j \in \mathbb{N}$ such that $h_n^j(x) < 0$ for all $x \in B_{\varepsilon_j}(x_0)$ and all $n \ge n_j$. Suppose that this is not true, then there is a sequence $(x_{n_k})_k$ converging to x_0 with $h_{n_k}^j(x_{n_k}) \ge 0$ and it follows that $\limsup_{k\to\infty} h_{n_k}^j(x_{n_k}) \ge 0 > h^j(x_0)$. Because of Lemma ... this is a contradiction to $\limsup_{n\to\infty} \operatorname{hypo} h_n^j \subset \operatorname{hypo} h^j$ (resp. $\operatorname{graph} h_n^1 \overset{\tau_{\text{Fell}}}{\to} \operatorname{graph} h^1$). Let $\varepsilon := \min\{\varepsilon_j : j = 1,\dots,r\}$ and $n' = \max\{n_j : j = 1,\dots,r\}$, then

$$h_n^j(x) < 0 \text{ for all } n \ge n',\ x \in B_\varepsilon(x_0) \text{ and } j = 1,\dots,r.$$

Since $\operatorname{graph} h_n^1 \to \operatorname{graph} h^1$ there is a sequence $\left((x_n, h_n^1(x_n))\right)_n$ with $(x_n, h_n^1(x_n)) \in \operatorname{graph} h_n^1$ and $(x_n, h_n^1(x_n)) \to (x_0, h^1(x_0))$. Now $x_n \to x_0$ implies that $x_n \in B_\varepsilon(x_0)$ for all $n \ge n''$. By choosing $\widetilde{n} := \max\{n', n''\}$ we obtain

$$x_n \in \left\{x : h_n^j(x) < 0,\ j = 1,\dots,r\right\}$$

and thus

$$x_n \in \operatorname{cl}\left\{x : h_n^j(x) < 0,\ j = 1, \ldots, r\right\}$$

for all $n \geq \widetilde{n}$.

Now let $x \in \operatorname{cl}\{x : h^j(x) < 0,\ j = 1, \ldots, r\}$. Then there is a sequence $\left(x_0^k\right)_k$ with $x_0^k \in \{x : h^j(x) < 0,\ j = 1, \ldots, r\}$ and $x_0^k \to x$. As we have seen, there are sequences $\left(x_n^k\right)_n$ with $x_n^k \in \operatorname{cl}\{x : h_n^j(x) < 0,\ j = 1, \ldots, r\}$ and $x_n^k \to x_0^k$ for $n \to \infty$. Thus we can find a sequence $(\widetilde{x}_n)_n$ such that $\widetilde{x}_n \in \operatorname{cl}\{x : h_n^j(x) < 0,\ j = 1, \ldots, r\}$ and $\widetilde{x}_n \to x$.

If $\{x : h^j(x) < 0,\ j = 1, \ldots, r\} = \emptyset$, then $\operatorname{cl}\{x : h^j(x) < 0,\ j = 1, \ldots, r\} = \emptyset$ and we have $\operatorname{cl}\{x : h_n^j(x) < 0,\ j = 1, \ldots, r\} \stackrel{\tau_H}{\to} \operatorname{cl}\{x : h^j(x) < 0,\ j = 1, \ldots, r\}$, because each sequence of closed sets converges to the empty set with respect to τ_H. □

Under additional assumptions we obtain a result for our initial definition of the restriction sets from the beginning of this section.

Corollary 3.5 *If additionally, to the assumptions of Theorem 3.4,* $\{x : g^j(x) \leq 0,\ j = 1, \ldots, r\} \subset \operatorname{cl}\{x : g^j(x) < 0,\ j = 1, \ldots, r\}$ *almost surely, and if each* g_n^j *is continuous, then*

$$\left\{x : g_n^j(x) \leq 0,\ j = 1, \ldots, r\right\} \stackrel{D_{\tau_H}}{\to} \left\{x : g^j(x) \leq 0,\ j = 1, \ldots, r\right\}.$$

Proof. From the continuity of the g_n^j it follows that $\operatorname{cl}\{x : g_n^j(x) < 0,\ j = 1, \ldots, r\} \subset \{x : g_n^j(x) \leq 0,\ j = 1, \ldots, r\}$.

For each τ_H-open U we have

$$\begin{aligned}
&\liminf_{n\to\infty} P\left(\left\{x : g_n^j(x) \leq 0,\ j = 1, \ldots, r\right\} \in U\right) \\
&\geq \liminf_{n\to\infty} P\left(\operatorname{cl}\left\{x : g_n^j(x) < 0,\ j = 1, \ldots, r\right\} \in U\right) \\
&\geq P\left(\operatorname{cl}\left\{x : g^j(x) < 0,\ j = 1, \ldots, r\right\} \in U\right) \\
&\geq P\left(\left\{x : g^j(x) \leq 0,\ j = 1, \ldots, r\right\} \in U\right).
\end{aligned}$$

□

Unfortunately the additional condition in 3.5 is not always fulfilled, even in the case of continuous functions. Let for example $g(x) = 0$ for all $x \in \mathbb{R}$, then $\{x : g(x) \leq 0\} = \mathbb{R} \not\subset \operatorname{cl}\{x : g(x) < 0\} = \emptyset$.

To overcome this problem, stronger assumptions on the g^j are imposed and it is demanded that the set of Slater points $\{x : g^j(x) < 0,\ j = 1, \ldots, m\}$ is not empty. The following result is contained in Theorem 3.1.6 of [2]. We give a short proof because of the notational differences to [2]

Lemma 3.6 *If each g^j is convex and continuous and if there is z such that $g^j(z) < 0,\ j = 1, \dots, r$, then*

$$\left\{x : g^j(x) \le 0,\ j = 1, \dots, r\right\} = \mathrm{cl}\left\{x : g^j(x) < 0,\ j = 1, \dots, r\right\}.$$

Proof. It remains to show that $\{x : g^j(x) \le 0,\ j = 1, \dots, r\} \subset \mathrm{cl}\,\{x : g^j(x) < 0,\ j = 1, \dots, r\}$. Let $g^j(x) \le 0,\ j = 1, \dots, r$. If $g^j(x) < 0$ for all j, then $x \in \mathrm{cl}\,\{x : g^j(x) < 0,\ j = 1, \dots, r\}$. Now assume that $g^j(x) = 0$ for at least one j. Let $J = \{j : g^j(x) = 0\}$. From $g^j(z) < 0$ and the convexity of g^j it follows that $g^j(y) < g^j(x) = 0,\ j \in J$, for all y on the line segment connecting z and x. From $g^j(x) < 0$ for all $j \notin J$ we obtain that $g^j(x) < 0$ for all those x and for $j \notin J$. It follows that we can find $(y_k)_k$ on this line segment with $y_k \to z$ and $g^j(y_k) < 0,\ j = 1, \dots, r$. This shows $z \in \mathrm{cl}\,\{x : g^j(x) < 0,\ j = 1, \dots, r\}$. □

The example $g_n(x) = g(x) = 0$, for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$ shows that the condition in the above corollary is not a necessary condition.

3.2. Minimum and Argmin

In this section we investigate stability of stochastic optimisation problems in the setting of τ_M- and τ_H-convergence in distribution. In [34] Fell convergence in distribution was considered. Vogel dealt with one-sided versions of almost sure convergence and convergence in probability in [39]. In [43] Vogel investigated τ_M convergence in distribution, while Lachout in [22] established sensitivity results for nets of random functions in combination with the localisation concepts from [21]. We will extend the results from [43] to points of ε_n-optimality and provide corresponding τ_H-results.

For the minimal values in the presence of random restriction sets Vogel in [43] has obtained a result under the following stochastic compactness condition.

Definition 3.7 (Definition 4.1 in [43]) The sequence $((f_n, C_n))_n$ is called equi-inf bounded if for each $\omega \in \Omega$ and each $y \in \mathbb{R}$ there is a compact set $K(\omega, y)$ such that

$$\lim_{n\to\infty} P\left(\{x \in C_n(\omega) : f_n(x, \omega) \le y\} \subset K(\omega, y)\right) = 1.$$

Theorem 3.8 (Theorem 4.2 in [43]) *Let f_n, f be random lsc-functions, let C_n, C be random closed sets. Additionally, assume that the sequence $(f_n, C_n)_n$ is equi-inf bounded. Then*

$$(\text{epi}\, f_n, C_n) \overset{D_{\tau_M \times \tau_M}}{\longrightarrow} (\text{epi}\, f, C)$$

implies

$$\limsup_{n\to\infty} P(\Phi_n \le c) \le P(\Phi \le c)$$

for all $c \in \mathbb{R}$ with $P(\Phi = c) = 0$.

An inspection of the proof shows that the condition $P(\Phi = c) = 0$ is not necessary.

The assertion of the theorem is an 'in distribution' version of the lower semicontinuity property for the minimal value function. We will now provide a corresponding version of the upper semicontinuity property. As Theorem 4.2.1. in [2] shows we cannot expect to obtain upper semicontinuity of the optimal value function, if only $\text{epi}\, f_n \overset{\tau_M}{\to} \text{epi}\, f$ for lower semicontinuous functions f_n, f. Hence we work with the hypographs of upper semicontinuous functions in the following theorem.

Theorem 3.9 *Let f_n, f be random upper semicontinuous functions, let C_n, C be random closed sets.*
Then

$$(\text{hypo}\, f_n, C_n) \overset{D_{\tau_M \times \tau_H}}{\longrightarrow} (\text{hypo}\, f, C)$$

implies

$$\limsup_{n\to\infty} P(\Phi_n \ge c) \le P(\Phi \ge c)$$

for all $c \in \mathbb{R}$.

Proof. It suffices to show, that for deterministic upper semicontinuous functions g and closed sets D the mapping

$$S(\text{hypo}\, g, D) = \inf\{g(x) : x \in D\}$$

is upper semicontinuous with respect to $\tau_M \times \tau_H$ and to apply the Usc-Mapping Theorem 1.28(b). Let $(\text{hypo}\, g_n, D_n) \overset{\tau_M \times \tau_H}{\longrightarrow} (\text{hypo}\, g, D)$, then $\text{hypo}\, g_n \overset{\tau_M}{\to} \text{hypo}\, g$ and $D_n \overset{\tau_H}{\to} D$. First assume that $S(\text{hypo}\, g, D) > -\infty$. For $\varepsilon > 0$ let $x \in D$ such that $g(x) \le S(\text{hypo}\, g, D) + \varepsilon$. Because of $D_n \overset{\tau_H}{\to} D$ there is a sequence $(x_n)_n$ with $x_n \to x$ and $x_n \in D_n$. It follows that

$$\begin{aligned}\limsup_{n\to\infty} S\left(\operatorname{hypo} g_n, D_n\right) &\leq \limsup_{n\to\infty} g_n(x_n) \\ &\leq g(x) \\ &\leq S\left(\operatorname{hypo} g, D\right) + \varepsilon.\end{aligned}$$

Here $\operatorname{hypo} g_n \overset{\tau_M}{\to} \operatorname{hypo} g$ was used to obtain the second inequality. It remains to let $\varepsilon \to 0$. Now assume that $S\left(\operatorname{hypo} g, D\right) = -\infty$. For $\varepsilon > 0$ choose $x \in D$ such that $g(x) \leq -\frac{1}{\varepsilon}$. There is a sequence $(x_n)_n$ with $x_n \to x$ and $x_n \in D_n$. We obtain

$$\begin{aligned}\limsup_{n\to\infty} S\left(\operatorname{hypo} g_n, D_n\right) &\leq \limsup_{n\to\infty} g_n\left(x_n\right) \\ &\leq g(x) \\ &\leq -\frac{1}{\varepsilon}\end{aligned}$$

and with $\varepsilon \to 0$ it follows that $\limsup_{n\to\infty} S\left(\operatorname{hypo} g_n, D_n\right) = -\infty = S\left(\operatorname{hypo} g, D\right).$ $\square$

The theorem allows an extension to ε_n-optimal values.

Corollary 3.10 *Let $(\varepsilon_n)_n$ be a sequence of random variables with $\varepsilon_n > 0$ and $\varepsilon_n \overset{p}{\to} 0$, then under the assumptions of the above theorem,*

$$\left(\operatorname{hypo} f_n, C_n\right) \overset{D_{\tau_M \times \tau_H}}{\to} \left(\operatorname{hypo} f, C\right)$$

implies

$$\limsup_{n\to\infty} P\left(\Phi_n^{\varepsilon_n} \geq c\right) \leq P\left(\Phi \geq c\right)$$

for all c.

Proof. Because of the continuity of the probability measure it follows that for each $\eta > 0$ there is $k \in \mathbb{N}$ with

$$P\left(\Phi \geq c - \frac{1}{k}\right) \leq P\left(\Phi \geq c\right) + \eta.$$

From $\varepsilon_n \overset{P}{\to} 0$ it follows that to $\frac{1}{k}$ there is n_0 such that $P\left(\varepsilon_n > \frac{1}{k}\right) \leq \eta$ for all $n \geq n_0$.

We obtain

$$
\begin{aligned}
&\limsup_{n\to\infty} P\left(\Phi_n^{\varepsilon_n} \geq c\right) \\
&= \limsup_{n\to\infty} P\left(\Phi_n + \varepsilon_n \geq c\right) \\
&= \limsup_{n\to\infty} \begin{pmatrix} P\left(\Phi_n + \varepsilon_n \geq c,\ \varepsilon_n > \frac{1}{k}\right) \\ +P\left(\Phi_n + \varepsilon_n \geq c,\ \varepsilon_n \leq \frac{1}{k}\right) \end{pmatrix} \\
&\leq \limsup_{n\to\infty} P\left(\Phi_n + \varepsilon_n \geq c,\ \varepsilon_n \leq \frac{1}{k}\right) + \eta \\
&\leq \limsup_{n\to\infty} P\left(\Phi_n + \frac{1}{k} \geq c\right) + \eta \\
&\leq P\left(\Phi \geq c - \frac{1}{k}\right) + \eta \\
&\leq P\left(\Phi \geq c\right) + 2\eta
\end{aligned}
$$

and it remains to let $\eta \to 0$. □

We now turn to the sets of minimising points (argmin sets) Ψ and Ψ_n. In [43] Vogel has obtained the following result, note that continuous functions and their graphs are used.

Theorem 3.11 (Theorem 4.2 in [43]) *Let f_n, f be random continuous functions, let C_n, C be random closed sets, then*

$$(\operatorname{graph} f_n, C_n) \overset{D_{\tau_{Fell} \times \tau_{Fell}}}{\rightarrow} (\operatorname{graph} f, C)$$

implies

$$\Psi_n \overset{D_{\tau_M}}{\rightarrow} \Psi.$$

For optimisation problems with a fixed restriction set it is sufficient to work with the epigraphs of lower semicontinuous functions.

Theorem 3.12 *Let f_n, f be random lsc-functions, then*

$$\operatorname{epi} f_n \overset{D_{\tau_{Fell}}}{\rightarrow} \operatorname{epi} f$$

implies

$$\Psi_n \overset{D_{\tau_M}}{\rightarrow} \Psi.$$

We will now extend Theorem 3.11 to the sets of points of $\varepsilon-$optimality ($\varepsilon-$argmin sets). Since we deal with convergence in the space of closed sets we first require a suitable definition for closed versions of the $\varepsilon-$argmin sets. In the case of τ_M convergence we can work (as is done for example in [33]) with

$$\Psi^{\varepsilon} := \{x \in C : \ f(x) \leq \Phi + \varepsilon\}.$$

For lower semicontinuous f and $\varepsilon > 0$ the set Ψ^{ε} is closed as the intersection of a closed set and a $\leq$-sublevel set. The following lemma holds in the deterministic case under the stronger assumption of uniform boundedness. We will discuss this assumption following Lemma 3.21.

Lemma 3.13 *Let $(f_n)_n, \ f$ be uniformly bounded continuous functions. Let $(C_n)_n, \ C$ be closed subsets of a compact set K.*
If

$$\operatorname{graph} f_n \overset{\tau_{Fell}}{\to} \operatorname{graph} f \ \textit{and}\ C_n \overset{\tau_{Fell}}{\to} C,$$

then

$$\Psi_n^{\varepsilon} \overset{\tau_M}{\to} \Psi^{\varepsilon}$$

for all $\varepsilon > 0$.

Proof. Assume that the assertion is not true, then there is a sequence $(x_{n_k})_k$ with $x_{n_k} \in \Psi_{n_k}^{\varepsilon}$ and $x_{n_k} \to x$ for some $x \notin \Psi^{\varepsilon}$. Since $C_n \overset{\tau_M}{\to} C$ it follows that $x \in C$, hence $x \notin \Psi^{\varepsilon}$ means $f(x) > \Phi + \varepsilon$. Thus there is $\delta > 0$ with $f(x) - \Phi = \varepsilon + \delta$. We choose $z \in \Psi$, i.e. $z \in C$ and $f(z) = \Phi$. Because of $C_n \overset{\tau_H}{\to} C$ there is a sequence $(z_n)_n$ with $z_n \in C_n$ for n sufficiently large and $z_n \to z$. It follows that $f_n(z_n) \to f(z)$, hence there is n_0 such that $f_n(z_n) \leq f(z) + \frac{\delta}{4}$ for all $n \geq n_0$. Together with $f_{n_k}(x_{n_k}) \to f(x)$, which follows from Lemma 3.21 we obtain

$$\begin{aligned} f_{n_k}(x_{n_k}) - f_{n_k}(z_{n_k}) \geq & f(x) - \frac{\delta}{4} - \left(f(z) + \frac{\delta}{4}\right) \\ = & f(x) - \Phi - \frac{\delta}{2} \\ = & \varepsilon + \frac{\delta}{2} \\ > & \varepsilon \end{aligned}$$

in contradiction to $x_{n_k} \in \Psi_{n_k}^{\varepsilon}$. $\square$

Now in the random setting as a generalisation of Theorem 3.11 we have

Theorem 3.14 *Let $(f_n)_n,\ f$ be random uniformly bounded continuous functions. Let $(C_n)_n,\ C$ be random closed subsets of a compact set K.*
If

$$(\operatorname{graph} f_n, C_n) \overset{D_{\tau_{Fell}\times\tau_{Fell}}}{\longrightarrow} (\operatorname{graph} f, C)\,,$$

then

$$\Psi_n^{\varepsilon} \overset{D_{\tau_M}}{\longrightarrow} \Psi^{\varepsilon}$$

for all $\varepsilon > 0$.

Proof. In the preceding Lemma we have shown, that the mapping
$S : \left(A \times \mathcal{F}\left(\mathbb{R}^d\right), \tau_{\text{Fell}} \times \tau_{\text{Fell}}\right) \to \left(\mathcal{F}\left(\mathbb{R}^d\right), \tau_M\right),\ S\left(\operatorname{graph} f, C\right) = \Psi^{\varepsilon}$ is continuous, if A is a set consisting of graphs of uniformly bounded continuous functions $f : \mathbb{R}^d \to \mathbb{R}$. It remains to apply the Continuous Mapping Theorem. □

Remark 3.15 Lemma 3.13 and Theorem 3.14 remain valid for a fixed restriction set, if the objective functions are only lower semicontinuous and $\operatorname{epi} f_n \overset{\tau_{\text{Fell}}}{\to} \operatorname{epi} f$ resp. $\operatorname{epi} f_n \overset{D_{\tau_{\text{Fell}}}}{\to} \operatorname{epi} f$. This follows immediately with Theorem 3.12.

We can further improve Theorem 3.14 to the ε_n-argmin case, where $(\varepsilon_n)_n$ is a sequence of nonnegative random numbers converging to 0 in probability.

Theorem 3.16 *From*

$$\Psi_n^{\varepsilon} \overset{D_{\tau_M}}{\longrightarrow} \Psi^{\varepsilon}$$

for all $\varepsilon > 0$, it follows that

$$\Psi_n^{\varepsilon_n} \overset{D_{\tau_M}}{\longrightarrow} \Psi$$

if $\varepsilon_n \geq 0,\ \varepsilon_n \overset{P}{\to} 0$.

Proof. We have to show that $\liminf\limits_{n\to\infty} P\left(\Psi_n^{\varepsilon_n} \in U\right) \geq P\left(\Psi \in U\right)$ for all τ_M-open U. We can assume, that $U = \bigcup\limits_{i=1}^{\infty} M(K_i)$ with compact K_i. Because of the continuity of the probability measure, to each $\eta > 0$ there is $m \in \mathbb{N}$ such that with $U_m = \bigcup\limits_{i=1}^{m} M(K_i)$ we have $P\left(\Psi \in U_m\right) \geq P\left(\Psi \in U\right) - \eta$. Next we show that $\{\Psi \in U_m\} = \bigcup\limits_{k=1}^{\infty} \left\{\Psi^{\frac{1}{k}} \in U_m\right\}$. From $\Psi \subset \Psi^{\frac{1}{k}}$ for all k it follows that $\bigcup\limits_{k=1}^{\infty} \left\{\Psi^{\frac{1}{k}} \in U_m\right\} \subset \{\Psi \in U_m\}$ for all $k \in \mathbb{N}$. Now assume that there is a fixed ω such that $\omega \in \{\Psi \in U_m\}$, but $\omega \notin \bigcup_{k=1}^{\infty} \left\{\Psi^{\frac{1}{k}} \in U_m\right\}$. Then there is a sequence $(x_k)_k$ with $x_k \in \Psi^{\frac{1}{k}} \cap K_{i_k}$ for some $i_k \in \{1, \dots, m\}$. Without loss of generality we have $i_{k(l)} = 1$ for a subsequence $(i_{k(l)})_l$. Since K_1 is compact, there is a subsequence of $\left(x_{k(l)}\right)_l$, without loss of generality the sequence $\left(x_{k(l)}\right)_l$ itself, converging to some $x \in K_1$. Now with $\Psi = \bigcap_{k=1}^{\infty} \Psi^{\frac{1}{k}}$ it follows that $x \in \Psi \cap K_1$ in contradiction to $\omega \in \{\Psi \in U_m\}$. It follows that to $\eta > 0$ from above there is $k \in \mathbb{N}$ such that $P\left(\Psi^{\frac{1}{k}} \in U_m\right) \geq P\left(\Psi \in U_m\right) - \eta$. Note that $\Psi_n^{\varepsilon_n} \subset \Psi_n^{\frac{1}{k}}$, if $\varepsilon_n \leq \frac{1}{k}$. Thus with $\varepsilon_n \xrightarrow{P} 0$ for n sufficiently large we have

$$\begin{aligned}P\left(\Psi_n^{\frac{1}{k}} \in U_m\right) =& P\left(\Psi_n^{\frac{1}{k}} \in U_m,\ \varepsilon_n \leq \frac{1}{k}\right) + P\left(\Psi_n^{\frac{1}{k}} \in U_m,\ \varepsilon_n > \frac{1}{k}\right)\\ \leq& P\left(\Psi_n^{\varepsilon_n} \in U_m\right) + \eta.\end{aligned}$$

With $\Psi_n^{\frac{1}{k}} \overset{D_{\tau_M}}{\to} \Psi^{\frac{1}{k}}$ we obtain

$$\begin{aligned}\liminf_{n\to\infty} P\left(\Psi_n^{\varepsilon_n} \in U\right) \geq& \liminf_{n\to\infty} P\left(\Psi_n^{\varepsilon_n} \in U_m\right)\\ \geq& \liminf_{n\to\infty} P\left(\Psi_n^{\frac{1}{k}} \in U_m\right) - \eta\\ \geq& P\left(\Psi^{\frac{1}{k}} \in U_m\right) - \eta\\ \geq& P\left(\Psi \in U_m\right) - 2\eta\\ \geq& P\left(\Psi \in U\right) - 3\eta\end{aligned}$$

and it remains to let $\eta \to 0$. □

Next we will investigate τ_H-convergence of argmin and $\varepsilon-$argmin sets.

Example 3.17 Let

$$f(x) = 0,\ f_n(x) = \begin{cases} \frac{1}{n} & ,\ x \in (-\infty, -1] \cup [1, \infty) \\ -\frac{1}{n}x & ,\ -1 < x \leq 0 \\ \frac{1}{n}x & ,\ 0 < x < 1 \end{cases}$$

and observe that f_n converges uniformly to f on $\mathbb{R}$. If follows that graph $f_n \overset{\tau_{\text{Fell}}}{\to}$ graph f. Because of $\Psi_n = \{0\}$ and $\Psi = \mathbb{R}$ it is clear that $\Psi_n \overset{\tau_H}{\not\to} \Psi$.

The example however suggests that each $x \in \Psi$ can be approximated by a sequence

$(x_n)_n$ with $f_n(x_n) \leq \Phi_n + \varepsilon$, where $\varepsilon > 0$. We will show, that this is true in a more general case. While the Ψ^ε concept of $\varepsilon-$argmins works well in the τ_M setting, it leads to difficulties, when we deal with convergence in distribution with respect to τ_H. Thus as an alternative to Ψ^ε we will choose

$$\widetilde{\Psi}^\varepsilon := \mathrm{cl}\left(\{x : f(x) < \Phi + \varepsilon\}\right)$$

as a closed $\varepsilon-$argmin set for the τ_H case.
If f is lower semicontinuous, then $\widetilde{\Psi}^\varepsilon \subset \Psi^\varepsilon$. For the verification we can assume that $\widetilde{\Psi}^\varepsilon \neq \emptyset$. Let $x \in \widetilde{\Psi}^\varepsilon$, then there is a sequence $(x_k)_k \subset \{x : f(x) < \Phi + \varepsilon\}$ with $x_k \to x$. With the lower semicontinuity of f we obtain $f(x) \leq \liminf_{k\to\infty} f(x_k) \leq \Phi + \varepsilon$, i.e. $x \in \Psi^\varepsilon$.

The following example shows the limitations of the use of $\widetilde{\Psi}^\varepsilon$ in the τ_M-case. It turns out that in general $(\mathrm{graph}\, f_n, C_n) \overset{\tau_{\mathrm{Fell}} \times \tau_{\mathrm{Fell}}}{\longrightarrow} (\mathrm{graph}\, f, C)$ (compare Theorem 3.14) does not imply $\widetilde{\Psi}^\varepsilon_n \overset{\tau_M}{\to} \widetilde{\Psi}^\varepsilon$.

Example 3.18 Let $C_n = C = [-3, 3]$. We choose

$$f(x) = \begin{cases} -x & , x \leq 0 \\ x & , 0 \leq x \leq 1 \\ 1 & , x \geq 1 \end{cases} , \quad f_n(x) = \begin{cases} -x + \frac{1}{n} & , x \leq 0 \\ x + \frac{1}{n} & , 0 \leq x \leq 1 \\ -\frac{1}{n}x + 1 + \frac{2}{n} & , 1 \leq x \leq 2 \\ 1 & , x \geq 2 \end{cases} .$$

Let $\varepsilon = 1$. We note that $\Phi = 0$ and $\widetilde{\Psi}^\varepsilon = \mathrm{cl}(\{x : f(x) < \Phi + \varepsilon\}) = \mathrm{cl}((-1, 1)) = [-1, 1]$. We have $\Phi_n = \frac{1}{n}$ and $\widetilde{\Psi}^\varepsilon_n = \mathrm{cl}(\{x : f(x) < \Phi_n + \varepsilon\}) = \mathrm{cl}((-1, 1) \cup (1, 3]) = [-1, 3]$. Thus if we choose $x_n = 2$ for all $n \in \mathbb{N}$, we obtain $x_n \in \widetilde{\Psi}^\varepsilon_n$ and $x_n \to 2 \notin \widetilde{\Psi}^\varepsilon$.

On the other hand we see that $\widetilde{\Psi}^\varepsilon_n \overset{\tau_M}{\to} \widetilde{\Psi}^\varepsilon$ for $\varepsilon < \frac{1}{2}$. The ε, for which τ_M-convergence occurs depend on f_n and f. This shows that in applications, where usually the accepted tolerance $\varepsilon > 0$ is chosen a priori, the $\widetilde{\Psi}^\varepsilon$ concept is not very well suited for the τ_M convergence concept.

The following lemmas contain set convergence versions of known results (see e.g. [2]) on the lower semicontinuity of the optimal values which will be used in the proofs of the main results on τ_H-convergence of argmin sets.

Lemma 3.19 *Let $K \subset U \subset \mathbb{R}^d$, where K is compact and U is an open set. Let f_n, f be continuous, real valued functions on U. Let C_n, C be closed subsets of K. Then it follows from*

$$\operatorname{graph} f_n \overset{\tau_{Fell}}{\to} \operatorname{graph} f$$

and

$$C_n \overset{\tau_M}{\to} C,$$

that

$$\liminf_{n\to\infty} \Phi_n \geq \Phi.$$

Proof. Assume that $\liminf_{n\to\infty} \Phi_n = a < \Phi$ for some a. Then there is a subsequence $(\Phi_{n_k})_k$ with $\Phi_{n_k} \to a$. There are $x_{n_k} \in C_{n_k}$ with $f_{n_k}(x_{n_k}) = \Phi_{n_k}$. It follows that a subsequence of $(x_{n_k})_k$ (without loss of generality $(x_{n_k})_k$ itself) converges to some $x \in K$.
In the first case let $a > -\infty$. We have $(x_{n_k}, \Phi_{n_k}) \in \operatorname{graph} f_{n_k}$ and $(x_{n_k}, \Phi_{n_k}) \to (x, a)$, thus it follows from $\operatorname{graph} f_n \overset{\tau_{\mathrm{Fell}}}{\to} \operatorname{graph} f$ that $(x, a) \in \operatorname{graph} f$. Hence $f(x) = a$. Together with $x \in C$, which follows from $C_n \overset{\tau_M}{\to} C$, this yields a contradiction to $a < \Phi$.
Now let $a = -\infty$. We choose b with $a < b < f(x)$. To $(x, f(x)) \in \operatorname{graph} f$ there is a sequence $(z_n, f_n(z_n))_n$ with $(z_n, f_n(z_n)) \to (x, f(x))$. It follows that $f_{n_k}(z_{n_k}) > b$ for all $k \geq k_0$. From $\Phi_{n_k} \to -\infty$ we obtain $f_{n_k}(x_{n_k}) < b$ for all $k \geq k_1$. With the openness of U we can find $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$. Because of $x_{n_k} \to x$ and $z_{n_k} \to x$ we have $x_{n_k} \in B_\varepsilon(x)$ and $z_{n_k} \in B_\varepsilon(x)$ for all $k \geq k_2$. Since f_{n_k} is continuous and the line segment between x_{n_k} and z_{n_k} is connected and belongs to U we can, by the Mean Value Theorem, find w_{n_k} on this segment with $f_{n_k}(w_{n_k}) = b,\ k \geq \max(k_0, k_1, k_2)$. From $x_{n_k} \to x$ and $z_{n_k} \to x$ it follows that $w_{n_k} \to b$. With $\operatorname{graph} f_n \overset{\tau_{\mathrm{Fell}}}{\to} \operatorname{graph} f$ this leads to $(x, b) \in \operatorname{graph} f$ in contradiction to $f(x) > b$. □

For uniformly bounded functions τ_M-convergence of the graphs is sufficient.

Lemma 3.20 *Let f_n, f be continuous uniformly bounded functions on a compact set K, then*

$$\operatorname{graph} f_n \overset{\tau_M}{\to} \operatorname{graph} f$$

together with

$$C_n \overset{\tau_M}{\to} C$$

implies

$$\liminf_{n\to\infty} \Phi_n \geq \Phi.$$

Proof. Assume that $\liminf_{n\to\infty} \Phi_n = a$ for some $a < \Phi$. Then there is a subsequence $(\Phi_{n_k})_k$ with $\Phi_{n_k} \to a$. There are $x_{n_k} \in C_{n_k}$ with $f_{n_k}(x_{n_k}) = \Phi_{n_k}$. It follows that a subsequence of $(x_{n_k})_k$ (without loss of generality $(x_{n_k})_k$ itself) converges to some $x \in K$. From $C_n \overset{\tau_M}{\to} C$ it then follows that $x \in C$. Because of the uniform boundedness of $(f_n)_n$ it is true that $a > -\infty$. We have $(x_{n_k}, \Phi_{n_k}) \to (x, a)$ and from $(x_{n_k}, \Phi_{n_k}) \in \operatorname{graph} f_{n_k}$ and $\operatorname{graph} f_n \overset{\tau_M}{\to} \operatorname{graph} f$ it follows that $(x, a) \in \operatorname{graph} f$, which means $f(x) = a < \Phi$ in contradiction to $\Phi = \min\{f(x) : x \in C\}$. □

Note that the condition $\operatorname{graph} f_n \overset{\tau_M}{\to} \operatorname{graph} f$ is only apparently weaker than $\operatorname{graph} f_n \overset{\tau_{\text{Fell}}}{\to} \operatorname{graph} f$ under the settings of the above lemma. The following lemma is inspired by Corollary 5.45 of [33].

Lemma 3.21 *Let f_n, f be continuous uniformly bounded functions on a compact set K. The following are equivalent:*

(i) $\operatorname{graph} f_n \overset{\tau_M}{\to} \operatorname{graph} f$

(ii) $\operatorname{graph} f_n \overset{\tau_{Fell}}{\to} \operatorname{graph} f$

(iii) $\lim_{n\to\infty} f_n(x_n) = f(x)$, *for all* $(x_n)_n \subset K,\ x \in K$ *with* $x_n \to x$.

Proof. We show $(ii) \implies (i) \implies (iii) \implies (ii)$.
Since convergence in the Fell topology implies convergence with respect to τ_M, it is clear that $(ii) \implies (i)$ holds.
Now let $\operatorname{graph} f_n \overset{\tau_M}{\to} \operatorname{graph} f$ and let $(x_n)_n \subset K,\ x \in K$ with $x_n \to x$. Because of the uniform boundedness of the sequence $(f_n)_n$ it follows that $\{f_n(x_n) : n \in \mathbb{N}\}$ is contained in a compact set. Let $(f_{n_k}(x_{n_k}))_k$ be an arbitrary convergent subsequence of $(f_n(x_n))_n$ with limit c. From $(x_{n_k}, f_{n_k}(x_{n_k})) \to (x, c)$ and $(x_{n_k}, f_{n_k}(x_{n_k})) \in \operatorname{graph} f_{n_k}$ it follows with $\operatorname{graph} f_n \overset{\tau_M}{\to} \operatorname{graph} f$ that $(x, c) \in \operatorname{graph} f$, i.e. $f(x) = c$. Consequently all cluster points of the sequence $(f_n(x_n))_n$ are equal to $f(x)$. It follows that $\lim_{n\to\infty} f_n(x_n) = f(x)$.
Finaly we assume that (iii) holds. First let $(x_{n_k}, f_{n_k}(x_{n_k})) \in \operatorname{graph} f_{n_k}$ and let $(x_{n_k}, f_{n_k}(x_{n_k})) \to (x, c)$ it follows from (iii) that $f_{n_k}(x_{n_k}) \to f(x)$, i.e. $c = f(x)$ and thus $(x, c) \in \operatorname{graph} f$.
It remains to show that for each $(x, f(x)) \in \operatorname{graph} f$ there is a sequence $((x_n, f_n(x_n)))_n$ with $(x_n, f_n(x_n)) \in \operatorname{graph} f_n$ and $(x_n, f_n(x_n)) \to (x, f(x))$. We choose $x_n = x$ for all $n \in \mathbb{N}$, then obviously $x_n \to x$ and $f_n(x_n) \to f(x)$ because of (iii). □

Starting with 3.13 we have encountered the assumption of uniform boundedness on compact sets for the objective functions. The assumption (or an almost sure version, which used instead) is fulfilled by many classes of functions, e.g. empirical distribution functions or (random) probability density functions. It should however be noted that even in the case of continuous functions the assumption can in general not be dispensed with. The following example shows that Lemma 3.20 does not remain true, if the uniform boundedness condition is dropped.

Example 3.22 Let $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Then K is a compact subset of $\mathbb{R}$, since K is bounded and closed. Let

$$f(x) = 0,\ f_n(x) = \begin{cases} 0 & , x \neq \frac{1}{n} \\ -n & , x = \frac{1}{n} \end{cases}$$

Each of these functions is continuous on K. This follows, because the subspace topology, which K inherits from $\mathbb{R}$ equipped with the usual topology, is the discrete topology. Each function defined on a topological space, equipped with the discrete topology is continuous.

To verify $\operatorname{graph} f_n \overset{\tau_{\text{Fell}}}{\to} \operatorname{graph} f$, first let $(x, f(x)) \in \operatorname{graph} f$. It follows that $f(x) = 0$. For f_n we have $f_n(x) = 0$, whenever $\frac{1}{n} < x$ or $x = 0$. This shows $(x, 0) \in \operatorname{graph} f_n$ for $n \geq n_0$ and thus $\operatorname{graph} f_n \overset{\tau_H}{\to} \operatorname{graph} f$. Now to show $\operatorname{graph} f_n \overset{\tau_M}{\to} \operatorname{graph} f$ let $(x_n, f_n(x_n)) \in \operatorname{graph} f_n$ and let $(x_n, f_n(x_n)) \to (x, y)$. We have to show that $y = f(x)$. As $f_n(x_n)$ can only take the values 0 and $-n$ we can only have $(x_n, f_n(x_n)) \to (x, y)$, if $f_n(x_n) = 0$ for all $n \geq n_0$. Otherwise $f_n(x_n)$ diverges. It follows that $y = 0$ and with the closedness of K we obtain $(x, y) \in \operatorname{graph} f$.

In contrast to Lemma 3.20 we have $\liminf_{n\to\infty} \Phi_n = \liminf_{n\to\infty} -n = -\infty < 0 = \Phi$.

It was essential in the example that the functions f, f_n were defined on a set which is not connected. We will now see that the situation is different if we deal with the frequent case of continuous functions on a region. In this case uniform boundedness on compact sets imediately follows from convergence of the graphs.

Lemma 3.23 *Let G be a region, let $(f_n)_n, f$ be continuous real valued functions on G. Then*

$$\operatorname{graph} f_n \overset{\tau_{Fell}}{\to} \operatorname{graph} f$$

implies that the sequence $(f_n)_n$ is uniformly bounded on each compact $K \subset G$.

Proof. Assume that $(f_n)_n$ is not uniformly bounded on K. Then there is a sequence $(x_{n_k})_{n_k} \subset K$ such that without loss of generalisation $f_{n_k}(x_{n_k}) \to \infty$. Since K is compact a subsequence $(x_{n_{k(l)}})_l$ converges to some $x \in K$. Because $f(x)$ is finite it follows from

$f_{n_k}(x_{n_k}) \to \infty$ that $f_{n_{k(l)}}(x_{n_{k(l)}}) \geq f(x) + 3$ for all $l \geq l_0$. From graph $f_n \overset{\tau_H}{\to}$ graph f it follows that there is a sequence $(z_{n_{k(l)}}, f_{n_{k(l)}}(z_{n_{k(l)}}))$ with $(x_{n_{k(l)}}, f_{n_{k(l)}}(z_{n_{k(l)}})) \in$ graph $f_{n_{k(l)}}$ and $(x_{n_{k(l)}}, f_{n_{k(l)}}(z_{n_{k(l)}})) \to (x, f(x))$. Thus $f_{n_{k(l)}}(z_{n_{k(l)}}) \leq f(x)+1$ for all $l \geq l_1$. Since G is open we can find $\varepsilon > 0$ such that $U_\varepsilon(x) \subset G$. For all $l \geq l_2 \geq \max(l_0, l_1)$ we have $x_{n_{k(l)}} \in U_\varepsilon(x)$ and $z_{n_{k(l)}} \in U_\varepsilon(x)$. By the Mean Value Theorem because of $f_{n_{k(l)}}(x_{n_{k(l)}}) \geq f(x)+3$ and $f_{n_{k(l)}}(z_{n_{k(l)}}) \geq f(x) + 3$ there is $w_{n_{k(l)}} \in U_\varepsilon(x)$ such that $f_{n_{k(l)}}(w_{n_{k(l)}}) = f(x) + 2$. With $\varepsilon \to 0$ we obtain $(w_{n_{k(l)}}, f_{n_{k(l)}}(w_{n_{k(l)}})) \to (x, f(x) + 2) \notin$ graph f, which yields a contradiction to graph $f_n \overset{\tau_M}{\to}$ graph f □

Remark 3.24 It is well known from calculus that for a sequence of continuous functions on a compact set, uniform convergence implies continuous convergence in the sense of (iii). This is very helpful for finding a sufficient condition for graph $f_n \overset{D_{\tau_{\text{Fell}}}}{\to}$ graph f. In combination with Lemma 3.21 and the Continuous Mapping Theorem one can use the convergence criteria for convergence in distribution in the space $(C, |.|_\infty)$, which can for example be found in [5] and [30].

The first result on τ_H convergence shows that for a constant, compact and deterministic restriction set it is sufficient to work with the epigraphs of lower semicontinuous functions.

Theorem 3.25 *Let f_n, f be lower semicontinuous functions, let $C_n = C = K$ for a compact set K. Then*

$$\text{epi}\, f_n \overset{\tau_{Fell}}{\to} \text{epi}\, f$$

implies

$$\widetilde{\Psi}_n^\varepsilon \overset{\tau_H}{\to} \widetilde{\Psi}^\varepsilon$$

for all $\varepsilon > 0$.

Proof. First we show that for each $z \in \{x \in K : f(x) < \Phi + \varepsilon\}$ there is a sequence $(z_n)_n$ with $z_n \in \{x \in K : f_n(x) < \Phi_n + \varepsilon\}$ and $z_n \to z$. Assume that this was not true, then for each sequence $(z_n)_n \subset K$ with $z_n \to z$ there is a subsequence $(z_{n_k})_k$ such that $f_{n_k}(z_{n_k}) \geq \Phi_{n_k} + \varepsilon$. From $(z, f(z)) \in$ epi f it follows with epi $f_n \overset{\tau_{\text{Fell}}}{\to}$ epi f that there is a sequence $((z_n, u_n))_n$ with $(z_n, u_n) \in$ epi f_n and $(z_n, u_n) \to (z, f(z))$. Let $(n_k)_k$ be such that $f_{n_k}(z_{n_k}) \geq \Phi_{n_k} + \varepsilon$.
Then with $\liminf_{n\to\infty} \Phi_n \geq \Phi$, which follows from Lemma 3.19

we obtain

$$\begin{aligned} f(z) &= \lim_{k\to\infty} u_{n_k} \\ &\geq \liminf_{k\to\infty} f_{n_k}(z_{n_k}) \\ &\geq \liminf_{k\to\infty} \Phi_{n_k} + \varepsilon \\ &\geq \Phi + \varepsilon \end{aligned}$$

in contradiction to $z \in \{x \in K : f(x) < \Phi + \varepsilon\}$.
Now let $z \in \widetilde{\Psi}^{\varepsilon}$, then there is a sequence $(w_k)_k$ with $w_k \in \{x \in K : f(x) < \Phi + \varepsilon\}$ and $w_k \to z$.
Since by the above, for each w_k there is a sequence $\left(x_n^k\right)_n$ such that
$x_n^k \in \{x \in K : f_n(x) < \Phi_n + \varepsilon\}$ and $x_n^k \to w_k$. It follows that we can find a sequence $(x_n)_n$ with $x_n \in \{x \in K : f_n(x) < \Phi_n + \varepsilon\} \subset \widetilde{\Psi}_n^{\varepsilon}$ and $x_n \to z$. □

Next we additionaly allow non constant restriction sets. It turns out, that in this case we need stronger conditions on $(f_n)_n$ and f.

Theorem 3.26 *Let f_n, f be continuous functions, which are uniformly bounded on a compact set K. Let $C_n, C \subset K$ be closed sets, then*

$$\operatorname{graph} f_n \stackrel{\tau_M}{\to} \operatorname{graph} f,\ C_n \stackrel{\tau_{\text{Fell}}}{\to} C,$$

implies

$$\widetilde{\Psi}_n^{\varepsilon} \stackrel{\tau_H}{\to} \widetilde{\Psi}^{\varepsilon}.$$

Proof. Let $x \in \widetilde{\Psi}^{\varepsilon}$ and let $\eta > 0$.
Then there is $z \in \{x \in C : f(x) < \Phi + \varepsilon\}$ with $z \in B_\eta(x)$. We can find $\delta > 0$ such that $B_\delta(z) \subset B_\eta(x)$ and $f_n(w) < \Phi_n + \varepsilon$ for all $w \in B_\delta(z) \cap C_n$ and all $n \geq n_1$.
Assume that this was not true, then we could find $(w_{n_k})_k$ with $w_{n_k} \in C_{n_k},\ w_{n_k} \to z$ and $f_{n_k}(w_{n_k}) \geq \Phi_{n_k} + \varepsilon$. Since $(f_n)_n$ is uniformly bounded, there is a subsequence $\left(f_{n_{k(l)}}(w_{n_{k(l)}})\right)_l$ converging to some $a \in \mathbb{R}$. With $\left(w_{n_{k(l)}}, f_{n_{k(l)}}(w_{n_{k(l)}})\right) \in \operatorname{graph} f_{n_{k(l)}}$ and $\left(w_{n_{k(l)}}, f_{n_{k(l)}}(w_{n_{k(l)}})\right) \to (z, a)$ it follows from $\operatorname{graph} f_n \stackrel{\tau_M}{\to} \operatorname{graph} f$, that $(z, a) \in \operatorname{graph} f$ and $a = f(z)$. Together with Lemma 3.20 we obtain

$$f(z) = \liminf_{l\to\infty} f_{n_{k(l)}}(w_{n_{k(l)}}) \geq \liminf_{l\to\infty} \Phi_{n_{k(l)}} + \varepsilon \geq \Phi + \varepsilon$$

in contradiction to $z \in \{x \in C : f(x) < \Phi + \varepsilon\}$.
Now from $z \in C$ and $C_n \xrightarrow{\tau_H} C$ it follows that there is $(z_n)_n$ with $z_n \in C_n$ for all $n \geq n_2$ and $z_n \to z$. Thus for $n \geq n_3$ we have $z_n \in B_\delta(z)$ and with $n_0 := \max\{n_1, n_2, n_3\}$ we obtain $z_n \in \{x \in C_n : f_n(x) < \Phi_n + \varepsilon\}$ and $z_n \in B_\eta(x)$ for all $n \geq n_0$. Now let $\eta = \frac{1}{k}$, then we can find n_k with $z_n \in B_{\frac{1}{k}}(x)$ for $n \geq n_k$. It follows that $z_n \to x$. □

In the following theorem we do not require the explicit assumption of uniform boundedness.

Theorem 3.27 *Let f_n, f be continuous functions on an open set U. Let $C_n, C \subset K \subset U$ for a deterministic compact set K. Assume that*

$$\operatorname{graph} f_n \xrightarrow{\tau_{Fell}} \operatorname{graph} f,\ C_n \xrightarrow{\tau_{Fell}} C,$$

then

$$\widetilde{\Psi}_n^\varepsilon \xrightarrow{\tau_H} \widetilde{\Psi}^\varepsilon,$$

for all $\varepsilon > 0$.

Proof. Let $x \in \widetilde{\Psi}^\varepsilon$ and $\eta > 0$. Then there is $z \in \{x \in C : f(x) < \Phi + \varepsilon\}$ with $z \in B_\eta(x)$. We can find $\delta > 0$ such that $B_\delta(z) \subset B_\eta(x)$ and $f_n(w) < \Phi_n + \varepsilon$ for all $w \in B_\delta(z)$ and all $n \geq n_1$.
Assume that this was not true, then we could find $(w_{n_k})_k$ with $w_{n_k} \to z$ and $f_{n_k}(w_{n_k}) \geq \Phi_{n_k} + \varepsilon$. From Lemma 3.19 it follows that $\liminf_{k\to\infty} \Phi_{n_k} \geq \Phi$. We have $\lim_{k\to\infty} f_{n_k}(w_{n_k}) = f(z)$ and obtain

$$f(z) = \liminf_{k\to\infty} f_{n_k}(w_{n_k}) \geq \liminf_{l\to\infty} \Phi_{n_k} + \varepsilon \geq \Phi + \varepsilon$$

in contradiction to $z \in \{x \in C : f(x) < \Phi + \varepsilon\}$.
Now from $z \in C$ and $C_n \xrightarrow{\tau_H} C$ it follows that there is $(z_n)_n$ with $z_n \in C_n$ for all $n \geq n_2$ and $z_n \to z$. Thus for $n \geq n_3$ we have $z_n \in B_\delta(z)$ and with $n_0 := \max\{n_1, n_2, n_3\}$ it follows that $z_n \in \{x \in C_n : f_n(x) < \Phi_n + \varepsilon\}$ and $z_n \in B_\eta(x)$ for all $n \geq n_0$. By letting $\eta \to 0$ we see, that a sequence $(\widetilde{z}_n)_n$ with $\widetilde{z}_n \in \{x \in C_n : f_n(x) < \Phi_n + \varepsilon\}$ and $\widetilde{z}_n \to x$ can be found. □

Following the preparations in the deterministic case we can state the main results for τ_H convergence in distribution of the $\varepsilon-$argmin sets. In the case of a fixed restriction set we have

Theorem 3.28 *Let $f_n,\ f$ be random lower semicontinuous functions. Then*

$$\operatorname{epi} f_n \overset{D_{\tau_{Fell}}}{\to} \operatorname{epi} f$$

implies

$$\widetilde{\Psi}_n^\varepsilon \overset{D_{\tau_H}}{\to} \widetilde{\Psi}^\varepsilon$$

for each $\varepsilon > 0$.

Proof. Since we have shown in Theorem 3.25 that the function which maps $\operatorname{epi} f$ to $\widetilde{\Psi}^\varepsilon$ is continuous (with respect to the particular topologies) the assertions of the theorem follow with the Continuous Mapping Theorem. □

For the general case with convergent restriction sets we have

Theorem 3.29 *(a) Let $f_n,\ f$ be random continuous functions, which are uniformly bounded, let $C_n,\ C$ be random closed sets, then*

$$(\operatorname{graph} f_n, C_n) \overset{D_{\tau_M \times \tau_{Fell}}}{\to} (\operatorname{graph} f, C)$$

implies

$$\widetilde{\Psi}_n^\varepsilon \overset{D_{\tau_H}}{\to} \widetilde{\Psi}^\varepsilon,$$

for all $\varepsilon > 0$.
(b) Let f_n, f be random continuous functions and let C_n, C be random closed sets with $C_n, C \subset K$ for a deterministic compact set K, then

$$(\operatorname{graph} f_n, C_n) \overset{D_{\tau_{Fell} \times \tau_{Fell}}}{\to} (\operatorname{graph} f, C)$$

implies

$$\widetilde{\Psi}_n^\varepsilon \overset{D_{\tau_H}}{\to} \widetilde{\Psi}^\varepsilon,$$

for all $\varepsilon > 0$.

Proof. This is proven as in the proof of Theorem 3.28 with the help of Theorems 3.26 and 3.27 in combination with the Continuous Mapping Theorem. □

The conditions in part (a) provide an example for the applicability of Theorem 2.67. It is sufficient, that f and C, f_n and C and f_n and C_n are independent. Then from $\operatorname{graph} f_n \overset{D_{\tau_M}}{\to} \operatorname{graph} f$ and $C_n \overset{D_{\tau_{\text{Fell}}}}{\to} C$ it follows that the conditions of part (a) are fulfilled.

When we dealt with τ_M-convergence in distribution, we could find general results for convergence of $\Psi_n^{\varepsilon_n}$ in the case, that the sequence $(\varepsilon_n)_n$ converges to 0 in probability, see Theorem 3.16. The following example shows that the situation is different in the τ_H-case. We can in general not expect, that $\widetilde{\Psi}_n^{\varepsilon_n} \overset{D_{\tau_H}}{\to} \widetilde{\Psi}$, for $\varepsilon_n \to 0$ under the assumptions of the preceding theorem. A deterministic counterexample is sufficient.

Example 3.30 Let

$$f(x) = 0, \;\; f_n(x) = \begin{cases} \frac{2}{n} & , \; x \in (-\infty, -1] \cup [1, \infty) \\ -\frac{2}{n}x & , \; x \in (-1, 0] \\ \frac{2}{n}x & , \; x \in (0, 1) \end{cases} .$$

Let $C_n = C = [-3, 3]$ and let $\varepsilon_n = \frac{1}{n}$. It is easy to see, that the assumptions of Theorem 3.26 are fulfilled. We have $\widetilde{\Psi} = [-3, 3]$ and $\widetilde{\Psi}_n^{\varepsilon_n} = \left[-\frac{1}{2}, \frac{1}{2}\right]$, which shows that $\widetilde{\Psi}_n^{\varepsilon_n} \overset{\tau_H}{\not\to} \widetilde{\Psi}$.

Given $(f_n)_n$ and f it is the choice of the sequence ε_n that prevents τ_H convergence. Note that for a different choice (e.g. $\varepsilon_n = \frac{2}{n}$) we would obtain τ_H convergence. The fact that ε_n has to be chosen accordingly to f_n and that the dependence structure between f_n and ε_n is usually not known shows that τ_H-convergence of $\widetilde{\Psi}_n^{\varepsilon_n}$ is not applicable to problems, in which an a priori sequence of errors $(\varepsilon_n)_n$ is given.

3.3. Argmin Continuous Mapping Theorems

In [12] Ferger recently developed an extension of the well known Argmax Continuous Mapping Theorem (cf. Theorem 3.2.2 of [36]) to the case of non unique argmax sets. He shows that besides the traditional use in Maximum Likelihood Estimation the theory has statistical applications in change point estimation. In this section we will show, that we can obtain Ferger's results (after the canonical transformation from lower semicontinuous functions and minimisation problems to upper semicontinuous functions and maximisation problems) with the methods developed in [43] and in the preceding sections. In the case of a unique argmin, it is often possible to determine the approximate distribution for a (normalised) sequence of argmins. We cannot expect to achieve such strong results from one-sided convergence of argmin sets. Instead the obtainable results provide one-sided bounds for probabilities, which hold uniformly for all sequences of minimisers. A central role in the investigation is played by the smallest and the largest minimisers, which are always measurable. Throughout this section we assume that Ψ_n,

resp. Ψ are subsets of $\mathbb{R}$ and possess smallest and largest elements $s_n,\ l_n$, resp. $s,\ l$. In the same way s_n^ε and l_n^ε denote the smallest/largest element of Ψ_n^ε (or if needed elements of $\widetilde{\Psi}_n^\varepsilon$), for $\varepsilon > 0$. The existence of the smallest/largest minimisers is guaranteed, if C_n and C are compact sets.
In the following we require one-sided versions of the stochastic boundedness property $O_p(1)$.

Definition 3.31 Let $(X_n)_n$ be a sequence of real valued random variables.

(i) The sequence $(X_n)_n$ is called stochastically bounded from below, if for each $\eta > 0$ there are $a \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $P\left(X_n < a\right) \leq \eta$ for all $n \geq n_0$.

(ii) The sequence $(X_n)_n$ is called stochastically bounded from above, if for each $\eta > 0$ there are $b \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $P\left(X_n > b\right) \leq \eta$ for all $n \geq n_0$.

Theorem 3.32 *Let $\Psi_n^{\varepsilon_n} \stackrel{D_{\tau_M}}{\to} \Psi$ and let $(l_n^{\varepsilon_n})_n$ be stochastically bounded from above. Then for every sequence $(x_n)_n$ with measurable $x_n \in \Psi_n^{\varepsilon_n}$ and all $c \in \mathbb{R}$*

$$\liminf_{n\to\infty} P\left(x_n < c\right) \geq P\left(l < c\right).$$

Proof. Let $\eta > 0$ and let b be the constant from the stochastical boundedness from above property of $(l_n^{\varepsilon_n})_n$. By enlarging b if necessary, we can assume that $c \leq b$. For $n \geq n_0$ we obtain

$$\begin{aligned}
P\left(\Psi_n^{\varepsilon_n} \in M([c,b])\right) =& P\left(\Psi_n^{\varepsilon_n} \in M([c,b]),\ l_n^{\varepsilon_n} \leq b\right) + P\left(\Psi_n^{\varepsilon_n} \in M([c,b]),\ l_n^{\varepsilon_n} > b\right) \\
=& P\left(l_n^{\varepsilon_n} < c\right) + P\left(\Psi_n^{\varepsilon_n} \in M([c,b]),\ l_n^{\varepsilon_n} > b\right) \\
\leq& P\left(l_n^{\varepsilon_n} < c\right) + P\left(l_n^{\varepsilon_n} > b\right) \\
\leq& P\left(l_n^{\varepsilon_n} < c\right) + \eta.
\end{aligned}$$

It follows that

$$\begin{aligned}
\liminf_{n\to\infty} P\left(x_n < c\right) \geq& \liminf_{n\to\infty} P\left(l_n^{\varepsilon_n} < c\right) \\
\geq& \liminf_{n\to\infty} P\left(\Psi_n^{\varepsilon_n} \in M([c,b])\right) - \eta \\
\geq& P\left(\Psi \in M([c,b])\right) - \eta \\
\geq& P\left(l < c\right) - \eta.
\end{aligned}$$

Now let $\eta \to 0$. $\square$

Theorem 3.33 *Let* $\Psi_n^{\varepsilon_n} \stackrel{D_{\tau_M}}{\to} \Psi$ *and let* $(s_n^{\varepsilon_n})_n$ *be stochastically bounded from below. Then for every sequence* $(x_n)_n$ *with measurable* $x_n \in \Psi_n^{\varepsilon_n}$ *and all* $c \in \mathbb{R}$

$$\limsup_{n\to\infty} P\left(x_n \leq c\right) \leq P\left(s \leq c\right).$$

Proof. Let a be the constant obtained from the stochastic boundedness from below property of $(s_n^{\varepsilon_n})_n$. By choosing a smaller a if necessary, we can assume that $a < c$. We obtain

$$\begin{aligned}
\limsup_{n\to\infty} P\left(x_n \leq c\right) \leq & \limsup_{n\to\infty} P\left(s_n^{\varepsilon_n} \leq c\right) \\
= & \limsup_{n\to\infty} \left(P\left(s_n^{\varepsilon_n} \leq c,\ s_n^{\varepsilon_n} \geq a\right) + P\left(s_n^{\varepsilon_n} \leq c,\ s_n^{\varepsilon_n} < a\right)\right) \\
\leq & \limsup_{n\to\infty} P\left(s_n^{\varepsilon_n} \leq c,\ s_n^{\varepsilon_n} \geq a\right) + \eta \\
= & \limsup_{n\to\infty} P\left(\Psi_n^{\varepsilon_n} \in H([a,c])\right) + \eta \\
\leq & P\left(\Psi \in H([a,c])\right) + \eta \\
\leq & P\left(s \leq c\right) + \eta
\end{aligned}$$

and it remains to let $\eta \to 0$. □

Remark 3.34 If the solution sets $\Psi,\ \Psi_n$ are derived from random lower semicontinuous objective functions which fulfill the measurability assumptions from Section 1.3, then the existence of a sequence of measurable minimisers $(x_n)_n$ in the preceding theorems follows with Theorem 14.37 of [33]. The measurability assumption for the x_n can be dropped, if we work with inner/outer probability as is done in [12].

As the following example shows, the assertions in general do not remain valid without the stochastic boundedness conditions.

Example 3.35 Let

$$f(x) = \begin{cases} 0 & ,\ x \neq 0 \\ -1 & ,\ x = 0 \end{cases},\quad f_n(x) = \begin{cases} 0 & ,\ x \notin \{0,n\} \\ -1 & ,\ x \in \{0,n\} \end{cases}$$

Then $\Psi = \{0\},\ \Psi_n = \{0,n\}$ and we have $\Psi_n \stackrel{D_{\tau_M}}{\to} \Psi$. On the other hand we note that with $x_n = n \in \Psi_n,\ c = 1$

$$\liminf_{n\to\infty} P\left(x_n < c\right) = 0 < P\left(l < c\right) = 1.$$

A similar example can be constructed with Theorem 3.33 in mind.

Next we show, how Ferger's results for the non-unique case can be obtained with the set-convergence approach to stochastic optimisation. The assertion of Theorem 2.1 in [12] follows (after the canonical transformation to upper semicontinuous functions and maximisation problems) from Theorems 3.32 and 3.33 in combination with 3.15 and 3.16. Regarding the assumptions of 3.15 it is to note that for stochastic processes with lower semicontinuous trajectories convergence in distribution in the sense of uniform convergence on compact sets implies convergence in distribution of their epigraphs with respect to the Fell topology. This follows from the deterministic case (see A.7) and the Continuous Mapping Theorem. In the parallel work [44] Vogel has shown in a similar way how Ferger's results can be achieved with set convergence methods. Note that [12] and [44] do not contain the case of ε_n-argmins in the case of Skorohod convergence in distribution. We are now able to provide this generalisation. The sufficient conditions from Chapter 2 together with 3.15, 3.16 and the universal Theorems 3.32 and 3.33 can be used as building blocks to obtain ε_n-argmin results for the various types of convergence in distribution of random functions (stochastic processes). Take for example 3.32 , 3.15 and 3.16 in combination with 2.32 to receive the following generalisation in the case of Skorohod convergence in distribution.

Theorem 3.36 *Let f_n, f be random elements of $D[0,\infty)$ with $f_n \stackrel{D_{\tau_S}}{\to} f$. Let $\Psi_n^{\varepsilon_n}$, Ψ be derived from the lower semicontinuous modifications $\widetilde{f}_n, \widetilde{f}$. Assume that the sequence $(l_n^{\varepsilon_n})_n$ is stochastically bounded from above and that $(x_n)_n$ is a sequence of measurable $x_n \in \Psi_n^{\varepsilon_n}$.*
Then

$$\liminf_{n\to\infty} P\left(x_n < c\right) \geq P\left(l < c\right),$$

for all $c \in \mathbb{R}$ and all $\varepsilon_n \geq 0$ with $\varepsilon_n \stackrel{p}{\to} 0$.

By combining Theorems 3.32 and 3.33 we obtain

Theorem 3.37 *If $\Psi_n^{\varepsilon_n} \stackrel{D_{\tau_M}}{\to} \Psi$ and if the sequences $(s_n^{\varepsilon_n})_n$, resp. $(l_n^{\varepsilon_n})_n$ are stochastically bounded from below resp. from above, then for every sequence $(x_n)_n$ with measurable $x_n \in \Psi_n^{\varepsilon_n}$ and for all $a \leq b$*

$$\liminf_{n\to\infty} P\left(x_n \in (a,b)\right) \geq P\left(l < b\right) - P\left(s \leq a\right).$$

Proof.

$$\begin{aligned}\liminf_{n\to\infty} P\left(x_n \in (a,b)\right) =& \liminf_{n\to\infty} \left(P\left(x_n < b\right) - P\left(x_n \le a\right)\right)\\ \ge& \liminf_{n\to\infty} P\left(x_n < b\right) + \liminf_{n\to\infty}\left(-P\left(x_n \le a\right)\right)\\ =& \liminf_{n\to\infty} P\left(x_n < b\right) - \limsup_{n\to\infty} P\left(x_n \le a\right)\\ \ge& P\left(l < b\right) - P\left(s \le a\right)\end{aligned}$$

□

The theorem can be applied to find approximate confidence intervals in the sense of Pflug [27] (see the following section):

Corollary 3.38 *Let the assumptions of Theorem 3.37 be fulfilled, let $\alpha \in [0,1]$ and let $a,\ b$ be such that $P\left(s \le a\right) \le \frac{\alpha}{2}$ and $P\left(l < b\right) \ge 1 - \frac{\alpha}{2}$, then*

$$\liminf_{n\to\infty} P\left(x_n \in (a,b)\right) \ge 1 - \alpha.$$

By using the concept of τ_H-convergence in distribution the following theorem provides universal bounds for the probabilities $P(x < c)$ and $P(x > c)$, where $x \in \Psi$ and $c \in \mathbb{R}$.

Theorem 3.39 *Let $\widetilde{\Psi}_n^\varepsilon \overset{D_{\tau_H}}{\to} \widetilde{\Psi}^\varepsilon$ for an $\varepsilon > 0$ and let $x \in \widetilde{\Psi}^\varepsilon$ be measurable, then*

$$P\left(x < c\right) \le \liminf_{n\to\infty} P(s_n^\varepsilon < c)$$

and

$$P\left(x > c\right) \le \liminf_{n\to\infty} P(l_n^\varepsilon > c)$$

for all $c \in \mathbb{R}$.

Proof. Since the sets $H((-\infty,c))$ and $H((c,\infty))$ are τ_H-open, we have

$$\begin{aligned}P(x < c) \le& P\left(\widetilde{\Psi}^\varepsilon \in H\left((-\infty,c)\right)\right)\\ \le& \liminf_{n\to\infty} P\left(\widetilde{\Psi}_n^\varepsilon \in H((-\infty,c))\right)\\ =& \liminf_{n\to\infty} P\left(s_n^\varepsilon < c\right)\end{aligned}$$

and

$$\begin{aligned} P(x > c) &\leq P\left(\widetilde{\Psi}^{\varepsilon} \in H((c,\infty))\right) \\ &\leq \liminf_{n\to\infty} P\left(\widetilde{\Psi}^{\varepsilon}_n \in H((c,\infty))\right) \\ &= \liminf_{n\to\infty} P\left(l^{\varepsilon}_n > c\right). \end{aligned}$$ □

3.4. Approximate Confidence Regions

In this section we show how the stability theory from Section 3.2 can be applied to derive approximate confidence regions for the argmin sets. The first result was obtained by Pflug:

Theorem 3.40 (Theorem 1.4 in [27]) *Let f_n, f be random lower semicontinuous functions such that* $\operatorname{epi} f_n \overset{D_{\tau_{Fell}}}{\to} \operatorname{epi} f$. *Assume further that there are a compact set K such that*

$$\liminf_{n\to\infty} P\left(\Psi_n \subset K\right) \geq 1 - \frac{\alpha}{2}$$

and an open set $D \subset K$ with $P\left(\Psi \subset D\right) \geq 1 - \frac{\alpha}{2}$. Then D is an asymptotic confidence set for Ψ_n in the sense that

$$\liminf_{n\to\infty} P\left(\Psi_n \subset D\right) \geq 1 - \alpha$$

We can use the idea of Pflug's proof to obtain the following ε_n-optimality version:

Theorem 3.41 *Assume that $\Psi^{\varepsilon_n}_n \overset{D_{\tau_M}}{\to} \Psi$ for $\varepsilon_n \geq 0$ and $\varepsilon_n \overset{p}{\to} 0$. For an $\alpha \in [0,1]$ let there be a compact set K such that*

$$\liminf_{n\to\infty} P\left(\Psi^{\varepsilon_n}_n \subset K\right) \geq 1 - \frac{\alpha}{2}.$$

If there is an open set $D \subset K$ with $P\left(\Psi \subset D\right) \geq 1 - \frac{\alpha}{2}$, then D is an asymptotic confidence set for $\Psi^{\varepsilon_n}_n$ in the sense that

$$\liminf_{n\to\infty} P\left(\Psi^{\varepsilon_n}_n \subset D\right) \geq 1 - \alpha.$$

Proof. The set $K \setminus D$ is compact as the intersection of a closed and a compact set. Hence $M(K \setminus D)$ is a τ_M-open set. We have $P\left(\Psi^{\varepsilon_n}_n \in M(K\setminus D)\right) = P\left(\Psi^{\varepsilon_n}_n \subset D\right) + P\left(\Psi^{\varepsilon_n}_n \subset K^C\right)$ and thus $P\left(\Psi^{\varepsilon_n}_n \subset D\right) \geq P\left(\Psi^{\varepsilon_n}_n \in M(K\setminus D)\right) - \frac{\alpha}{2}$.

With $\Psi_n^{\varepsilon_n} \overset{D_{\tau_M}}{\to} \Psi$ we obtain

$$\begin{aligned}
\liminf_{n\to\infty} P\left(\Psi_n^{\varepsilon_n} \subset D\right) &\geq \liminf_{n\to\infty} P\left(\Psi_n^{\varepsilon_n} \in M(K \setminus D)\right) - \frac{\alpha}{2} \\
&\geq P\left(\Psi \in M(K \setminus D)\right) - \frac{\alpha}{2} \\
&= P\left(\Psi \subset D\right) + P\left(\Psi \subset K^C\right) - \frac{\alpha}{2} \\
&\geq 1 - \frac{\alpha}{2} - \frac{\alpha}{2} \\
&= 1 - \alpha
\end{aligned}$$

□

Together with Theorems 3.14 and 3.16 this theorem provides a generalisation of Pflug's result to the case of optimisation problems with restrictions and to ε_n-argmin sets.

With Theorem 3.41 one can derive information about the location of the approximating argmin sets from properties of the approximated set. We can provide a sort of opposite result, where location properties of the approximated argmin set are obtained from the properties of the approximating ε-argmin sets.

Theorem 3.42 *If* $\widetilde{\Psi}_n^\varepsilon \overset{D_{\tau_H}}{\to} \widetilde{\Psi}^\varepsilon$ *and* $\limsup_{n\to\infty} P\left(\widetilde{\Psi}_n^\varepsilon \subset A\right) \geq 1 - \alpha$ *for a closed set* A *and an* $\alpha \in [0,1]$, *then*

$$P\left(\widetilde{\Psi}^\varepsilon \subset A\right) \geq 1 - \alpha.$$

Proof. Since A^C is open, the set $M\left(A^C\right)$ is τ_H-closed. With $\widetilde{\Psi}_n^\varepsilon \overset{D_{\tau_H}}{\to} \widetilde{\Psi}^\varepsilon$ this leads to

$$\begin{aligned}
P\left(\widetilde{\Psi}^\varepsilon \subset A\right) &= P\left(\widetilde{\Psi}^\varepsilon \in M\left(A^C\right)\right) \\
&\geq \limsup_{n\to\infty} P\left(\widetilde{\Psi}_n^\varepsilon \in M\left(A^C\right)\right) \\
&= \limsup_{n\to\infty} P\left(\widetilde{\Psi}_n^\varepsilon \subset A\right) \\
&\geq 1 - \alpha.
\end{aligned}$$

□

4. Multiobjective Optimisation

In this final chapter we show, how the concept of convergence in distribution for random closed sets can be applied to multiobjective optimisation problems. Multiobjective optimisation problems occur in a variety of settings. In economics for example the producer of multiple goods wants to optimise the input for the factors of production. In finance a multiobjective optimisation problem is given by the goal to maximise the expected return of a portfolio, while minimising its variance (see e.g. [42]).
We will deal with random optimisation problems of the form

$$\text{minimise} f(x,\omega), \text{ subject to } x \in C(\omega),$$

where $f(x) = (f^1(x,\omega),\dots,f^p(x,\omega))$ with $f^i(.,\omega) : \mathbb{R}^d \to \mathbb{R},\ i = 1,\dots,p$ and $C(\omega)$ is a closed set. As in the one-dimensional case approximations of the objective function and the restriction set are taken into consideration. After choosing a suitable optimality concept we will see that in contrast to the one-dimensional case the optimal values can consist of closed sets with more than one element and thus allow the application of set convergence methods.
Throughout this chapter we rely on the deterministic result from [35] and provide convergence in distribution versions as well as extensions to $\varepsilon-$optimality.
Solving the optimisation problem shall be equivalent to finding the optimal values of $f\,(C(\omega),\omega) \subset \mathbb{R}^p$. Since there exists no natural order on $\mathbb{R}^p$, we have to choose an optimality concept on $\mathbb{R}^p$. A standard method is to generate a semi-order on $\mathbb{R}^p$ by using a cone $D \subset \mathbb{R}^p$ and writing $y < z$ if and only if $z - y \in D \setminus \{0\}$. Througout this section we will only work with the cones $D = \mathbb{R}^p_+ = \{y \in \mathbb{R}^p :\ y^i \geq 0,\ i = 1,\dots,p\}$ and $D = \text{int}(\mathbb{R}^p_+)$. These lead to the well known concepts of Pareto– and weak Pareto–efficiency, which are frequently used in economics and finance. For the general case see [16] and [35]. In [40] sufficient conditions for the coincidence of Pareto– and weakly Pareto–efficient points are collected.

Definition 4.1 Let $A \subset \mathbb{R}^d,\ \varepsilon \geq 0.$

(i) The set $\mathcal{E}(A)$ of efficient points of A with respect to the cone semi-order generated by $\mathbb{R}^p_+$ is called the set of Pareto–efficient points of A.

(ii) The set $\mathcal{E}_w(A)$ of efficient points of A with respect to the cone semi-order generated by $\mathrm{int}(\mathbb{R}^p_+)$ is called the set of weakly Pareto–efficient points of A.

(iii) The set $\mathcal{E}^\varepsilon(A) := \{z \in A : \exists y \in \mathcal{E}(A) : \|z - y\| \leq \varepsilon\}$ is called the set of $\varepsilon-$Pareto–efficient points of A.

(iv) The set $\mathcal{E}^\varepsilon_w(A) := \{z \in A : \exists y \in \mathcal{E}_w(A) : \|z - y\| < \varepsilon\ \}$ is called the set of weakly $\varepsilon-$Pareto–efficient points of A.

For a collection of various other ways to define a set of $\varepsilon-$efficient points see [17]. We have chosen the above definition because it is topologically accessible and fits into our framework.
Note that $\mathcal{E}^0_w(A) = \mathcal{E}_w(A)$. Further we note that $y \in \mathcal{E}(A)$ if and only if there is no $z \in A \setminus \{y\}$ such that $z^i \leq y^i,\ i = 1, \ldots, p$ and that $y \in \mathcal{E}_w(A)$ if and only if there is no $z \in A$ such that $z^i < y^i,\ i = 1, \ldots, p$. It immediately follows that $\mathcal{E}(A) \subset \mathcal{E}_w(A)$.
As in the preceding chapter we are interested in stability results in the setting of convergence in distribution, when an original optimisation problem with random objective functions and restriction sets is approximated by a sequence of surrogate optimisation problems. We do not take approximations of the semi-order generating cone into consideration. The work could however be extended in this direction. Recall that we require sequences of closed sets, when we deal with convergence (in distribution) with respect to the topologies $\tau_M,\ \tau_H$ and τ_{Fell}. The following example shows that unfortunately even for compact $A \subset \mathbb{R}^p$ the set $\mathcal{E}(A)$ is not always closed.

Example 4.2 Let $A = \{(x, 2 - x) : x \in [0, 1]\} \cup \{(1, y) : y \in [0, 1]\} \subset \mathbb{R}^2$, then $\mathcal{E}(A) = \{(x, 2 - x) : x \in [0, 1)\} \cup \{(1, 0)\}$ and this set is not closed.

In contrast for the weakly Pareto–efficient points we have

Lemma 4.3 *For each closed $A \subset \mathbb{R}^p$, the set $\mathcal{E}_w(A)$ is closed.*

Proof. Let $(y_n)_n \subset \mathcal{E}_w(A)$ and $y_n \to y$. It suffices to show that $y \in \mathcal{E}_w(A)$. Assume that $y \notin \mathcal{E}_w(A)$, then there is $z \in A$ with $z < y$. We can find $\alpha > 0$ such that $z^i < y^i - \alpha,\ i = 1, \ldots, p$. From $y^i_n \to y^i$ it follows that $y^i_n \geq y^i - \frac{\alpha}{2}$ for all $n \geq n_0,\ i = 1, \ldots, p$. This yields $z^i < y^i - \alpha < y^i - \frac{\alpha}{2} \leq y^i_n,\ n \geq n_0,\ i = 1, \ldots, d$, i.e. $z < y_n,\ n \geq n_0$ in contradiction to $y_n \in \mathcal{E}_w(A)$. $\square$

Corollary 4.4 *For each closed $A \subset \mathbb{R}^p$ and each $\varepsilon > 0$, the set $\mathcal{E}_w^{\varepsilon}(A)$ is closed.*

As a consequence, we will either work with $\mathcal{E}_w(A)$ or with $\mathrm{cl}(\mathcal{E}(A))$. We will see, that $\mathcal{E}_w(A)$ is well suited for dealing with τ_M-convergence, whereas $\mathrm{cl}(\mathcal{E}(A))$ will be used when we are interersted in τ_H-convergence. At first it may seem unusual, that we allow the optimality concept to depend on the type of convergence under consideration. This is however totally in line with Theorems 4.2.1 and 4.2.2 in [35], where the choice of the domination structure depends on the type of semicontinuity investigated in the theorems. To apply τ_M, τ_H and τ_{Fell} convergence methods to our optimisation problem, we have to make sure that the sets of weakly efficient points of $f(C)$ are closed. $\mathrm{cl}\left(\mathcal{E}(f(C))\right)$ is a closed set by definition.

Lemma 4.5 *If each function $f^i : \mathbb{R}^d \to \mathbb{R},\ i = 1, \dots, p$ is continuous and if C is compact, then $\mathcal{E}_w\left(f(C)\right)$ is closed.*

Proof. This follows immediately with Lemma 4.3, because $f(C)$ is closed as the image of a compact set under a continuous mapping. □

For the random case we require in the remainder of this chapter, that the mappings $\omega \mapsto \mathcal{E}_w(f(C(\omega), \omega))$, resp. $\omega \mapsto \mathrm{cl}\left(\mathcal{E}(f(C(\omega), \omega))\right)$ are $\mathcal{B}_{\tau_{\mathrm{Fell}}}$-measurable. As in the one-dimensional case we generally assume that the mappings $(x, \omega) \mapsto f^i(x, \omega)$ is jointly measurable in x and ω. In Lemma 6.1 of [39] Vogel has proven sufficient additional conditions on f^i and C which guarantee that our measurability requirements are fulfilled. Measurability of $\mathrm{cl}\left(\mathcal{E}(f(C(\omega), \omega))\right)$ then follows with Proposition 14.2 of [33]. Here we can conveniently choose continuity of each f^i and compactness of C as a sufficient condition for the measurability of the sets of efficient points. As we have just seen the set $\mathcal{E}_w\left(f(C)\right)$ is closed under these conditions.

We obtain the following main results for convergence in distribution of the optmial values.

Theorem 4.6 *Let f_n, f be random continuous functions with values in $\mathbb{R}^p$, which are uniformly bounded on a compact set $K \subset \mathbb{R}^d$. Let $C_n, C \subset K$ be random closed sets.*

(i) If

$$\left(\operatorname{graph} f_n^1, \ldots, \operatorname{graph} f_n^p, C_n\right) \stackrel{D_\tau}{\to} \left(\operatorname{graph} f^1, \ldots, \operatorname{graph} f^p, C\right),$$

where $\tau = \tau_{Fell} \times \ldots \times \tau_{Fell}$, then

$$\mathcal{E}_w^\varepsilon\left(f_n\left(C_n\right)\right) \stackrel{D_{\tau_M}}{\to} \mathcal{E}_w^\varepsilon\left(f\left(C\right)\right)$$

for all $\varepsilon \geq 0$.

(ii) If additionally $\varepsilon_n \stackrel{P}{\to} 0$ for a sequence of random nonnegative ε_n, then

$$\mathcal{E}_w^{\varepsilon_n}\left(f_n\left(C_n\right)\right) \stackrel{D_{\tau_M}}{\to} \mathcal{E}_w\left(f\left(C\right)\right)$$

Regarding τ_H we have

Theorem 4.7 *Let f_n, f be random continuous functions with values in $\mathbb{R}^p$, which are uniformly bounded on a compact set $K \subset \mathbb{R}^d$. Let $C_n, C \subset K$ be random closed sets. If*

$$(\operatorname{graph} f_n^1, \ldots, \operatorname{graph} f_n^p, C_n) \stackrel{D_{\tau_{Fell}}}{\to} (\operatorname{graph} f^1, \ldots, \operatorname{graph} f^p, C),$$

where $\tau = \tau_{Fell} \times \ldots \times \tau_{Fell} \times \tau_{Fell}$, then

(i)

$$\operatorname{cl}(\mathcal{E}(f_n(C_n))) \stackrel{D_{\tau_H}}{\to} \operatorname{cl}(\mathcal{E}(f(C)))$$

(ii)

$$\operatorname{cl}(\mathcal{E}^\varepsilon(f_n(C_n))) \stackrel{D_{\tau_H}}{\to} \operatorname{cl}(\mathcal{E}^\varepsilon(f(C)))$$

for all $\varepsilon > 0$.

As to the uniform boundedness condition we refer to the investigation following Lemma 3.21. The condition can be waived, if convergence of the graphs occurs on a region

containing the compact set K, e.g. $\mathbb{R}^d$.
We now turn to the solution sets

$$\Psi(f, C) = \{x \in C : \; f(x) \in \mathrm{cl}(\mathcal{E}(f(C)))\},$$

the weak solution sets

$$\Psi_w(f, C) = \{x \in C : \; f(x) \in \mathcal{E}_w(f(C))\}$$

and their $\varepsilon-$extensions

$$\Psi^\varepsilon(f, C) = \{x \in C : \; f(x) \in \mathrm{cl}(\mathcal{E}^\varepsilon(f(C)))\}$$

$$\Psi^\varepsilon_w(f, C) = \{x \in C : \; f(x) \in \mathcal{E}^\varepsilon_w(f(C))\}.$$

and note that all these sets are closed as the preimage of a closed set under a continuous mapping. $\mathcal{B}_{\tau_{\mathrm{Fell}}}$-measurability of the solution sets follows with Lemma 6.1 of [39].
As main results for the solution sets we have

Theorem 4.8 *If the assumptions of Theorem 4.6 are fulfilled, then*

(i)

$$\Psi^\varepsilon_w(f_n, C_n) \stackrel{D_{\tau_M}}{\to} \Psi^\varepsilon_w(f, C)$$

for all $\varepsilon \geq 0$.

(ii) If additionally $\varepsilon_n \stackrel{P}{\to} 0$ *for a sequence of random nonnegative* ε_n*, then*

$$\Psi^{\varepsilon_n}_w(f_n, C_n) \stackrel{D_{\tau_M}}{\to} \Psi_w(f, C)$$

Theorem 4.9 *If the assumptions of Theorem 4.7 are fulfilled, then for all* $\varepsilon > 0$

(i)

$$\Psi^\varepsilon(f_n, C_n) \stackrel{D_{\tau_H}}{\to} \Psi^\varepsilon(f, C)$$

(ii)

$$\Psi^\varepsilon(f_n, C_n) \stackrel{D_{\tau_H}}{\to} \Psi(f, C)$$

As in the one-dimensional case, we will rely heavily on results from parametric optimisation, found in [40] and [35], and transfer these results to the convergence in distribution setting with the help of the Continuous Mapping Theorem. The following theorem extends Theorem 4.2.1 of [35] to the case of ε-efficient points. Due to the notational differences in [35] we give a detailed proof, using the ideas of the original proof.

Theorem 4.10 *Let f_n, f be continuous functions with values in $\mathbb{R}^p$, which are uniformly bounded on a compact set $K \subset \mathbb{R}^d$. Let $C_n, C \subset K$ be random closed sets. If*

$$\left(\operatorname{graph} f_n^1, \ldots, \operatorname{graph} f_n^p, C_n\right) \xrightarrow{\tau} \left(\operatorname{graph} f^1, \ldots, \operatorname{graph} f^p, C\right),$$

where $\tau = \tau_{Fell} \times \ldots \times \tau_{Fell}$, then

$$\mathcal{E}_w^\varepsilon\left(f_n\left(C_n\right)\right) \xrightarrow{\tau_M} \mathcal{E}_w^\varepsilon\left(f\left(C\right)\right)$$

for all $\varepsilon \geq 0$.

Proof. Let $y_{n_k} \in \mathcal{E}_w^\varepsilon\left(f_{n_k}\left(C_{n_k}\right)\right)$ and $y_{n_k} \to y$. First we show that $y \in f\left(C_n\right)$. For each y_{n_k} there is at least one $x_{n_k} \in C_{n_k}$ with $f_{n_k}\left(x_{n_k}\right) = y_{n_k}$ and there is $w_{n_k} \in C_{n_k}$ such that $z_{n_k} := f_{n_k}\left(w_{n_k}\right) \in \mathcal{E}\left(f_{n_k}\left(C_{n_k}\right)\right)$ and $\left\|y_{n_k} - z_{n_k}\right\| \leq \varepsilon$. By using the compactness of K, we can find $\left(n_{k(l)}\right)_l$ such that $x_{n_{k(l)}} \to x$ and $w_{n_{k(l)}} \to w$ for some $x, w \in K$. From $\limsup\limits_{n\to\infty} C_n \subset C$ it follows that $x \in C$ and $w \in C$. Let $i \in \{1, \ldots, p\}$. From the uniform boundedness of $\left(f_n^i\right)_n$ and from $\operatorname{graph} f_n^i \xrightarrow{\tau_{\text{Fell}}} \operatorname{graph} f^i$ we obtain $y_{n_{k(l)}}^i = f_{n_{k(l)}}^i\left(x_{n_{k(l)}}\right) \to f^i\left(x\right)$ and thus $f^i\left(x\right) = y^i$ because of $y_{n_{k(l)}}^i \to y^i$. This shows $y \in f\left(C\right)$.
Likewise we have $z_{n_{k(l)}}^i = f_{n_{k(l)}}^i\left(w_{n_{k(l)}}\right) \to f^i\left(w\right)$. We set $z = \left(f^1\left(w\right), \ldots, f^p\left(w\right)\right)$, then $z \in \left\{f\left(x\right) : x \in C\right\}$. From $\left\|y_{n_{k(l)}} - z_{n_{k(l)}}\right\| \leq \varepsilon$ it follows that $\left\|y - z\right\| \leq \varepsilon$. It remains to show that there is $u \in \mathcal{E}_w\left(f\left(C\right)\right)$ with $\left\|y - u\right\| \leq \varepsilon$. Assume that this is not the case, then $z \notin \mathcal{E}_w\left(\left\{f\left(x\right) : x \in C\right\}\right)$. This implies that there is $\widetilde{x} \in C$ such that $\widetilde{z} := f\left(\widetilde{x}\right) < z$. Because of $C \subset \liminf\limits_{n\to\infty} C_n$ we can find a sequence $\left(\widetilde{x}_n\right)_n$ with $\widetilde{x}_n \in C_n$, for $n \geq n_0$ and $\widetilde{x}_n \to \widetilde{x}$. For each i we have $f_n^i\left(\widetilde{x}_n\right) \to f^i$ and with $\widetilde{z}_{n_{k(l)}}^i := f_{n_{k(l)}}^i\left(\widetilde{x}_{n_{k(l)}}\right)$ we obtain

$$\begin{aligned}
&\liminf_{l\to\infty}\left(z_{n_{k(l)}}^i - \widetilde{z}_{n_{k(l)}}^i\right) \\
&\geq \liminf_{l\to\infty} z_{n_{k(l)}}^i + \liminf_{l\to\infty}\left(-\widetilde{z}_{n_{k(l)}}^i\right) \\
&= z^i - \widetilde{z}^i \\
&> 0.
\end{aligned}$$

This shows that there is l_0 such that $\widetilde{z}^i_{n_{k(l)}} < z^i_{n_{k(l)}}$ for all $l \geq l_0$ in contradiction to $z_{n_{k(l)}} \in \mathcal{E}_w\left(\left\{f_{n_{k(l)}}(x) : x \in C_{n_{k(l)}}\right\}\right)$. □

The following example shows that the concept of weak Pareto–efficient points does in general not lead to the desired results for τ_H-convergence.

Example 4.11 Let $C_n = C = [0,1]$. Let $f^1_n(x) = x,\ f^2_n(x) = \frac{1}{n}x,\ f^1(x) = x,\ f^2(x) = 0$. Then $\operatorname{graph} f^i_n \overset{\tau_{\text{Fell}}}{\to} \operatorname{graph} f^i$, but we have $\mathcal{E}_w(f_n(C_n)) = \{(0,0)\}$ and $\mathcal{E}_w(f(C)) = [0,1] \times \{0\}$, which shows that τ_H-convergence cannot hold. As to τ_H-convergence in distribution take the τ_H-open set $A = H(\mathbb{R}^2 \setminus \{(0,0)\})$. Then $P(\mathcal{E}_w(f_n(C_n)) \in U) = 0$ and $P(\mathcal{E}_w(f(C)) \in U) = 1$.

It is thus reasonable to change the optimality concept for the investigation of τ_H-convergence. We will work with the closures of the sets of Pareto-efficient points. First note that the results of the previous theorem and those of Theorem 4.6 do in general not hold for the sets $\operatorname{cl}(\mathcal{E}(f_n(C_n)))$ and $\operatorname{cl}(\mathcal{E}(f(C)))$.

Example 4.12 Let $C_n = C = [0,1],\ f^1_n(x) = x,\ f^2_n(x) = -\frac{1}{n}x,\ f^1(x) = x,\ f^2(x) = 0$, then $\operatorname{graph} f^i_n \overset{\tau_{\text{Fell}}}{\to} \operatorname{graph} f^i$, $i = 1,2$. We have $\left(1, -\frac{1}{n}\right) \in \operatorname{cl}(\mathcal{E}(f_n(C_n)))$ and $\left(1, -\frac{1}{n}\right) \to (1,0) \notin \operatorname{cl}(\mathcal{E}(f(C))) = \{(0,0)\}$. For the case of τ_M-convergence in distribution consider the τ_M-open set $U = M\left([-1,0] \times \left\{\frac{1}{2}\right\}\right)$ to obtain $P(\mathcal{E}(f_n(C_n)) \in U) = 0$ and $P(\mathcal{E}(f(C)) \in U) = 1$.

Theorem 4.13 *If the assumptions of Theorem 4.10 are fulfilled, then*

(i)

$$cl\left(\mathcal{E}\left(f_n\left(C_n\right)\right)\right) \overset{\tau_H}{\to} cl\left(\mathcal{E}\left(f\left(C\right)\right)\right).$$

(ii)

$$cl\left(\mathcal{E}^{\varepsilon}\left(f_n\left(C_n\right)\right)\right) \overset{\tau_H}{\to} cl\left(\mathcal{E}^{\varepsilon}\left(f\left(C\right)\right)\right),$$

for all $\varepsilon > 0$

Proof. (i) First let $y \in \mathcal{E}(f(C))$. Because of $\operatorname{graph} f^i_n \overset{\tau_{\text{Fell}}}{\to} \operatorname{graph} f^i$ there is a sequence $(y_n)_n$ with $y_n \in f_n(C_n)$ and $y_n \to y$. To each y_n there is $\widetilde{y}_n$ with $\widetilde{y}_n \leq y_n$ and $\widetilde{y}_n \in \mathcal{E}(f_n(C_n))$, this immediately follows with theorem 3.2.9 (external stability) of [35], note that $f_n(C_n)$ is compact and that the closed cone $\mathbb{R}^p_+$ is used to determine the Pareto-efficient points. From the uniform boundedness of $(f_n)_n$ it follows that $\widetilde{y}_{n_k} \to \widetilde{y}$ for some subsequence $(\widetilde{y}_{n_k})_k$ and some $\widetilde{y}$. We have $\widetilde{y} \in f(C)$ because of graph $f_n^i \overset{\tau_{\text{Fell}}}{\to}$ graph f^i. From $\widetilde{y}_{n_k} \leq y_{n_k}$, $\widetilde{y}_{n_k} \to \widetilde{y}$ and $y_{n_k} \to y$ it follows that $\widetilde{y} \leq y$, which in view of $y \in \mathcal{E}(f(C))$ implies $\widetilde{y} = y$. Furthermore we obtain $\widetilde{y}_n \to y$, because the above shows that for the sequence $(\widetilde{y}_n)_n$, which is contained in a compact set, every cluster point coincides with y.

Now let $y \in cl(\mathcal{E}(f(C)))$, then y is the limit of a sequence $(z_l)_l$ with $z_l \in \mathcal{E}(f(C))$. Since each z_l can be approximated as in the first part of the proof, it is clear, that we can find a sequence $(y_n)_n$ with $y_n \in \mathcal{E}(f_n(C_n)) \subset \mathrm{cl}(\mathcal{E}(f_n(C_n)))$ and $y_n \to y$.

(ii) Let $y \in \mathcal{E}^\varepsilon(f(C))$, then there is $z \in \mathcal{E}(f(C))$ such that $\|y - z\| < \varepsilon$. Let $\alpha := \varepsilon - \|y - z\|$. Because of (i) we can find $(z_n)_n$ such that $z_n \in \mathcal{E}(f_n(C_n))$ and $z_n \to z$. From the convergence assumptions it follows that there is a sequence $(y_n)_n$ with $y_n \in f_n(C_n)$ and $y_n \to y$. Consequently there is n_0 such that $\|z_n - z\| < \frac{\alpha}{3}$ and $\|y_n - y\| < \frac{\alpha}{3}$ for all $n \geq n_0$. We obtain

$$\begin{aligned}
\|y_n - z_n\| &\leq \|y_n - y\| + \|y - z\| + \|z - z_n\| \\
&\leq \frac{2}{3}\alpha + \|y - z\| \\
&= \frac{2}{3}\alpha + \varepsilon - \alpha < \varepsilon,
\end{aligned}$$

for $n \geq n_0$, which shows that $y_n \in \mathcal{E}^\varepsilon(f(C))$ for all $n \geq n_0$

The argumentation for the case $y \in \mathrm{cl}(\mathcal{E}^\varepsilon(f(C)))$ is the same as that in the last step in the proof of (i). □

Theorem 4.14 *If the assumptions of Theorem 4.10 are fulfilled, then*

$$\Psi_w^\varepsilon(f_n, C_n) \overset{\tau_M}{\to} \Psi_w^\varepsilon(f, C,)$$

for all $\varepsilon \geq 0$.

Proof. First we deal with the case $\varepsilon = 0$. Let $x_{n_k} \in \Psi_w(f_{n_k}(C_{n_k}))$ and $x_{n_k} \to x$. It is to show that $x \in \Psi_w(f(C))$. Because of $\limsup_{n\to\infty} C_n \subset C$ it is clear, that $x \in C$. Assume that $x \notin \Psi_w(f(C))$, then there is $\widetilde{x} \in C$ such that $f(\widetilde{x}) < f(x)$. Now $C \subset \liminf_{n\to\infty} C_n$ implies that there is a sequence $(\widetilde{x}_n)_n$ with $\widetilde{x}_n \in C_n$, $n \geq n_0$ and $\widetilde{x}_n \to \widetilde{x}$. We have $f_n(x_n) \to f(x)$ and $f_n(\widetilde{x}_n) \to f(\widetilde{x})$. It follows that

$$\lim_{k\to\infty} \left(f_{n_k}^i(x_{n_k}) - f_{n_k}^i(\widetilde{x}_{n_k})\right) = f^i(x) - f^i(\widetilde{x}) > 0$$

for $i = 1, \ldots, p$. Thus there is k_0 such that $f^i_{n_k}(x_{n_k}) - f^i_{n_k}(\widetilde{x}_{n_k}) > 0,\ i = 1, \ldots, p$ and all $k \geq k_0$. We obtain $f^i_{n_k}(\widetilde{x}_{n_k}) < f^i_{n_k}(x_{n_k}),\ i = 1, \ldots, p,\ k \geq k_0$ in contradiction to $x_{n_k} \in \Psi_w(f_{n_k}, C_{n_k})$.

Now let $\varepsilon > 0$ and let $x_{n_k} \in \Psi^\varepsilon_w(f_{n_k}, C_{n_k})$ with $x_{n_k} \to x$. We show that $x \in \Psi^\varepsilon_w(f, C)$. To x_{n_k} there is $u_{n_k} \in C_{n_k}$ with $f_{n_k}(u_{n_k}) \in \Psi_w(f_{n_k}, C_{n_k})$ and $\|f_{n_k}(x_{n_k}) - f_{n_k}(u_{n_k})\| \leq \varepsilon$. For a subsequence $\left(u_{n_{k(l)}}\right)_l$ we have $u_{n_{k(l)}} \to u$ for some $u \in C$. This follows from the compactness of K and from $\limsup_{n\to\infty} C_n \subset C$. From $f_{n_{k(l)}}\left(x_{n_{k(l)}}\right) \to f(x)$ and $f_{n_{k(l)}}\left(u_{n_{k(l)}}\right) \to f(u)$ we obtain $\|f(x) - f(u)\| \leq \varepsilon$. The first part of the proof yields $u \in \Psi_w(f, C)$, and thus $x \in \Psi_w(f, C)$. □

In the one-dimensional case we have already seen, that in general not every solution of the original problem is the limit of a sequence of solutions of the approximating problems. We have also seen, that a positive result can be obtained, if we allow a $\varepsilon-$perturbation of the optimal values and consider the corresponding solution sets. We will now show, that this also holds true in the vector valued case.

Theorem 4.15 *If the assumptions of Theorem 4.10 are fulfilled, then*

$$\Psi^\varepsilon(f_n, C_n) \stackrel{\tau_H}{\to} \Psi^\varepsilon(f, C)$$

for all $\varepsilon > 0$.

Proof. First let x be such that $\|f(x) - f(\widetilde{x})\| < \varepsilon$ for some $f(\widetilde{x}) \in \mathrm{cl}\,(\mathcal{E}(f(C)))$. Let $\alpha = \varepsilon - \|f(x) - f(\widetilde{x})\|$. We can find a sequence $(x_n)_n$ with $x_n \to x$ and it follows that $f_n(x_n) \to f(x)$. Thus $\|f_n(x_n) - f(x)\| \leq \frac{\alpha}{3}$ for all $n \geq n_0$. Following the proof of 4.13 there is a sequence $(f_n(\widetilde{x}_n))_n$ with $f_n(\widetilde{x}_n) \in \mathcal{E}(f_n(C_n))$ and $f_n(\widetilde{x}_n) \to f(\widetilde{x})$, i.e. $\|f_n(\widetilde{x}_n) - f(\widetilde{x})\| \leq \frac{\alpha}{3}$ for all $n \geq n_1$.
This yields

$$\begin{aligned}
&\|f_n(x_n) - f_n(\widetilde{x}_n)\| \\
&\leq \|f_n(x_n) - f(x)\| + \|f(x) - f(\widetilde{x})\| + \|f(\widetilde{x}) - f_n(\widetilde{x}_n)\| \\
&\leq \|f(x) - f(\widetilde{x})\| + \frac{2}{3}\alpha \\
&< \varepsilon
\end{aligned}$$

for all $n \geq \max(n_0, n_1)$, which shows that $x_n \in \Psi^\varepsilon(f_n, C_n)$.
Now let $x \in \Psi(f, C)$, then there is a sequence $(u_l)_l$ with $u_l \in \{x \in C : \; f(x) \in \operatorname{cl}(\mathcal{E}(f(C)))\}$ and $u_l \to x$. From the first part of the proof it follows that to each u_l there is a sequence $\left(v_n^l\right)_n$ with $v_n^l \to u_l$ and $v_n^l \in \Psi^\varepsilon(f_n(C_n)$. With this we can choose a sequence $(x_n)_n$ such that $x_n \to x$ and $x_n \in \Psi^\varepsilon(f, C)$. □

Proof. (of Theorem 4.6) (i) Theorem 4.10 shows that the mapping
$S : \left(\mathrm{GRA}(\mathbb{R}^d) \times \ldots \times \mathrm{GRA}(\mathbb{R}^d) \times \mathcal{F}(\mathbb{R}^d), \tau\right) \to \left(\mathcal{F}(\mathbb{R}^p), \tau_M\right)$,

$$S\left(\operatorname{graph} f^1, \ldots, \operatorname{graph} f^p, C\right) = \mathcal{E}_w^\varepsilon\left(\{f(x) : x \in C\}\right)$$

is continuous. It remains to apply the Continuous Mapping Theorem. Now (ii) follows from (i) with the help of Lemma A.8 from the appendix. □

Proof. (of Theorem 4.7) In Theorem 4.13 it was shown that the mappings

$$S_1((\operatorname{graph} f^1, \ldots, \operatorname{graph} f^p, C) = \operatorname{cl}(\mathcal{E}(f(C)))$$

and

$$S_2((\operatorname{graph} f^1, \ldots, \operatorname{graph} f^p, C) = \operatorname{cl}(\mathcal{E}^\varepsilon(f(C)))$$

are continuous with respect to τ and τ_H. It remains to apply the Continuous Mapping Theorem. □

Proof. (of Theorem 4.8) We have seen in Theorem 4.14 that the mapping $(\operatorname{graph} f^1, \ldots, \operatorname{graph} f^p, C) \mapsto \Psi_w^\varepsilon(f, C)$ is continuous with respect to τ and τ_M. Part (i) now follows with the Continuous Mapping Theorem. For (ii) note that $\Psi_w = \bigcap_{k=1}^{\infty} \Psi_w^{\frac{1}{k}}$. The proof can now, by using (i), be carried out as in the proof of Theorem 3.16. □

Proof. (of Theorem 4.9) We have shown in Theorem 4.15 that the mapping $(\operatorname{graph} f^1, \ldots, \operatorname{graph} f^p, C) \mapsto \Psi^\varepsilon(f, C)$ is $\tau - \tau_H$ continuous, now (i) follows with the Continuous Mapping Theorem. For (ii) note that $\Psi(f, C) \subset \Psi^\varepsilon(f, C)$, which together with (i) for all τ_H-open U implies

$$\liminf_{n\to\infty} P(\Psi^\varepsilon(f_n, C_n) \in U) \geq P(\Psi^\varepsilon(f, C) \in U) \geq P(\Psi(f, C) \in U). \qquad \square$$

So far, we have only dealt with convergence (in distribution) of the sets of efficient and weakly-efficient points with respect to the topologies τ_M and τ_H. We have seen, that we require rather strong assumptions, namely continuous, uniformly bounded functions and $\tau_{\text{Fell}}-$convergence of their graphs. It is well known and easy to show that the results for the sets of efficient points, if applied in the one-dimensional case, lead to convergence of the optimal values, which is much stronger than the usually investigated semicontinuity of the optimal values. This was one reason for Penot and Sterna-Karwat (see [26]) to introduce order semicontinuity, which reduces to semicontinuity in the one-dimensional case.

In our context order-lower semicontinuous behaviour of the sets of efficient points means that

$$\mathcal{E}(f_n(C_n)) \stackrel{\tau_M}{\to} \mathcal{E}(f(C)) + \mathbb{R}_+^p$$

and we have order-upper semicontinuous behaviour, if

$$\mathcal{E}(f_n(C_n)) + \mathbb{R}_+^p \stackrel{\tau_H}{\to} \mathcal{E}(f(C)).$$

We will see, that to establish the obviously weaker order semicontinuous behaviour we require weaker assumptions on the objective functions f_n and f. For example we only need semicontinuity of the f_n^i, f^i and τ_M-, resp. τ_H-convergence of their epi- resp. hypographs. Also the boundedness assumptions can be weakened. In the deterministic case we have

Theorem 4.16 *Let $f_n^i,\ f^i,\ i = 1, \ldots, p,\ n \in \mathbb{N}$ be lower semicontinuous on a subset of $\mathbb{R}^d$. Let $C_n,\ n \in \mathbb{N},\ C$ be closed subsets of a compact set $K \subset \mathbb{R}^d$.*
Under the assumptions

(a) $\mathcal{E}(f_n(C_n))$ and $\mathcal{E}(f(C)) + \mathbb{R}_+^p$ are closed.

(b) $f(C) \subset \mathcal{E}(f(C)) + \mathbb{R}_+^p$

(c)

$$(\operatorname{epi} f_n^1, \ldots, \operatorname{epi} f_n^p, C_n) \stackrel{\tau}{\to} (\operatorname{epi} f^1, \ldots, \operatorname{epi} f^p, C),$$

where $\tau = \tau_M \times \ldots \times \tau_M \times \tau_H$

it follows that

$$\mathcal{E}(f_n(C_n)) \stackrel{\tau_M}{\to} \mathcal{E}(f(C)) + \mathbb{R}_+^p$$

Proof. This theorem follows from Theorem 5.1 of [42] or Theorem 3.1 of [26]. □

The theorem does not allow a direct transfer to an 'in distribution' version, since it does not provide a $\tau - \tau_M-$continuous mapping. Under the additional assumption of uniform boundedness from below we obtain the following modification. Here boundedness from below for a function $f : K \to \mathbb{R}^p$ means that there are $m_1, \ldots, m_p \in \mathbb{R}$ such that $f^i(x) \geq m_i,\ i = 1, \ldots, p$ for all $x \in K$.

Theorem 4.17 *Let $f_n^i,\ f^i,\ i = 1, \ldots, p,\ n \in \mathbb{N}$ be lower semicontinuous on a subset of $\mathbb{R}^d$. Let $C_n,\ n \in \mathbb{N},\ C$ be closed subsets of a compact set $K \subset \mathbb{R}^d$.*
Under the assumptions

(a) The functions $f_n,\ n \in \mathbb{N},\ f$ are uniformly bounded from below on K.

(b) $\mathcal{E}(f_n(C_n)) + \mathbb{R}^p_+,\ n \in \mathbb{N}$ and $\mathcal{E}(f(C)) + \mathbb{R}^p_+$ are closed.

(c) $f_n(C_n) \subset \mathcal{E}(f_n(C_n)) + \mathbb{R}^p_+,\ n \in \mathbb{N},\ f(C) \subset \mathcal{E}(f(C)) + \mathbb{R}^p_+$

(d)

$$(\operatorname{epi} f_n^1, \ldots, \operatorname{epi} f_n^p, C_n) \xrightarrow{\tau} (\operatorname{epi} f^1, \ldots, \operatorname{epi} f^p, C),$$

where $\tau = \tau_M \times \ldots \times \tau_M \times \tau_M$

it follows that

$$\mathcal{E}(f_n(C_n)) + \mathbb{R}^p_+ \xrightarrow{\tau_M} \mathcal{E}(f(C)) + \mathbb{R}^p_+$$

Proof. Let $y_n \in \mathcal{E}(f_n(C_n)) + \mathbb{R}^p_+$ and $y_n \to y$. It follows that $y \in \mathbb{R}^p$. We have $y_n = u_n + v_n$ with $u_n \in \mathcal{E}(f_n(C_n)$ and $v_n \in \mathbb{R}^p_+$. This yields the boundedness from above of the sequence $(u_n)_n$, because otherwise y_n can not converge to an element of $\mathbb{R}^p$. With the uniform boundedness of $(f_n)_n$ from below it follows that $(u_n)_n$ is bounded. Consequently there is a subsequence $(u_{n_k})_k$ converging to some $u \in \mathbb{R}^d$. More precisely we have $u \in \mathcal{E}(f(C)) + \mathbb{R}^p_+$, because of Theorem 4.16 . It follows that $v_{n_k} = y_{n_k} - u_{n_k}$ converges to $v := y - u$. We have $v \in \mathbb{R}^p_+$ and thus $y = u + v \in \mathcal{E}(f(C)) + \mathbb{R}^p_+$ □

Remark 4.18 The Theorems 4.16 and 4.17 remain valid, if we substitute the sets of Pareto-efficient points $\mathcal{E}$ with sets of weakly Pareto-efficient points $\mathcal{E}_w$ in all assumptions and assertions. Indeed an inspection of the proofs of the Theorems 4.16 (resp. Theorem 5.1 of [42]) and 4.17 shows that the Pareto-efficiency property of the points $y_n \in \mathcal{E}(f_n(C_n))$ is not used. The proofs only make use of $y_n \in f_n(C_n)$. Consequently Theorems and and the following theorems also hold, if they are formulated for the sets of (weakly-)$\varepsilon-$efficient points $\mathcal{E}^\varepsilon_w$, resp. $\mathcal{E}^\varepsilon$.

Theorem 4.19 *Let $f_n^i,\ f^i,\ i = 1, \ldots, p,\ n \in \mathbb{N}$ be random lower semicontinuous functions, let $C_n,\ n \in \mathbb{N},\ C$ be random closed subsets of a compact set K.*
Under the the assumptions

(a) The functions $f_n,\ n \in \mathbb{N},\ f$ are uniformly bounded from below on K.

(b) $\mathcal{E}(f_n(C_n)) + \mathbb{R}_+^p,\ n \in \mathbb{N}$ and $\mathcal{E}(f(C)) + \mathbb{R}_+^p$ are closed.

(c) $f_n(C_n) \subset \mathcal{E}(f_n(C_n)) + \mathbb{R}_+^p,\ n \in \mathbb{N},\ f(C) \subset \mathcal{E}(f(C)) + \mathbb{R}_+^p$

(d)

$$(\operatorname{epi} f_n^1, \ldots, \operatorname{epi} f_n^p, C_n) \xrightarrow{D_\tau} (\operatorname{epi} f^1, \ldots, \operatorname{epi} f^p, C),$$

where $\tau = \tau_M \times \ldots \times \tau_M \times \tau_M$

it follows that

$$\mathcal{E}_w(f_n(C_n)) \xrightarrow{D_{\tau_M}} \mathcal{E}_w(f(C)) + \mathbb{R}_+^p$$

Proof. Let U be τ_M-open, then

$$\begin{aligned}
&\liminf_{n\to\infty} P\left(\mathcal{E}_w(f_n(C_n)) \in U\right) \\
\geq &\liminf_{n\to\infty} P\left(\mathcal{E}_w(f_n(C_n)) + \mathbb{R}_+^p \in U\right) \\
\geq &P\left(\mathcal{E}_w(f(C)) + \mathbb{R}_+^p \in U\right)
\end{aligned}$$

□

Theorem 4.20 *Let $f_n^i,\ f^i,\ i = 1, \ldots, p,\ n \in \mathbb{N}$ be upper semicontinuous, let $C_n,\ n \in \mathbb{N},\ C$ be closed subsets of a compact set K.*
Under the assumptions

(a) The sets $\mathcal{E}\left(f_n\left(C_n\right)\right) + \mathbb{R}_+^p,\ n \in \mathbb{N}$ and $\mathcal{E}\left(f\left(C\right)\right) + \mathbb{R}_+^p$ are closed.

(b) $f_n(C_n) \subset \mathcal{E}(f_n(C_n)) + \mathbb{R}_+^p,\ n \in \mathbb{N},\ f(C) \subset \mathcal{E}(f(C)) + \mathbb{R}_+^p$

(c)

$$(\operatorname{hypo} f_n^1, \ldots, \operatorname{hypo} f_n^p, C_n) \xrightarrow{\tau} (\operatorname{hypo} f^1, \ldots, \operatorname{hypo} f^p, C),$$

where $\tau = \tau_M \times \ldots \times \tau_M \times \tau_H$

it follows that

$$\mathcal{E}\left(f_n\left(C_n\right)\right) + \mathbb{R}_+^p \xrightarrow{\tau_H} \mathcal{E}\left(f\left(C\right)\right) + \mathbb{R}_+^p.$$

Proof. Let $y \in \mathcal{E}\left(f\left(C\right)\right) + \mathbb{R}^p_+$, then $y = u + v$ with $u \in \mathcal{E}\left(f\left(C\right)\right)$ and $v \in \mathbb{R}^p_+$. From Theorem 5.2 of [42] it follows that there is a sequence $(u_n)_n$ with $u_n \in \mathcal{E}\left(f\left(C\right)\right) + \mathbb{R}^p_+$ and $u_n \to u$. Let $y_n = u_n + v$, then $y_n \in \mathcal{E}\left(f\left(C\right)\right) + \mathbb{R}^p_+$ and $y_n \to y$. $\square$

Theorem 4.21 *Let $f^i_n,\ f^i,\ i = 1, \ldots, p,\ n \in \mathbb{N}$ be random upper semicontinuous functions on a subset of $\mathbb{R}^d$. Let $C_n,\ n \in \mathbb{N},\ C$ be random closed subsets of a compact set $K \subset \mathbb{R}^d$.*
Under the assumptions

(a) $\mathcal{E}(f_n(C_n)) + \mathbb{R}^p_+,\ n \in \mathbb{N}$ *and* $\mathcal{E}(f(C)) + \mathbb{R}^p_+$ *are closed.*

(b) $f_n(C_n) \subset \mathcal{E}(f_n(C_n)) + \mathbb{R}^p_+,\ n \in \mathbb{N},\ f(C) \subset \mathcal{E}(f(C)) + \mathbb{R}^p_+$

(c)

$$(\operatorname{hypo} f^1_n, \ldots, \operatorname{hypo} f^p_n, C_n) \stackrel{D_\tau}{\to} (\operatorname{hypo} f^1, \ldots, \operatorname{hypo} f^p, C),$$

where $\tau = \tau_M \times \ldots \times \tau_M \times \tau_H$

it follows that

$$\mathcal{E}(f_n(C_n)) + \mathbb{R}^p_+ \stackrel{D_{\tau_H}}{\to} \mathcal{E}(f(C))$$

Proof. Let U be τ_H-open, then

$$\begin{aligned}
&\liminf_{n\to\infty} P(\mathcal{E}(f_n(C_n)) + \mathbb{R}^p_+ \in U) \\
\geq &P(\mathcal{E}(f(C)) + \mathbb{R}^p_+ \in U) \\
\geq &P(\mathcal{E}(f(C)) \in U)
\end{aligned}$$

$\square$

A. Appendix

Lemma A.1 *Let (X, τ) be a topological space with base $\mathcal{U}$. Let $\mathcal{A}$ be a countable collection of subsets of X. If each element of $\mathcal{U}$ is the countable union of elements of $\mathcal{A}$, then $\mathcal{A}$ is a countable base for X.*

Proof. Let $U \subset X$ be open. Then $U = \bigcup_{i \in I} B_i$, with $B_i \in \mathcal{U}$ and an arbitrary index set I. For each B_i we can find a sequence $(A_n^i)_n \subset \mathcal{A}$ such that $B_i = \bigcup_{n=1}^{\infty} A_n^i$. We obtain

$$U = \bigcup_{i \in I} \bigcup_{n=1}^{\infty} A_n^i = \bigcup_{n=1}^{\infty} \bigcup_{i \in I} A_n^i.$$

Since $A_n^i \in \mathcal{A}$ for all $n \in \mathbb{N},\ i \in I$ and $\mathcal{A}$ has only countable many elements, it follows that U is the countable union of elements of $\mathcal{A}$. Since U was an arbitrary open set, $\mathcal{A}$ is a countable base. $\square$

Lemma A.2 *Let (X, τ) be a regular topological space, let $K \subset X$ be compact. If $K = \bigcap_{m=1}^{\infty} L_m$ with compact L_m and $L_{m+1} \subset L_m$, then*

$$M(K) = \bigcup_{m=1}^{\infty} M(L_m).$$

Proof. First let $F \in \bigcup_{m=1}^{\infty} M(L_m)$, then there is m_0 such that $F \in M(L_{m_0})$. Because of $L_m \subset L_{m_0}$ for all $m \geq m_0$ it follows that $F \in M(L_m)$ for all $m \geq m_0$ and thus $F \in M\left(\bigcap_{m=1}^{\infty} L_m\right) = M(K)$.
Now let $F \in M(K)$. We can find open sets U and V such that $K \subset U$, $F \subset V$ and $U \cap V = \emptyset$ (see Theorem 3.1.6 of [10]). There is (see [10] Corrollary 3.1.5) m_0 such that $L_{m_0} \subset U$. It follows that $F \in M(L_{m_0})$ and thus $F \in \bigcup_{m=1}^{\infty} M(L_m)$. $\square$

Lemma A.3 *The mapping* $S : (\mathcal{F}([0,\infty)), \tau_M) \to [0,\infty]$, $S(A) = \min\{x : x \in A\}$ *with* $min\{x : x \in \emptyset\} = \infty$ *is lower semicontinuous.*

Proof. Assume that S is not lower semicontinuous in F. We distinguish two cases. First let $F \neq \emptyset$. There is a sequence $(F_n)_n \subset \mathcal{F}([0,\infty))$ with $F_n \stackrel{\tau_M}{\to} F$ and $\liminf_{n\to\infty} S(F_n) < S(F)$. We can thus find a subsequence $(F_{n_k})_k$ with $S(F_{n_k}) \to a$ for some $a \geq 0$ with $a < S(F)$. For each k there is $x_{n_k} \in F_{n_k}$ such that $S(F_{n_k}) = x_{n_k}$. It follows that $x_{n_k} \to a$, which because of $F_n \stackrel{\tau_M}{\to} F$ yields $a \in F$ in contradiction to $a < S(F)$.
Now let $F = \emptyset$, then from $F_n \stackrel{\tau_M}{\to} F$ it follows that for each $b \geq 0$ we have $F_n \subset [b,\infty)$ for all $n \geq n_0$. This implies $S(F_n) \to \infty = S(F)$.

Lemma A.4 *If* $A = \bigcup_{n=1}^{\infty} A_n$ *with* $A_n \subset A_{n+1}$, *then for each* $\delta > 0$ *there is* $n_0 \in$ *such that for all* $n \geq n_0$:

(i)

$$P(A_n) \geq P(A) - \delta$$

and

(ii)

$$P(A_n \cap B) \geq P(A \cap B) - \delta, \text{ for all } B.$$

Proof. The first part is the well known continuity from below for probability measures. Let $n_0 \in$ such that (i) holds for $n \geq n_0$. Assume that there is $n \geq n_0$ and B such that

$$P(A_n \cap B) < P(A \cap B) - \delta,$$

then together with $A_n \subset A$ we obtain

$$\begin{aligned} P(A) &= P(A \cap B) + P\left(A \cap B^C\right) \\ &> P(A_n \cap B) + \delta + P\left(A \cap B^C\right) \\ &\geq P(A_n \cap B) + P\left(A_n \cap B^C\right) + \delta \\ &= P(A_n) + \delta \end{aligned}$$

and thus $P(A_n) < P(A) - \delta$ in contradiction to (i). □

Lemma A.5 *Let $(X_n)_n$, X be random variables with values in a metric space (S, d). Then the following are equivalent*

(i)

$$X_n \xrightarrow{P} X$$

(ii)

$$\lim_{n\to\infty} P\left(X_n \notin U, X \in U\right) = 0$$

for all open $U \subset S$.

Proof. First assume that (ii) holds. Let $\eta > 0$. From the regularity of P_X it follows that there is a compact set K such that $P(X \in K) \geq 1 - \eta$. We obtain

$$\begin{aligned} P\left(d(X_n, X) \geq \varepsilon\right) =& P\left(d(X_n, X) \geq \varepsilon, X \in K\right) + P\left(d(X_n, X) \geq \varepsilon, X \notin K\right) \\ \leq& P\left(d(X_n, X) \geq \varepsilon, X \in K\right) + \eta. \end{aligned}$$

Because of the compactness of K we can choose finitely many points $z_i \in K,\ i = 1, \dots, N$ and open balls $B_{\frac{\varepsilon}{4}}(z_i)$ such that $K \subset \bigcup_{i=1}^{N} B_{\frac{\varepsilon}{4}}(z_i)$. Since the open ball have diameter $\frac{\varepsilon}{2}$ the event $(d(X_n, X), X \in K$ can only occur, if for some $i \in \{1, \dots, N\}:\ X \in B_{\frac{\varepsilon}{4}}(z_i)$ and $X_n \notin B_{\frac{\varepsilon}{4}}(z_i)$. This yields

$$P\left(d(X_n, X) \geq \varepsilon\right) \leq \sum_{i=1}^{N} P\left(X_i \notin B_{\frac{\varepsilon}{4}}(z_i), X \in B_{\frac{\varepsilon}{4}}(z_i)\right) + \eta$$

and with (ii) it follows that the finite sum converges to 0. It remains to let $\eta \to 0$ to obtain (i).

Now assume that (i) holds. Let $U \subset S$ be open. We can assume that $U \neq \emptyset$ and $U \neq S$. Otherwise (ii) is obviously fulfilled. It follows that $\mathrm{bdy}(U) \neq \emptyset$. As in the first part of the proof for $\eta > 0$ we can find a compact set $K \subset S$ with $P(X \in K) \geq 1 - \eta$.

Because of the continuity of the probability measure we can find an open set $V \subset U$ with $P(X \in V) \geq P(X \in U) - \eta$ and $V \subset U$ such that $\mathrm{bdy}(U) \cap \mathrm{bdy}(V) = \emptyset$. We assume that $V \subset K$, because of $P(X \in V \setminus K) \leq \eta$ and $\eta > 0$ can be chosen arbitrarily small.

Now let $\varepsilon = \mathrm{dist}\left(\mathrm{bdy}(V), \mathrm{bdy}(U)\right)$, then $\varepsilon > 0$, because $\mathrm{bdy}(V)$ is compact, $\mathrm{bdy}(U)$ is closed and $\mathrm{bdy}(V) \cap \mathrm{bdy}(U) = \emptyset$. It follows that $(X_n \notin U, X \in V)$ implies $d(X_n, X) \geq \varepsilon$.

We obtain

$$\begin{aligned}P\left(X_n \notin U, X \in U\right) \leq & P\left(X_n \notin U, X \in V\right) + \eta \\ \leq & P\left(d(X_n, X) \geq \varepsilon\right) + \eta\end{aligned}$$

and with (i) and $\eta \to 0$ it follows that (ii) holds. $\square$

Lemma A.6 *Let (T, τ) be a second countable topological space.*
Let $(X_i)_i$ and $(X_i^n)_i$, $n \in \mathbb{N}$, $i \in \mathbb{N}$ be random variables with values in T, then

$$(X_1^n, X_2^n, \ldots) \overset{D_{\tau^\infty}}{\to} (X_1, X_2, \ldots)$$

if

$$(X_{i_1}^n, \ldots, X_{i_k}^n) \overset{D_{\tau^k}}{\to} (X_{i_1}, \ldots, X_{i_k})$$

for all $k \in \mathbb{N}$ and all $i_1, \ldots, i_k \in \mathbb{N}$.

Proof. Let $U \subset T^\infty$ be open with respect to τ^∞. We have to show that $\liminf_{n\to\infty} P\left((X_1^n, X_2^n, \ldots) \in U\right) \geq P\left((X_1, X_2, \ldots) \in U\right)$. Because of the properties of the product topology we can assume that $U = \bigcup_{j=1}^{\infty} B_j$, where each set B_j is of the form $B_j = \prod_{l=1}^{\infty} A_j^l$ such that each $A_j^l \subset T$ is τ open and $A_j^l \neq T$ for only finitely many $l_1^j, \ldots, l_{s_j}^j$. Because of the continuity of the probability measure, for $\varepsilon > 0$ we can find m such that $P\left((X_1, X_2, \ldots) \in \bigcup_{j=1}^{m} B_j\right) \geq P\left((X_1, X_2, \ldots) \in U\right) - \varepsilon$.

Now

$$
\begin{aligned}
&\liminf_{n\to\infty} P\left((X_1^n, X_2^n, \ldots) \in U\right)\\
\geq &\liminf_{n\to\infty} P\left((X_1^n, X_2^n, \ldots) \in \bigcup_{j=1}^{m} B_j\right)\\
= &\liminf_{n\to\infty} P\left((X_1^n, X_2^n, \ldots) \in \bigcup_{j=1}^{m} \prod_{l=1}^{\infty} A_j^l\right)\\
= &\liminf_{n\to\infty} P\left((X_{l_1^1}^n, \ldots, X_{l_{s_1}^1}^n, \ldots, X_{l_1^j}^n, \ldots, X_{l_{s_j}^j}^n) \in \left(\begin{array}{c} \left(A_{l_1^1} \times \ldots \times A_{l_{s_1}^1} \times T \times \ldots \times T\right) \\ \cup \ldots \cup \left(T \times \ldots \times T \times A_{l_1^j} \times \ldots \times A_{l_{s_j}^j}\right) \end{array}\right)\right)\\
\geq &P\left((X_{l_1^1}, \ldots, X_{l_{s_1}^1}, \ldots, X_{l_1^j}, \ldots, X_{l_{s_j}^j}) \in \left(\begin{array}{c} \left(A_{l_1^1} \times \ldots \times A_{l_{s_1}^1} \times T \times \ldots \times T\right) \\ \cup \ldots \cup \left(T \times \ldots \times T \times A_{l_1^j} \times \ldots \times A_{l_{s_j}^j}\right) \end{array}\right)\right)\\
= &P\left((X_1, X_2, \ldots) \in \bigcup_{j=1}^{m} B_j\right)\\
\geq &P\left((X_1, X_2, \ldots) \in U\right) - \varepsilon
\end{aligned}
$$

and it remains to let $\varepsilon \to 0$. $\square$

Lemma A.7 *Let $(f_n)_n$ be a sequence of lower semicontinuous functions, let f be lower semicontinuous. If $(f_n)_n$ converges to f uniformly on all compact sets, then*

$$\operatorname{epi} f_n \stackrel{\tau_{Fell}}{\to} \operatorname{epi} f$$

Proof. First let $(x_n, y_n) \in \operatorname{epi} f_n$ with $(x_n, y_n) \to (x, y)$ for some (x, y). It is to show that $(x, y) \in \operatorname{epi} f$. Let K be a compact neighbourhood of x. Then it follows from the uniform convergence on K, that for each $\varepsilon > 0$ there is n_0 such that $f_n(z) \geq f(z) - \varepsilon$ for all $z \in K$. By enlarging n_0 if necessary it follows that $x_n \in K$ for all $n \geq n_0$. We have

$$
\begin{aligned}
y = &\liminf_{n\to\infty} y_n\\
\geq &\liminf_{n\to\infty} f_n(x_n)\\
\geq &\liminf_{n\to\infty} f(x_n) - \varepsilon\\
\geq &f(x) - 2\varepsilon.
\end{aligned}
$$

With $\varepsilon \to 0$ it follows that $y \geq f(x)$, i.e. $(x, y) \in \operatorname{epi} f$.

Now let $(x,y) \in \operatorname{epi} f$. We have to find a sequence $((x_n,y_n))_n$ with $(x_n,y_n) \in \operatorname{epi} f_n$ and $(x_n,y_n) \to (x,y)$. We can choose $(x_n,y_n) = (x, f_n(x) + (y - f(x)))$. $\square$

For a set $F \subset \mathbb{R}^d$ and $\varepsilon > 0$ let $\varepsilon F = \left\{x \in \mathbb{R}^d \ : \ \exists z \in F : \ \|x - z\| \leq \varepsilon\right\}$, i.e. εF is a ε-neighbourhood of F.

Lemma A.8 *If $\varepsilon F_n \overset{D_{\tau_M}}{\to} \varepsilon F$ for all $\varepsilon > 0$ and if $\varepsilon_n \overset{P}{\to} 0$ for a sequence of nonnegative ε_n, then*

$$\varepsilon_n F_n \overset{D_{\tau_M}}{\to} F$$

Proof. Let $U = \bigcup\limits_{i=1}^{\infty} M(K_i)$ be a τ_M-open set, let $\eta > 0$, then because of the continuity of the probability measure, there is m, such that $P\left(F \in \bigcup\limits_{i=1}^{m} M(K_i)\right) \geq P\left(F \in U\right) - \eta$. From $M(K_i) = \bigcup\limits_{\varepsilon > 0} M(\varepsilon K_i)$ it follows that there is $\varepsilon > 0$ such that $P\left(F \in \bigcup\limits_{i=1}^{m} M(\varepsilon K_i)\right) \geq P\left(F \in U\right) - 2\eta$.
Now we have

$$\begin{aligned}
\liminf_{n\to\infty} P\left(\varepsilon_n F_n \in U\right) &\geq \liminf_{n\to\infty} P\left(\varepsilon_n F_n \in U, \ \varepsilon_n \leq \varepsilon\right) + \liminf_{n\to\infty} P\left(\varepsilon_n F_n \in U, \ \varepsilon_n > \varepsilon\right) \\
&= \liminf_{n\to\infty} P\left(\varepsilon_n F_n \in U, \ \varepsilon_n \leq \varepsilon\right) \\
&\geq \liminf_{n\to\infty} P\left(\varepsilon F_n \in U, \ \varepsilon_n \leq \varepsilon\right) \\
&= \liminf_{n\to\infty} P\left(\varepsilon F_n \in U\right) \\
&\geq \liminf_{n\to\infty} P\left(\varepsilon F_n \in \bigcup_{i=1}^{m} M(K_i)\right) \\
&\geq P\left(\varepsilon F \in \bigcup_{i=1}^{m} M(K_i)\right) \\
&= P\left(F \in \bigcup_{i=1}^{m} M(\varepsilon K_i)\right) \\
&\geq P(F \in U) - 2\eta
\end{aligned}$$

and it remains to let $\eta \to 0$. $\square$

Lemma A.9 *Let $(f_n)_n$, f be lower semicontinuous, then $\limsup\limits_{n\to\infty} \operatorname{epi} f_n \subset \operatorname{epi} f$ implies $\liminf\limits_{n\to\infty} f_n(x_n) \geq f(x)$ for all $(x_n)_n$, x with $x_n \to x$.*

Lemma A.10 *Let* $(f_n)_n$, f *be upper semicontinuous, then* $\limsup_{n\to\infty} \operatorname{hypo} f_n \subset \operatorname{hypo} f$ *implies* $\limsup_{n\to\infty} f_n(x_n) \leq f(x)$ *for all* $(x_n)_n$, x *with* $x_n \to x$.

Lemma A.11 *Let* $\mathcal{F}(\mathbb{R}^d)$ *be equipped with the topology* τ_M . *If* $K \subset \mathbb{R}^d$ *is nonempty and compact, then* $\operatorname{bdy} M(K) = H(K) = \mathcal{F} \setminus M(K)$.

Proof. Let $F \in H(K)$, let U be τ_M−open with $F \in U$. Then $F \in U \cap (\mathcal{F} \setminus M(K))$ and it remains to show that there is $G \in U \cap M(K)$. We can assume, that $U = M(J)$ with some compact J. We can find a closed set $G \subset \mathbb{R}^d$ such that $G \cap K = \emptyset$ and $G \cap J = \emptyset$ and obtain $G \in U \cap M(K)$. □

B. Notations

A^C	complement of the set A
$B_r(x)$	open ball with radius r and center x
$\overline{B}_r(x)$	closed ball with radius r and center x
$\mathcal{B}_\tau$	Borel σ-field generated by the τ-open sets
$\mathrm{bdy}(A)$	boundary of the set A
$\mathrm{cl}(A)$	closure of the set A
$EPI(A)$	all closed subsets of $A \times \mathbb{R}$, which are epigraphs of functions $f : A \to \overline{\mathbb{R}}$
$EPI(p)$	all closed subsets of $\mathbb{R}^{p+1}$, which are epigraphs of functions $f : \mathbb{R}^p \to \overline{\mathbb{R}}$
$\mathcal{F}(X)$	space of all closed subsets of X
$GRA(p)$	all closed subsets of $\mathbb{R}^{p+1}$, which are graphs of functions $f : \mathbb{R}^p \to \mathbb{R}$
$HYP(p)$	all closed subsets of $\mathbb{R}^{p+1}$, which are hypographs of functions $f : \mathbb{R}^p \to \overline{\mathbb{R}}$
$\mathrm{int}(A)$	interior of the set A
τ_H	hit-topology on $\mathcal{F}(X)$, see Definition 1.3
τ_M	miss-topology on $\mathcal{F}(X)$, see Definition 1.3
$U[a,b]$	uniform distribution on $[a,b]$
$\mathcal{U}(x)$	system of all neighbourhoods of x
1_A	indicator function of the set A
$\xrightarrow{a.s.}$	convergence almost surely
$\xrightarrow{p}$	convergence in probability
$\xrightarrow{D_\tau}$	convergence in distribution with respect to the topology τ

C. Kurzzusammenfassung in deutscher Sprache

Diese Arbeit beschäftigt sich mit der einseitigen Konvergenz in Verteilung für zufällige abgeschlossene Mengen und deren Anwendung in der Stabilitätstheorie stochastischer Optimierungsprobleme. Einseitige Mengenkonvergenz, im Sinne innerer/äußerer Konvergenz, wird benötigt, da z.B. bei der Approximation eines Optimierungsproblemes durch einfachere Ersatzprobleme die Lösungsmengen der Ersatzprobleme in der Regel nur eine Teilmenge der Lösungsmenge des Originalproblemes approximieren. Konvergenz in Verteilung wird verwendet, um bei stochastischen Originalproblemen Informationen über die Verteilung der optimalen Werte und Minimalstellen zu gewinnen.

Die Arbeit ist folgendermaßen aufgebaut:
In Kapitel 1 werden Eigenschaften der Topologien, die als Basis für die Definition der einseitigen Konvergenz dienen, zusammengetragen. Es folgen grundlegende Definitionen und Messbarkeitsvoraussetzungen.

Da die Bedingung in der Definiton der Konvergenz in Verteilung bezüglich der betrachteten Topologien für eine gegebene Folge zufälliger abgeschlossener Mengen schwer direkt zu überprüfen ist, werden im zweiten Kapitel nützliche hinreichende Bedingungen für die (einseitige) Konvergenz in Verteilung zufälliger abgeschlossener Mengen bewiesen. Dabei wird zum einen Konvergenz in Verteilung aus anderen Konvergenzarten abgeleitet. Diese Konvergenzkriterien dienen dazu, Anwendungen der Mengenkonvergenz in Verteilung für wichtige Klassen stochastischer Prozesse (z.B. $D[0, \infty)$) zugänglich zu machen. Ein Hauptresultat des zweiten Kapitels ist ein neues Konvergenzkriterium für stochastische Prozesse mit unterhalbstetigen Trajektorien, welches der klassischen Methode der Konvergenz endlichdimensionaler Verteilungen folgt und dabei das Konzept der stochastischen gleichgradigen Unterhalbstetigkeit verwendet.

Das dritte Kapitel behandelt Anwendungen der einseitigen Konverenz in Verteilung für abgeschlossene Mengen in der Stabilitätstheorie stochastischer Optimierungsprobleme.

Hierbei werden Probleme mit zufälligen Zielfunktionen und zufälligen Restriktionen betrachtet. Es werden bekannte Resultate für die innere Konvergenz (etwa aus [43]) auf den Fall der ε- bzw. ε_n-Optimalität ausgedehnt. Zudem werden entsprechende Aussagen für die äussere Konverenz bereitgestellt. Wir zeigen, dass sich mit der einseitigen Konvergenz ähnliche argmax/argmin Continuous Mapping Theoreme für Folgen von Minimalstellen im nicht eindeutigen Fall wie in [12] erzielen lassen. Die einseitige Konvergenz in Verteilung liefert dabei nicht die Verteilung der Grenzwerte, sondern einseitige Abschätzungen, die z.B. für approximative Konfidenzgebiete verwendet werden können.

Im vierten Kapitel wird die einseitige Konvergenz in Verteilung für zufällige Mengen benutzt, um 'in Verteilung' Versionen von Stabilitätsaussagen in der Vektoroptimierung herzuleiten. Hierbei sind Methoden der Mengenkonvergenz besonders hilfreich, da in der Vektoroptimierung im allgemeinen auch die optimalen Werte/effizienten Punkte aus mehrelementigen Mengen bestehen. Es werden dieselben Techniken wie im eindimensionalen Fall aus Kapitel 3 verwendet: Bekannte Stabilitätsaussagen aus der parametrischen Optimierung (etwa aus [2] im eindimensionalen Fall und aus [35] für die Vektoroptimierung) werden mit Hilfe des Continuous Mapping Theorems und dessen halbstetigen Varianten auf die Konvergenz in Verteilung übertragen.

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Convergence in Distribution of Random Closed Sets and Applications in Stability Theory of Stochastic Optimisation

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Zusammenfassung:

In dieser Dissertation wird die einseitige Konvergenz in Verteilung fuer abgeschlossene zufaellige Mengen und deren Anwendung auf stochastische Optimierungsprobleme untersucht. Ausgehend von den Konvergenzbegriffen von Kuratowski-Painleve wird Konvergenz in Verteilung basierend auf Hit- und Miss-Topologien definiert.
Wichtige Hilfsmittel wie das Continuous Mapping Theorem und halbstetige Verallgemeinerungen werden bereitgestellt.
Es wird eine Vielzahl von hinreichenden Bedingungen fuer die Konvergenz der Epigraphen zufaelliger unterhalbstetiger Funktionen bewiesen.
Dabei wird gezeigt, wie Klassen stochastischer Prozesse dem Mengenkonvergenzansatz zugaenglich gemacht werden koennen. Neben der unterhalbstetigen Modifikation der Skorohod-Raeume D wird mit Hilfe der Methode der Konvergenz endlichdimensionaler Verteilungen ein neues Konvergenzkriterium fuer die Konvergenz stochastischer Prozesse mit unterhalbstetigen Trajektorien bewiesen.
Aussagen ueber die Konvergenz in Verteilung der optimalen Werte und der Loesungsmengen stochastischer Optimierungsprobleme werden hergeleitet und fuer einseitige Abschaetzungen und Konfidenzbereiche angewendet.
Im letzen Kapitel wird gezeigt, wie sich das Konzept der einseitigen Mengenkonvergenz in Verteilung auf die Menge der effizienten Punkte und die Loesungsmengen stochastischer Vektoroptimierungsprobleme anwenden laesst. Hierbei wird wie in der eindimensionalen Optimierung auch die naeherungsweise Optimalitaet (epsilon optimality) betrachtet.

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Convergence in Distribution of Random Closed Sets and Applications in Stability Theory of Stochastic Optimisation

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Summary:

In this dissertation one sided convergence in distribution of random closed sets and its application to stochastic optimisation problems are investigated. Starting with the convergence concepts of Kuratowski-Painleve convergence in distribution based on hit- and miss-topologies is defined.
Important tools like the Continuous Mapping Theorem and semicontinuous versions are provided.
A variety of sufficient conditions for the convergence of the epigraphs of random lower semicontinuous functions is proven.
It is shown, how classes of stochastic processes can be made accessible to the concept of set convergence. Besides the lower semicontinuous modification of the Skorohod-spaces D a new convergence criterion for stochastic processes with lower semicontinuous trajectories is proven by using the method of convergence of finite dimensional distributions.
Results for the convergence in distribution of optimal values and solution sets of stochastic optimisation problems are derived and applied to one-sided estimates and confidence regions.
In the final chapter it is shown, how the concept of one-sided set convergence in distribution can be applied to the set of efficient points and the solutions of vector optimisation problems. Like in the one-dimensional case epsilon optimality is taken into consideration.

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Erklärung

Ich erkläre hiermit, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Ilmenau, 18.04.2006 Oliver Gersch