Technische Universität Ilmenau
Fakultät für Mathematik
und Naturwissenschaften
Institut für Mathematik

Postfach 10 05 65
98684 Ilmenau
Germany
Tel.: 03677/69-3253
Fax : 03677/69-3270
Telex: 33 84 23 tuil d.
e-mail: knobloch@mathematik.tu-ilmenau.de

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Chaotic behaviour near non-transversal homoclinic points with quadratic tangencies

Jürgen Knobloch

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Abstract

We consider a family of discrete dynamical systems having, for a certain parameter value, a homoclinic point with quadratic tangency. We use Lin's method for the construction of a Poincaré-map which is topologically conjugated to a shift map. We discuss the effects of changing the parameter.

Key words: Discrete dynamical systems, quadratic tangency, Lin's method, shift dynamics.

AMS Classification: 39A12, 58F03, 58F13, 58F14

1 Introduction

We consider discrete dynamical systems

$$x(n+1) = f(x(n), \lambda), \tag{1.1}$$

where $f : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$ is smooth. In particular $f(\cdot, \lambda)$ is a diffeomorphism for all λ. For $\lambda = 0$ system (1.1) has a degenerate homoclinic orbit Γ,

$$\Gamma := \{\gamma(n) := f^n(q), n \in \mathbb{Z}\}. \tag{1.2}$$

The orbit Γ is assumed to be asymptotic to a hyperbolic fixed point p. Degeneracy is synonymic to non-transversality of the corresponding homoclinic points. In particular we will assume that (for $\lambda = 0$) the tangent spaces of the stable and unstable manifolds at q intersect in a one-dimensional space U,

$$U := T_q W^s_{\lambda=0}(p) \cap T_q W^u_{\lambda=0}(p). \tag{1.3}$$

Furthermore we suppose that the stable and unstable manifolds have a quadratic tangency in q (over U). Generically this phenomenon forms a codimension one problem - $\lambda \in \mathbb{R}$. The bifurcation of the homoclinic points by changing the parameter λ is well known. Moving λ through the critical point ($\lambda = 0$) two transversal homoclinic points q_1, q_2 (existing for $\lambda > 0$) merge and disappear if λ becomes negative.

In this paper we address the recurrent dynamics of (1.1) - of course also for parameter values $\lambda \neq 0$. The main goal of this paper is the construction of a Poincaré-map, involving two different transversal homoclinic points, which is topologically conjugated to shift dynamics - see Theorem 3.7 below. Afterwards we discuss the behaviour of the dynamics of this map in dependence on the parameter λ. In particular we show that (under certain conditions) the shift dynamics survives for parameter values for which transversal homoclinic points no more do exist - see Theorem 3.10 and Remark 3.11.

Our tool in constructing the Poincaré-map is Lin's method. This approach for discovering dynamics near non-transversal homoclinic points is more analytically oriented than the one described in [8] which is geometric in nature. The basic idea is to join several orbits following Γ for one revolution to one orbit staying for all time near Γ. Maybe our method is helpful for a numerical computation of orbits nearby a (non-transversal) homoclinic orbit - similar to the procedure presented in [1] and [2] which is based on [7]. Note that the Shadowing Lemma as used in [7] does not longer work in the present case. The reason is that the

set $\Gamma \cup \{p\}$ is not hyperbolic. However, we should mention that numerical investigation of non-transversal connecting orbits (for discrete systems) has been started already - see [4].
The paper is organized as follows. In Section 2 we give a concise summary of Lin's method and introduce notations and previous results which we will use for our analysis. For the detailed description of Lin's method we refer to [5]. Here we restrict to explain the idea of this method briefly and to present the main existence result. However we cannot avoid giving some details which we need for our further analysis.
In Section 3 we give a precise formulation of the assumptions and of the problem under consideration. The main object is the construction of the Poincaré-map. Furthermore we formulate the major results which state that this map is conjugated to a shift map.
The rest of the paper contains (very technical and lengthy) proofs - each in a separate section.

2 Previous results concerning Lin's method

In [5] we carried Lin's method forward to discrete systems. In particular this method is suitable for revealing recurrent dynamics near connecting orbits. The main idea of this method is as follows: We look for solutions x_i, $i \in \mathbb{Z}$, of (1.1) starting in a neighborhood of q following the forward orbit Γ^+ of the orbit Γ, passing p, following the backward orbit Γ^- of Γ and arriving after N_i steps finally in the same neighborhood of q we started from. In this way we find solutions staying for all time close to Γ by stringing together the single solutions x_i. Analytically that means that we have to solve the bifurcation equation

$$\Xi := (\Xi_i)_{i\in\mathbb{Z}} = 0, \quad \Xi_i := x_{i+1}(0) - x_i(N_i). \tag{2.1}$$

Of course the jumps Ξ_i depend on λ, $\mathcal{N} = (N_i)_{i\in\mathbb{Z}}$ and in general on an additional parameter $\mathbf{u} = (u_i)_{i\in\mathbb{Z}}$, $u_i \in U$. The space U has been defined in (1.3).
For the further analysis we use the following direct sum decomposition of $\mathbb{R}^k$:

$$\mathbb{R}^k = U \oplus W^+ \oplus W^- \oplus Z, \tag{2.2}$$

where $U \oplus W^{+(-)} = T_q W^{s(u)}_{\lambda=0}(p)$.
Now, let $\mathcal{N} := (N_i)_{i\in\mathbb{Z}}$, $N_i \in \mathbb{N}$ and $\mathbf{u} := (u_i)_{i\in\mathbb{Z}}$, $u_i \in U$, be any sequences. We define

$$N_i^+ := \left\lfloor \frac{N_i}{2} \right\rfloor, \; N_i^- := N_i - N_i^+, \tag{2.3}$$

where $\lfloor n \rfloor$ denotes the integer part of n.
We can prove the following theorem, cf. [5, Section 4]:

Theorem 2.1 *There are constants c and $\tilde{N}$ such that for each $\mathbf{u}$, λ with $\|\mathbf{u}\| := \sup |u_i| < c$, $|\lambda| < c$ and each $\mathcal{N}$ with $N_i^{+(-)} > 2\tilde{N}$ there are unique solutions $x_i^+(\mathcal{N}, \mathbf{u}, \lambda)(\cdot)$, $x_i^-(\mathcal{N}, \mathbf{u}, \lambda)(\cdot)$ of (1.1) satisfying*

(i) $x_i^+(\mathcal{N}, \mathbf{u}, \lambda)(\cdot) : [0, N_{i+1}^+] \cap \mathbb{Z} \to \mathbb{R}^k, \quad x_i^-(\mathcal{N}, \mathbf{u}, \lambda)(\cdot) : [-N_i^-, 0] \cap \mathbb{Z} \to \mathbb{R}^k,$

(ii) the orbits of x_i^+ and x_i^- are close to the forward and backward orbit Γ^+ and Γ^-, respectively,

(iii) $x_i^+(\mathcal{N},\mathbf{u},\lambda)(N_{i+1}^+) = x_{i+1}^-(\mathcal{N},\mathbf{u},\lambda)(-N_{i+1}^-)$,

(iv) $x_i^+(\mathcal{N},\mathbf{u},\lambda)(0)$, $x_i^-(\mathcal{N},\mathbf{u},\lambda)(0)$ *are close to* $\gamma(0)$,

(v) $x_i^+(\mathcal{N},\mathbf{u},\lambda)(0) - x_i^-(\mathcal{N},\mathbf{u},\lambda)(0) \in Z$,

(vi) $x_i^{+(-)}(\mathcal{N},\mathbf{u},\lambda)(0) - \gamma^{+(-)}(u_i,\lambda)(0) \in W^+ \oplus W^- \oplus Z$. ■

The item (iii) can be seen as a coupling condition which allows to define solutions x_i by

$$x_i(\mathcal{N},\mathbf{u},\lambda)(n) := \begin{cases} x_{i-1}^+(\mathcal{N},\mathbf{u},\lambda)(n) & , \quad n \in [0, N_i^+] \cap \mathbb{Z}, \\ x_i^-(\mathcal{N},\mathbf{u},\lambda)(n - N_i) & , \quad n \in [N_i^+, N_i] \cap \mathbb{Z}. \end{cases} \tag{2.4}$$

To prove Theorem 2.1 we seek $x_i^{+(-)}$ in the form

$$x_i^\pm(\mathcal{N},\mathbf{u},\lambda)(\cdot) = \gamma^\pm(u_i,\lambda)(\cdot) + \bar{v}_i^\pm(\mathcal{N},\mathbf{u},\lambda)(\cdot). \tag{2.5}$$

Here $\gamma^\pm$ are uniquely determined solutions whose orbits lie in the stable and unstable manifold of p, respectively, and for which $\gamma^+(u,\lambda)(0) - \gamma^-(u,\lambda)(0) \in Z$ holds true. Moreover we know that the U-component (with respect to the direct sum decomposition (2.2)) of $\gamma^\pm(u,\lambda)$ is equal to u. We refer to [5, Lemma 2.3] for the exact statement. With that we see that for fixed $\mathcal{N}$, $\mathbf{u}$ and λ the i^{th} jump Ξ_i is defined as follows

$$\Xi_i(\mathcal{N},\mathbf{u},\lambda) := \xi^\infty(u_i,\lambda) + \xi_i(\mathcal{N},\mathbf{u},\lambda), \tag{2.6}$$

where ξ^∞ and ξ_i are given by

$$\begin{aligned} \xi^\infty(u,\lambda) \quad &:= \quad \gamma^+(u,\lambda)(0) - \gamma^-(u,\lambda)(0), \\ \xi_i(\mathcal{N},\mathbf{u},\lambda) \quad &:= \quad \bar{v}_i^+(\mathcal{N},\mathbf{u},\lambda)(0) - \bar{v}_i^-(\mathcal{N},\mathbf{u},\lambda)(0). \end{aligned} \tag{2.7}$$

The quantity ξ^∞ measures the distance of the stable and unstable manifold over U. From [5] we know that

$$\xi^\infty(0,0) = 0, \quad D_1\xi^\infty(0,0) = 0. \tag{2.8}$$

3 Dynamics near Homoclinic Tangencies

We start with giving a precise formulation of the assumptions we are working with. First of all we want to notice that, due to the hyperbolicity of the fixed point p, we may assume that p is fixed point of $f(\cdot,\lambda)$ for all sufficiently small $|\lambda|$.
Next we want to describe the homoclinic tangency analytically:

(HT) (i) $\dim U = 1$,

(ii) $\operatorname{rank} \left. \dfrac{\partial(\xi^\infty, D_1\xi^\infty)}{\partial(u,\lambda)} \right|_{(u,\lambda)=(0,0)} = 2.$

So (HT)(ii) can be seen as a transversality condition which states that the mapping $(u,\lambda) \mapsto (\xi^\infty(u,\lambda), D_1\xi^\infty(u,\lambda))$ is transversal to $(0,0) \in Z \times Z$. An immediate consequence of (HT) is

$$D_1^2\xi^\infty(0,0) \neq 0, \quad D_2\xi^\infty(0,0) \neq 0. \tag{3.1}$$

So $\xi^\infty(u,\lambda)$ can be transformed into $\lambda \pm u^2$. For our further considerations we will assume

$$\xi^\infty(u,\lambda) = \lambda - u^2. \tag{3.2}$$

Anyway the "+" case does not qualitatively differ from that one we consider.
The recurrent dynamics near a homoclinic orbit depends - besides others - on the spectrum of the linearization at the fixed point. We will make the following assumption concerning the principal eigenvalues. Let $\mu^s(\lambda)$ and $\mu^u(\lambda)$ be the principal stable and unstable eigenvalues of $D_1 f(p,\lambda)$, respectively. Then the spectrum of $D_1 f(p,\lambda)$ can be represented by

$$\begin{array}{l} \sigma(D_1 f(p,\lambda)) = \sigma^{ss}(\lambda) \cup \{\mu^s(\lambda), \mu^u(\lambda)\} \cup \sigma^{uu}(\lambda), \\ \text{where} \\ 0 < |\mu| < \alpha^{ss} < |\mu^s(\lambda)| < \alpha^s < 1 < \alpha^u < |\mu^u(\lambda)| < \alpha^{uu} < |\tilde{\mu}| \end{array} \tag{3.3}$$

for all $\mu \in \sigma^{ss}(\lambda)$, $\tilde{\mu} \in \sigma^{uu}(\lambda)$. We will suppose:

(EV_R) The principal eigenvalues $\mu^{s(u)}(\lambda)$ are simple and real.

To exclude non-generic cases we will also assume

(HOM_A) Γ does not approach p within the strong stable and strong unstable manifold, respectively.

The next assumption requires a similar behaviour for special solutions of the (formally) adjoint of the variational equation along γ:

(SUB) Z is neither in the strong stable nor in the strong unstable subspace of $x(n+1) = (D_1 f(\gamma(n),0)^{-1})^T x(n)$.

Condition (HOM_A) is geometrically reasonable and also its consequences are clear. Namely, assuming (HOM_A) we exclude an effect which is called homoclinic orbit flip in theory of homoclinic orbits in differential equations. Condition (SUB) is related to the so-called strong inclination property and prevents inclination flip bifurcations (at least in the vector field case). However, both assumptions (SUB) and (HOM_A) are used in [6] for the calculation of the leading terms of ξ_i.
The homoclinic tangency condition (HT) together with (HOM_A) and (SUB) implies that such a homoclinic orbit arises generically in one-parameter families. This justifies to choose $\lambda \in \mathbb{R}$.

Henceforth we will concentrate on solutions with a fixed number N of revolution steps. In our language this means $\mathcal{N} = (N_i)_{i\in\mathbb{Z}}$, $N_i = N$, for all $i \in \mathbb{Z}$. The bifurcation equation for such solutions reads

$$\Xi_i(N, \mathbf{u}, \lambda) := \Xi_i((N_i)_{i\in\mathbb{Z}}, \mathbf{u}, \lambda) = 0,\ i \in \mathbb{Z}, \tag{3.4}$$

We consider $\Xi := (\Xi_i)_{i\in\mathbb{Z}}$ as a map

$$\Xi(\cdot,\cdot,\cdot) : (\mathbb{N} \cup \{\infty\}) \times l^\infty_U \times \mathbb{R}^1 \to l^\infty_Z. \tag{3.5}$$

By $l^\infty_{U(Z)}$ we denote the spaces of bounded (bi-infinite) sequences in $U(Z)$. These spaces are equipped with the supremum norm. We will rewrite equation (3.4) as a fixed point equation

- see (3.10) below. Then we will solve this by means of the Banach fixed point theorem. To do so we need a few more information about ξ_i: Under the given assumptions (including $N_i = N, \forall i \in \mathbb{Z}$) the following corollary is a direct conclusion of [6, Theorem 3.1].

Corollary 3.1 *If the principal eigenvalues are simple and real - (EV_R) - the tangent spaces at q of the stable and unstable manifold intersect in one-dimensional space - (HT) - if the non-degeneracy conditions (SUB) and (HOM_A) are fulfilled and if additionally $N_i = N, \forall i \in \mathbb{Z}$ holds true then the jump ξ_i can be written as*

$$\begin{aligned}\xi_i(N, \mathbf{u}, \lambda) = \; & c^s(u_{i-1}, u_i, \lambda)(\mu^s(\lambda))^N + c^u(u_i, u_{i+1}, \lambda)(\mu^u(\lambda))^{-N} \\ & + o\Big((\mu^s(\lambda))^N\Big) + o\Big((\mu^u(\lambda))^{-N}\Big).\end{aligned}$$

Here $c^{s(u)}$ are continuous functions and moreover $c^{s(u)}(0,0,0) \neq 0$. ■

Next we claim that the norm of $D_2\Big(\xi_i(N, \mathbf{u}, \lambda)\Big)_{i\in\mathbb{Z}}$ decays exponentially (uniformly in $\mathbf{u}$ and λ) as N increases. More precisely we can prove

Lemma 3.2 *Assume that the hypotheses of Corollary 3.1 are fulfilled. Then, for sufficiently large N and sufficiently small $\mathbf{u}$ and λ, there are continuous $c^+, c^- : U \times U \times \mathbb{R} \to \mathbb{R}$, such that*

$$\begin{aligned}\|D_2\xi_i(N, (u_j), \lambda)\| = \; & c^+(u_i, u_{i+1}, \lambda)(\mu^u(\lambda))^{-N} + c^-(u_{i-1}, u_i, \lambda)(\mu^s(\lambda))^N \\ & + o\Big(\|\mu^u(\lambda)\|^{-N}\Big) + o\Big(\|\mu^s(\lambda)\|^N\Big).\end{aligned}$$

Because the proof of this lemma is somewhat lengthy and very technical and we do not want to destroy the train of thought for constructing the Poincaré-map, we shift the proof to Section 4.

With Lemma 3.2 we have all necessary ingredients to discuss $\Xi(N, \mathbf{u}, \lambda) = 0$. Let $\mathbf{u}^*$ and λ_o be chosen such that $\lambda_o > 0$ and

$$\mathbf{u}^* := (u_i^*)_{i\in\mathbb{Z}} : u_i^* \in \{u_1(\lambda_o), u_2(\lambda_o)\}, \text{ where } \xi^\infty(u_j(\lambda_o), \lambda_o) = 0, \; j = 1, 2. \tag{3.6}$$

Due to (3.2) this makes sense. We will solve $\Xi = 0$ for $\mathbf{u} = \mathbf{u}(N, \lambda)$ near $(N, \mathbf{u}, \lambda) = (\infty, \mathbf{u}^*, \lambda_o)$ by means of a procedure which takes its pattern from the implicit function theorem - cf. [10].
Note that the solutions $(u_i(\lambda_o), \lambda_o)$ correspond to transversal homoclinic points $q_i(\lambda_o)$, $i = 1, 2$, of the equation (1.1). For $(N, \mathbf{u}, \lambda) = (\infty, \mathbf{u}^*, \lambda_o)$ we have

$$\Xi_i(\infty, \mathbf{u}^*, \lambda_o) = \xi^\infty(u_i^*, \lambda_o) = 0. \tag{3.7}$$

Hence, assuming (HT) we have $D_2\Xi_i(\infty, (u_j^*), \lambda_o) = D_1\xi^\infty(u_i^*, \lambda_o) \neq 0$. Therefore - see Subsection 6.1 - $D_2\Xi(\infty, \mathbf{u}^*, \lambda_o)$ is invertible and we have the following equivalence

$$\Xi(N, \mathbf{u}, \lambda) = 0 \Leftrightarrow D_2\Xi(\infty, \mathbf{u}^*, \lambda_o)^{-1}\Xi(N, \mathbf{u}, \lambda) = 0. \tag{3.8}$$

On the other hand - starting from the representation (2.6) of Ξ and using the Taylor expansion

$$\xi^\infty(u_i,\lambda) = D_1\xi^\infty(u_i^*,\lambda_o)(u_i-u_i^*) + D_2\xi^\infty(u_i^*,\lambda_o)(\lambda-\lambda_o) + \xi_r^\infty(u_i,\lambda) \tag{3.9}$$

we see that the second equation in (3.8) is equivalent to

$$\begin{aligned} (u_i)_{i\in\mathbb{Z}} &= \Big(u_i^* - D_1\xi^\infty(u_i^*,\lambda_o)^{-1}[D_2\xi^\infty(u_i^*,\lambda_o)(\lambda-\lambda_o) \\ &\quad +\xi_r^\infty(u_i,\lambda) + \xi_i(N,(u_j)_{j\in\mathbb{Z}},\lambda)]\Big)_{i\in\mathbb{Z}} \\ &=: \Xi_R(N,(u_i),\lambda). \end{aligned} \tag{3.10}$$

We will apply the Banach fixed point theorem to this equation:

Lemma 3.3 *Assume that the hypotheses of Corollary 3.1 are fulfilled. Let* $\mathbf{u}^*$ *be a sequence according to (3.6). Then there are* ϵ_u, $\tilde\epsilon_N$, $\tilde\epsilon_\lambda$ *such that for all* $N > 1/\tilde\epsilon_N$, *all* λ, $|\lambda-\lambda_o| < \tilde\epsilon_\lambda$, *the mapping* $\Xi_R(N,\cdot,\lambda)$ *is a contraction of* $\mathrm{cl}\, B(\mathbf{u}^*,\epsilon_u)$ *into itself.* $B(\mathbf{u}^*,\epsilon_u)$ *is the ball in* l_U^∞ *around* $\mathbf{u}^*$ *with radius* ϵ_u.

Proof Note that ξ_r^∞ comprises only higher order terms. Further it is clear that the derivative $D_2\Xi_R(\infty,\mathbf{u}^*,\lambda_o) = 0$ and, due to Lemma 3.2 and our considerations in Subsection 6.1 - see also Remark 3.5 below

$$D_2\Xi_R(N,\mathbf{u},\lambda) \to 0 \quad \text{as} \quad (N,\mathbf{u},\lambda) \to (N,\mathbf{u}^*,\lambda_o). \tag{3.11}$$

First we will make clear that Ξ_R is contractive. For that we invoke the mean value theorem which tells:

$$\Xi_R(N,\mathbf{u}^1,\lambda) - \Xi_R(N,\mathbf{u}^2,\lambda) = \int_0^1 D_2\Xi_R(N,\mathbf{u}^2 + t(\mathbf{u}^1-\mathbf{u}^2),\lambda)dt(\mathbf{u}^1-\mathbf{u}^2). \tag{3.12}$$

Because of (3.11) there is, for all $c > 0$, an $\epsilon = (\epsilon_u,\epsilon_\lambda,\epsilon_N)$ such that for all N, λ, $\mathbf{u}$ with $N > 1/\epsilon_N$, $|\lambda-\lambda_o| < \epsilon_\lambda$ and $\|\mathbf{u}-\mathbf{u}^*\|_{l^\infty} < \epsilon_u$ it holds

$$\|D_2\Xi_R(N,\mathbf{u},\lambda)\| < c. \tag{3.13}$$

In particular we find a corresponding ϵ for $c < 1$. This means nothing else but $\Xi_R(N,\cdot,\lambda)$ is contractive on $\mathrm{cl}\, B(\mathbf{u}^*,\epsilon_u)$ for all $N > 1/\epsilon_N$ and$|\lambda-\lambda_o| < \epsilon_\lambda$. Next we show that for sufficiently large N and sufficiently small $|\lambda-\lambda_o|$ the mapping $\Xi_R(N,\cdot,\lambda)$ maps this ball into itself. Let $\mathbf{u} \in B(\mathbf{u}^*,\epsilon_u)$. Then

$$\begin{aligned} \|\Xi_R(N,\mathbf{u},\lambda) - \mathbf{u}^*\| &\le c\epsilon_u + \|\Xi_R(N,\mathbf{u}^*,\lambda) - \mathbf{u}^*\| \\ &\le c\epsilon_u + \|D_2\Xi(\infty,\mathbf{u}^*,\lambda_o)^{-1}\|\,\|\Xi(N,\mathbf{u}^*,\lambda)\| \end{aligned} \tag{3.14}$$

Because $\|\Xi(N,\mathbf{u}^*,\lambda)\|$ tends to zero as $N\to\infty$ and $\lambda\to\lambda_o$ there are $\tilde\epsilon_N < \epsilon_N$ and $\tilde\epsilon_\lambda < \epsilon_\lambda$ such that for all $N > 1/\tilde\epsilon_N$ and all λ, $|\lambda-\lambda_o| < \tilde\epsilon_\lambda$ the right-hand side in (3.14) is less than ϵ_u - recall $c < 1$. ■

So, for a fixed sequence $\mathbf{u}^*$, we can apply the Banach fixed point theorem to the fixed point equation (3.10) and get $\mathbf{u} = \Xi_R(N, \mathbf{u}, \lambda)$ if and only if $\mathbf{u} = \mathbf{u}(N, \lambda; \mathbf{u}^*)$. The associated solution of (1.1) we denote by $x(N, \mathbf{u}^*, \lambda)(\cdot)$. The mapping $\mathbf{u} = \mathbf{u}(N, \cdot; \mathbf{u}^*)$ is even differentiable. This can be seen by applying the implicit function theorem to the fixed point equation (3.10) at a solution point $(N, \mathbf{u}(N, \lambda; \mathbf{u}^*), \lambda)$. Moreover we can prove continuity at $(N, \lambda) = (\infty, \lambda_o)$:

Lemma 3.4 *The (unique) solution* $\mathbf{u}$ *of the fixed point equation* $\mathbf{u} = \Xi_R(N, \mathbf{u}, \lambda)$ *is "continuous" in* $(N, \lambda) = (\infty, \lambda_o)$*. More exactly:* $\mathbf{u}(N, \lambda; \mathbf{u}^*) \to \mathbf{u}^*$ *as* $N \to \infty$ *and* $\lambda \to \lambda_o$*.*

Proof Also this proof takes its pattern from the proof of the implicit function theorem. $\mathbf{u}(\cdot, \cdot; \mathbf{u}^*)$ solves the fixed point problem (3.10). Therefore we can write

$$\begin{aligned} \|\mathbf{u}(N, \lambda; \mathbf{u}^*) - \mathbf{u}^*\| &= \|\Xi_R(N, \mathbf{u}(N, \lambda; \mathbf{u}^*), \lambda) - \mathbf{u}^*\| \\ &\leq \|\Xi_R(N, \mathbf{u}(N, \lambda; \mathbf{u}^*), \lambda) - \Xi_R(N, \mathbf{u}^*, \lambda)\| + \|\Xi_R(N, \mathbf{u}^*, \lambda) - \mathbf{u}^*\|. \end{aligned}$$

The second term (on the right-hand side) can be estimated as in (3.14). The first term can be estimated by means of (3.12) and (3.13). Exploiting $c < 1$ we finally get

$$\|\mathbf{u}(N, \lambda; \mathbf{u}^*) - \mathbf{u}^*\| \leq \frac{1}{1-c} \|D_2\Xi(\infty, \mathbf{u}^*, \lambda_o)^{-1}\| \, \|\Xi(N, \mathbf{u}^*, \lambda_)\|.$$

Now the lemma follows because $\|\Xi(N, \mathbf{u}^*, \lambda)\| \to 0$ as $(N, \lambda) \to (\infty, \lambda_o)$. This again becomes clear by using Corollary 3.1 and Equation (3.7). ∎

Remark 3.5 A somewhat closer look at the limit (3.11) reveals that the pair (c, ϵ) can be chosen independently of the sequence $\mathbf{u}^*$ which was fixed at the beginning. With our considerations in Subsection 6.1 we find

$$\|D_2\Xi_R(N, \mathbf{u}, \lambda)\| \leq \sup_{i \in \mathbb{Z}} \|D_1\xi^\infty(u_i^*, \lambda_o)^{-1}\| \Big(\|D_1\xi_r^\infty(u_i, \lambda)\| + \|D_2\xi_i(N, \mathbf{u}, \lambda)\| \Big).$$

The first term within the parenthesis tends to zero as $(u_i, \lambda) \to (u_i^*, \lambda_o)$ while the second term can be estimated in accordance with Lemma 3.2 - i.e. uniformly in $\mathbf{u}$ and λ. That means, for all sequences $\mathbf{u}^*$ according to (3.6) the estimate (3.13) asks for the same ϵ. □

Remark 3.6 Let $(c, \epsilon_u, \epsilon_\lambda, \epsilon_N)$ be chosen in accordance with Remark 3.5. Then also $\tilde{\epsilon}_\lambda$ and $\tilde{\epsilon}_N$ can be chosen independently of the special choice of (u_i^*) - as long as $u_i^* \in \{u_1(\lambda_o), u_2(\lambda_o)\}$: According to the estimate (3.14) we have only to make sure that $\Xi(N, \mathbf{u}^*, \lambda)$ tends to zero (as $N \to \infty$ and $\lambda \to \lambda_o$) uniformly in $\mathbf{u}^*$:

$$\begin{aligned} \|\Xi(N, \mathbf{u}^*, \lambda)\| = &\ \sup_{i \in \mathbb{Z}} \{\|\xi^\infty(u_i^*, \lambda) + \xi_i(N, \mathbf{u}^*, \lambda)\|\} \\ \leq &\ \max\{\|\xi^\infty(u_1(\lambda_o), \lambda)\|, \|\xi^\infty(u_2(\lambda_o), \lambda)\|\} + \sup_{i \in \mathbb{Z}} \{\|\xi_i(N, \mathbf{u}^*, \lambda)\|\}. \end{aligned}$$

The first term on the right-hand side of the last inequality does not depend on the choice of $\mathbf{u}^*$ at all. The norm of ξ_i tends to zero as $N \to \infty$ uniformly in $\mathbf{u}^*$ and λ - see Corollary 3.1. □

Due to Remarks 3.5 and 3.6 we can also look from another point of view at the things - namely: Fix $N > 1/\tilde{\epsilon}_N$ and λ, $|\lambda - \lambda_o| < \tilde{\epsilon}_\lambda$. Then we can consider the solution $\mathbf{u}$ of (3.10) as a quantity depending on $\mathbf{u}^*$. This we denote shortly by

$$\mathbf{u}(\mathbf{u}^*) := \mathbf{u}(N, \mathbf{u}^*, \lambda).$$

Let $\mathcal{S}^2 := \{1,2\}^{\mathbb{Z}}$ be the set of all sequences $\mathbf{s} = (s_i)_{i\in\mathbb{Z}}$ of elements of $\{1,2\}$ - that is $s_i \in \{1,2\}$. In $\mathcal{S}^2$ we introduce the usual metric ρ: $\rho(\mathbf{s^1}, \mathbf{s^2}) := \sum_{i\in\mathbb{Z}} 1/2^{|i|}\|s_i^1 - s_i^2\|$ - see [9] or [3]. There is a canonical 1-1-map

$$\begin{array}{rccc} \iota: & \mathcal{S}^2 & \to & \{\mathbf{u}^* : u_i^* \in \{u_1(\lambda), u_2(\lambda)\}\} \\ & (s_i) & \mapsto & (u_{s_i}(\lambda)) =: \mathbf{u}^*_{\mathbf{s}}. \end{array}$$

The left shift ζ on $\mathcal{S}^2$ is defined as follows: $\zeta : \mathcal{S}^2 \to \mathcal{S}^2,\ \mathbf{s} \mapsto \zeta\mathbf{s},\ (\zeta\mathbf{s})_i := s_{i+1}$.

Theorem 3.7 *Consider the dynamical system (1.1) under the assumptions (HT), (EV_R), (HOM_ A) and (SUB). Then, depending on the relation of the signs of $D_1^2\xi^\infty(0,0)$ and $D_2\xi^\infty(0,0)$, for each sufficiently small $\lambda > 0$ (or $\lambda < 0$) and each sufficiently large $N \in \mathbb{N}$ there is an invariant set $D_{N,\lambda}$ on which $f^N(\cdot,\lambda)$ is topologically conjugated to the left shift ζ on $\mathcal{S}^2$.*

Proof We assume that $\operatorname{sgn}(D_2\xi^\infty(0,0)) = 1$ and $\operatorname{sgn}(D_1^2\xi^\infty(0,0)) = -1$ - see also (3.2). Then the statement of the theorem is true for $\lambda > 0$.
Let $\epsilon = (\epsilon_u, \tilde{\epsilon}_N, \tilde{\epsilon}_\lambda)$ be as figured out in the proof of Lemma 3.3 and let $N > 1/\tilde{\epsilon}_N$ and λ, $|\lambda - \lambda_o| < \tilde{\epsilon}_\lambda$. If, additionally, we choose $\epsilon_u < 1/3\|u_1(\lambda_o) - u_2(\lambda_o)\|_{\mathbb{R}^k}$ we get a unique allocation of sequences $\mathbf{u}^*$ to solutions $\mathbf{u}(\mathbf{u}^*)$ of $\Xi(N, \mathbf{u}, \lambda) = 0$ and hence to solutions $x(\mathbf{u}^*)(\cdot) := x(N, \mathbf{u}^*, \lambda)(\cdot)$ of $x(n+1) = f(x(n), \lambda)$ staying for all time near the primary homoclinic orbit Γ. So our results allow a map

$$\begin{array}{rccl} \Phi: & \mathcal{S}^2 & \to & q + (W^+ \oplus W^- \oplus Z) + (B(u_1(\lambda_o), \epsilon_u) \cup B(u_2(\lambda_o), \epsilon_u)) \\ & \mathbf{s} & \mapsto & x(\mathbf{u}^*_{\mathbf{s}})(0), \end{array}$$

again $B(u_i(\lambda_o), \epsilon_u) \subset U$, $i = 1,2$, denote corresponding balls. Finally we can define a Poincaré-map $\Pi(\lambda)$ on $D := \operatorname{im}\Phi$ by

$$\Pi(\lambda) : D \to D, \quad x \mapsto f^N(x, \lambda). \tag{3.15}$$

Indeed, the set $D_{N,\lambda} := D$ is invariant under $f^N(\cdot, \lambda)$ - because the uniqueness of $\mathbf{u}(N, \lambda, \cdot) = \mathbf{u}(\cdot)$ provides

$$x(\mathbf{u}^*_{\zeta\mathbf{s}})(0) = x(\mathbf{u}^*_{\mathbf{s}})(N). \tag{3.16}$$

Note that $x(\mathbf{u}^*)(N) = f^N(x(\mathbf{u}^*)(0), \lambda)$. This gives immediately the invariance of D. Moreover, (3.16) yields $f^N(\cdot,\lambda) \circ \Phi = \Phi \circ \zeta$. This means that (on D) $f^N(\cdot, \lambda)$ is conjugated to the left shift on $\mathcal{S}^2$.
So it remains to prove that Φ is a homeomorphism. This will be done in the following lemma. ■

Lemma 3.8 $\Phi : \mathcal{S}^2 \to D = \operatorname{im}\Phi$ *is a homeomorphism.*

The proof of this lemma will be given in Section 5.

Remark 3.9 Of course we are far away to describe the recurrent dynamics near the transversal homoclinic orbits Γ_1, Γ_2 of $q_1(\lambda)$ and $q_2(\lambda)$ completely. It is well known that in a neighborhood of a single transversal homoclinic orbit there is a very complicated recurrent dynamics - see for instance [7], [8] or [5]. There the existence of shift dynamics has been proved.
Here we constructed shift dynamics by incorporating both homoclinic orbits Γ_1 and Γ_2. This has been done by prescribing the way in which a recurrent solution has to follow alternately either Γ_1 or Γ_2. □

Now we show that under certain conditions also for $\lambda \leq 0$ shift dynamics does exist. A further assumption concerning the principal eigenvalues allows to perceive the sign of ξ_i:

(EV_R$^+$) Additionally to (EV_R) let $\mu^s(\lambda) > |\mu^u(\lambda)^{-1}|$.

Theorem 3.10 *Let all assumptions of Theorem 3.7 be met. Additionally we will assume*

(i) $\xi^\infty(u, \lambda) = \lambda - u^2$,

(ii) the principal eigenvalues satisfy (EV_R$^+$),

(iii) $c^s(0, 0, 0) > 0$ *- see Corollary 3.1,*

(iv) that there is a $K \in \mathbb{N}$ *such that* $\|D_2\xi_i(N, \mathbf{u}, 0)\| \leq K\|\xi_i(N, \mathbf{u}, 0)\|$.

Then there is an $N \in \mathbb{N}$ *and an invariant set* D_N *on which* $f^N(\cdot, 0)$ *is conjugated to shift dynamics.*

Proof We will show that there are appropriate N, λ_o and ϵ_u such that the fixed point equation (3.10) has a unique solution $\mathbf{u}(\mathbf{u}^*)$ for $\lambda = 0$. For that we have to prove that $\Xi_R(N, \cdot, 0)$ is a contraction on $\operatorname{cl} B(\mathbf{u}^*, \epsilon_u)$.
Taking (EV_R$^+$) into consideration Corollary 3.1 tells that

$$\xi_i(N, \mathbf{u}, 0) = c^s(u_{i-1}, u_i, 0)(\mu^s)^N + \xi_i^r(N, \mathbf{u}),$$

where $\xi_i^r(N, \mathbf{u}) = o((\mu^s)^N)$ uniformly in $\mathbf{u}$ and i.
Let $\delta > 0$ and let $B_\delta(0) \subset U$ be a closed ball around 0 with radius δ. The function $c^s(\cdot, \cdot, 0) : B_\delta(0) \times B_\delta(0) \to \mathbb{R}$ is continuous. Therefore $c^s(\cdot, \cdot, 0)$ takes its maximum C and its minimum c on $B_\delta(0) \times B_\delta(0)$. Because of $c^s(0, 0, 0) > 0$ the quantity δ can be chosen such that $c > 0$ and moreover $3/2c > C$. Hence we will find a $k \in \mathbb{N}$ such that $C + 1/k < 2(c - 1/k)$. Then there is an N_o such that for all $N > N_o$ we have the estimate

$$2\tilde{c}(N) > \xi_i(N, \mathbf{u}, 0) > \tilde{c}(N), \quad \tilde{c}(N) := (c - 1/k)(\mu^s)^N. \tag{3.17}$$

To verify this estimate mind that $\xi_i(N, \mathbf{u}, 0) = c^s(u_{i-1}, u_i, 0)(\mu^s)^N + o((\mu^s)^N)$. Finally we choose $N > N_o$ as large such that

$$2\sqrt{\tilde{c}(N)} < \min\{\delta, 1/K\} \tag{3.18}$$

With that N we define

$$\lambda_o := 9/4\tilde{c}(N), \quad \epsilon_u := 1/3\sqrt{\lambda_o}. \tag{3.19}$$

With this setting we prove that $\Xi_R(N,\cdot,\lambda_o)$ is indeed a contraction on $B(\mathbf{u}^*,\epsilon_u)$: In the present context we have

$$\Xi_R(N,\mathbf{u},0)=\Big(u_i^*+\frac{1}{2u_i^*}[-\lambda_o-(u_i-u_i^*)^2+\xi_i(N,\mathbf{u},0)]\Big)_{i\in\mathbb{Z}}$$

and - again exploiting the results of Subsection 6.1 -

$$D_2\Xi_R(N,\mathbf{u},0)=\Big(\frac{1}{2u_i^*}[-2(u_i-u_i^*)+D_2\xi_i(N,\mathbf{u},0)]\Big)_{i\in\mathbb{Z}}.$$

Then $\Xi_R(N,\cdot,0)$ is a contractive mapping from $\mathrm{cl}\,B(\mathbf{u}^*,\epsilon_u)$ into itself if

$$\sup_{i\in\mathbb{Z}}\frac{1}{2\sqrt{\lambda_o}}\|-\lambda_o-(u_i-u_i^*)^2+\xi_i(N,\mathbf{u},0)\|\le\epsilon_u. \tag{3.20}$$

and

$$\frac{1}{2\sqrt{\lambda_o}}\Big(2\epsilon_u+\sup_{i\in\mathbb{Z}}\|D_2\xi_i(N,\mathbf{u},0)\|\Big)\le c<1 \tag{3.21}$$

- mind that $|u_i^*|=\sqrt{\lambda_o}$. An easy computation shows that in our setting the constant c in formula (3.21) can be chosen as 2/3. Further, equation (3.20) is fulfilled if for all i

$$\lambda_o>\xi_i(N,\mathbf{u},0)>4/9\lambda_o.$$

This is guaranteed by the estimate (3.17).
So, for all $\mathbf{u}^*$ with $u_i^*=\pm\sqrt{\lambda_o}$ we find solutions of the bifurcation equation $\Xi(N,\mathbf{u},0)=0$. The the shift dynamics which is conjugated to $f^N(\cdot,0)$ can be constructed as above. ■

Remark 3.11 1. The robustness of our construction shows that Theorem 3.10 remains true for small $\lambda<0$.

2. Note that for $\lambda\le 0$ no transversal homoclinic orbit does exist nearby the primary one.
3. The proof that for $\lambda\le 0$ the shift dynamics is even topologically conjugated to f^N calls for more delicate estimates which we will not carry out here.
4. If assumption (iii) of Theorem 3.10 will be replaced by $c^s(0,0,0)<0$ then there is no shift dynamics (as constructed above) for $\lambda\le 0$. This is due to the fact that there are no 1-periodic orbits of period N for those λ - see also [6].
5. Finally we will comment the assumption (iv): $D_2\xi_i$ decays at least with the same exponential rate as ξ_i as $N\to\infty$ - see Corollary 3.1 and the estimates concerning $D_2\xi_i$ within this section. So, the number K does exist and does not depend on N.

□

As announced above we will finish this paper by giving the proofs of the Lemmas 3.2 and 3.8.

4 Proof of Lemma 3.2

In proving Lemma 3.2 we will often fall back on assertions made in [6]. For that reason we repeat further details concerning Lin's method.
Theorem 2.1 has been proved by revealing $\bar{v}^{\pm}$ as unique solutions of a fixed point problem

$$\mathbf{v} = \hat{\mathbf{v}}_{\mathcal{N}}(\mathbf{u}, \lambda, (h^+(\cdot, u_i, v_i^+(\cdot), \lambda), h^-(\cdot, u_i, v_i^-(\cdot), \lambda))_{i\in\mathbb{Z}}, (d_i(\mathcal{N}, \mathbf{u}, \lambda))_{i\in\mathbb{Z}}), \tag{4.1}$$

$$d_i(\mathcal{N}, \mathbf{u}, \lambda) := \gamma^-(u_i, \lambda)(-N_i^-) - \gamma^+(u_{i-1}, \lambda)(N_i^+), \quad \mathbf{d} := (d_i)_{i\in\mathbb{Z}},$$
$$h^{\pm}(n, u, v, \lambda) := f(\gamma^{\pm}(u, \lambda)(n) + v, \lambda) - f(\gamma^{\pm}(u, \lambda)(n), \lambda) - D_1 f(\gamma^{\pm}(u, \lambda)(n), \lambda)v.$$

For that purpose we conceive - for fixed $\mathbf{u}$ and λ - equation (4.1) as a fixed point equation in $\mathcal{V}$, where the space $\mathcal{V}$ is defined as follows:
Let S_N and S_{-N} be the spaces of functions mapping $\{0, \dots, N\}$ and $\{-N, \dots, 0\}$, respectively, into $\mathbb{R}^k$. These spaces are equipped with the maximum norm. By $\mathcal{V}$ we denote the space of all sequences $v := ((v_i^+, v_i^-))_{i\in\mathbb{Z}}$, $(v_i^+, v_i^-) \in S_{N_{i+1}^+} \times S_{-N_i^-}$. The norm in $\mathcal{V}$ is $\|v\| := \max\{\sup_{i\in\mathbb{Z}} \|v_i^+\|_{S_{N_{i+1}^+}}, \sup_{i\in\mathbb{Z}} \|v_i^-\|_{S_{-N_i^-}}\}$. See altogether [5, Definition 4.5].
For the detailed explanation of $\hat{\mathbf{v}}_{\mathcal{N}}$ we refer to [5]. The only thing we want to mention here is that $\hat{\mathbf{v}}_{\mathcal{N}}(\mathbf{u}, \lambda, \cdot, \cdot)$ is linear.
Finally we want to remark that for fixed $\mathcal{N}$ the mapping $\bar{\mathbf{v}} : l_U^\infty \times \mathbb{R} \to \mathcal{V}$, $(\mathbf{u}, \lambda) \mapsto \bar{\mathbf{v}}(\mathbf{u}, \lambda) = (\bar{v}_i^+(\mathbf{u}, \lambda)(\cdot), \bar{v}_i^-(\mathbf{u}, \lambda)(\cdot))_{i\in\mathbb{Z}}$ is smooth - cf. [5, Lemma 4.13].
An essential role in dealing with Lin's method play variational equations along $\gamma^{\pm}$,

$$x(n+1) = D_1 f(\gamma^{\pm}(u, \lambda), \lambda)x(n),$$

in each case considered on $\mathbb{Z}^{\pm}$. The transition matrices of these equations we denote by $\Phi^{\pm}(u, \lambda, \cdot, \cdot)$. It is important to know that these equations possess exponential dichotomies on $\mathbb{Z}^{\pm}$. Let $P^{\pm}(u, \lambda, \cdot)$ be the projections associated to the exponential dichotomies. The images of the projections are determined to be the corresponding tangent spaces at the stable (unstable) manifold. For our analysis we laid down the kernel of the projection according to the $\mathbb{R}^k$-decomposition (2.2). However, the projections are completely settled by defining

$$\begin{aligned}
\operatorname{im} P^{+(-)}(u, \lambda, 0) &= T_{\gamma^{+(-)}(u,\lambda)(0)} W_\lambda^{s(u)}(p), \\
\ker P^{+(-)}(u, \lambda, 0) &= W^{-(+)} \oplus Z, \\
P^{\pm}(u, \lambda, n)\Phi^{\pm}(u, \lambda, n, m) &= \Phi^{\pm}(u, \lambda, n, m)P^{\pm}(u, \lambda, m).
\end{aligned}$$

Lemma 4.4 in [5] states that for sufficiently large N and sufficiently small $\mathbf{u}$, λ it holds

$$\mathbb{R}^k = \operatorname{im}(id - P^+(u_i, \lambda, N^+)) \oplus \operatorname{im}(id - P^-(u_{i+1}, \lambda, -N^-)).$$

The corresponding projection with range $\operatorname{im}(id - P^+(u_i, \lambda, N^+))$ we name $\tilde{P}(u_i, u_{i+1}, \lambda, N^+)$.
Finally we assign sequences of Nemitskii operators $\mathcal{H} = (H_i^+, H_i^-)_{i\in\mathbb{Z}}$ to $h^{\pm}$ as follows:

$$\begin{aligned}
\mathcal{H} : \quad l_U^\infty \times \mathcal{V} \times \mathbb{R} &\to \mathcal{V} \\
(\mathbf{u}, \mathbf{v}, \lambda) &\mapsto (H_i^+, H_i^-), \qquad \text{where} \\
H_i^{\pm}(\mathbf{u}, \mathbf{v}, \lambda)(n) &:= h^{\pm}(n, u_i, v_i^{\pm}(n), \lambda).
\end{aligned}$$

Proof of Lemma 3.2

Because of $\dim Z = 1$ we get $\xi_i = \langle z, \xi_i\rangle z$, where we put $z \in Z : \|z\| = 1$. Hence $D_2\xi_i = (D_2\langle z, \xi_i\rangle)z$. To compute the expression on the right-hand side we use the representation [6, (5.7)] of the scalar product $\langle z, \xi_i\rangle$. We write up this representation here again:

$$
\begin{aligned}
&\langle z, \xi_i(N, \mathbf{u}, \lambda)\rangle = \\
&\quad \Big\langle \Phi^+(u_i, \lambda, 0, N^+)^T (id - P^+(u_i, \lambda, 0))^T z, \\
&\qquad\qquad \tilde{P}(u_i, u_{i+1}, \lambda, N)\Big(\gamma^-(u_{i+1}, \lambda)(-N^-) - \gamma^+(u_i, \lambda)(N^+) \\
&\qquad\qquad\qquad +P^-(u_{i+1}, \lambda, -N^-)\bar{v}_{i+1}^-(N, \mathbf{u}, \lambda)(-N^-) \\
&\qquad\qquad\qquad -P^+(u_i, \lambda, N^+)\bar{v}_i^+(N, \mathbf{u}, \lambda)(N^+)\Big)\Big\rangle \\
&\quad -\Big\langle z, \sum_{j=1}^{N^+} \Phi^+(u_i, \lambda, 0, j)(id - P^+(u_i, \lambda, j))h^+(j-1, u_i, \bar{v}_i^+(N, \mathbf{u}, \lambda)(j-1), \lambda)\Big\rangle \\
&\quad +\Big\langle \Phi^-(u_i, \lambda, 0, -N^-)^T (id - P^-(u_i, \lambda, 0))^T z, \\
&\qquad\qquad (id - \tilde{P}(u_{i-1}, u_i, \lambda, N))\Big(-\gamma^+(u_{i-1}, \lambda)(N^+) + \gamma^-(u_i, \lambda)(-N^-) \\
&\qquad\qquad\qquad -P^+(u_{i-1}, \lambda, N^+)\bar{v}_{i-1}^+(N, \mathbf{u}, \lambda)(N^+) \\
&\qquad\qquad\qquad +P^-(u_i, \lambda, -N^-)\bar{v}_i^-(N, \mathbf{u}, \lambda)(-N^-)\Big)\Big\rangle \\
&\quad -\Big\langle z, \sum_{j=0}^{-N^-+1} \Phi^-(u_i, \lambda, 0, j)(id - P^-(u_i, \lambda, j))h^-(j-1, u_i, \bar{v}_i^-(N, \mathbf{u}, \lambda)(j-1), \lambda)\Big\rangle.
\end{aligned}
\tag{4.2}
$$

This yields

$$
\begin{aligned}
D_2(\langle z, \xi_i\rangle) = \; & \langle D_{\mathbf{u}}((\Phi^+)^T (id - P^+)^T)z, \tilde{P}[\gamma^- - \gamma^+ + P^-\bar{v}_{i+1}^- - P^+\bar{v}_i^+]\rangle \\
& +\langle (\Phi^+)^T (id - P^+)^T z, (D_{\mathbf{u}}\tilde{P})[\gamma^- - \gamma^+ + P^-\bar{v}_{i+1}^- - P^+\bar{v}_i^+]\rangle \\
& +\langle (\Phi^+)^T (id - P^+)^T z, \tilde{P}[D_{\mathbf{u}}\gamma^- - D_{\mathbf{u}}\gamma^+ + D_{\mathbf{u}}(P^-\bar{v}_{i+1}^-) - D_{\mathbf{u}}(P^+\bar{v}_i^+)]\rangle \\
& +\langle z, D_{\mathbf{u}} \sum_{j=1}^{N^+} \Phi^+(id - P^+)h^+(\ldots, u_i, \bar{v}_i^+, \ldots)\rangle \\
& + \ldots
\end{aligned}
\tag{4.3}
$$

With a view to greater clarity we omitted the independent variables in the above equation. These can be taken from (4.2). The dots in the last line of (4.3) represent the corresponding terms containing Φ^-. The estimates of the non-differentiated terms can be taken from [6]. Here we restrict to give estimates of the partial derivatives with respect to $\mathbf{u}$ of the individual terms involved in (4.3). We give only the estimates of terms corresponding to Φ^+. The rest yields by similar considerations.

The derivative of $\boldsymbol{(\Phi^+(u_i, \ldots)^T(id - P^+(u_i, \ldots))^T}$ with respect to $\mathbf{u}$ is actually the partial derivative with respect to u_i. This again can be estimated by means of Lemma [6, Lemma 6.12]. Note that the λ used there stands for (u, λ) in the present context. However, with our assumptions the estimate given there provides that there is a $K > 0$ such that

$$\|D_{u_i}\Big(\Phi^+(u_i, \lambda, 0, N^+)^T (id - P^+(u_i, \lambda, 0))^T\Big)\| \leq K(\alpha^u)^{-N^+}. \tag{4.4}$$

The latter estimate can be rendered more severe by the following considerations: For shortening we put $A(u, \lambda, n) := (D_1 f(\gamma^+(u, \lambda)(n), \lambda)^{-1})^T$. Then the equation

$$X(n+1) = \begin{pmatrix} A(u,\lambda,n) & D_u A(u,\lambda,n) \\ 0 & A(u,\lambda,n) \end{pmatrix} X(n) \tag{4.5}$$

has an exponential dichotomy on $\mathbb{Z}^+$. This becomes clear by writing the matrix on the right-hand side of (4.5) as:

$$\begin{pmatrix} A(u,\lambda,n) & D_u A(u,\lambda,n) \\ 0 & A(u,\lambda,n) \end{pmatrix} = \begin{pmatrix} (D_1 f(p,\lambda)^{-1})^T & 0 \\ 0 & (D_1 f(p,\lambda)^{-1})^T \end{pmatrix} + \mathcal{R}(u,\lambda,n).$$

$\mathcal{R}(u, \lambda, n)$ decays exponentially as n tends to infinity - see also derivative $D_u\gamma^+$.
On the other hand $\begin{pmatrix} D_u(\Phi^+(u,\lambda,0,n)^T(id - P^+(u,\lambda,0))^T) \\ \Phi^+(u,\lambda,0,n)^T(id - P^+(u,\lambda,0))^T \end{pmatrix}$ solves (4.5) - see also [6, Corollary 5.2] - and decays exponentially as n tends to infinity - see (4.4) and [6, (5.13)] which contains the corresponding estimate of $\Phi^+(\dots, n)(id - P^+(\dots))$. Now, exploiting the exponential dichotomy of equation (4.5) we get the following estimate - see also [6, Lemma 6.6]:

$$\begin{aligned} \|D_{u_i}(\Phi^+(u_i, \lambda, 0, N^+)^T (id - P^+(u_i, \lambda, 0))^T)\| = {} & \hat{c}^+(u_i, \lambda)(\mu^u)^{-N^+} \\ & + O((\max\{(\alpha^{uu})^{-1}, (\alpha^u)^{-2}\})^{N^+}). \end{aligned} \tag{4.6}$$

Note that actually [6, Lemma 6.6] requires that the principal eigenvalue μ^u is simple. In the case under consideration μ^u is a principal eigenvalue of $(D_1 f(p, \lambda)^{-1})^T$ and therefore a double eigenvalue of $\begin{pmatrix} (D_1 f(p,\lambda)^{-1})^T & 0 \\ 0 & (D_1 f(p,\lambda)^{-1})^T \end{pmatrix}$ - but still semi-simple. Under this condition the statement of the quoted lemma remains true.

The partial derivative of $\tilde{\mathbf{P}}(\boldsymbol{u_i}, \boldsymbol{u_{i+1}}, \dots)$ with respect to $\mathbf{u}$ is nothing else but the sum of the partial derivatives of $\tilde{P}$ with respect to u_i and u_{i+1}. [6, Corollary 6.21] tells that these remain bounded as N tends to infinity. Therefore there is a $C > 0$ such that

$$\begin{aligned} \|(D_{\mathbf{u}}\tilde{P})[\gamma^- - \gamma^+ + P^-\bar{v}_{i+1}^- - P^+\bar{v}_i^+]\| \leq {} & C(\mu^u(\lambda)^{-N^-} + \mu^s(\lambda)^{N^+}) \\ & + O\Big((\max\{\alpha^s, (\alpha^u)^{-1}\})^{2N^+}\Big). \end{aligned} \tag{4.7}$$

See also [6, (5.15) and (5.30)] for the estimate of the terms within the brackets on the left-hand side of (4.7).

The derivative $\boldsymbol{D_u\gamma^{+(-)}(u, \lambda)}$ of $\gamma^{+(-)}$ with respect to the sequence $\mathbf{u}$ is nothing else but the partial derivative of $\gamma^{+(-)}$ with respect to u which can be estimated by means of

[6, (6.61)] which tells that there is a $C(u,\lambda)$ depending continuously on (u,λ) and satisfying $C(0,0)\neq 0$:

$$D_u\gamma^+(u,\lambda)(n) = C(u,\lambda)(\mu^s)^n + O\Big((\max\{\alpha^{ss},(\alpha^s)^2\})^n\Big). \tag{4.8}$$

Next we will estimate **the derivatives $D_u(P^-\bar{\mathbf{v}}_{i+1}^-)$ and $D_u(P^+\bar{\mathbf{v}}_i^+)$.** We will do it exemplarily for $D_u(P^+\bar{v}_i^+)$. For that we use the following representation of $P^+\bar{v}_i^+$ - see [6]:

$$\begin{aligned} P^+(u_i,\lambda,N^+)\bar{v}_i^+(N,\mathbf{u},\lambda)(N^+) \; &= \Phi^+(u_i,\lambda,N^+,0)P^+(u_i,\lambda,0)\bar{v}_i^+(N,\mathbf{u},\lambda)(0) \\ &+\sum_{j=1}^{N^+}\Phi^+(u_i,\lambda,N^+,j)P^+(u_i,\lambda,j)H_i^+(u_i,\bar{v}_i^+,\lambda)(N^+). \end{aligned} \tag{4.9}$$

Differentiation of (4.9) with respect to $\mathbf{u}$ yields (we omit the arguments of all functions)

$$\begin{aligned} D_{\mathbf{u}}(P^+\bar{v}_i^+) = \; & D_{u_i}(\Phi^+P^+)\bar{v}_i^+ + (\Phi^+P^+)D_{\mathbf{u}}(P^+\bar{v}_i^+) \\ &+\sum_{j=1}^{N^+}D_{u_i}(\Phi^+P^+)H_i^+ + \sum_{j=1}^{N^+}(\Phi^+P^+)[D_1H_i^+ + D_2H_i^+D_{\mathbf{u}}\bar{v}_i^+]. \end{aligned} \tag{4.10}$$

An easy computation - similar to that leading to the estimate of $H^\pm$ - see [6, (5.29)] - shows

$$D_1H_i^+(\dots) = O\Big((\max\{\alpha^s,(\alpha^u)^{-1}\})^{2N^+}\Big), \quad D_2H_i^+(\dots) = O\Big((\max\{\alpha^s,(\alpha^u)^{-1}\})^{N^+}\Big).$$

If besides this also $D_{\mathbf{u}}\bar{v}_i^+ = O((\max\{\alpha^s,(\alpha^u)^{-1}\})^{N^+})$ then $D_{\mathbf{u}}(P^+\bar{v}_i^+)$ can be estimated along the same lines as $P^+\bar{v}_i^+$ - [6, Section 5]. It turns out that

$$D_{\mathbf{u}}(P^+\bar{v}_i^+) = O\Big((\max\{\alpha^s,(\alpha^u)^{-1}\})^{2N^+}\Big) \tag{4.11}$$

So it remains to prove that the estimate assumed estimate for $D_{\mathbf{u}}\bar{v}_i^+$ actually holds true. This runs completely parallel to the proof of the corresponding estimate for $\bar{v}_i^+$ - see again [6, Section 5].

Now we can tackle **the estimate of the derivative $D_u\sum_{j=1}^{N^+}\Phi^+(id-P^+)h^+$.** Replacing $(id-P^+)$ by $(id-P^+)^2$ we get

$$D_{\mathbf{u}}\sum_{j=1}^{N^+}\Phi^+(id-P^+)h^+ = \sum_{j=1}^{N^+}D_{\mathbf{u}}(\Phi^+(id-P^+))h^+ + \sum_{j=1}^{N^+}\Phi^+(id-P^+)D_{\mathbf{u}}((id-P^+)h^+).$$

With the previous considerations the first term on the right-hand side of the last equation can be estimated similar to $\sum\Phi^+(id-P^+)h^+$ - see [6, Section 5]. But also for the second term we can use these considerations: Combining [6, (5.38) and (5.39)] provides a representation of $(id-P^+)h^+$. Partial differentiation of this expression with respect to $\mathbf{u}$ and factoring out the unstable part $v_i^{+,u}$ of v_i^+ - as it was defined in [6] - and $D_{\mathbf{u}}v_i^{+,u}$, respectively, leads to an estimate similar to [6, (5.40)]:

$$\begin{aligned} &\|D_{\mathbf{u}}(id-P^+(u_i,\lambda,j))h^+(j-1,u_i,v_i^+(j-1),\lambda)\|_{\mathbb{R}^k} \\ &\quad\leq C\Big(\|v_i^{+,u}(j-1)\|_{\mathbb{R}^k}(\|D_{\mathbf{u}}v_i^{+,s}(j-1)\|_{\mathbb{R}^k}+\|D_{\mathbf{u}}v_i^{+,u}(j-1)\|_{\mathbb{R}^k}) \\ &\quad+\|D_{\mathbf{u}}v_i^{+,u}(j-1)\|_{\mathbb{R}^k}(\|v_i^{+,s}(j-1)\|_{\mathbb{R}^k}+\|v_i^{+,u}(j-1)\|_{\mathbb{R}^k})\Big). \end{aligned} \tag{4.12}$$

Now we can proceed as in [6, Section 5] and we get the final estimate

$$\|D_{\mathbf{u}} \sum_{j=1}^{N^+} \Phi^+(id - P^+)h^+\| \leq C(\delta\alpha^u)^{-N^+}(\max\{\alpha^s, (\alpha^u)^{-1}\})^{2N^+}. \tag{4.13}$$

Summarizing our considerations we get the lemma. ■

5 Proof of Lemma 3.8

As in the proof of Lemma 3.2 we will make use of estimates derived in [6]. For that we need more informations regarding the fixed point equation (4.1). Beyond the notations introduced at the beginning of Section 4 we will use the following facts. From [5] we know that $\hat{\mathbf{v}}_{\mathcal{N}}$ has the form

$$\begin{aligned} &\hat{\mathbf{v}}_{\mathcal{N}}(\mathbf{u}, \lambda, \mathbf{g}, \mathbf{d}) = \mathbf{v}_{\mathcal{N}}(\mathbf{u}, \lambda, \mathbf{g}, \mathbf{a}(\mathbf{u}, \lambda, \mathbf{g}, \mathbf{d})), \quad \text{where} \\ &\mathbf{a} : l^\infty_U \times \mathbb{R} \times \mathcal{V} \times l^\infty_{\mathbb{R}^k} \to l^\infty_{\mathbb{R}^k}. \end{aligned}$$

Furthermore we define - see also [5]

$$a_i^+ := (id - P^+(u_{i-1}, \lambda, N^+))a_i, \quad a_i^- := (id - P^+(u_i, \lambda, -N^-))a_i.$$

The operator $\mathbf{v}_{\mathcal{N}}$ represents again a sequence (v_i^+, v_i^-). However, two properties will be crucial for our further analysis. First, the quantities $v_i^\pm$ depend only on u_i, a_{i+1}^+, a_i^- and not on the entire sequences $\mathbf{u}$ and $\mathbf{a}$. Secondly, our considerations in [5] yield $a_i = a_i(u_{i-1}, u_i, \lambda, \mathbf{g}, \mathbf{d})$. That means, a_i depends only on u_{i-1}, u_i and not on the whole sequence $\mathbf{u}$.

Proof of Lemma 3.8

In [5] we constructed a similar Poincaré-map related to the dynamics near transversal homoclinic points. Also that one was proved to be topologically conjugated to the left shift on sequences in two symbols. That sequences were built by describing $\mathcal{N}$ while here we are playing with the sequences $\mathbf{u}^*$. However, the proofs of the present lemma and of the corresponding [5, Lemma 5.1] are similar. But the sequences $\mathbf{u}^*$ which are involved here make the proof much more complicated. The difficulty in proving such type of lemma is based on the following: The distances between sequences $\mathbf{u}^*$ have to be measured by means of the metric in $\mathcal{S}^2$ - while the other analysis calls for the l^∞-norm.
Since in the present context N and λ are fixed we will omit them in all quantities where they usually occur - as long as they are not used for estimates.
Because Φ is a one-to-one map and $\mathcal{S}^2$ is compact it suffices to show that Φ is continuous.
Two sequences (u_i^{*1}) and (u_i^{*2}) are close if they coincide on a large set of indices centered at $i = 0$. So we assume

$$u_i^{*1} = u_i^{*2}, \quad i \in [-i_o(i_o+1), i_o(i_o+1)] \cap \mathbb{Z}. \tag{5.1}$$

Then the lemma has proved if $x(\mathbf{u^{*1}})(0) - x(\mathbf{u^{*2}})(0) = O(1/i_o)$. By the construction of x the following condition (5.2) is sufficient for that

$$\begin{aligned} \gamma^+(u_o(\mathbf{u^{*1}})) - \gamma^+(u_o(\mathbf{u^{*2}})) &= O(1/i_o) \\ \text{and} \qquad & \\ \bar{v}_o^+(\mathbf{u}(\mathbf{u^{*1}}))(0) - \bar{v}_o^+(\mathbf{u}(\mathbf{u^{*2}}))(0) &= O(1/i_o). \end{aligned} \tag{5.2}$$

In future we will write shortly $\mathbf{u^j}$ for $\mathbf{u}(\mathbf{u^j}) = \mathbf{u}(N, \lambda, \mathbf{u^j})$, $j = 1, 2$.

To prove (5.2) we show in a first step that

$$u_i^1 - u_i^2 = O(1/i_o) \quad \text{for} \quad i \in [-i_o, i_o] \cap \mathbb{Z}. \tag{5.3}$$

For that purpose we remember that $\mathbf{u}(\mathbf{u}^*)$ has been gained from the fixed equation (3.10). It turns out that for $|i| \leq i_o$ we have

$$u_i^1 - u_i^2 = D_1\xi^\infty(u_i^{*1}, \lambda_o)^{-1}[\xi_r^\infty(u_i^2, \lambda) - \xi_r^\infty(u_i^1, \lambda) + \xi_i(N, \mathbf{u^2}, \lambda) - \xi_i(N, \mathbf{u^1}, \lambda)] \tag{5.4}$$

We will treat the two differences within the brackets separately. First we consider $\xi_r^\infty(u_i^2, \lambda) - \xi_r^\infty(u_i^1, \lambda)$. ξ_r^∞ is defined by equation (3.9). Using Taylor expansion of $\xi^\infty(\cdot, \lambda)$ with second order residual terms we find

$$\begin{aligned} \xi_r^\infty(u_i^2, \lambda) - \xi_r^\infty(u_i^1, \lambda) = \; & \int_0^1 D_2 D_1 \xi^\infty(u_i^{*1}, \lambda_o + \tau(\lambda - \lambda_o))d\tau(\lambda - \lambda_o)(u_i^2 - u_i^1) \\ & + \int_0^1 (1-\tau) D_1^2 \xi^\infty(u_i^{*1} + \tau(u_i^2 - u_i^{*1}), \lambda)d\tau(u_i^2 - u_i^{*1})^2 \\ & - \int_0^1 (1-\tau) D_1^2 \xi^\infty(u_i^{*1} + \tau(u_i^1 - u_i^{*1}), \lambda)d\tau(u_i^1 - u_i^{*1})^2 \end{aligned} \tag{5.5}$$

Applying the mean value theorem to the last integral in the above equation leads to

$$\xi_r^\infty(u_i^2, \lambda) - \xi_r^\infty(u_i^1, \lambda) = E(u_i^1, u_i^2, \lambda)(u_i^2 - u_i^1). \tag{5.6}$$

As a consequence of Lemma 3.4 and $u_i^{*1} = u_i^{*2}$ (for i under consideration) $\|E(\dots)\|$ can be made arbitrarily small by choosing $|\lambda - \lambda_o|$ small enough and N large enough. Together with (5.4) this provides the estimate

$$(2/3)|u_i^1 - u_i^2| \leq \|D_1\xi^\infty(u_i^{*1}, \lambda_o)^{-1}\| \, \|\xi_i(N, \mathbf{u^2}, \lambda) - \xi_i(N, \mathbf{u^1}, \lambda)\|. \tag{5.7}$$

Now we turn towards the estimate of $\|\xi_i(N, \mathbf{u^2}, \lambda) - \xi_i(N, \mathbf{u^1}, \lambda)\|$: The representation (2.7) of ξ_i provides

$$\begin{aligned} \xi_i(N, \mathbf{u^2}, \lambda) - \xi_i(N, \mathbf{u^1}, \lambda) = \; & \bar{v}_i^+(N, \mathbf{u^2}, \lambda)(0) - \bar{v}_i^+(N, \mathbf{u^1}, \lambda)(0) \\ & + \bar{v}_i^-(N, \mathbf{u^1}, \lambda)(0) - \bar{v}_i^-(N, \mathbf{u^2}, \lambda)(0) \end{aligned} \tag{5.8}$$

For the next steps the dependence on N and λ of the quantities arising in (5.8) is not significant. So again we will omit these variables and we will remember them only if they are needed.

Let $\Delta\bar{\mathbf{v}} := \bar{\mathbf{v}}(\mathbf{u^1}) - \bar{\mathbf{v}}(\mathbf{u^2})$. Then the components of $\Delta\bar{\mathbf{v}}$ are defined by $\triangle\bar{v}_i^{+(-)}(n) := \bar{v}_i^{+(-)}(\mathbf{u^1})(n) - \bar{v}_i^{+(-)}(\mathbf{u^2})(n)$. In accordance with (4.1) and with the notations introduced at the beginning of Section 4 we have

$$\Delta\bar{\mathbf{v}} = \hat{\mathbf{v}}(\mathbf{u^1}, \mathcal{H}(\mathbf{u^1}, \bar{\mathbf{v}}(\mathbf{u^1})), \mathbf{d}(\mathbf{u^1})) - \hat{\mathbf{v}}(\mathbf{u^2}, \mathcal{H}(\mathbf{u^2}, \bar{\mathbf{v}}(\mathbf{u^2})), \mathbf{d}(\mathbf{u^2})).$$

Actually we are interested in $\triangle\bar{v}_i(0)$. Applying the mean value theorem and exploiting the linearity of $\hat{\mathbf{v}}(\mathbf{u}, \lambda, \cdot, \cdot)$ (see Section 2) we get finally:

$$\begin{aligned} \triangle\bar{v}_i(0) = \; & \hat{v}_i(\mathbf{u^1}, \Delta\mathcal{H}, \Delta\mathbf{d})(0) \\ & - \int_0^1 D_1\hat{v}_i(\mathbf{u^1} + \tau(\mathbf{u^2} - \mathbf{u^1}), \mathcal{H}(\mathbf{u^2}, \bar{\mathbf{v}}(\mathbf{u^2})), \mathbf{d}(\mathbf{u^2}))(0)d\tau(\mathbf{u^2} - \mathbf{u^1}). \end{aligned} \tag{5.9}$$

$\Delta\mathcal{H}$ and $\Delta\mathbf{d}$ are defined like $\Delta\bar{\mathbf{v}}$. Note that $\triangle\bar{v}_i(0)$ represents a pair $(\triangle\bar{v}_i^+(0), \triangle\bar{v}_i^-(0))$. Hence estimates of $\|\triangle\bar{v}_i(0)\|$ comprise both estimates of $\|\triangle\bar{v}_i^+(0)\|$ and $\|\triangle\bar{v}_i^-(0)\|$. So, in view of (5.8), the estimate of $\|\xi_i(N,\mathbf{u^2},\lambda)-\xi_i(N,\mathbf{u^1},\lambda)\|$ can be substituted by $2\|\triangle\bar{v}_i(0)\|$. Hence instead of (5.7) we use

$$(1/3)|u_i^1-u_i^2| \le \|D_1\xi^\infty(u_i^{*1},\lambda_o)^{-1}\|\,\|\triangle\bar{v}_i(0)\| \tag{5.10}$$

to show (5.3).

The first term on the right-hand side of (5.9) can be estimated similar to the corresponding term $\triangle v_o$ in the proof of [5, Lemma 5.1]. We have to take care just at one point: Here d_i is not zero for $|i|\le i_o$. Just this point asks for more effort.

It is true we get in the same way as in [5]

$$\begin{aligned} \|\hat{v}_i(\mathbf{u^1},\Delta\mathcal{H},\Delta\mathbf{d})(0)\| &= \|v_i(\mathbf{u^1},\Delta\mathcal{H},\mathbf{a}(\mathbf{u},\Delta\mathcal{H},\Delta\mathbf{d}))(0)\| \\ &< Ce^{-\alpha N}(\|a_{i+1}^+\|+\|a_i^-\|). \end{aligned} \tag{5.11}$$

C and α are positive constants. But unlike [5] here we get only

$$\|a_j^-\|+\|a_j^+\| \le c\|\triangle d_j\| + (1/4)(\|a_{j-1}^-\|+\|a_{j+1}^+\|). \tag{5.12}$$

For the derivation of (5.12) we refer again to [5]. Applying Corollary 6.5 and combining this with (5.10) we obtain:

Let i be any integer between $-i_o^2$ and i_o^2 then there exist positive constants C (C will differ from that one in (5.11)) and α such that

$$\begin{aligned} \tfrac{1}{3}|u_i^1-u_i^2| \le\ & \|D_1\xi^\infty(u_i^{*1},\lambda_o)^{-1}\|\Big(Ce^{-\alpha N}\max\{\|\triangle d_k\|,\ k\in[-i-i_o,i+i_o]\cap\mathbb{Z}\} \\ & +\tfrac{1}{2^{i_o}}\sup_{j\in\mathbb{Z}}\|a_j^\pm\| \\ & +\|\int_0^1 D_1\hat{v}_i(\mathbf{u^1}+\tau(\mathbf{u^2}-\mathbf{u^1}),\mathcal{H}(\mathbf{u^2},\bar{\mathbf{v}}(\mathbf{u^2})),\mathbf{d}(\mathbf{u^2}))(0)d\tau(\mathbf{u^2}-\mathbf{u^1})\|\Big). \end{aligned} \tag{5.13}$$

In [5] we have shown that $\sup_{j\in\mathbb{Z}}\|a_j^\pm\|$ does exist. Further, again invoking the mean value theorem provides

$$\begin{aligned} \|\triangle d_k\| \le\ & \sup_{\tau\in[0,1]}\|D_1\gamma^-(u_k^1+\tau(u_k^2-u_k^1),\lambda)(-N^-)\|\,\|\triangle u_k\| \\ & +\sup_{\tau\in[0,1]}\|D_1\gamma^+(u_{k-1}^1+\tau(u_{k-1}^2-u_{k-1}^1),\lambda)(N^+)\|\,\|\triangle u_{k-1}\|. \end{aligned} \tag{5.14}$$

The estimate (4.8) of $D_1\gamma^\pm$ ensures that we can choose N large enough such that

$$\begin{aligned} & 3Ce^{-\alpha N}\|D_1\xi^\infty(u_i^{*1},\lambda_o)^{-1}\|\ \max\{\|\triangle d_k\|,\ k\in[-i-i_o,i+i_o]\cap\mathbb{Z}\} \\ & \le \tfrac{1}{4}\max\{\|\triangle u_k\|,\ k\in[-i-i_o-1,i+i_o]\cap\mathbb{Z}\} \end{aligned} \tag{5.15}$$

Now we turn towards the estimate of the last term in (5.13): As we established at the beginning of this section $v_i^\pm$ depends only on u_i and a_{i+1}^+, a_i^- and not on the entire sequences $\mathbf{u}$ and $\mathbf{a}$. This yields $\hat{v}_i(\mathbf{u},\mathcal{H},\mathbf{d}) = v_i(u_i,\mathcal{H},a_{i+1}^+(\mathbf{u},\dots),a_i^-(\mathbf{u},\dots))$. This again provides

$$\begin{aligned} D_1\hat{v}_i(\mathbf{u^1}+\tau(\mathbf{u^2}-\mathbf{u^1}),\mathcal{H}(\dots),\mathbf{d}(\dots))(0)(\mathbf{u^2}-\mathbf{u^1}) &= D_1v_i(u_i^1+\tau(u_i^2-u_i^1),\dots)\triangle u_i \\ & +D_3v_i(\dots)(0)D_1a_{i+1}^+(\mathbf{u^1}+\tau(\mathbf{u^2}-\mathbf{u^1}),\dots)(\mathbf{u^2}-\mathbf{u^1}) \\ & +D_4v_i(\dots)(0)D_1a_i^-(\mathbf{u^1}+\tau(\mathbf{u^2}-\mathbf{u^1}),\dots)(\mathbf{u^2}-\mathbf{u^1}). \end{aligned}$$

Because of $a_i^{\pm}(\mathbf{u},\dots)=a_i^{\pm}(u_{i-1},u_i,\dots)$ - see again at the beginning of this section - we have

$$D_1 a_i^{\pm}(\mathbf{u},\dots)(\mathbf{u^2}-\mathbf{u^1}) = D_{u_{i-1}} a_i^{\pm}(u_{i-1},u_i,\dots)\triangle u_{i-1} + D_{u_i} a_i^{\pm}(u_{i-1},u_i,\dots)\triangle u_i.$$

Together with our results obtained in Section 6 - especially the derivatives of v_i given by (6.3) - we find a sufficiently large N such that

$$\begin{aligned} 3\|D_1\xi^{\infty}(u_i^{*1},\lambda_o)^{-1}\|\,\|\sup_{\tau\in[0,1]} D_1\hat{v}_i(\mathbf{u^1}+\tau(\mathbf{u^2}-\mathbf{u^1}),\mathcal{H}(\dots),\mathbf{d}(\dots))(0)(\mathbf{u^2}-\mathbf{u^1})\| \\ \le \tfrac{1}{4}\max\{|\triangle u_k|,\ k\in[-i-1,i+1]\cap\mathbb{Z}\}. \end{aligned} \tag{5.16}$$

Now, combining the estimates (5.13), (5.15) and (5.16), we see that

$$|\triangle u_i| \le \frac{1}{2}\max\{|\triangle u_k|,\ k\in[-i-i_o-1,i+i_o]\cap\mathbb{Z}\} + \frac{1}{2^{i_o}}M. \tag{5.17}$$

The constant M is defined by $M:=\sup\limits_{j\in\mathbb{Z}}\|a_j^{\pm}\|\sup\limits_{j\in\mathbb{Z}}\|D_1\xi^{\infty}(u_j^{*1},\lambda_o)^{-1}\|$. To finish the first step in proving the lemma we choose any $j\in\mathbb{Z}$, $|j|\le i_o$. Reapplying the estimate (5.17) we arrive at:

$$|\triangle u_j| \le \frac{1}{2^{i_o-1}}\max\{|\triangle u_k|,\ k\in[-i_o(i_o+1),i_o(i_o+1)]\cap\mathbb{Z}\} + \frac{i_o-1}{2^{i_o}}M. \tag{5.18}$$

We mentioned already that $\sup_{j\in\mathbb{Z}}\|a_j^{\pm}\|$ does exist. As well the derivation of $\mathbf{u^{1(2)}}$ provides that $\max\{|\triangle u_k|, k\in\mathbb{Z}\}$ does exist. This together with (5.18) gives immediately that

$$|\triangle u_j| = O\left(\frac{1}{i_o}\right),\quad \forall j,\ |j|\le i_o. \tag{5.19}$$

In the second step we will prove the actual statement of the lemma by using estimate (5.19). At the beginning of the proof we made clear that we have done, if (5.2) is proved. Putting $j=0$ in (5.19) and again applying the mean value theorem we have directly

$$\gamma^{+}(u_o^1)-\gamma^{+}(u_o^2) = O\left(\frac{1}{i_o}\right).$$

To verify the second equation in (5.2) we repeat the estimates of $\triangle v_i^{\pm}$ for $i=0$. In doing so and thereby using the O-property (5.19) of $\triangle u_j$, $|j|\le i_o$, we attain to

$$\bar{v}_o^{+}(\mathbf{u^1})(0)-\bar{v}_o^{+}(\mathbf{u^2})(0) = O\left(\frac{1}{i_o}\right).$$

■

6 Some technical results

In this section we compile some results we used in Section 3.

6.1 Differentiability of mappings $l^\infty \to l^\infty$

We start with presenting some comments concerning the derivative of Ξ_R.

Lemma 6.1 *Let $F : l^\infty \to l^\infty$, $x \mapsto (f^i(x))_{i\in\mathbb{Z}}$, be a differentiable map. Then the maps $f^i : l^\infty \to \mathbb{R}$, $i \in \mathbb{Z}$, are differentiable and $DF(x_o) = (Df^i(x_o))_{i\in\mathbb{Z}}$.*

Proof The projection $P^i : l^\infty \to \mathbb{R}$, $(x_i) \mapsto x_i$ is bounded (and linear). Hence $P^i \circ DF(x_o) = D(P^i F)(x_o)$. ∎

Let $\bar{\xi}_N : l^\infty \times \mathbb{R} \to l^\infty$, $(\mathbf{u}, \lambda) \mapsto (\xi_i(N, \mathbf{u}, \lambda))_{i\in\mathbb{Z}}$. Together with the notations introduced in Section 2 the mapping $\bar{\xi}_N$ can be understood as $\bar{\xi}_N(\mathbf{u}, \lambda) = P_1 \bar{\mathbf{v}}(\mathbf{u}, \lambda)(0) - P_2 \bar{\mathbf{v}}(\mathbf{u}, \lambda)(0)$, where $P_j : l^\infty \times l^\infty \to l^\infty$, $(x_1, x_2) \mapsto x_j$. The projections P_j are bounded (and linear). So the smoothness of $\bar{\mathbf{v}}$ - see Section 2 - implies the smoothness of $\bar{\xi}_N$. Finally, the above lemma tells that $D_1 \bar{\xi}_N(\mathbf{u}, \lambda) = (D_2 \xi_i(N, \mathbf{u}, \lambda))_{i\in\mathbb{Z}}$. The next lemma gives a sufficient condition for the differentiability of $F : l^\infty \to l^\infty$, $x \mapsto (f^i(x))_{i\in\mathbb{Z}}$, if all f^i are differentiable.

Lemma 6.2 *Consider $F : l^\infty \to l^\infty$, $x \mapsto (f^i(x))_{i\in\mathbb{Z}}$, where all f^i are differentiable. If furthermore hold*

(i) there is a $K > 0$ such that $\|Df^i(x_o)\| < K$, $\forall i \in \mathbb{Z}$ and

(ii) $Df^i(\cdot)$ are continuous in x_o, uniformly in i,

then F is differentiable in x_o.

Proof We write $F(x_o + h)$ as $F(x_o + h) = F(x_o) + (Df^i(x_o)h)_{i\in\mathbb{Z}} + R(h)$. Assumption (i) of the lemma ensures that $h \mapsto (Df^i(x_o)h)_{i\in\mathbb{Z}}$ is a bounded linear mapping from l^∞ into l^∞. So it remains to prove $R(h) = o(\|h\|)$. Let $R^i(h) := P^i R(h)$. P^i is defined as in the proof of Lemma 6.1. Actually we have $\sup_{i\in\mathbb{Z}} R^i(h) = o(\|h\|)$ what is an easily follows with mean value theorem. ∎

Finally we apply Lemma 6.2 to the mapping Ξ - see Section 3.
Let again P^i be as above. Then $\Xi(\infty, \cdot, \lambda_o)$ can be rewritten as $\Xi(\infty, \cdot, \lambda_o) : l^\infty \to l^\infty$, $\mathbf{u} \mapsto (f^i(\mathbf{u}))_{i\in\mathbb{Z}} := \xi^\infty(P^i(\mathbf{u}), \lambda_o)$. With that we get

$$\begin{aligned} \|D_2\Xi_i(\infty, \mathbf{u}^* + \mathbf{h}, \lambda_o) - D_2\Xi_i(\infty, \mathbf{u}^*, \lambda_o)\| = & \ \|(D_1\xi^\infty(u_i + h_i, \lambda_o) - D_1\xi^\infty(u_i, \lambda_o))P^i\| \\ \leq & \ \int_0^1 \sup_{i\in\mathbb{Z}} \|D_1^2\xi^\infty(u_i + \tau h_i, \lambda_o)\| d\tau C \|\mathbf{h}\|. \end{aligned}$$

To achieve the latter estimate we used the mean value theorem. The constant C is the norm of P^i which is equal for all i. The continuity of $D^2\xi^\infty$ ensures that Lemma 6.2 works in our case.

6.2 Estimates used in the proof of Lemma 3.8

In this subsection we itemize some results, purely technical in nature, we used in the proof of Lemma 3.8.

Lemma 6.3 *Let $(a_i^\pm)_{i\in\mathbb{Z}}$ and $(d_i)_{i\in\mathbb{Z}}$ be sequences of positive numbers such that for all $j\in\mathbb{Z}$ $a_j^- + a_j^+ \le d_j + \frac{1}{4}(a_{j-1}^- + a_{j+1}^+)$. Then*

$$a_{-i}^- + a_i^+ \le \sum_{k=0}^{i} 2^{1+k-i}(d_{-k}+d_k) + \frac{1}{2}(a_{-i-1}^- + a_{i+1}^+). \tag{6.1}$$

Proof We show that even $(a_{-i}^- + a_i^+) + (a_{-i}^+ + a_i^-) \le \sum\limits_{k=0}^{i} 2^{1+k-i}(d_{-k}+d_k) + \frac{1}{2}(a_{-i-1}^- + a_{i+1}^+)$. The latter equation can be proved by induction. ■

Corollary 6.4 *Let $(a_i^\pm)_{i\in\mathbb{Z}}$ and $(d_i)_{i\in\mathbb{Z}}$ be sequences as in Lemma 6.3. Then*

$$a_o^- + a_o^+ \le \sum_{k=0}^{i} 2^{2-k}(d_{-k}+d_k) + \frac{1}{2^{i+1}}(a_{-i-1}^- + a_{i+1}^+). \tag{6.2}$$

Proof We start by writing up (6.1) for $i=0$. At the right-hand side we replace $a_{-1}^- + a_1^+$ by the estimate given by (6.1). We continue by replacing in each case the the expression $a_{-j}^- + a_j^+$. This procedure finally gives the lemma. ■

By means of straight forward computations we get from (6.2)

Corollary 6.5 *Let $(a_i^\pm)_{i\in\mathbb{Z}}$ and $(d_i)_{i\in\mathbb{Z}}$ be sequences as in Lemma 6.3. Then*
$a_j^- + a_j^+ \le 16\max\{d_k,\, k\in[-j-i, j+i]\cap\mathbb{Z}\} + \frac{1}{2^{i+1}}(a_{j-i-1}^- + a_{j+i+1}^+)$. ■

In the end we want to remark that similar to considerations in [6] and the proof of Lemma 3.2 we get estimates of the derivatives of v_i. In particular we can prove

$$\begin{aligned} D_a v_i(\ldots)(0) &= O((\max\{\alpha^s, (\alpha^u)^{-1}\})^{N^+}),\\ D_u v_i(\ldots, H, \ldots)(0) &= O((\max\{\alpha^s, (\alpha^u)^{-1}\})^{N^+}). \end{aligned} \tag{6.3}$$

References

[1] Beyn, W.-J. and Kleinkauf, J.-M., *The Numerical Computation of Homoclinic Orbits for Maps*, SIAM J. Numer. Anal. 34, 1207-1236, 1997.

[2] Beyn, W.-J. and Kleinkauf, J.-M., *Numerical approximation of homoclinic chaos*, Numer. Algorithms 14, 25-52, 1997.

[3] Katok, A. and Hasselblatt, B. *Introduction to the Modern Theory of Dynamical Systems* Cambridge University Press, 1995

[4] Kleinkauf, J.-M., *The numerical computation and geometrical analysis of heteroclinic tangencies* Preprint Universität Bielefeld 1998 No. 98-048.

[5] Knobloch, J., *Lin's method for discrete dynamical systems,* Journal of Difference Equations and Applications, Vol. 00, pp. 1-47, 2000

[6] Knobloch, J., *Jump estimates for Lin's method,* Preprint TU Ilmenau No M31/99, 1999.

[7] Palmer, K. J., *Exponential Dichotomies, the Shadowing Lemma and Transversal Homoclinic Points,* in: Dynamics Reported, Vol. I, pp. 265 - 306, John Wiley & Sons and B. G. Teubner 1988.

[8] Palis, J. and Takens, F., *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations,* Cambridge University Press 1993.

[9] Wiggins, S. *Introduction to applied nonlinear dynamical systems and chaos,* Springer Verlag 1990.

[10] Zeidler, E., *Nonlinear Functional Analysis and its Applications, Vol. I Fixed point theorems,* Springer Verlag 1993.