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Jump estimates for Lin's method

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Abstract

We consider discrete dynamical systems having a homoclinic orbit asymptotic to a hyperbolic fixed point. After we carried forward the ideas of Lin's method to discrete systems - cf. [5] - we provide now the necessary estimates for solving the bifurcation equation.
We apply our results to study 1-periodic orbits near homoclinic orbits having quadratic tangencies.

1 Introduction

In a first paper [5] we carried forward Lin's method to discrete systems

$$x(n+1) = f(x(n), \lambda) \tag{1.1}$$

having, for $\lambda = 0$, a homoclinic orbit Γ,

$$\Gamma := \{\gamma(n) := f^n(q), n \in \mathbb{Z}\} \tag{1.2}$$

asymptotic to a hyperbolic fixed point p. The main idea of this method is as follows: We look for solutions x_i of (1.1) starting in a neighborhood of q following the forward orbit Γ^+ of the orbit Γ, passing p, following the backward orbit Γ^- of Γ and arriving finally after N_i steps again in the same neighborhood of q. In this way we find solutions staying for all time close to Γ by solving the bifurcation equations

$$\Xi_i := x_{i+1}(0) - x_i(N_i) = 0. \tag{1.3}$$

Of course Ξ_i will depend on λ, $\mathcal{N} = (N_i)_{i\in\mathbb{Z}}$ and in general on some additional parameters $\mathbf{u} = (u_i)_{i\in\mathbb{Z}}$ which we will introduce below. Hence $\Xi_i = \Xi_i(\mathcal{N}, \mathbf{u}, \lambda)$. However, Ξ_i takes the form

$$\Xi_i(\mathcal{N}, \mathbf{u}, \lambda) = \xi^\infty(u_i, \lambda) + \xi_i(\mathcal{N}, \mathbf{u}, \lambda). \tag{1.4}$$

Here

$$\xi^\infty(u, \lambda) = 0 \tag{1.5}$$

is the bifurcation equation for detecting homoclinic orbits of (1.1) which are close to Γ. In some cases it is relatively easy to discuss equation (1.5) - see for instance [5] Section 3.
However, the discussion of the full bifurcation equation $(\Xi_i)_{i\in\mathbb{Z}} = 0$ needs a good knowledge of the term ξ_i. Because ξ_i becomes exponentially small with respect to $\min\{N_i, i \in \mathbb{Z}\}$ each Ξ_i can be seen as a pertubation of ξ^∞. But this is not sufficient to know, because we need the sign of ξ_i for a comprehensive discussion of the bifurcation equation - see Section 7 and in the forthcoming paper [4]. So the primary object of this paper is the determination of the leading terms of ξ_i. It turns out that these terms are governed by the velocity of $\gamma(\cdot)$ while approaching the fixed point and by the velocity by which solutions of the of the adjoint of the variational equation along Γ tend to zero. In Section 3 we give the precise formulation of the main theorems.

In the section before we compile notations and results we introduced and achieved, respectively, in the previous paper [5]. The analysis we will carry out here is based on these as well as the formulation of the main theorem in Section 3.
However, we start considering the jumps ξ_i by performing transformations making the stable and unstable manifolds, respectively, flat in a neighborhood of Γ. This will be done in Section 4. The actual estimate of ξ_i will be performed in Section 5. For that we give an appropriate representation of ξ_i and figure out the leading terms.
In Section 6 we provide assertions concerning convergence properties of solutions starting in the stable manifold (of a discrete dynamical system) or the stable subspace (of a linear difference equation), respectively. Although we use these results only to estimate ξ_i they are of some relevance as independent results in the theory of discrete systems. So we prove that solutions in the stable manifold not starting within the strong stable manifold, decay like $(\mu^s)^n + o((\mu^s)^n)$, where μ^s is the principal stable eigenvalue. - see Lemma 6.2. This is a stronger estimate than this one given in [6]. Lemma 6.6 makes an analoge assertion for linear (non-autonomous) difference equations. To define strong stable subspaces for such equations we carry over the concept of exponential trichotomies to difference equations - see Lemma 6.8 and Remark 6.9.
In the last part we apply our results to the case that q is a non-transversal homoclinic point. We assume that the tangent spaces at q of the stable and unstable manifold intersect in an one-dimensional subspace. Generically this scenario forms a codimension one problem ($\lambda \in \mathbb{R}^1$). Moving the parameter λ through the critical point ($\lambda = 0$) we observe (the well known phenomenon) that two transversal homoclinic points q_1, q_2 (existing for $\lambda > 0$) merge at $\lambda = 0$ into q and disappear afterwards. Exploiting the bifurcation equation we study the behaviour of 1-periodic solutions. Here 1-periodicity means that the corresponding orbit hits a sufficiently small neighborhood of q exactly once. In the forthcoming paper [4] we prove the existence of shift dynamics involving both transversal homoclinic points q_1 and q_2.

2 Some previous results

Now we will make things we mentioned in the introduction more precise. We consider the discrete system (1.1) where $f : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$ is smooth. In particular $f(\cdot, \lambda)$ is a diffeomorphism for all λ. Further we will assume that $x = p$ is a hyperbolic fixed point of (1.1) - i.e.

(HYP) $f(p, \lambda) = p,$
$\sigma(D_1 f(p, \lambda)) \cap S^1 = \emptyset.$

$\sigma(D_1 f(p, \lambda))$ denotes the spectrum of $D_1 f$ - the derivative of f with respect to the first variable; S^1 is the unite circle line (in $\mathbb{C}$). Finally we assume that for $\lambda = 0$ (1.1) has a homoclinic solution $\gamma(\cdot)$ asymptotic to $x = p$:

(HOM) $\gamma(n+1) = f(\gamma(n), 0),$
$\lim_{n \to \pm\infty} \gamma(n) = p.$

The orbit $\{\gamma(n), n \in \mathbb{Z}\}$ of the homoclinic solution $\gamma(\cdot)$ we denote by Γ. Let $q := \gamma(0)$. Finally we assume that the local stable and unstable manifolds are flat

(MAN) $W^{s(u)}_{loc,\lambda}(p) \subset T_p W^{s(u)}_{\lambda=0}(p)$.

$W^{s(u)}_{loc,\lambda}(p)$ is the local stable (unstable) manifold of the fixed point p with respect to $x(n+1) = f(x(n), \lambda)$. $T_p W^{s(u)}_{loc,\lambda}(p)$ is the tangent space of this manifold at p. See also [5, Section 4.1] to make sure that this assumption make sense.
For the further analysis we use the following direct sum decomposition of $\mathbb{R}^k$:

$$\mathbb{R}^k = U \oplus W^+ \oplus W^- \oplus Z, \tag{2.1}$$

where $U = T_q W^s_{\lambda=0}(p) \cap T_q W^u_{\lambda=0}(p)$ and $U \oplus W^{+(-)} = T_q W^{s(u)}_{\lambda=0}(p)$. Finally we denote the projector from $\mathbb{R}^k$ on U along $W^+ \oplus W^- \oplus Z$ by P_U. Now we can prove the following lemma - see [5, Lemma 2.4]:

Lemma 2.1 *For each* $(u, \lambda) \in U \times \mathbb{R}^l$ *close to* $(0, 0)$ *there is a unique pair* $(\gamma^+(u, \lambda)(\cdot), \gamma^-(u, \lambda)(\cdot))$ *of solutions of (1.1)satisfying*

(i) $\gamma^{+(-)} : \mathbb{Z}^{+(-)} \to \mathbb{R}^k$,

(ii) the orbits of γ^+ *and* γ^- *are close to* Γ,

(iii) $\lim\limits_{n\to\infty} \gamma^+(n) = p$, $\lim\limits_{n\to-\infty} \gamma^-(n) = p$,

(iv) $\gamma^+(0)$, $\gamma^-(0)$ *are close to* $\gamma(0)$,

(v) $\gamma^+(0) - \gamma^-(0) \in \mathbb{Z}$ *and*

(vi) $P_U(\gamma^{+(-)}(u, \lambda)(0) - \gamma(0)) = u$. ■

Consequently the bifurcation equation for detecting homoclinic orbits close to Γ runs

$$\xi^\infty(u, \lambda) := \gamma^+(u, \lambda)(0) - \gamma^-(u, \lambda)(0) = 0. \tag{2.2}$$

Now, let $\mathcal{N} := (N_i)_{i\in\mathbb{Z}}$, $N_i \in \mathbb{N}$ and $\mathbf{u} := (u_i)_{i\in\mathbb{Z}}$, $u_i \in U$, any sequences. We define

$$N_i^+ := \left\lfloor \frac{N_i}{2} \right\rfloor, N_i^- := N_i - N_i^+, \tag{2.3}$$

where $\lfloor n \rfloor$ denotes the integer part of n.
The solutions x_i we mentioned in the introduction we compose of pertubations of γ^+ and γ^-, cf. [5, Section 4]:

Theorem 2.2 *There are constants* c *and* $\tilde{N}$ *such that for each* $\mathbf{u}$, λ *with* $\|\mathbf{u}\| := \sup |u_i| < c$, $|\lambda| < c$ *and each* $\mathcal{N}$ *with* $N_i^{+(-)} > 2\tilde{N}$ *there are unique solutions* $x_i^+(\mathcal{N}, \mathbf{u}, \lambda)(\cdot)$, $x_i^-(\mathcal{N}, \mathbf{u}, \lambda)(\cdot)$ *of (1.1) satisfying*

(i) $x_i^+(\mathcal{N}, \mathbf{u}, \lambda)(\cdot) : [0, N_{i+1}^+] \cap \mathbb{Z} \to \mathbb{R}^k$,
$x_i^-(\mathcal{N}, \mathbf{u}, \lambda)(\cdot) : [-N_i^-, 0] \cap \mathbb{Z} \to \mathbb{R}^k$,

(ii) the orbits of x_i^+ *and* x_i^- *are close to the forward and backward orbit through* q*, respectively,*

(iii) $x_i^+(\mathcal{N}, \mathbf{u}, \lambda)(N_{i+1}^+) = x_{i+1}^-(\mathcal{N}, \mathbf{u}, \lambda)(-N_{i+1}^-)$,

(iv) $x_i^+(\mathcal{N}, \mathbf{u}, \lambda)(0)$, $x_i^-(\mathcal{N}, \mathbf{u}, \lambda)(0)$ *are close to* $\gamma(0)$,

(v) $x_i^+(\mathcal{N}, \mathbf{u}, \lambda)(0) - x_i^-(\mathcal{N}, \mathbf{u}, \lambda)(0) \in Z$,

(vi) $x_i^{+(-)}(\mathcal{N},\mathbf{u},\lambda)(0) - \gamma^{+(-)}(u_i,\lambda)(0) \in W^+ \oplus W^- \oplus Z$. ■

With that the solutions x_i are defined by

$$x_i(\mathcal{N},\mathbf{u},\lambda)(n) := \begin{cases} x_{i-1}^{+}(\mathcal{N},\mathbf{u},\lambda)(n) & , \quad n \in [0, N_i^+] \cap \mathbb{Z}, \\ x_i^{-}(\mathcal{N},\mathbf{u},\lambda)(n-N_i) & , \quad n \in [N_i^+, N_i] \cap \mathbb{Z}. \end{cases} \tag{2.4}$$

In particular we find $x_i^{+(-)}$ in the form

$$x_i^{+(-)}(\mathcal{N},\mathbf{u},\lambda)(\cdot) = \gamma^{+(-)}(u_i,\lambda)(\cdot) + \bar{v}_i^{+(-)}(\mathcal{N},\mathbf{u},\lambda)(\cdot). \tag{2.5}$$

With these notations the bifurcation equations $\Xi_i = 0$ take the form

$$\Xi_i(\mathcal{N},\mathbf{u},\lambda) = x_i^{+}(\mathcal{N},\mathbf{u},\lambda)(0) - x_i^{-}(\mathcal{N},\mathbf{u},\lambda)(0) = 0. \tag{2.6}$$

Replacing here $x_i^{+(-)}$ by the representation (2.5) we see that Ξ_i actually take the form (1.4). Beyond it we see that

$$\xi_i(\mathcal{N},\mathbf{u},\lambda) = \bar{v}_i^{+}(\mathcal{N},\mathbf{u},\lambda)(0) - \bar{v}_i^{-}(\mathcal{N},\mathbf{u},\lambda)(0) = 0. \tag{2.7}$$

Indeed, in [5] Theorem 2.2 was verified by proving the existence of corresponding $\bar{v}_i^{+(-)}$. To be able to estimate the jump ξ_i we have to recall the derivation of $\bar{v}_i^{+(-)}$. In [5] we have shown that $\bar{v}_i^{+(-)}$ are solutions of the following boundary value problem

$$\begin{aligned} & v_i^{+(-)}(n+1) = D_1 f(\gamma^{+(-)}(u_i,\lambda)(n),\lambda) v_i^{+(-)}(n) + h^{+(-)}(n, u_i, v_i^{+(-)}(n), \lambda), \\ & v_i^{+}(N_{i+1}^{+}) - v_{i+1}^{-}(-N_{i+1}^{-}) = \gamma^{-}(u_{i+1},\lambda)(-N_{i+1}^{-}) - \gamma^{+}(u_i,\lambda)(N_{i+1}^{+}) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =: d_{i+1}(N_{i+1}, u_i, u_{i+1}, \lambda), \\ & v_i^{+}(0), v_i^{-}(0) \in W^+ \oplus W^- \oplus Z \quad \text{close to zero}, \\ & v_i^{+}(0) - v_i^{-}(0) \in Z. \end{aligned} \tag{2.8}$$

$h^{+(-)}$ are defined as follows:

$$\begin{aligned} h^{+(-)}(n,u,v,\lambda) := \; & f(\gamma^{+(-)}(u,\lambda)(n) + v, \lambda) - f(\gamma^{+(-)}(u,\lambda)(n),\lambda) \\ & - D_1 f(\gamma^{+(-)}(u,\lambda)(n),\lambda) v. \end{aligned} \tag{2.9}$$

The boundary value problem (2.8) can be solved in two steps - see again [5, Section 4] for the details:
First we consider the inhomogeneous equation

$$v_i^{+(-)}(n+1) = D_1 f(\gamma^{+(-)}(u_i,\lambda)(n),\lambda) v_i^{+(-)}(n) + g_i^{+(-)}(n). \tag{2.10}$$

Again as in [5] we write shortening $\mathbf{g} := ((g_i^{+}, g_i^{-}))_{i\in\mathbb{Z}}$. Let $\Phi^{+(-)}(u_i,\lambda,n,m)$ be the transition matrix of the corresponding homogeneous equation

$$v_i^{+(-)}(n+1) = D_1 f(\gamma^{+(-)}(u_i,\lambda)(n),\lambda) v_i^{+(-)}(n). \tag{2.11}$$

These equations have exponential dichotomies on $\mathbb{Z}^+$ and $\mathbb{Z}^-$ with projectors $P^+(u_i,\lambda,\cdot)$ and $P^-(u_i,\lambda,\cdot)$, respectively. $P^{+(-)}$ are defined as in [5]:

$$\begin{aligned}
&\operatorname{im} P^{+(-)}(u_i,\lambda,0) = T_{\gamma^{+(-)}(u_i,\lambda)(0)} W_\lambda^{s(u)}(p),\\
&\ker P^{+(-)}(u_i,\lambda,0) = W^{-(+)} \oplus Z,\\
&P^{+(-)}(u_i,\lambda,n)\Phi^{+(-)}(u_i,\lambda,n,m) = \Phi^{+(-)}(u_i,\lambda,n,m)P^{+(-)}(u_i,\lambda,m).
\end{aligned} \tag{2.12}$$

Then [5, Lemma 4.6] tells that for $\mathcal{N}$, $\mathbf{u}$ and λ as in Theorem 2.2 the system (2.10) has a unique solution $\mathbf{v}_\mathcal{N}(\mathbf{u},\lambda,\mathbf{g},\mathbf{a}) := (v_i^+(\cdot), v_i^-(\cdot))_{i\in\mathbb{Z}}$ satisfying

(i) $(id - P^+(u_{i-1},\lambda,N_i^+))v_{i-1}^+(N_i^+) = a_i^+$,
$(id - P^-(u_i,\lambda,-N_i^-))v_i^-(-N_i^-) = a_i^-$,

(ii) $v_i^+(0), v_i^-(0) \in W^+ \oplus W^- \oplus Z$,

(iii) $v_i^+(0) - v_i^-(0) \in Z$.

Here $a_i^{+(-)}$ are any given quantities within the images of the corresponding projectors. $\mathbf{a} := (a_i^+, a_i^-)_{i\in\mathbb{Z}}$. Exploiting this result we can prove that for $\mathcal{N}$, $\mathbf{u}$ and λ as in Theorem 2.2 the system (2.10) has a unique solution $\hat{\mathbf{v}}_\mathcal{N}(\mathbf{u},\lambda,\mathbf{g},\mathbf{d}) := (v_i^+(\cdot), v_i^-(\cdot))_{i\in\mathbb{Z}}$ satisfying

(i) $v_{i-1}^+(N_i^+) - v_i^-(-N_i^-) = d_i$,

(ii) $v_i^+(0), v_i^-(0) \in W^+ \oplus W^- \oplus Z$,

(iii) $v_i^+(0) - v_i^-(0) \in Z$.

d_i are any given quantities. Indeed we prove that there is an $\hat{\mathbf{a}}_\mathcal{N}(\mathbf{u},\lambda,\mathbf{g},\mathbf{d})$ such that $\hat{\mathbf{v}}_\mathcal{N}(\mathbf{u},\lambda,\mathbf{g},\mathbf{d}) = \mathbf{v}_\mathcal{N}(\mathbf{u},\lambda,\mathbf{g},\hat{\mathbf{a}}_\mathcal{N}(\mathbf{u},\lambda,\mathbf{g},\mathbf{d}))$. See [5, Lemma 4.8] for the details. However, it is worth to mention that $\hat{\mathbf{v}}$ depends linearly on $(\mathbf{g},\mathbf{d})$.
The second step in solving the boundary value problem (2.8) comprises the following: With the foregoing results we rewrite (2.8) into a fixed point problem

$$\begin{aligned}
\mathbf{v} &= \hat{\mathbf{v}}_\mathcal{N}(\mathbf{u},\lambda,(h^+(\cdot,u_i,v_i^+(\cdot),\lambda),h^-(\cdot,u_i,v_i^-(\cdot),\lambda))_{i\in\mathbb{Z}},(d_i(\mathcal{N},\mathbf{u},\lambda))_{i\in\mathbb{Z}})\\
&=: \mathcal{F}_\mathcal{N}(\mathbf{u},\mathbf{v},\lambda).
\end{aligned} \tag{2.13}$$

$$d_i(\mathcal{N},\mathbf{u},\lambda) := \gamma^-(u_i,\lambda)(-N_i^-) - \gamma^+(u_{i-1},\lambda)(N_i^+), \quad \mathbf{d} := (d_i)_{i\in\mathbb{Z}}. \tag{2.14}$$

Now, let S_N and S_{-N} be the spaces of functions mapping $\{0,\dots,N\}$ and $\{-N,\dots,0\}$, respectively, into $\mathbb{R}^k$. These spaces are equipped with the maximum norm. The space $\mathcal{V}$ of all sequences $v := ((v_i^+, v_i^-))_{i\in\mathbb{Z}}$, $(v_i^+, v_i^-) \in S_{N_{i+1}^+} \times S_{-N_i^-}$ is equipped with the norm $\|v\| := \max\{\sup_{i\in\mathbb{Z}} \|v_i^+\|_{S_{N_{i+1}^+}}, \sup_{i\in\mathbb{Z}} \|v_i^-\|_{S_{-N_i^-}}\}$. See altogether [5, Definition 4.5]. With these notations equation (2.13) can (for fixed $\mathbf{u}$ and λ) be seen as a fixed point equation in $\mathcal{V}$. This fixed point problem has for fixed $\mathcal{N}$ the unique solution

$$\bar{\mathbf{v}}(\mathbf{u},\lambda) = (\bar{v}_i^+(\mathbf{u},\lambda)(\cdot), \bar{v}_i^-(\mathbf{u},\lambda)(\cdot))_{i\in\mathbb{Z}}. \tag{2.15}$$

Moreover, the mapping $\bar{\mathbf{v}} : l_U^\infty \times \mathbb{R}^l \to \mathcal{V}, \quad (\mathbf{u},\lambda) \mapsto \bar{\mathbf{v}}(\mathbf{u},\lambda)$ is smooth - cf. [5, Lemma 4.13]. Now, after we have briefely repeated the steps leading to the fixed point equation (2.13) we will reveal somewhat more of the structure of $\mathbf{v}_{\mathcal{N}}$ - for the details we refer again to [5]. The quantities $v_i^{+(-)}$ defining $\mathbf{v}_{\mathcal{N}}$ take the form

$$\begin{aligned} v_i^+(\mathbf{u},\lambda,\mathbf{g},\mathbf{a})(n) = & \ \Phi^+(u_i,\lambda,n,0)[L(u_i,\lambda)^{-1}F(u_i,\lambda,a_{i+1}^+,a_i^-,N_{i+1}^+,N_i^-]_1 \\ & + \sum_{j=1}^{n} \Phi^+(u_i,\lambda,n,j)g_i^+(j-1) \end{aligned}$$

and

$$\begin{aligned} v_i^-(\mathbf{u},\lambda,\mathbf{g},\mathbf{a})(n) = & \ \Phi^-(u_i,\lambda,n,0)[L(u_i,\lambda)^{-1}F(u_i,\lambda,a_{i+1}^+,a_i^-,N_{i+1}^+,N_i^-]_2 \\ & - \sum_{j=0}^{n+1} \Phi^-(u_i,\lambda,n,j)g_i^-(j-1). \end{aligned}$$

Here $[L(u,\lambda)^{-1}\cdot]_j$ is the jth component of the image of the inverse of the linear operator

$$\begin{aligned} L(u,\lambda): \quad (W^+\oplus W^-)\times Z\times Z \quad & \rightarrow \quad (W^-\oplus Z)\times(W^+\oplus Z) \\ (w,z^+,z^-) \quad & \mapsto \quad \begin{pmatrix} (id-P^+(u_i,\lambda,o))(w+z^+) \\ (id-P^-(u_i,\lambda,o))(w+z^-) \end{pmatrix} \end{aligned}$$

while F is a function we will not further specify. However, we want emphasize that $v_i^{+(-)}$ actually depend only on u_i and a_{i+1}^+, a_i^- and not on the entire sequences $\mathbf{u}$ and $\mathbf{a}$.

3 The main result

As already mentioned the leading terms of ξ_i depend on the manner how the homoclinic orbit and solutions of the adjoint of the variational equation along this orbit approach the fixed point and zero, respectively. We will determine this behaviour by the following assumptions. First we will make an assumption concerning the principal eigenvalues. Let $\mu^s(\lambda)$ and $\mu^u(\lambda)$ be the principal stable and unstable eigenvalues of $D_1f(p,\lambda)$, respectively. Then the spectrum of $D_1f(p,\lambda)$ can be represented by

$$\begin{aligned} & \sigma(D_1f(p,\lambda)) = \sigma^{ss}(\lambda)\cup\{\mu^s(\lambda),\mu^u(\lambda)\}\cup\sigma^{uu}(\lambda), \\ & \text{where} \\ & 0<|\mu|<\alpha^{ss}<|\mu^s(\lambda)|<\alpha^s<1<\alpha^u<|\mu^u(\lambda)|<\alpha^{uu}<|\tilde{\mu}| \end{aligned} \tag{3.1}$$

for all $\mu\in\sigma^{ss}(\lambda)$, $\tilde{\mu}\in\sigma^{uu}(\lambda)$. We will suppose:

(EV_R) The principal eigenvalues $\mu^{s(u)}(\lambda)$ are simple and real.

Beyond it we will assume that the homoclinic orbit Γ approaches the fixed point generically:

(HOM_A) $\eta^s(0,0),\eta^u(0,0)\neq 0$

or equivalently:

Γ does not approach p within the strong stable and strong unstable manifold, respectively.

Here $\eta^{s(u)}(u,\lambda)$ are defined by $\eta^s(u,\lambda) := \lim_{n\to\infty} D_1 f(p,\lambda)^n P\gamma^+(u,\lambda)(n)$ and $\eta^u(u,\lambda) := \lim_{n\to-\infty} D_1 f(p,\lambda)^n (id-P)\gamma^-(u,\lambda)(n)$. The projector P projects $\mathbb{R}^k$ on $T_pW^s(p)$ along $T_pW^u(p)$. Lemma 6.2 and the remarks following this lemma ensure that assumption (HOM_A) makes sense.
Further we will assume that the homoclinic point is non-transversal one but the non-transversality should be is as small as possible.

(DIM_U) $\dim U = 1$.

This implies that also $\dim Z = 1$. For the Z generating element z^1 we will assume

(SUB) $\eta^+((id - P^+(u_i,\lambda,0),\lambda)^T z^1) \neq 0$, $\eta^-((id - P^-(u_i,\lambda,0),\lambda)^T z^1) \neq 0$
or equivalently:
$(id - P^+(u_i,\lambda,0))^T z^1$ and $(id - P^-(u_i,\lambda,0))^T z^1$ are not in the strong stable subspace $X^{ss}_\lambda(0)$ and strong unstable subspace $X^{uu}_\lambda(0)$, respectively.

In this connection $\eta^{+(-)}$ are defined by $\eta^+(w(0),\lambda) := \lim_{n\to\infty} \Big((D_1 f(p,\lambda)^{-1})^T\Big)^n (id-P)^T w(n)$ and $\eta^-(w(0),\lambda) := \lim_{n\to\infty} \Big((D_1 f(p,\lambda)^{-1})^T\Big)^n P^T w(n)$, respectively. The functions $w(\cdot)$ are here solutions of the variational equations $w(n+1) = (D_1 f(\gamma^+(u_i,\lambda)(n),\lambda)^{-1})^T w(n)$, $n \in \mathbb{Z}^+$ and $w(n+1) = (D_1 f(\gamma^-(u_i,\lambda)(n),\lambda)^{-1})^T w(n)$, $n \in \mathbb{Z}^-$, lying within the stable and unstable subspace, respectively, of these equations. See Lemma 6.6 for the proof that these limits indeed exist. After the proof of Lemma 6.6 the notions (un)stable and strong (un)stable subspaces $X^{ss(uu)}_\lambda(0)$ of these equations will be explained as well as their connection to the assumption (SUB).
Under the above conditions we have the following result concerning the leading terms of ξ_i:

Theorem 3.1 *If the principal eigenvalues are simple and real - (EV_R) - the tangent spaces at q of the stable and unstable manifold intersect in one-dimensional space - (DIM_U) - and if additionally the non-degeneracy conditions (SUB) and (HOM_A) are fulfilled then the jump ξ_i can be written as*

$$\begin{aligned}\xi_i(\mathcal{N},\mathbf{u},\lambda) = \;& c^s(u_{i-1},u_i,\lambda)(\mu^s(\lambda))^{N_i} + c^u(u_i,u_{i+1},\lambda)(\mu^u(\lambda))^{-N_{i+1}} \\ &+ o\Big((\mu^s(\lambda))^{N_i}\Big) + o\Big((\mu^u(\lambda))^{-N_{i+1}}\Big).\end{aligned} \tag{3.2}$$

4 Transformations

To carry out our analysis in [5] we performed a transformation we did not mention in this paper as yet. This transformation effects that around $\gamma^+(u,\lambda)(0)$ ($\gamma^-(u,\lambda)(0)$) the stable (unstable) manifold $W^{s(u)}_\lambda(p)$ in the direction $W^{+(-)}$ is flat. If, for fixed λ, the transformation is denoted by $\mathcal{T}_\lambda$ we have more precisely - cf.[5, (4.12), (4.13)]:

$$\mathcal{T}_\lambda(\gamma^{+(-)}(u,\lambda)(0) + w^{+(-)}) \in W^{s(u)}_\lambda(p) \cap \{\gamma^{+(-)}(u,\lambda)(0) + (W^+ \oplus W^- \oplus Z)\}. \tag{4.1}$$

To perform estimates in Section 5 we need similar transformations in neighborhoods of $\gamma^+(0,\lambda)(n)$ and $\gamma^-(0,\lambda)(-n)$, respectively, for all $n \geq 1$. We will give transformations mapping the stable (unstable) manifold around $\gamma^{+(-)}(0,\lambda)(n)$ into its tangent space. Because the stable and unstable manifold are both flat in a neigborhood of p (cf. **(MAN)**), we have to perform these transformations only in a finite number of points - say for $n \in \{1, \dots N_{\text{flat}}\}$. Let for $n \in \mathbb{Z}^{+(-)} \cap \{-N_{\text{flat}}, \dots, N_{\text{flat}}\}$

$$U_n := \Phi^{+(-)}(0,0,(-)n,0)U. \tag{4.2}$$

W_n^+, W_n^- and Z_n will be defined in the same way. Further, let $\hat{h}^s_{\lambda,n}$ and $\hat{h}^u_{\lambda,n}$ be mappings whose graphs locally represent the stable and unstable manifold, respectively. More precisely: For $n \geq 1$:

$$\hat{h}^s_{\lambda,n} : U_n \times W_n^+ \to W_n^- \oplus Z$$

such that for small $(u, w^+) \in U_n \times W_n^+$ and for small ϵ

$$\gamma^+(0,\lambda)(n) + u + w^+ + \hat{h}^s_{\lambda,n} = W^s_\lambda(p) \cap B(\gamma^+(0,\lambda)(n), \epsilon).$$

Here $B(x,r)$ denotes a ball around x with radius r.

Remark 4.1 $T_{\gamma(0)}W^s_{\lambda=0}(p) = U \oplus W^+$. So by definition $U_n \oplus W_n^+ = T_{\gamma(n)}W^s_{\lambda=0}(p)$, $n \geq 1$, cf. [6, Proposition 5.4]. Therefore exists indeed such a mapping $\hat{h}^s_{\lambda,n}$. Moreover, $D\hat{h}^s_{0,n}(0,0) = 0$. □

Similar for $n \leq -1$.

$$\hat{h}^u_{\lambda,n} : U_n \times W_n^- \to W_n^+ \oplus Z$$

such that for small $(u, w^-) \in U_n \times W_n^-$ and for small ϵ

$$\gamma^-(0,\lambda)(n) + u + w^- + \hat{h}^u_{\lambda,n} = W^u_\lambda(p) \cap B(\gamma^-(0,\lambda)(n), \epsilon).$$

Further for $n \in \mathbb{Z}^{+(-)} \cap \{-N_{\text{flat}}, \dots, N_{\text{flat}}\}$, $n \neq 0$, we define:

$$\begin{array}{rlcl} H^{+(-)}_{\lambda,n}: & U_n \oplus W_n^+ \oplus W_n^- \oplus Z_n & \to & \mathbb{R}^k \\ & u + w^+ + w^- + z & \mapsto & u + w^+ + w^- + z + \hat{h}^{s(u)}_{\lambda,n}(u, w^{+(-)}), \end{array}$$

$$\begin{array}{rlcl} J^{+(-)}_{\lambda,n}: & U_n \oplus W_n^+ \oplus W_n^- \oplus Z_n & \to & \mathbb{R}^k \\ & u + w^+ + w^- + z & \mapsto & \gamma^{+(-)}(0,\lambda)(n) + u + w^+ + w^- + z \end{array}$$

and finally let

$$\mathcal{T}^{+(-)}_{\lambda,n} : B(\gamma^{+(-)}(0,\lambda)(n), \epsilon) \to \mathbb{R}^k$$

be defined by

$$\mathcal{T}^{+(-)}_{\lambda,n} := J^{+(-)}_{\lambda,n} \circ H^{+(-)}_{\lambda,n} \circ (J^{+(-)}_{\lambda,n})^{-1}.$$

Remark 4.2 Due to $D\hat{h}^{s(u)}_{0,n}(0,0) = 0$ $H^{+(-)}_{\lambda,n}$ is a diffeomorphism of a neighborhood of the origin in $\mathbb{R}^k$ onto a neighborhood of the origin in $\mathbb{R}^k$. With it $\mathcal{T}^{+(-)}_{\lambda,n}$ is indeed a (local) transformation. □

By construction we have for sufficiently small ϵ

$$\mathcal{T}^{+(-)}_{\lambda,n}\left(\gamma^{+(-)}(0,\lambda)(n) + T_{\gamma^{+(-)}(0,\lambda)(n)}W^{s(u)}_{\lambda}(p)\right) = W^{s(u)}_{\lambda}(p) \cap B(\gamma^{+(-)}(0,\lambda)(n),\epsilon). \tag{4.3}$$

Similar to the line of action for globalizing $\mathcal{T}$ to $\hat{\mathcal{T}}$ in [5] we can globalize $\mathcal{T}^{+(-)}_{\lambda,n}$ to $\hat{\mathcal{T}}^{+(-)}_{\lambda,n}$. This can be done in a way such that the individual transformations do not influence each other. Now we define (using f_λ for $f(\cdot,\lambda)$)

$$\tilde{f}_\lambda := (\hat{\mathcal{T}}^{+}_{\lambda,N_{\text{flat}}} \circ \hat{\mathcal{T}}^{-}_{\lambda,-N_{\text{flat}}} \circ \ldots \circ \hat{\mathcal{T}}^{+}_{\lambda,1} \circ \hat{\mathcal{T}}^{-}_{\lambda,-1})^{-1} \circ f_\lambda \circ (\hat{\mathcal{T}}^{+}_{\lambda,N_{\text{flat}}} \circ \ldots \circ \hat{\mathcal{T}}^{-}_{\lambda,-1})$$

and afterwards rename $\tilde{f}$ to f. After this transformation locally around $\gamma^{+(-)}(0,\lambda)(n)$, $n \neq 0$, the stable (unstable) manifold $W^{s(u)}_{\lambda}(p)$ coincides with $\gamma^{+(-)}(0,\lambda)(n) + T_{\gamma^{+(-)}(0,\lambda)(n)}W^{s(u)}_{\lambda}(p)$. Hence, for sufficiently small u $\gamma^{+(-)}(u,\lambda)(n)$ lies in such a flat area of the stable (unstable) manifold. Altogether we have for sufficiently small u, λ and ϵ and for all $n \in \mathbb{Z}$, $n \neq 0$

$$W^{s(u)}_{\lambda}(p) \cap B(\gamma^{+(-)}(u,\lambda)(n),\epsilon) \subset \gamma^{+(-)}(u,\lambda)(n) + \operatorname{im} P^{+(-)}(u,\lambda,n). \tag{4.4}$$

5 Estimates of the jump ξ_i

In this section we will deduce estimates for ξ_i and finally prove Theorem 3.1. We start with giving an appropriate representation of $\xi_i(\mathcal{N},\mathbf{u},\lambda)$ which permits the estimates of ξ_i. For that we decompose $\bar{v}^+_i(\mathcal{N},\mathbf{u},\lambda)(0)$ and $\bar{v}^-_i(\mathcal{N},\mathbf{u},\lambda)(0)$ by means of $P^+(u_i,\lambda,0)$ and $P^-(u_i,\lambda,0)$, respectively. Taking into consideration [5, Equation (4.25)], this is nothing else but the decomposition of $\bar{v}^{+(-)}_i(\mathcal{N},\mathbf{u},\lambda)(0)$ in its components of $W^{+(-)}$ and $(W^{-(+)} \oplus Z)$. Now, let $\langle\cdot,\cdot\rangle$ be any scalar product in $\mathbb{R}^k$ such that the direct sum decomposition (2.1) is an orthogonal one. Then ξ_i can be written as

$$\xi_i(\mathcal{N},\mathbf{u},\lambda) = \sum_{m=1}^{\dim Z} \langle z^m, \xi_i(\mathcal{N},\mathbf{u},\lambda)\rangle z^m, \tag{5.1}$$

where $Z = \operatorname{lin}\{z^m, m = 1,\ldots,\dim Z\}$ and $\{z^m, m = 1,\ldots,\dim Z\}$ is an orthonormal system. The above considerations show

$$\begin{aligned}\langle z^m, \xi_i(\mathcal{N},\mathbf{u},\lambda)\rangle &= \langle z^m, (id - P^+(u_i,\lambda,0))\bar{v}^+_i(\mathcal{N},\mathbf{u},\lambda)(0)\rangle \\ &\quad -\langle z^m, (id - P^-(u_i,\lambda,0))\bar{v}^-_i(\mathcal{N},\mathbf{u},\lambda)(0)\rangle.\end{aligned} \tag{5.2}$$

Next we derive representations of $(id - P^{+(-)}(u_i,\lambda,0))\bar{v}^{+(-)}_i(\mathcal{N},\mathbf{u},\lambda)(0)$ appropriated for the estimates. We will do this exemplarily for $(id - P^+(u_i,\lambda,0))\bar{v}^+_i(\mathcal{N},\mathbf{u},\lambda)(0)$. For that purpose we start with the representation [5, (4.34)]:

$$\begin{aligned}(id - P^+(u_i,\lambda,0))\bar{v}^+_i(\mathcal{N},\mathbf{u},\lambda)(0) &= \Phi^+(u_i,\lambda,0,N^+_{i+1})a^+_{i+1}(u_i,\lambda,N^+_{i+1}) \\ &\quad - \sum_{j=1}^{N^+_{i+1}} \Phi^+(u_i,\lambda,0,j)(id - P^+(u_i,\lambda,j))h^+(j-1,u_i,\bar{v}^+_i(\mathcal{N},\mathbf{u},\lambda)(j-1),\lambda).\end{aligned} \tag{5.3}$$

In (5.3) we substitute a_{i+1}^+ by using [5, (4.46)]:

$$a_{i+1}^+ - a_{i+1}^- = d_{i+1} + P^-(u_{i+1},\lambda,-N_{i+1}^-)\bar{v}_{i+1}^-(\mathcal{N},\mathbf{u},\lambda)(-N_{i+1}^-) \\ -P^+(u_i,\lambda,N_{i+1}^+)\bar{v}_i^+(\mathcal{N},\mathbf{u},\lambda)(N_{i+1}^+). \quad (5.4)$$

d_{i+1} we write as it is defined - see (2.13). Further we make use of

$$a_{i+1}^+ \in id - P^+(u_i,\lambda,N_{i+1}^+) \quad \text{and} \quad a_{i+1}^- \in id - P^-(u_{i+1},\lambda,-N_{i+1}^-).$$

[5, Lemma 4.4] states that for sufficiently large N_{i+1} and sufficiently small $\mathbf{u}$, λ

$$\mathbb{R}^k = \operatorname{im}(id - P^+(u_i,\lambda,N_{i+1}^+)) \oplus \operatorname{im}(id - P^-(u_{i+1},\lambda,-N_{i+1}^-)). \quad (5.5)$$

Let $\tilde{P}(u_i,u_{i+1},\lambda,N_{i+1})$ be the corresponding projector with range $\operatorname{im}(id - P^+(u_i,\lambda,N_{i+1}^+))$. Then $(id - P^+(u_i,\lambda,0))\bar{v}_i^+(\mathcal{N},\mathbf{u},\lambda)(0)$ can be written as:

$$(id - P^+(u_i,\lambda,0))\bar{v}_i^+(\mathcal{N},\mathbf{u},\lambda)(0) = \Phi^+(u_i,\lambda,0,N_{i+1}^+)(id - P^+(u_i,\lambda,N_{i+1}^+)) \circ \\ \tilde{P}(u_i,u_{i+1},\lambda,N_{i+1})\Big(\gamma^-(u_{i+1},\lambda)(-N_{i+1}^-) - \gamma^+(u_i,\lambda)(N_{i+1}^+) \\ +P^-(u_{i+1},\lambda,-N_{i+1}^-)\bar{v}_{i+1}^-(\mathcal{N},\mathbf{u},\lambda)(-N_{i+1}^-) \\ -P^+(u_i,\lambda,N_{i+1}^+)\bar{v}_i^+(\mathcal{N},\mathbf{u},\lambda)(N_{i+1}^+)\Big) \\ -\sum_{j=1}^{N_{i+1}^+}\Phi^+(u_i,\lambda,0,j)(id - P^+(u_i,\lambda,j))h^+(j-1,u_i,\bar{v}_i^+(\mathcal{N},\mathbf{u},\lambda)(j-1),\lambda). \quad (5.6)$$

In a similar way we get an expression for $(id - P^-(u_i,\lambda,0))\bar{v}_i^-(\mathcal{N},\mathbf{u},\lambda)(0)$. Plugging this into the scalar product (5.2) and thereby taking into consideration $\Phi^{+(-)}(\ldots,0,n)P^{+(-)}(\ldots,n) = P^{+(-)}(\ldots,0)\Phi^{+(-)}(\ldots,0,n)$ we obtain

$$\langle z^m, \xi_i(\mathcal{N},\mathbf{u},\lambda)\rangle = \\ \Big\langle \Phi^+(u_i,\lambda,0,N_{i+1}^+)^T(id - P^+(u_i,\lambda,0))^T z^m, \\ \tilde{P}(u_i,u_{i+1},\lambda,N_{i+1})\Big(\gamma^-(u_{i+1},\lambda)(-N_{i+1}^-) - \gamma^+(u_i,\lambda)(N_{i+1}^+) \\ +P^-(u_{i+1},\lambda,-N_{i+1}^-)\bar{v}_{i+1}^-(\mathcal{N},\mathbf{u},\lambda)(-N_{i+1}^-) \\ -P^+(u_i,\lambda,N_{i+1}^+)\bar{v}_i^+(\mathcal{N},\mathbf{u},\lambda)(N_{i+1}^+)\Big)\Big\rangle \\ -\Big\langle z^m, \sum_{j=1}^{N_{i+1}^+}\Phi^+(u_i,\lambda,0,j)(id - P^+(u_i,\lambda,j))h^+(j-1,u_i,\bar{v}_i^+(\mathcal{N},\mathbf{u},\lambda)(j-1),\lambda)\Big\rangle \\ +\Big\langle \Phi^-(u_i,\lambda,0,-N_i^-)^T(id - P^-(u_i,\lambda,0))^T z^m, \\ (id - \tilde{P}(u_{i-1},u_i,\lambda,N_i))\Big(\gamma^+(u_{i-1},\lambda)(N_i^+) - \gamma^-(u_i,\lambda)(-N_i^-) \\ +P^+(u_{i-1},\lambda,N_i^+)\bar{v}_{i-1}^+(\mathcal{N},\mathbf{u},\lambda)(N_i^+) \\ -P^-(u_i,\lambda,-N_i^-)\bar{v}_i^-(\mathcal{N},\mathbf{u},\lambda)(-N_i^-)\Big)\Big\rangle \\ -\Big\langle z^m, \sum_{j=0}^{-N_i^-+1}\Phi^-(u_i,\lambda,0,j)(id - P^-(u_i,\lambda,j))h^-(j-1,u_i,\bar{v}_i^-(\mathcal{N},\mathbf{u},\lambda)(j-1),\lambda)\Big\rangle. \quad (5.7)$$

By A^T we denote the adjoint of A with respect to $\langle\cdot,\cdot\rangle$.
Next we will estimate the individual terms in (5.7). We start with the

Estimate of $(\Phi^{+(-)})^T(id - P^{+(-)})^T z^m$

We will consider exemplarily $(\Phi^+(u_i,\lambda,0,N^+_{i+1}))^T(id - P^+(u_i,\lambda,0))^T z^m$:

Lemma 5.1 *Let $\Phi(\cdot,\cdot)$ be the transition matrix of $x(n+1) = A(n)x(n)$. Then $\Psi(n,m) := \Phi(m,n)^T = (\Phi(n,m)^{-1})^T$ is the transition matrix of the formally adjoint equation $y(n+1) = (A(n)^{-1})^T y(n)$.*

Proof

$$\Phi(n,m) = \begin{cases} A(n-1)\cdot\ldots\cdot A(m) & , \quad n > m \\ id & , \quad n = m \\ A(n)^{-1}\cdot\ldots\cdot A(m-1)^{-1} & , \quad n < m. \end{cases}$$

Therefore

$$\Psi(n,m) = \begin{cases} (A(n-1)^{-1})^T\cdot\ldots\cdot(A(m)^{-1})^T & , \quad n > m \\ id & , \quad n = m \\ A(n)^T\cdot\ldots\cdot A(m-1)^T & , \quad n < m. \end{cases}$$

Hence $\Psi(n+1,m) = (A(n)^{-1})^T\Psi(n,m)$. ■

Corollary 5.2 *$\Psi^+(u_i,\lambda,n,m) := \Phi^+(u_i,\lambda,m,n)^T$ is the transition matrix of*

$$w^+(n+1) = (D_1 f(\gamma^+(u_i,\lambda)(n),\lambda)^{-1})^T w^+(n). \tag{5.8}$$

See also (2.11). ■

Therefore $\Phi^+(u_i,\lambda,0,n)^T(id - P^+(u_i,\lambda,0))^T z^m$ solves (5.8). Equation (5.8) has an exponential dichotomy. This is true because of

$$\sigma(D_1 f(p,\lambda)^{-1})^T = \left\{\frac{1}{\mu} : \mu \in \sigma(D_1 f(p,\lambda))\right\}. \tag{5.9}$$

Moreover we have

$$\{w : \sup_{n\in\mathbb{N}} \|\Psi^+(u_i,\lambda,n,0)w\| < \infty\} = (T_{\gamma^+(u_i,\lambda)(0)} W^s_\lambda(p))^\perp. \tag{5.10}$$

The proof of this runs completely parallel to that of Lemma 1.6 in [8]. Furthermore - see (2.12) it holds true

$$(T_{\gamma^+(u_i,\lambda)(0)} W^s_\lambda(p))^\perp = \operatorname{im}(id - P^+(u_i,\lambda,0))^T. \tag{5.11}$$

Let $\mu^s(\lambda)$ and $\mu^u(\lambda)$ again be the principal stable and unstable eigenvalues of $D_1 f(p,\lambda)$, respectively. According to Lemma 6.6 we will assume that

(EV) the eigenvalues $\mu^s(\lambda)$ and $\mu^u(\lambda)$ are simple (but possibly complex).

Besides we use the notation introduced in (3.1). (5.9) provides that $(\mu^u(\lambda))^{-1}$ is the principal stable eigenvalue of $(D_1 f(p,\lambda)^{-1})^T$. Now applying Lemma 6.6 to (5.8) yields:

$$w^+(n) = ((D_1 f(p,\lambda)^{-1})^T)^n \eta^+(\lambda, w^+(0)) + O((\max\{(\alpha^{uu})^{-1}, (\alpha^u)^{-2}\})^n). \tag{5.12}$$

$\eta^+(\lambda, w^+(0))$ is defined as in (6.10). The quantity β we used in Lemma 6.6 may be replaced by $(\alpha^u)^{-1}$ in the case under consideration. This becomes clear by invoking Lemma 6.2. Especially for $w^+(n) = \Phi^+(u_i,\lambda,0,n)^T(id - P^+(u_i,\lambda,0))^T z^m$ (5.12) reads for $n = N^+_{i+1}$

$$\begin{aligned}&\Phi^+(u_i,\lambda,0,N^+_{i+1})^T(id - P^+(u_i,\lambda,0))^T z^m \\ &= ((D_1 f(p,\lambda)^{-1})^T)^{N^+_{i+1}} \eta^+(\lambda, (id - P^+(u_i,\lambda,0))^T z^m) \\ &\quad + O((\max\{(\alpha^{uu})^{-1}, (\alpha^u)^{-2}\})^{N^+_{i+1}}).\end{aligned} \tag{5.13}$$

Completely analogously we obtain

$$\begin{aligned}&\Phi^-(u_i,\lambda,0,-N^-_i)^T(id - P^-(u_i,\lambda,0))^T z^m \\ &= ((D_1 f(p,\lambda)^{-1})^T)^{-N^-_i} \eta^-(\lambda, (id - P^-(u_i,\lambda,0))^T z^m) \\ &\quad + O((\max\{(\alpha^{ss}), (\alpha^s)^2\})^{N^-_i}).\end{aligned} \tag{5.14}$$

Now we proceed in each case to estimate the terms in the second component of $\langle\cdot,\cdot\rangle$ in (5.7).

Estimate of $\tilde{\mathbf{P}}(\gamma^- + \gamma^+)$

Assuming the eigenvalue condition (EV) Lemma 6.2 provides that there are eigenvectors (generalized ones - in the case of complex eigenvalues) $v^s(u,\lambda)$ and $v^u(u,\lambda)$ of $\mu^s(\lambda)$ and $\mu^u(\lambda)$ such that

$$\begin{aligned}&\gamma^+(u,\lambda)(n) = D_1 f(p,\lambda)^n \eta^s(u,\lambda) + O((\max\{\alpha^{ss}, (\alpha^s)^2\})^n) \qquad \text{and} \\ &\gamma^-(u,\lambda)(-n) = D_1 f(p,\lambda)^{-n} \eta^u(u,\lambda) + O((\max\{(\alpha^{uu})^{-1}, (\alpha^u)^{-2}\})^n),\end{aligned} \tag{5.15}$$

respectively.

Remark 5.3 If $\mu^{s(u)}(\lambda)$ are complex eigenvalues and $\tilde{\eta}^{s(u)}(u,\lambda)$ corresponding eigenvectors then $\eta^{s(u)}(u,\lambda) \in \mathrm{lin}\{\Re\,\tilde{\eta}^{s(u)}(u,\lambda), \Im\,\tilde{\eta}^{s(u)}(u,\lambda)\}$. □

Remark 5.4 If $\mu^{s(u)}(\lambda)$ are real (and still simple) then

$$D_1 f(p,\lambda)^n \eta^s(u,\lambda) = (\mu^s(\lambda))^n \eta^s(u,\lambda) \quad \text{and} \quad D_1 f(p,\lambda)^{-n} \eta^u(u,\lambda) = (\mu^u(\lambda))^{-n} \eta^u(u,\lambda).$$

With that (5.15) reads

$$\begin{aligned}&\gamma^+(u,\lambda)(n) = (\mu^s(\lambda))^n \eta^s(u,\lambda) + O((\max\{\alpha^{ss}, (\alpha^s)^2\})^n) \qquad \text{and} \\ &\gamma^-(u,\lambda)(-n) = (\mu^u(\lambda))^{-n} \eta^u(u,\lambda) + O((\max\{(\alpha^{uu})^{-1}, (\alpha^u)^{-2}\})^n)\end{aligned} \tag{5.16}$$

as n tends to infinity. Together with Lemma 6.18 and Lemma 6.19 and taking into consideration (MAN) we get finally

$$\begin{aligned}&\tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1})\gamma^-(u_{i+1},\lambda)(-N^-_{i+1}) = (\mu^u(\lambda))^{-N^-_{i+1}} \eta^u(u_{i+1},\lambda) \\ &\quad + O((\max\{\alpha^s, (\alpha^u)^{-1}\}(\mu^u(\lambda))^{-1})^{N^-_{i+1}}) + O((\max\{(\alpha^{uu})^{-1}, (\alpha^u)^{-2}\})^{N^-_{i+1}})\end{aligned} \tag{5.17}$$

and

$$\tilde{P}(u_i,u_{i+1},\lambda,N_{i+1})\gamma^+(u_i,\lambda)(N_{i+1}^+) = O((\max\{\alpha^s,(\alpha^u)^{-1}\}\mu^s(\lambda))^{N_{i+1}^+}) \\ + O((\max\{\alpha^s,(\alpha^u)^{-1}\}\max\{\alpha^{ss},(\alpha^s)^2\})^{N_{i+1}^+}). \tag{5.18}$$

Similar we get

$$(id-\tilde{P}(u_{i-1},u_i,\lambda,N_i))\gamma^+(u_{i-1},\lambda)(N_i^+) = (\mu^s(\lambda))^{N_i^+}\eta^s(u_{i-1},\lambda) \\ + O((\max\{\alpha^s,(\alpha^u)^{-1}\}\mu^s(\lambda))^{N_i^+}) + O((\max\{\alpha^{ss},(\alpha^s)^2\})^{N_i^+}) \tag{5.19}$$

and

$$(id-\tilde{P}(u_{i-1},u_i,\lambda,N_i))\gamma^-(u_i,\lambda)(-N_i^-) = O((\max\{\alpha^s,(\alpha^u)^{-1}\}(\mu^u(\lambda))^{-1})^{N_i^-}) \\ + O((\max\{\alpha^s,(\alpha^u)^{-1}\}\max\{(\alpha^{uu})^{-1},(\alpha^u)^{-2}\})^{N_i^-}). \tag{5.20}$$

In each case is $O(\dots)$ as N_i and N_{i+1}, respectively, tend to infinity. □

Estimate of $\tilde{\mathbf{P}}(\mathbf{P}^-\bar{\mathbf{v}}_{i+1}^- + \mathbf{P}^+\bar{\mathbf{v}}_i^+)$

Now we take up the estimate of $P^-(u_{i+1},\lambda,-N_{i+1}^-)\bar{v}_{i+1}^-(\mathcal{N},\mathbf{u},\lambda)(-N_{i+1}^-)$ $-P^+(u_i,\lambda,N_{i+1}^+)$ $\bar{v}_i^+(\mathcal{N},\mathbf{u},\lambda)(N_{i+1}^+)$ - the next term in (5.7): Let $\hat{H}_i^+(u_i,\bar{v}_i^+,\lambda)(\cdot)\in S_{N_{i+1}^+}$ and $\hat{H}_i^-(u_i,\bar{v}_i^-,\lambda)(\cdot)\in S_{-N_i^-}$ be defined by $\hat{H}_i^{+(-)}(u,v,\lambda)(n) := h^{+(-)}(n,u,v(n),\lambda)$. Then $\hat{\mathcal{H}} := ((\hat{h}_i^+,\hat{h}_i^-))_{i\in\mathbb{Z}}\in\mathcal{V}$. According to [5, (4.32)] we have

$$\|P^-(u_{i+1},\lambda,-N_{i+1}^-)\bar{v}_{i+1}^-(\mathcal{N},\mathbf{u},\lambda)(-N_{i+1}^-) - P^+(u_i,\lambda,N_{i+1}^+)\bar{v}_i^+(\mathcal{N},\mathbf{u},\lambda)(N_{i+1}^+)\|_{\mathbb{R}^k} \\ \le C_1e^{-2\alpha\bar{N}}\|\hat{\mathbf{a}}_{\mathcal{N}}(\mathbf{u},\lambda,\hat{\mathcal{H}},\mathbf{d}(\mathcal{N},\mathbf{u},\lambda))\|_{l^\infty_{\mathbb{R}^k}} + C_2\|\hat{\mathcal{H}}(\mathbf{u},\bar{\mathbf{v}},\lambda)\|_{\mathcal{V}}. \tag{5.21}$$

$2\bar{N}$ is a lower bound of $\{N_i, i\in\mathbb{Z}\}$. Looking carefully at the proof of this estimate we see that (5.21) may be specified to

$$\|P^-(u_{i+1},\lambda,-N_{i+1}^-)\bar{v}_{i+1}^-(\mathcal{N},\mathbf{u},\lambda)(-N_{i+1}^-) - P^+(u_i,\lambda,N_{i+1}^+)\bar{v}_i^+(\mathcal{N},\mathbf{u},\lambda)(N_{i+1}^+)\|_{\mathbb{R}^k} \\ \le C_1e^{2\max\{\ln\alpha^s,-\ln\alpha^u\}N_{i+1}^+} + C_2(\|\hat{H}_i^+(u_i,\bar{v}_i^+,\lambda)\|_{S_{N_{i+1}^+}} + \|\hat{H}_{i+1}^-(u_{i+1},\bar{v}_{i+1}^-,\lambda)\|_{S_{-N_{i+1}^-}}). \tag{5.22}$$

From the definition we see $\hat{H}_i^+(u_i,0,\lambda)=0$ and $D_2\hat{H}_i^+(u_i,0,\lambda)=0$, see also [5, Lemma 4.10] for the latter equality. Hence we have uniformly in u, λ

$$\hat{H}_i^+(u_i,\bar{v}_i^+,\lambda) = O(\|\bar{v}_i^+\|^2). \tag{5.23a}$$

Analogously we satisfy as to

$$\hat{H}_{i+1}^-(u_{i+1},\bar{v}_{i+1}^-,\lambda) = O(\|\bar{v}_{i+1}^-\|^2). \tag{5.23b}$$

Next we give appropriate estimates for $\bar{v}_i^{+(-)}$. To it we benefit that $\bar{\mathbf{v}}(\mathbf{u},\lambda)$ solves the fixed point problem (2.13). According to [5, (4.43)] we have uniformly in $\mathbf{u}$ and λ

$$\|\mathbf{v}_{\mathcal{N}}(\mathbf{u},\lambda,\hat{\mathcal{H}},\mathbf{d})\| \le C(\|\hat{\mathcal{H}}\|_{\mathcal{V}} + \|\mathbf{d}\|_{l^\infty_{\mathbb{R}^k}}). \tag{5.24}$$

Again going carefully through the proof given in [5] we see that in particular

$$\begin{aligned}\|(\bar{v}_i^+, \bar{v}_{i+1}^-)\|_{S_{N_{i+1}^+} \times S_{-N_{i+1}^-}} &\leq C(\|(\hat{H}_i^+(u_i, \bar{v}_i^+, \lambda), \hat{H}_{i+1}^-(u_{i+1}, \bar{v}_{i+1}^-, \lambda))\|_{S_{N_{i+1}^+} \times S_{-N_{i+1}^-}} \\ &\quad + \|d_{i+1}(\mathcal{N}, \mathbf{u}, \lambda)\|_{\mathbb{R}^k}). \end{aligned} \tag{5.25}$$

Due to (5.23) there is a $\epsilon > 0$ such that for $\|\bar{\mathbf{v}}\| < \epsilon$

$$C\|(\hat{H}_i^+(u_i, \bar{v}_i^+, \lambda), \hat{H}_{i+1}^-(u_{i+1}, \bar{v}_{i+1}^-, \lambda))\|_{S_{N_{i+1}^+} \times S_{-N_{i+1}^-}} < \tfrac{1}{2}\|(\bar{v}_i^+, \bar{v}_{i+1}^-)\|_{S_{N_{i+1}^+} \times S_{-N_{i+1}^-}}. \tag{5.26}$$

Together with (5.25) this shows that there is a constant C such that

$$\|(\bar{v}_i^+, \bar{v}_{i+1}^-)\|_{S_{N_{i+1}^+} \times S_{-N_{i+1}^-}} \leq C\|d_{i+1}(\mathcal{N}, \mathbf{u}, \lambda)\|_{\mathbb{R}^k}. \tag{5.27}$$

By the definition of d_i - see (2.14) - and assuming the eigenvalue condition (EV) the representation (5.15) provides

$$(\bar{v}_i^+, \bar{v}_{i+1}^-) = O((\max\{\alpha^s, (\alpha^u)^{-1}\})^{N_{i+1}^+}). \tag{5.28}$$

Here we took into consideration (2.3). Hence by (5.23)

$$(\hat{H}_i^+(u_i, \bar{v}_i^+, \lambda), \hat{H}_{i+1}^-(u_{i+1}, \bar{v}_{i+1}^-, \lambda)) = O((\max\{\alpha^s, (\alpha^u)^{-1}\})^{2N_{i+1}^+}). \tag{5.29}$$

Together with (5.22) this yields

$$\begin{aligned}&\|P^-(u_{i+1}, \lambda, -N_{i+1}^-)\bar{v}_{i+1}^-(\mathcal{N}, \mathbf{u}, \lambda)(-N_{i+1}^-) - P^+(u_i, \lambda, N_{i+1}^+)\bar{v}_i^+(\mathcal{N}, \mathbf{u}, \lambda)(N_{i+1}^+)\| \\ &\leq C(\max\{\alpha^s, (\alpha^u)^{-1}\})^{2N_{i+1}^+}.\end{aligned} \tag{5.30}$$

Moreover, the corresponding proofs in [5] tell that there are constants $\bar{N}$ and c such that (5.30) remains true (with the same constant C) for all $\|\mathbf{u}\|, |\lambda| < c$ and all $\mathcal{N}$ with $N_i > 2\bar{N}$, $i \in \mathbb{Z}$. Cf. in particular [5, Lemma 4.6, Lemma 4.8 and Lemma 4.13]. (5.30) together with Lemma 6.17 provides

$$\begin{aligned}&\tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1})[P^-(u_{i+1}, \lambda, -N_{i+1}^-)\bar{v}_{i+1}^-(\mathcal{N}, \mathbf{u}, \lambda)(-N_{i+1}^-) \\ &\qquad - P^+(u_i, \lambda, N_{i+1}^+)\bar{v}_i^+(\mathcal{N}, \mathbf{u}, \lambda)(N_{i+1}^+)] = O((\max\{\alpha^s, (\alpha^u)^{-1}\})^{2N_{i+1}^+})\end{aligned} \tag{5.31a}$$

and

$$\begin{aligned}&(id - \tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1}))[P^+(u_{i-1}, \lambda, N_i^+)\bar{v}_{i-1}^+(\mathcal{N}, \mathbf{u}, \lambda)(N_i^+) \\ &\qquad - P^-(u_i, \lambda, -N_i^-)\bar{v}_i^-(\mathcal{N}, \mathbf{u}, \lambda)(-N_i^-)] = O((\max\{\alpha^s, (\alpha^u)^{-1}\})^{2N_i^-}).\end{aligned} \tag{5.31b}$$

Remark 5.5 (5.28) provides that $\xi_i(\mathcal{N}, \mathbf{u}, \lambda) = O((\max\{\alpha^s, (\alpha^u)^{-1}\})^{N_{i+1}^+})$. However, this estimate is unsuitable for solving the bifurcation equation because it contains no information regarding the sign of ξ_i. Our effort in giving estimates of ξ_i is devoted just this subject. □

A first summary

Before dealing with the estimates of the remaining terms we will briefly sum up our results up to this point. However we will do this by assuming the requirements (EV_R), (HOM_A), (DIM_U) and (SUB) we stated in Section 3. Linear algebra teaches

$$\begin{aligned} &\langle \eta^+((id - P^+(0,0,0))^T z^1, 0), \eta^u(0,0)\rangle \neq 0, \\ &\langle \eta^-((id - P^-(0,0,0))^T z^1, 0), \eta^s(0,0)\rangle \neq 0. \end{aligned} \tag{5.32}$$

Mind that $\eta^+((id - P^+(u_i, \lambda, 0))^T z^1, \lambda)$ and $\eta^-((id - P^-(u_i, \lambda, 0))^T z^1, \lambda)$ are (due to (EV_R)) eigenvectors of $(D_1 f(p,\lambda)^{-1})^T$ to the eigenvalues $\mu^u(\lambda)^{-1}$ and $\mu^s(\lambda)^{-1}$, respectively. As well $\eta^u(u,\lambda)$ and $\eta^s(u,\lambda)$ are eigenvectors of $D_1 f(p,\lambda)$ to the eigenvalues $\mu^u(\lambda)$ and $\mu^s(\lambda)$, respectively.
Now, taking into consideration (5.32), our previous estimates provide

$$\begin{aligned} \Big\langle &\Phi^+(u_i, \lambda, 0, N^+_{i+1})^T (id - P^+(u_i,\lambda,0))^T z^1, \\ &\tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1}) \Big(\gamma^-(u_{i+1},\lambda)(-N^-_{i+1}) - \gamma^+(u_i,\lambda)(N^+_{i+1}) \\ &\qquad + P^-(u_{i+1}, \lambda, -N^-_{i+1}))\bar{v}^-_{i+1}(\mathcal{N}, \mathbf{u}, \lambda)(-N^-_{i+1}) \\ &\qquad - P^+(u_i, \lambda, N^+_{i+1}))\bar{v}^+_i(\mathcal{N}, \mathbf{u}, \lambda)(N^+_{i+1}) \Big) \Big\rangle \\ = {} & c^u(u_i, u_{i+1}, \lambda)(\mu^u(\lambda))^{-N_{i+1}} + o(|(\mu^u(\lambda))^{-N_{i+1}}|), \end{aligned} \tag{5.33}$$

where $c^u(u_i, u_{i+1}, \lambda) := \langle \eta^+((id - P^+(u_i,\lambda,0))^T z^1, \lambda), \eta^u(u_{i+1},\lambda)\rangle$ and according to (5.32) $c^u(0,0,0) \neq 0$. And as well we get

$$\begin{aligned} \Big\langle &\Phi^-(u_i, \lambda, 0, -N^-_i)^T (id - P^-(u_i,\lambda,0))^T z^1, \\ &(id - \tilde{P}(u_{i-1}, u_i, \lambda, N_i)) \Big(\gamma^+(u_{i-1},\lambda)(N^+_i) - \gamma^-(u_i,\lambda)(-N^-_i) \\ &\qquad + P^+(u_{i-1}, \lambda, N^+_i))\bar{v}^+_{i-1}(\mathcal{N}, \mathbf{u}, \lambda)(N^+_i) \\ &\qquad - P^-(u_i, \lambda, -N^-_i))\bar{v}^-_i(\mathcal{N}, \mathbf{u}, \lambda)(-N^-_i) \Big) \Big\rangle \\ = {} & c^s(u_{i-1}, u_i, \lambda)(\mu^s(\lambda))^{N_i} + o(|(\mu^s(\lambda))^{N_i}|), \end{aligned} \tag{5.34}$$

where c^s is defined similar to c^u. Again we have $c^s(0,0,0) \neq 0$.
Comparing the results presented in (5.33) and (5.34) with representation (5.7) of the jump ξ_i we see that the proof of Theorem 3.1 is done if also the remaining terms $\sum \Phi^\pm(id - P^\pm)h^\pm$ are small o quantities with respect to $(\mu^s(\lambda))^{N_i}$ and $(\mu^u(\lambda))^{-N_{i+1}}$, respectively.

Remark 5.6 Also if we replace the eigenvalue condition (EV_R) by the somewhat weaker condition (EV) - so we admit the principal eigenvalues to be complex - still $\langle \Phi^+(\dots)^T(id - P^+(\dots)^T z^1, \tilde{P}(\dots)\gamma^-(\dots)\rangle$ and $\langle \Phi^-(\dots)^T(id - P^-(\dots)^T z^1, (id - \tilde{P}(\dots))\gamma^+(\dots)\rangle$ are the leading terms of ξ_i. □

Estimate of $\sum \Phi^+(id - P^+)h^+$

Now we start to estimate the remaining terms in (5.7). We will do the estimate exemplarily for $\sum_{j=1}^{N^+_{i+1}} \Phi^+(u_i, \lambda, 0, j)(id - P^+(u_i, \lambda, j))h^+(j-1, u_i, \bar{v}^+_i(\mathcal{N}, \mathbf{u}, \lambda)(j-1), \lambda)$. For it we define the stable and unstable part $v_i^{+,s}$ and $v_i^{+,u}$ of v_i^+

$$v_i^{+,u}(j) := (id - P^+(u_i,\lambda,j))v_i^+(j) \quad \text{and} \quad v_i^{+,s}(j) := P^+(u_i,\lambda,j)v_i^+(j), \tag{5.35}$$

respectively. So $v_i^+ = v_i^{+,s} + v_i^{+,u}$ and $v_i^{+,s},\ v_i^{+,u} \in S_{N_{i+1}^+}$. Now (2.9) provides

$$\begin{aligned}
&(id - P^+(u_i,\lambda,j))h^+(j-1,u_i,v_i^+(j-1),\lambda)\\
&= (id - P^+(u_i,\lambda,j))[f(\gamma^+(u_i,\lambda)(j-1) + v_i^{+,s}(j-1) + v_i^{+,u}(j-1),\lambda)\\
&\qquad - f(\gamma^{+(-)}(u_i,\lambda)(j-1),\lambda)\\
&\qquad - D_1 f(\gamma^{+(-)}(u_i,\lambda)(j-1),\lambda)(v_i^{+,s}(j-1) + v_i^{+,u}(j-1))].
\end{aligned} \tag{5.36}$$

For fixed j the mean value theorem provides

$$f(\gamma^+(u,\lambda)(j-1) + v,\lambda) - f(\gamma^{+(-)}(u,\lambda)(j-1),\lambda) = (\int_0^1 D_1 f(\gamma^+(u,\lambda)(j-1) + \tau v,\lambda)d\tau)v.$$

Plugging this into (5.36) and splitting v_i^+ into its stable and unstable part according to (5.35) we get

$$\begin{aligned}
&(id - P^+(u_i,\lambda,j))h^+(j-1,u_i,v_i^{+,s}(j-1) + v_i^{+,u}(j-1),\lambda)\\
&= (id - P^+(u_i,\lambda,j))[\int_0^1 D_1 f(\gamma^+(u_i,\lambda)(j-1) + \tau(v_i^{+,s}(j-1) + v_i^{+,u}(j-1)),\lambda)\\
&\qquad - D_1 f(\gamma^+(u_i,\lambda)(j-1),\lambda)d\tau](v_i^{+,s}(j-1) + v_i^{+,u}(j-1))\\
&= [\int_0^1 \tfrac{\partial}{\partial v^s}(id - P^+(u_i,\lambda,j))f(\gamma^+(u_i,\lambda)(j-1) + \tau(v_i^{+,s}(j-1) + v_i^{+,u}(j-1)),\lambda)\\
&\quad - \tfrac{\partial}{\partial v^s}(id - P^+(u_i,\lambda,j))f(\gamma^+(u_i,\lambda)(j-1),\lambda)d\tau]v_i^{+,s}(j-1)\\
&+[\int_0^1 \tfrac{\partial}{\partial v^u}(id - P^+(u_i,\lambda,j))f(\gamma^+(u_i,\lambda)(j-1) + \tau(v_i^{+,s}(j-1) + v_i^{+,u}(j-1)),\lambda)\\
&\quad - \tfrac{\partial}{\partial v^u}(id - P^+(u_i,\lambda,j))f(\gamma^+(u_i,\lambda)(j-1),\lambda)d\tau]v_i^{+,u}(j-1).
\end{aligned} \tag{5.37}$$

Applying again the mean value theorem (on the last item in (5.37)) yields

$$\begin{aligned}
&(id - P^+(u_i,\lambda,j))h^+(j-1,u_i,v_i^{+,s}(j-1) + v_i^{+,u}(j-1),\lambda)\\
&= [\int_0^1\int_0^1 \tfrac{\partial^2}{\partial(v^s)^2}(id - P^+(u_i,\lambda,j))f(\gamma^+(u_i,\lambda)(j-1)\\
&\qquad + \tau_1\tau_2(v_i^{+,s}(j-1) + v_i^{+,u}(j-1)),\lambda)d\tau_1 d\tau_2](v_i^{+,s}(j-1), v_i^{+,s}(j-1))\\
&+2[\int_0^1\int_0^1 \tfrac{\partial^2}{\partial v^s \partial v^u}(id - P^+(u_i,\lambda,j))f(\gamma^+(u_i,\lambda)(j-1)\\
&\qquad + \tau_1\tau_2(v_i^{+,s}(j-1) + v_i^{+,u}(j-1)),\lambda)d\tau_1 d\tau_2](v_i^{+,s}(j-1), v_i^{+,u}(j-1))\\
&+[\int_0^1\int_0^1 \tfrac{\partial^2}{\partial(v^u)^2}(id - P^+(u_i,\lambda,j))f(\gamma^+(u_i,\lambda)(j-1)\\
&\qquad + \tau_1\tau_2(v_i^{+,s}(j-1) + v_i^{+,u}(j-1)),\lambda)d\tau_1 d\tau_2](v_i^{+,u}(j-1), v_i^{+,u}(j-1)).
\end{aligned} \tag{5.38}$$

For further investigation of the first item on the right hand side in (5.38) we need the transformation introduced in Section 4: Let $\|v_i^{+,s}(j-1)\|_{\mathbb{R}^k} < \epsilon$. Then we have $v_i^{+,s}(j-1) \in W_\lambda^s(p) \cap B(\gamma^+(u,\lambda)(j-1),\epsilon)$. For $j > 1$ this is true, since $v_i^{+,s}(j-1) \in \operatorname{im} P^+(u,\lambda,j-1)$ and since after the transformation we performed in Section 4 around $\gamma^+(u,\lambda)(j-1)$ the stable manifold $W^s_\lambda(p)$ coincides with $T_{\gamma^+(u,\lambda)(j-1)}W^s_\lambda(p) = \operatorname{im} P^+(u,\lambda,j-1)$. For $j=1$ we use $v_i^+(0) \in W^+ \oplus W^- \oplus Z$ and the transformation we performed in [5] - see also (4.1). Because $f(\cdot,\lambda)$ leaves $W^s_\lambda(p)$ invariant the same argument provides also

$$(id - P^+(u,\lambda,j)) \underbrace{f(\gamma^+(u,\lambda)(j-1) + v_i^{+,s}(j-1),\lambda)}_{\in W^s_\lambda(p)\cap B(\gamma^+(u,\lambda)(j),\epsilon)=\operatorname{im} P^+(u,\lambda,j)} = 0.$$

Therefore

$$\frac{\partial^2}{\partial(v^s)^2}(id - P^+(u,\lambda,j))f(\gamma^+(u,\lambda)(j-1)+v^s,\lambda) = 0$$

for all $v^s \in W^s_\lambda(p)\cap B(\gamma^+(u,\lambda)(j-1),\epsilon)$. So we can rewrite the first item on the right hand side in (5.38) and close-fitting apply the mean value theorem. This provides

$$\begin{aligned}
&\Big[\int_0^1\int_0^1 \tfrac{\partial^2}{\partial(v^s)^2}(id - P^+(u_i,\lambda,j))f(\gamma^+(u_i,\lambda)(j-1)\\
&\qquad + \tau_1\tau_2(v_i^{+,s}(j-1)+v_i^{+,u}(j-1)),\lambda)d\tau_1 d\tau_2\Big](v_i^{+,s}(j-1),v_i^{+,s}(j-1))\\
&= \Big[\int_0^1\int_0^1 \tfrac{\partial^2}{\partial(v^s)^2}(id - P^+(u_i,\lambda,j))f(\gamma^+(u_i,\lambda)(j-1)\\
&\qquad + \tau_1\tau_2(v_i^{+,s}(j-1)+v_i^{+,u}(j-1)),\lambda)d\tau_1 d\tau_2\\
&\quad - \int_0^1\int_0^1 \tfrac{\partial^2}{\partial(v^s)^2}(id - P^+(u_i,\lambda,j))f(\gamma^+(u_i,\lambda)(j-1)\\
&\qquad + \tau_1\tau_2 v_i^{+,s}(j-1),\lambda)d\tau_1 d\tau_2\Big](v_i^{+,s}(j-1),v_i^{+,s}(j-1))\\
&= \Big[\int_0^1\int_0^1\int_0^1 \tfrac{\partial^3}{\partial(v^s)^2\partial v^u}(id - P^+(u_i,\lambda,j))f(\gamma^+(u_i,\lambda)(j-1)\\
&\qquad + \tau_1\tau_2 v_i^{+,s}(j-1) + \tau_1\tau_2\tau_3 v_i^{+,u}(j-1),\lambda)d\tau_1 d\tau_2 d\tau_3\Big]\\
&\qquad\qquad (v_i^{+,s}(j-1),v_i^{+,s}(j-1),v_i^{+,u}(j-1)).
\end{aligned}\tag{5.39}$$

Exploiting the boundedness of $v_i^{+,s}(\dots)$ and of the partial derivatives of $(id - P^+(\dots,j)f(\dots)$ we get

$$\begin{aligned}
&\|(id - P^+(u_i,\lambda,j))h^+(j-1,u_i,v_i^+(j-1),\lambda)\|_{\mathbb{R}^k}\\
&\qquad \le C\|v_i^{+,u}(j-1)\|_{\mathbb{R}^k}(\|v_i^{+,s}(j-1)\|_{\mathbb{R}^k} + \|v_i^{+,u}(j-1)\|_{\mathbb{R}^k}).
\end{aligned}\tag{5.40}$$

Taking into consideration $(id - P^+)^2 = (id - P^+)$ and exponential dichotomy of Φ^+ we get

$$\begin{aligned}
&\|\sum_{j=1}^{N^+_{i+1}} \Phi^+(u_i,\lambda,0,j)(id - P^+(u_i,\lambda,j))^2 h^+(j-1,u_i,\bar v_i^+(\mathcal{N},\mathbf{u},\lambda)(j-1),\lambda)\|_{\mathbb{R}^k}\\
&\le (\alpha^u)^{-N^+_{i+1}} N^+_{i+1}\Big(\sup_{j\in\{1,\dots,N^+_{i+1}\}} C(\alpha^u)^{N^+_{i+1}-j}\\
&\qquad\qquad \cdot \|(id - P^+(u_i,\lambda,j))h^+(j-1,u_i,\bar v_i^+(\mathcal{N},\mathbf{u},\lambda)(j-1),\lambda)\|_{\mathbb{R}^k}\Big).
\end{aligned}\tag{5.41}$$

Choosing $0 < \delta < 1$ such that $(\delta\alpha^u)^{-1} < 1$ and N_{i+1}^+ as large as $\delta^{N_{i+1}^+} N_{i+1}^+ < 1$ (5.41) together with (5.40) yields

$$\begin{aligned}
&\| \sum_{j=1}^{N_{i+1}^+} \Phi^+(u_i, \lambda, 0, j)(id - P^+(u_i, \lambda, j))^2 h^+(j-1, u_i, \bar{v}_i^+(\mathcal{N}, \mathbf{u}, \lambda)(j-1), \lambda)\|_{\mathbb{R}^k} \\
&\leq (\delta\alpha^u)^{-N_{i+1}^+} C \|\bar{v}_i^+(\mathcal{N}, \mathbf{u}, \lambda)\|_{S_{N_{i+1}^+}} \Big(\sup_{j\in\{1,\dots,N_{i+1}^+\}} (\alpha^u)^{N_{i+1}^+ - j} \|\bar{v}_i^{+,u}(\mathcal{N}, \mathbf{u}, \lambda)(j-1)\|_{\mathbb{R}^k} \Big).
\end{aligned}$$

Now (5.28) provides

$$\begin{aligned}
&\| \sum_{j=1}^{N_{i+1}^+} \Phi^+(u_i, \lambda, 0, j)(id - P^+(u_i, \lambda, j))^2 h^+(j-1, u_i, \bar{v}_i^+(\mathcal{N}, \mathbf{u}, \lambda)(j-1), \lambda)\|_{\mathbb{R}^k} \\
&\leq C(\delta\alpha^u)^{-N_{i+1}^+} (\max\{\alpha^s, (\alpha^u)^{-1}\})^{N_{i+1}^+} \\
&\qquad \cdot \sup_{j\in\{1,\dots,N_{i+1}^+\}} (\alpha^u)^{N_{i+1}^+ - j} \|\bar{v}_i^{+,u}(\mathcal{N}, \mathbf{u}, \lambda)(j-1)\|_{\mathbb{R}^k}.
\end{aligned} \tag{5.42}$$

Finally we will show that $\sup_{j\in\{1,\dots,N_{i+1}^+\}} (\alpha^u)^{N_{i+1}^+ - j} \|\bar{v}_i^{+,u}(\mathcal{N}, \mathbf{u}, \lambda)(j-1)\|_{\mathbb{R}^k}$ can be estimated by $(\max\{\alpha^s, (\alpha^u)^{-1}\})^{-N_{i+1}^+}$:
According to [5, (4.42)] we have

$$\begin{aligned}
&\|\bar{v}_i^{+,u}(\mathcal{N}, \mathbf{u}, \lambda)(j-1)\|_{\mathbb{R}^k} = \|(id - P^+(u_i, \lambda, j-1))\bar{v}_i^+(\mathcal{N}, \mathbf{u}, \lambda)(j-1)\|_{\mathbb{R}^k} \\
&\leq \|\Phi^+(u_i, \lambda, j-1, N_{i+1}^+)(id - P^+(u_i, \lambda, N_{i+1}^+))\| \|a_{i+1}^+(u_i, \lambda, N_{i+1}^+)\| \\
&+ \| \sum_{n=j-1}^{N_{i+1}^+} \Phi^+(u_i, \lambda, j-1, n)(id - P^+(u_i, \lambda, n))^2 h^+(n-1, u_i, \bar{v}_i^+(\mathcal{N}, \mathbf{u}, \lambda)(n-1), \lambda)\|.
\end{aligned}$$

Therefore, using (5.40) and taking into consideration that Φ^+ has an exponential dichotomy we get

$$\begin{aligned}
&(\alpha^u)^{N_{i+1}^+ - j} \|\bar{v}_i^{+,u}(\mathcal{N}, \mathbf{u}, \lambda)(j-1)\|_{\mathbb{R}^k} \\
&\leq (\alpha^u)^{N_{i+1}^+ - j} (\alpha^u)^{-(N_{i+1}^+ - j + 1)} \|a_{i+1}^+(u_i, \lambda, N_{i+1}^+)\| \\
&+ C(\alpha^u)^{N_{i+1}^+ - j} \sum_{n=j-1}^{N_{i+1}^+} (\alpha^u)^{-n+j-1} \|v_i^{+,u}(n-1)\|_{\mathbb{R}^k} (\|v_i^{+,s}(n-1)\|_{\mathbb{R}^k} + \|v_i^{+,u}(n-1)\|_{\mathbb{R}^k}) \\
&\leq \|a_{i+1}^+(u_i, \lambda, N_{i+1}^+)\| + C \sup_{n\in\{j-1,\dots,N_{i+1}^+\}} \Big((\alpha^u)^{N_{i+1}^+ - n} \|\bar{v}_i^{+,u}(\mathcal{N}, \mathbf{u}, \lambda)(n-1)\| \\
&\qquad \cdot (\|\bar{v}_i^{+,s}(\mathcal{N}, \mathbf{u}, \lambda)(n-1)\| + \|\bar{v}_i^{+,u}(\mathcal{N}, \mathbf{u}, \lambda)(n-1)\|) N_{i+1}^+ \Big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sup_{j\in\{0,\dots,N_{i+1}^+\}} (\alpha^u)^{N_{i+1}^+ - j} \|\bar{v}_i^{+,u}(\mathcal{N}, \mathbf{u}, \lambda)(j-1)\| \\
&\leq \|a_{i+1}^+(u_i, \lambda, N_{i+1}^+)\| + C \sup_{j\in\{0,\dots,N_{i+1}^+\}} (\alpha^u)^{N_{i+1}^+ - j} \|\bar{v}_i^{+,u}(\mathcal{N}, \mathbf{u}, \lambda)(j-1)\| \\
&\qquad \cdot \sup_{j\in\{0,\dots,N_{i+1}^+\}} (\|\bar{v}_i^{+,s}(\mathcal{N}, \mathbf{u}, \lambda)(j-1)\| + \|\bar{v}_i^{+,u}(\mathcal{N}, \mathbf{u}, \lambda)(j-1)\|) N_{i+1}^+.
\end{aligned} \tag{5.43}$$

$\sup_{j\in\{0,\dots,N_{i+1}^+\}}(\|\bar{v}_i^{+,s}(j-1)\|+\|\bar{v}_i^{+,u}(j-1)\|)$ can be estimated by $C\|\bar{v}_i^+\|_{S_{N_{i+1}^+}}$ which again tends exponentially fast to zero as N_{i+1}^+ tends to infinity. Therefore for sufficiently large N_{i+1}^+ we have

$$\sup_{j\in\{0,\dots,N_{i+1}^+\}}(\|\bar{v}_i^{+,s}(\mathcal{N},\mathbf{u},\lambda)(j-1)\|+\|\bar{v}_i^{+,u}(\mathcal{N},\mathbf{u},\lambda)(j-1)\|)N_{i+1}^+ < \frac{1}{2C}. \tag{5.44}$$

Together with (5.43) this provides for sufficiently large N_{i+1}^+

$$\sup_{j\in\{0,\dots,N_{i+1}^+\}}(\alpha^u)^{N_{i+1}^+-j}\|\bar{v}_i^{+,u}(\mathcal{N},\mathbf{u},\lambda)(j-1)\| \le C\|a_{i+1}^+(u_i,\lambda,N_{i+1}^+)\|. \tag{5.45}$$

$\|a_{i+1}^+(u_i,\lambda,N_{i+1}^+)\|$ can be estimated by making use of (5.4), (5.30), the definition of d_{i+1} (cf. (2.8)) and finally the estimates of γ^+ and γ^- - see Remark 5.4. It turns out that

$$\|a_{i+1}^+(u_i,\lambda,N_{i+1}^+)\| \le C(\max\{\alpha^s,(\alpha^u)^{-1}\})^{N_{i+1}^+}. \tag{5.46}$$

Hence

$$\sup_{j\in\{0,\dots,N_{i+1}^+\}}(\alpha^u)^{N_{i+1}^+-j}\|\bar{v}_i^{+,u}(\mathcal{N},\mathbf{u},\lambda)(j-1)\| \le C(\max\{\alpha^s,(\alpha^u)^{-1}\})^{N_{i+1}^+} \tag{5.47}$$

and finally (see (5.42))

$$\begin{aligned}&\|\sum_{j=1}^{N_{i+1}^+}\Phi^+(u_i,\lambda,0,j)(id-P^+(u_i,\lambda,j))^2h^+(j-1,u_i,\bar{v}_i^+(\mathcal{N},\mathbf{u},\lambda)(j-1),\lambda)\|_{\mathbb{R}^k}\\ &\le C(\delta\alpha^u)^{-N_{i+1}^+}(\max\{\alpha^s,(\alpha^u)^{-1}\})^{2N_{i+1}^+}.\end{aligned} \tag{5.48}$$

We see that indeed the term at the left-hand side of (5.48) is of order $o((\mu^u)^{-N_{i+1}})$.

Similar we get $\sum_{j=0}^{-N_i^-+1}\Phi^-(u_i,\lambda,0,j)(id-P^-(u_i,\lambda,j))^2h^-(j-1,u_i,\bar{v}_i^-(\mathcal{N},\mathbf{u},\lambda)(j-1),\lambda) = o((\mu^s)^{N_i})$.

6 Estimates used in Section 4

In this section we will make available assertions needed for the estimates in Section 5. This section is divided into two subsections. In the first one we study the asymptotic behaviour of solutions of a discrete system starting in the stable (unstable) manifold of a hyperbolic fixed point. The main result in this direction is Lemma 6.2. This lemma gives the most weakly converging part of a solution in the stable manifold. For the proof we write the diffeomorphism f as $f = A + g$. A is a matrix and g represents the higher order terms. Replacingn g by a matrix B which depends on n, leads to Lemma 6.6. This lemma makes an assertion similar to that of Lemma 6.2 but for non-autonomous perturbed linear systems. Lemma 6.8 gives the justification to speak of strong stable (unstable) subspaces also in the context of non-autonomous linear systems.

In the second subsection we give estimates regarding $\tilde{P}$ which was introduced in Section 5.

6.1 Behaviour in the stable and unstable manifold

We start with the consideration of the behaviour in the stable manifold of a fixed point. Lemma 5.3 in [6] states that a solution (starting in the stable manifold) tends exponentially fast to the fixed point. Looking at the proof one perceives that the exponential decaying rate can be specified:

Corollary 6.1 *Let p be a hyperbolic fixed point of the smooth family of C^1 diffeomorphisms $f(\cdot,\lambda):\mathbb{R}^k\to\mathbb{R}^k$, $\lambda\in\mathbb{R}^l$. Further let $A(\lambda):=D_1f(p,\lambda)$ and let $K>0$ and $1>\mu_o>0$ be any numbers such that $\|A^nx\|\le K\mu_o^n\|x\|$ for all x in the stable subspace of $A:=A(0)$. Then for each $\alpha\in(0,-\ln\mu_o)$ there are constants C, ϵ, $\delta>0$ such that for all λ, $\|\lambda\|<\epsilon$ and for all $n\in\mathbb{N}$ it holds $f^n(x,\lambda)\in W^s_\lambda\cap B(p,\epsilon)$ if only $x\in W^s_\lambda\cap B(p,\delta)$. Moreover those x tend exponentially fast towards p under iteration of f: $\|f^n(x,\lambda)-p\|\le Ce^{-\alpha n}\|x\|$.* ■

The following lemma is the discrete version of Lemma 1.7 in [7].

Lemma 6.2 *We consider*

$$x(n+1)=f(x(n),\lambda)\quad \textit{with}\quad f(0,\lambda)=0 \tag{6.1}$$

for $f\in C^2(\mathbb{R}^k\times\mathbb{R}^l,\mathbb{R}^k)$. Let the spectrum $\sigma(D_1f(0,0))$ be inside the unite circle. Further we will assume that the principal eigenvalue $\mu^s(0)$ of $D_1f(0,0)$ is simple (but possibly complex). More precisely $\sigma(D_1f(0,\lambda))=\{\mu^s(\lambda)\}\cup\sigma^{ss}(\lambda)$, where $0<|\mu|<\alpha^{ss}<|\mu^s(\lambda)|<\alpha^s<1$ for all $\mu\in\sigma^{ss}(\lambda)$. We choose α^s such that $(\alpha^s)^2<|\mu^s(\lambda)|$ for sufficiently small $|\lambda|$. Let $X^s(\lambda)$ and $X^{ss}(\lambda)$ be the generalized eigenspaces of $D_1f(0,\lambda)$ associated to $\mu^s(\lambda)$ and $\sigma^{ss}(\lambda)$. Let $P_s(\lambda)$ be the projector projecting $\mathbb{R}^k$ on $X^s(\lambda)$ along $X^{ss}(\lambda)$. Finally let ϵ,δ be constants according to Corollary 6.1 for $\alpha=-\ln\alpha^s$.
Then for all solutions $x(\cdot)$ with $\|x(0)\|<\delta$ exists the limit

$$\eta(x(0),\lambda):=\lim_{n\to\infty}(D_1f(0,\lambda))^{-n}P_s(\lambda)x(n)\in X^s(\lambda). \tag{6.2}$$

Furthermore there is a constant c such that

$$\|x(n)-(D_1f(0,\lambda))^n\eta(x(0),\lambda)\|\le c(\max\{\alpha^{ss},(\alpha^s)^2\})^n. \tag{6.3}$$

Proof Let $A(\lambda):=D_1f(0,\lambda)$. $f(x,\lambda)=A(\lambda)x+g(x,\lambda)$, $g(0,\lambda)=D_1g(0,\lambda)=0$. By $\Phi(\lambda,\cdot,\cdot)$ we denote the transition matrix of $x(n+1)=A(\lambda)x(n)$. Then $x(\cdot)$ is a solution of (6.1) iff

$$\begin{aligned} x(n) &= \Phi(\lambda,n,m)x(m)+\sum_{j=m+1}^{n}\Phi(\lambda,n,j)g(x(j-1),\lambda)\\ &= A(\lambda)^{n-m}x(m)+\sum_{j=m+1}^{n}A(\lambda)^{n-j}g(x(j-1),\lambda). \end{aligned} \tag{6.4}$$

Next we prove that $A^{-n}(\lambda)(P_s(\lambda)x(n))$ is a fundamental sequence. Because of $A(\lambda)(X^s(\lambda))\subset X^s(\lambda)$ and $A(\lambda)(X^{ss}(\lambda))\subset X^{ss}(\lambda)$ we have $P_s(\lambda)A(\lambda)=A(\lambda)P_s(\lambda)$. This together with

(6.4) implies:

$$A(\lambda)^{-n}P_s(\lambda)x(n) = A(\lambda)^{-m}P_s(\lambda)x(m) + \sum_{j=m+1}^{n} A(\lambda)^{-j}P_s(\lambda)g(x(j-1),\lambda). \tag{6.5}$$

Then for any $n_1, n_2 \in \mathbb{N}$, $n_1 > n_2$, there is a constant K such that

$$\begin{aligned} \|A(\lambda)^{-n_1}P_s(\lambda)x(n_1) - A(\lambda)^{-n_2}P_s(\lambda)x(n_2)\| &\leq \sum_{j=n_2+1}^{n_1} \|A(\lambda)^{-j}P_s(\lambda)\|\|g(x(j-1),\lambda)\| \\ &\leq \sum_{j=n_2+1}^{n_1} K\left(\frac{1}{|\mu^s(\lambda)|}\right)^j (\alpha^s)^{2(j-1)} \end{aligned} \tag{6.6}$$

for all $x(\cdot)$ with $\|x(0)\| < \delta$. That estimate we obtain by exploiting that μ^s is simple, invoking Corollary 6.1 and using finally that (due to $g \in C^2$) there is a constant c_g such that

$$\|g(x,\lambda)\| \leq c_g\|x\|^2, \quad \forall x: \|x\| < \epsilon. \tag{6.7}$$

Due to the definition of g this estimate is uniform in λ. Because of $(\alpha^s)^2 < |\mu^s(\lambda)|$ formula (6.6) provides that $\{A(\lambda)^{-n}P_s(\lambda)x(n)\}_{n\in\mathbb{N}}$ is a fundamental sequence and therefore convergent. $\eta(x(0),\lambda) := \lim_{n\to\infty} A(\lambda)^{-n}P_s(\lambda)x(n)$. Because $A(\lambda)^{-1}$ leaves $X^s(\lambda)$ invariant and $X^s(\lambda)$ is closed we have $\eta(x(0),\lambda) \in X^s(\lambda)$.

Proof of the inequality (6.3):

For that we write (6.4) as a system taking into consideration that $A(\lambda)$ commutes with $P_s(\lambda)$:

$$P_s(\lambda)x(n) = A(\lambda)^{n-m}P_s(\lambda)x(m) + \sum_{j=m+1}^{n} A(\lambda)^{n-j}P_s(\lambda)g(x(j-1),\lambda) \tag{6.8a}$$

$$(id - P_s(\lambda))x(n) = A(\lambda)^{n-m}(id - P_s(\lambda))x(m) + \sum_{j=m+1}^{n} A(\lambda)^{n-j}(id - P_s(\lambda)) g(x(j-1),\lambda). \tag{6.8b}$$

First we consider (6.8a). The results of the first part of the proof show that for both addends on the right-hand side of (6.8a) the limit for m to infinity does exist - we get

$$P_s(\lambda)x(n) = A(\lambda)^n\eta(x(0),\lambda) + \sum_{j=n}^{\infty} A(\lambda)^{n-j}P_s(\lambda)g(x(j-1),\lambda).$$

This together with (6.8b) provides

$$\begin{aligned} &\|x(n) - A(\lambda)^n\eta(x(0),\lambda)\| \leq \|A(\lambda)^{n-m}(id - P_s(\lambda))x(m)\| \\ &\quad + \|\sum_{j=m+1}^{n} A(\lambda)^{n-j}(id - P_s(\lambda))g(x(j-1),\lambda)\| + \|\sum_{j=n}^{\infty} A(\lambda)^{n-j}P_s(\lambda)g(x(j-1),\lambda)\| \\ &\leq C\Big((\alpha^{ss})^{n-m}\|x(m)\| + \sum_{j=m+1}^{n} (\alpha^{ss})^{n-j}\|g(x(j-1),\lambda)\| + \sum_{j=n}^{\infty}(\alpha^s)^{n-j}\|g(x(j-1),\lambda)\|\Big) \\ &\leq C\Big((\alpha^{ss})^{n-m}\|x(m)\| + \sum_{j=m+1}^{n} (\alpha^{ss})^{n-j}\|x(j-1)\|^2 + \sum_{j=n}^{\infty}(\alpha^s)^{n-j}\|x(j-1)\|^2\Big). \end{aligned} \tag{6.9}$$

For the latter estimate we used (6.7). The constant C may differ from one to next step in our estimates. Now let $\|x(0)\| < \delta$. Hence $\|x(m)\| < \epsilon$, $\forall m \in \mathbb{N}$ and moreover $\|x(j-1)\| \leq C(\alpha^s)^{j-1}\|x(0)\|$ - cf. Corollary 6.1 and remind that here $p = 0$. So, for fixed m the terms in (6.9) can be estimated as follows:

$$(\alpha^{ss})^{n-m}\|x(m)\| \leq c_1(\alpha^{ss})^n,$$

$$\sum_{j=m+1}^{n}(\alpha^{ss})^{n-j}\|x(j-1)\|^2 \leq c_2(\alpha^{ss})^n \sum_{j=m+1}^{n}\left(\frac{(\alpha^s)^2}{\alpha^{ss}}\right)^j \leq c_2((\alpha^s)^{2n} + (\alpha^{ss})^n),$$

$$\sum_{j=n}^{\infty}(\alpha^s)^{n-j}\|x(j-1)\|^2 \leq c_3(\alpha^s)^n \sum_{j=n}^{\infty}(\alpha^s)^{-j}(\alpha^s)^{2j} \leq c_3(\alpha^s)^{2n}.$$

This together with (6.9) yields (6.3). ■

Remark 6.3 Estimate (6.6) is uniform with respect to $x(0)$ and λ. Hence the sequence $(A(\lambda)^{-n}P_s(\lambda)x(n))_{n\in\mathbb{N}}$ converges uniformly. Therefore $\eta(\cdot,\cdot)$ is continuous. □

Remark 6.4 In Lemma 6.2 we assumed that 0 is a asymptotically stable fixed point. If 0 is a hyperbolic fixed point then Lemma 6.2 describes the behaviour of solutions in the stable manifold. By reversing 'time' we obtain a similar lemma for solutions in the unstable manifold. □

Remark 6.5 Let the hypothesis of Lemma 6.2 hold. Then $\eta(x,\lambda) \neq 0$ iff $x \notin W^{ss}_\lambda(0)$. $W^{ss}_\lambda(0)$ denotes the strong stable manifold. This is due to $f^n(x,\lambda)\cdot|\mu^s(\lambda)|^{-n} \to 0$ as $n \to \infty$ if $x \in W^{ss}_\lambda(0)$ - cf. [9]. □

Lemma 6.6 *Let $A : \mathbb{R}^l \to \mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$ and for all $n \in \mathbb{N}$ $B(n,\cdot) : \mathbb{R}^l \to \mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$ be smooth. Furthermore we will assume:*

(i) $\sigma(A(0))$ *is hyperbolic.*

(ii) The principal stable eigenvalue $\mu^s(\lambda)$ *of* $A(\lambda)$ *is simple (but possibly complex).*

(iii) $\sigma(A(\lambda)) = \sigma^{ss}(\lambda) \cup \{\mu^s(\lambda)\} \cup \sigma^u(\lambda)$ *where* $0 < |\mu| < \alpha^{ss} < |\mu^s(\lambda)| < \alpha^s < 1 < \alpha^u < |\tilde{\mu}|$ *for all* $\mu \in \sigma^{ss}(\lambda)$ *and* $\tilde{\mu} \in \sigma^u(\lambda)$.

(iv) There is a $\beta \in (0,1)$ *such that* $\|B(n,\lambda)\| < \beta^n$ *and* $\alpha^s\beta < |\mu^s(\lambda)|$ *for small* $|\lambda|$.

Let $X^s(\lambda)$, $X^{ss}(\lambda)$ and $X^u(\lambda)$ be the generalized eigenspaces of $A(\lambda)$ associated to $\mu^s(\lambda)$, $\sigma^{ss}(\lambda)$ and $\sigma^u(\lambda)$, respectively. Further let $P_s(\lambda)$ be the projector projecting $\mathbb{R}^k$ on $X^s(\lambda)$ along $X^{ss}(\lambda) \oplus X^u(\lambda)$.
Then for every solution $x(\cdot)$ of $x(n+1) = A(\lambda)x(n) + B(n,\lambda)x(n)$ tending to zero as $n \to \infty$ there exists the limit

$$\eta(x(0),\lambda) := \lim_{n\to\infty} A(\lambda)^{-n}P_s(\lambda)x(n) \in X^s(\lambda). \tag{6.10}$$

Furthermore there is a constant c such that

$$\|x(n) - A(\lambda)^n\eta(x(0),\lambda)\| \leq c(\max\{\alpha^{ss},\alpha^s\beta\})^n. \tag{6.11}$$

Proof Not only the formulation of this lemma is very similar to that of Lemma 6.2 but also the proof. Only the details in the estimates have to be changed.
Similar to (2.8) in [5] we find the stable solutions of $x(n+1) = A(\lambda)x(n) + B(n,\lambda)x(n)$ as bounded solutions of

$$\begin{aligned} x(n) &= A(\lambda)^{n-m}x(m) + \sum_{j=m+1}^{n} A(\lambda)^{n-j}P(\lambda,j)B(j-1,\lambda)x(j-1) \\ &\quad - \sum_{j=n+1}^{\infty} A(\lambda)^{n-j}(id - P(\lambda,j))B(j-1,\lambda)x(j-1). \end{aligned} \tag{6.12}$$

$P(\lambda,n)$ are projectors concerned with the exponential dichotomy of $x(n+1) = A(\lambda)x(n)$ on $\mathbb{N}$. In particular let $P(\lambda,n)$ be chosen such that $\operatorname{im} P(\lambda,n) = X^s(\lambda) \oplus X^{ss}(\lambda)$ and $\ker P(\lambda,n) = X^u(\lambda)$. So we have $P(\lambda,n) = P(\lambda)$, $\forall n \in \mathbb{N}$. From this definition we get $P(\lambda)P_s(\lambda) = P_s(\lambda)P(\lambda)$.
The next step is again to prove that $(A(\lambda)^{-n}P_s(\lambda)x(n))_{n\in\mathbb{N}}$ is a fundamental sequence. Instead of (6.5) we have to consider

$$A(\lambda)^{-n}P_s(\lambda)x(n) = A(\lambda)^{-m}x(m) + \sum_{j=m+1}^{n} A(\lambda)^{-j}P(\lambda,j)(B(j-1,\lambda)x(j-1). \tag{6.13}$$

Mind that $\operatorname{im} P(\lambda) \subset \ker P_s(\lambda)$. Now, using the estimates of B assumed in the lemma and of $x(\cdot)$ caused by the exponential dichotomy of

$$x(n+1) = A(\lambda)x(n) + B(n,\lambda)x(n) \tag{6.14}$$

we get the same estimate as (6.6). To see that α^s is indeed an appropriate constant for the exponential dichotomy of (6.14) use Lemma A.5 in [5] and the Roughness Theorem - cf. Proposition 2.10 in [6]. So we get similar to the proof of Lemma 6.2 the validity of (6.10).
Also the proof of (6.11) runs (on principle) along the same lines as the proof of (6.3). This time we get

$$\begin{aligned} \|x(n) - A(\lambda)^n\eta(x(0),\lambda)\| &\le \|A(\lambda)^{n-m}(id - P_s(\lambda))x(m)\| \\ &\quad + \|\sum_{j=m+1}^{n} A(\lambda)^{n-j}P(\lambda)(id - P_s(\lambda))B(j-1,\lambda)x(j-1)\| \\ &\quad + \|\sum_{j=n+1}^{\infty} A(\lambda)^{n-j}(id - P(\lambda))(id - P_s(\lambda))B(j-1,\lambda)x(j-1)\| \\ &\quad + \|\sum_{j=n+1}^{\infty} A(\lambda)^{n-j}P(\lambda)(id - P_s(\lambda))B(j-1,\lambda)x(j-1)\|. \end{aligned} \tag{6.15}$$

In difference to (6.9) we have here an additional term, which can be estimated as

$$\begin{aligned} &\|\sum_{j=n+1}^{\infty} A(\lambda)^{n-j}(id - P(\lambda))(id - P_s(\lambda))B(j-1,\lambda)x(j-1)\| \\ &\qquad \le \sum_{j=n+1}^{\infty} c(\alpha^u)^{n+1-j}(\alpha^s\beta)^{j-1} \le c(\alpha^s\beta)^n. \end{aligned}$$

The remaining terms in (6.15) can be estimated as (6.9) - by using again the estimates of $B(\cdot,\cdot)$ and $x(\cdot)$. This provides (6.11). ■

Remark 6.7 Analogously to Remark 6.3 we get the continuity of $\eta(\cdot,\cdot)$. □

In accordance with Remark 6.5 we will show that $\eta(x,\lambda) \neq 0$ iff x is not in the 'strong stable subspace' of $x(n+1) = A(\lambda)x(n) + B(n,\lambda)x(n)$. Because this is a non-autonomous equation the notion of such subspaces has to be introduced. For equations having an exponential dichotomy on $\mathbb{N}$ the stable subspace is well defined:
Let $x(n+1) = A(n)x(n)$ have an exponential dichotomy on $\mathbb{N}$ defined by positive constants K, α and projectors $P(n)$ such that $\|\Phi(n,m)P(m)\| \leq Ke^{-\alpha(n-m)}$, $n \geq m$. Here $\Phi(\cdot,\cdot)$ is the transition matrix of the equation under consideration. Then we can define the stable subspace $E^s(n)$ (depending on n) by $E^s(n) := \operatorname{im} P(n)$. This means nothing else but just solutions of the initial value problems $x(n+1) = A(n)x(n)$, $x(m) \in E^s(m)$ decay exponentially and, what's more, with an exponential rate of at least $-\alpha$ - see in the appendix of [5]. To define strong stable subspaces $E^{ss}(n)$ we introduce the concept of exponential trichotomies as it is usual for ordinary differential equations - cf. [1] or [3]. Consider a linear equation

$$x(n+1) = A(n)x(n), \tag{6.16}$$

where for each $n \in \mathbb{N}$ $A(n)$ is invertible. The transition matrix of (6.16) we denote by $\Phi(\cdot,\cdot)$. We say that (6.16) has an exponential trichotomy on $\mathbb{N}$ if there exist positive constants $\alpha_s < \alpha_c < 1 < \alpha^c < \alpha_u$ and projectors $P_s(n)$, $P_c(n)$ and $P_u(n)$ satisfying $id = P_s(n) + P_c(n) + P_u(n)$, $\forall n \in \mathbb{N}$ and $\Phi(n,m)P_i(m) = P_i(n)\Phi(n,m)$, for $i = s, c, u$ and $\forall n, m \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$

$$\begin{aligned}
\|\Phi(n,m)P_s(m)\| &\leq K\alpha_s^{n-m} &&, n \geq m,\\
\|\Phi(n,m)P_c(m)\| &\leq K\alpha_c^{-(m-n)} &&, m \geq n,\\
\|\Phi(n,m)P_c(m)\| &\leq K(\alpha^c)^{n-m} &&, n \geq m,\\
\|\Phi(n,m)P_u(m)\| &\leq K\alpha_u^{-(m-n)} &&, m \geq n.
\end{aligned} \tag{6.17}$$

That means: A solution $x(\cdot)$ of (6.16) with $x(m) \in P_s(m)$ decayes exponentially with exponential rate $\ln \alpha_s$. If $x(m) \in P_u(m)$ then $x(\cdot)$ growes exponentially with exponential rate $\ln \alpha_u$. If $x(m) \in P_c(m)$ then $x(\cdot)$ does not decay faster than α_c^{n-m} and simultaneously it does not grow faster than $(\alpha^c)^{n-m}$.
Let $A(n) \equiv A$, $\forall n \in \mathbb{N}$. $\sigma(A) = \sigma^s \cup \sigma^c \cup \sigma^u$: $\sigma^s, \sigma^c, \sigma^u \neq \emptyset$ and

$$\begin{aligned}
\mu \in \sigma^s &\Leftrightarrow 0 < |\mu| < 1,\\
\mu \in \sigma^c &\Leftrightarrow |\mu| = 1,\\
\mu \in \sigma^u &\Leftrightarrow |\mu| > 1.
\end{aligned}$$

Then $x(n+1) = Ax(n)$ has an exponential trichotomy. But also if $\sigma^c = \emptyset$ we can allocate an exponential trichotomy to $x(n+1) = Ax(n)$. Let

$$\begin{aligned}
&\sigma(A) = \sigma^{ss} \cup \sigma^s \cup \sigma^u :\\
&0 < \mu^{ss} < \alpha_s < \alpha_c < \mu^s < 1 < \alpha^c < \alpha_u < \mu^u, \quad \mu^{ss} \in \sigma^{ss}, \mu^s \in \sigma^s, \mu^u \in \sigma^u.
\end{aligned} \tag{6.18}$$

Then $x(n+1) = Ax(n)$ has an exponential trichotomy with

$$\operatorname{im} P_s = X^{ss}, \quad \operatorname{im} P_c = X^s, \quad \operatorname{im} P_u = X^u \tag{6.19}$$

and constants $K = K_A$ (cf. (6.17)), α_s, α_c, α^c and α_u. X^{ss}, X^s and X^u are the generalized eigenspaces to σ^{ss}, σ^s and σ^u, respectively. To utilize exponential trichotomies for our application we have to prove a corresponding roughness theorem. Indeed we will prove only the first part of the roughness theorem - see also Remark 6.9 below. What we prove can also be seen as a generalization of the notion exponential dichotomy (as it is given in [6] for difference equations or in [2] for differential equations) in the sense that solutions starting in certain subspace decay exponentially (at least with an exponential rate $\ln\alpha_s$ and all other solutions do not decay faster than α_c^{n-m}, as $n \to \infty$ - if they decay at all. The usual notion of exponential dichotomy contains that these solutions grow exponentially fast.

Lemma 6.8 *Let $A \in \mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$ with spectrum as in (6.18). The above considerations show that $x(n+1) = Ax(n)$ has an exponential trichotomy with constants $K = K_A$ (cf. (6.17)), α_s, α_c, α^c and α_u. Further, for all $n \in \mathbb{N}$ let $B(n) \in \mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$ with*

$$\|B(n)\| < K_B \beta^n, \quad \beta \in (0,1). \tag{6.20}$$

The transition matrix of

$$x(n+1) = (A + B(n))x(n) \tag{6.21}$$

we denote by $\Phi_B(\cdot,\cdot)$. Then there exist projectors $Q_s(n)$ and a constant $\mathcal{K}$ such that for all $n, m \in \mathbb{N}$

$$\Phi_B(n,m)Q_s(m) = Q_s(n)\Phi_B(n,m) \tag{6.22}$$

and

$$\begin{aligned} \|\Phi_B(n,m)Q_s(m)\| &\leq \mathcal{K}\alpha_s^{n-m} & , n \geq m \\ \|\Phi_B(n,m)(id - Q_s(m))\| &\leq \mathcal{K}\alpha_c^{-(m-n)} & , m \geq n. \end{aligned} \tag{6.23}$$

Proof The proof is similar to [7, Lemma 1.1].
Exactly the solutions $x(\cdot, m)$ of (6.21) decaying for $n > m$ exponentially with a rate $\ln\alpha_s$ we find as solutions of the fixed point problem

$$\begin{aligned} x(n,m) &= \Phi(n,m)\eta + \sum_{j=m+1}^{n} \Phi(n,j)P_s B(j-1)x(j-1,m) \\ &\quad - \sum_{j=n+1}^{\infty} \Phi(n,j)(id - P_s)B(j-1)x(j-1,m) \\ &=: (T_s(x,\eta))(n,m), \end{aligned} \tag{6.24}$$

$n, m \in \mathbb{N}$, $n \geq m$, $\eta \in \operatorname{im} P_s$, $\sum_{j=m+1}^{m} \ldots =: 0$.
Here $\Phi(\cdot,\cdot)$ is the transition matrix of $x(n+1) = Ax(n)$ and P_s as introduced in (6.19). Equation (6.24) has to be solved in

$$\mathbb{S}_{\alpha_s} := \{x : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^k : \sup_{n \geq m} \alpha_s^{m-n} \|x(n,m)\|_{\mathbb{R}^k} < \infty\}.$$

$\mathbb{S}_{\alpha_s}$ equipped with the norm $\|x\| := \sup_{n\geq m} \alpha_s^{m-n}\|x(n,m)\|_{\mathbb{R}^k}$ is a Banach space. The equivalence of the corresponding solutions of (6.21) and (6.24) can be seen as follows: Equation (6.21) is equivalent to

$$\begin{aligned} x(n,m) = &\ \Phi(n,m)x(m,m) + \sum_{j=m+1}^{n} \Phi(n,j)B(j-1)x(j-1,m) \\ = &\ \Phi(n,m)P_s x(m,m) + \Phi(n,m)(id-P_s)x(m,m) \\ &+ \sum_{j=m+1}^{n} \Phi(n,j)B(j-1)x(j-1,m). \end{aligned} \tag{6.25}$$

Now let $x \in \mathbb{S}_{\alpha_s}$ be solution of (6.21). Therefore the representation of $x(N,m)$ can be taken from the first line in (6.25). Letting acting $\Phi(n,N)(id-P_s)$ from the left yields

$$\begin{aligned} \Phi(n,N)(id-P_s)x(N,m) = &\ \Phi(n,m)(id-P_s)x(m,m) \\ &+ \sum_{j=m+1}^{N} \Phi(n,j)(id-P_s)B(j-1)x(j-1,m). \end{aligned} \tag{6.26}$$

Combining (6.25) and (6.26) we get:

$$\begin{aligned} x(n,m) = &\ \Phi(n,m)P_s x(m,m) + \Phi(n,N)(id-P_s)x(N,m) \\ &- \sum_{j=n+1}^{N} \Phi(n,j)(id-P_s)B(j-1)x(j-1,m) \\ &+ \sum_{j=m+1}^{n} \Phi(n,j)P_s B(j-1)x(j-1,m). \end{aligned} \tag{6.27}$$

Exploiting (generalized) dichotomy of $x(n+1) = Ax(n)$ and $x \in \mathbb{S}_{\alpha_s}$ we get $\|\Phi(n,N)(id-P_s)x(N,m)\| \leq c(n,m)\left(\frac{\alpha_s}{\alpha_c}\right)^N$. Because of $0 < \alpha_s < \alpha_c$ this implies $\|\Phi(n,N)(id-P_s)x(N,m)\| \to 0$ as $N \to \infty$. Similar we see that $\lim_{N\to\infty} \sum_{j=n+1}^{N} \Phi(n,j)(id-P_s)B(j-1)x(j-1,m)$ does exist. Hence by tending $N \to \infty$ arises (6.24) from (6.27). On the other hand it is a easy computation to verify that each solution $x \in \mathbb{S}_{\alpha_s}$ of (6.24) satifies (6.21). So we can consider (6.24) in $\mathbb{S}_{\alpha_s}$ instead of (6.21).
Indeed maps $T_s(\cdot,\eta)$ $\mathbb{S}_{\alpha_s}$ into $\mathbb{S}_{\alpha_s}$: Let $x \in \mathbb{S}_{\alpha_s}$. Then there are corresponding constants such that

$$\begin{aligned} \|T_s(x,\eta)(n,m)\|_{\mathbb{R}^k} \leq &\ \|\Phi(n,m)\eta\| + \sum_{j=m+1}^{n} \|\Phi(n,j)P_s\|\|B(j-1)\|\|x(j-1,m)\| \\ &+ \sum_{j=n+1}^{\infty} \|\Phi(n,j)(id-P_s)\|\|B(j-1)\|\|x(j-1,m)\| \\ \leq &\ c\Big(\alpha_s^{n-m}\|\eta\| + \sum_{j=m+1}^{n} \alpha_s^{n-j}\beta^{j-1}\alpha_s^{j-1-m}\|x\| + \sum_{j=n+1}^{\infty} \alpha_c^{n-j}\beta^{j-1}\alpha_s^{j-1-m}\|x\|\Big) \\ \leq &\ C\alpha_s^{n-m}. \end{aligned}$$

Both mappings, $x \mapsto T_i x$, $i = 1,2$, where

$$\begin{aligned} (T_1 x)(n,m) &:= \sum_{j=m+1}^{n} \Phi(n,j)P_s B(j-1)x(j-1,m) \quad \text{and} \\ (T_2 x)(n,m) &:= \sum_{j=n+1}^{\infty} \Phi(n,j)(id-P_s)B(j-1)x(j-1,m) \end{aligned}$$

are linear and moreover $T_1, T_2 \in \mathbb{L}(\mathbb{S}_{\alpha_s}, \mathbb{S}_{\alpha_s})$. If

$$K_B < \frac{1}{2K_A} \min\{\alpha_c - \alpha_s, \alpha_s - \beta\alpha_s\} \tag{6.28}$$

then, together with (6.20), this provides that their norms are less than $\frac{1}{2}$. Hence (6.24) can be solved for x depending on η, $x(\cdot,\cdot) = x_s(\eta)(\cdot,\cdot)$. Here $x_s(\eta)(\cdot,\cdot)$ depends linearly on η. Hence $x_s(\eta)(m,m)$ depends linearly on η as well. This implies

$$x_s(0)(m,m) = 0 \tag{6.29}$$

We will show that

$$\begin{array}{rccc} \hat{Q}_s(m): & \mathbb{R}^k & \to & \mathbb{R}^k \\ & \xi & \mapsto & x_s(P_s\xi)(m,m) \end{array}$$

are projectors: We consider $\hat{Q}_s(m)^2\xi = x_s(P_s\hat{Q}_s(m)\xi)(m,m)$. If we have shown that $P_s\hat{Q}_s(m) = P_s$ then the above equation implies $\hat{Q}_s(m)^2 = \hat{Q}_s(m)$.

$$\begin{aligned} x_s(P_s\xi)(n,m) = \quad & \Phi(n,m)P_s\xi + \sum_{j=m+1}^{n} \Phi(n,j)P_sB(j-1)x_s(P_s\xi)(j-1,m) \\ & - \sum_{j=n+1}^{\infty} \Phi(n,j)(id - P_s)B(j-1)x_s(P_s\xi)(j-1,m). \end{aligned} \tag{6.30}$$

From that we see that indeed

$$P_s\hat{Q}_s(m)\xi = P_sx_s(P_s\xi)(m,m) = P_s\xi \tag{6.31}$$

- mind that $\sum_{j=m+1}^{m} \ldots = 0$. (6.31) together with (6.29) yields

$$\ker P_s = \ker \hat{Q}_s(m). \tag{6.32}$$

Altogether the above considerations show

$$x_s(P_s\xi)(n,m) = \Phi_B(n,m)\hat{Q}_s(m)\xi. \tag{6.33}$$

Now there is a constant $K > 0$ such that

$$\begin{aligned} \|\Phi_B(n,m)\hat{Q}_s(m)\| = \quad & \sup_{\|\xi\|=1} \|\Phi_B(n,m)\hat{Q}_s(m)\xi\| \\ = \quad & \sup_{\|\xi\|=1} \|x_s(P_s\xi)(n,m)\|_{\mathbb{R}^k} \le \sup_{\|\xi\|=1} \|x_s(P_s\xi)(\cdot,\cdot)\|_{\mathbb{S}_{\alpha_s}} \alpha_s^{n-m} \\ \le \quad & K\alpha_s^{n-m}, \quad n \ge m \end{aligned} \tag{6.34}$$

because $\{\xi : \|\xi\| = 1\}$ is compact and $\xi \mapsto x_s(P_s\xi)(\cdot,\cdot)$ is continuous.
Tieing up to (6.33) we can also prove

$$\operatorname{im} \hat{Q}_s(n) = \Phi_B(n,m)(\operatorname{im} \hat{Q}_s(m)). \tag{6.35}$$

Namely, let $\xi \in \operatorname{im} \hat{Q}_s(m)$. Then for $k > n > m$ we have

$$\|\Phi_B(k,n)\Phi_B(n,m)\xi\| = \|\Phi_B(k,m)\xi\| \le K\alpha_s^{k-m}\|\xi\| \le K\alpha_s^{k-n}\|\xi\|. \tag{6.36}$$

That means that the solution $\Phi_B(\cdot,\cdot)\Phi_B(n,m)\xi$ of (6.21) is in $\mathbb{S}_{\alpha_s}$. This again implies that $\Phi_B(n,m)\xi \in \operatorname{im} \hat{Q}_s(n)$. Hence $\Phi_B(n,m)\Big(\operatorname{im} \hat{Q}_s(m)\Big) \subset \operatorname{im} \hat{Q}_s(n)$. Since, due to (6.32) $\dim \operatorname{im} \hat{Q}_s(m) = \dim \operatorname{im} \hat{Q}_s(n)$ equation (6.35) is proved.
However, unfortunately Φ_B does not commute with $\hat{Q}_s$. So $\hat{Q}_s(n)$ are not the wanted projectors. But exploiting (6.35) we see that for each n

$$\mathbb{R}^k = \operatorname{im} \hat{Q}_s(n) \oplus \Phi_B(n,0)\Big(\ker \hat{Q}_s(0)\Big) \tag{6.37}$$

is a direct sum decomposition of $\mathbb{R}^k$. Let $Q_s(n)$ be the corresponding projector with $\operatorname{im} Q_s(n) = \operatorname{im} \hat{Q}_s(n)$. We will show that $Q_s(n)$ are the projectors we are looking for:
First, by construction we have

$$\Phi_B(n,m)Q_s(m) = Q_s(n)\Phi_B(n,m). \tag{6.38}$$

So it remains to prove (6.23). We start with developing a fixed point problem for detecting solutions of (6.21) starting in $\ker Q_s(m)$. This fixed point problem runs

$$\begin{aligned} x(n,m) = \;& \Phi(n,m)(id-P_s)\xi + \sum_{j=1}^{n} \Phi(n,j)P_sB(j-1)x(j-1,m) \\ & - \sum_{j=n+1}^{m} \Phi(n,j)(id-P_s)B(j-1)x(j-1,m) \\ =: \;& (T_u(x,\xi))(n,m). \end{aligned} \tag{6.39}$$

Indeed it is a easy computation to verify that each solution of (6.39) also satisties (6.21) - mind $\Phi(n+1,n) = A$. Vice versa, for $m \ge n$ we have

$$\Phi_B(n,m) = \Phi(n,m)\Phi_B(m,m) - \sum_{j=n+1}^{m} \Phi(n,j)B(j-1)\Phi_B(j-1,m). \tag{6.40}$$

Now putting $n = 0$ in (6.40) and letting acting $\Phi(n,0)P_s$ from the left in this equation and combining this result finally with (6.25) leads to (note that in (6.25) $x(n,m)$ stands for $\Phi_B(n,m)\xi$)

$$\begin{aligned} \Phi_B(n,m)\xi = \;& \Phi(n,m)(id-P_s)\Phi_B(m,m)\xi + \Phi(n,0)P_s\Phi_B(0,m)\xi \\ & + \sum_{j=1}^{n} \Phi(n,j)P_sB(j-1)\Phi_B(j-1,m)\xi \\ & - \sum_{j=n+1}^{m} \Phi(n,j)(id-P_s)B(j-1)\Phi_B(j-1,m)\xi. \end{aligned} \tag{6.41}$$

If $\xi \in \Phi_B(m,0)\Big(\ker Q_s(0)\Big) = \ker Q_s(m)$ then $\Phi_B(0,m)\xi \in \ker Q_s(0)$. Because of $P_sQ_s(0) = P_s$ (cf. (6.31) and definition of $Q_s(\cdot)$ via (6.37) which yields $Q_s(0) = \hat{Q}_s(0)$) we have

$P_s\Phi_B(0,m)\xi = 0$. That means $x(n,m) := \Phi_B(n,m)\xi$ satisfies (6.39). In the opposite direction (6.41) yields that $\Phi_B(n,m)\xi$ is a solution of (6.39) only if

$$P_s\Phi_B(0,m)\xi = 0 \Leftrightarrow \Phi_B(0,m)\xi \in \ker P_s = \ker Q_s(0) \Leftrightarrow \xi \in \ker Q_s(m).$$

This means for solutions $x(\cdot,\cdot)$ of (6.39) that $x(m,m) \in \ker Q_s(m)$.
We will solve (6.39) in the space

$$\mathbb{S}^{\alpha_c} := \{x : \mathbb{N}\times\mathbb{N} \to \mathbb{R}^k : \sup_{m\geq n} \alpha_c^{m-n}\|x(n,m)\|_{\mathbb{R}^k} < \infty\}.$$

$\mathbb{S}^{\alpha_c}$ equipped with the norm $\|x\| := \sup_{m\geq n} \alpha_c^{m-n}\|x(n,m)\|_{\mathbb{R}^k}$ is a Banach space. Similar to the corresponding assertions regarding T_s one can prove that T_u maps $\mathbb{S}^{\alpha_c}$ into itself and that (6.39) can be solved for $x(\cdot,\cdot) = x_u((id-P_s)\xi)(\cdot,\cdot)$. Again the mappings

$$\begin{array}{rccl} \hat{Q}_u(m): & \mathbb{R}^k & \to & \mathbb{R}^k \\ & \xi & \mapsto & x_u((id-P_s)\xi)(m,m) \end{array}$$

are projectors. If $x(\cdot,\cdot)$ is a solution of (6.39) then $x(0,0) = (id-P_s)\xi \in \mathrm{im}\,(id-P_s)$. With that our considerations show $\mathrm{im}\,\hat{Q}_u(m) = \Phi_B(m,0)\Big(\ker \hat{Q}_s(0)\Big)$ and - similar to (6.34)

$$\|\Phi_B(n,m)\hat{Q}_u(m)\| \leq K\alpha_c^{n-m}, \quad m \geq n, \tag{6.42}$$

$$K := \max\{\sup_{\|\xi\|=1} \|x_s(P_s\xi)(\cdot,\cdot)\|_{\mathbb{S}_{\alpha_s}}, \sup_{\|\xi\|=1} \|x_u((id-P_s)\xi)(\cdot,\cdot)\|_{\mathbb{S}^{\alpha_u}}\}. \tag{6.43}$$

Because $\mathrm{im}\,Q_s(m) = \mathrm{im}\,\hat{Q}_s(m)$ and $\mathrm{im}\,(id - Q_s(m)) = \mathrm{im}\,\hat{Q}_u(m)$ we have

$$\begin{array}{l} \Phi_B(n,m)Q_s(m) = \Phi_B(n,m)\hat{Q}_s(m)Q_s(m) \quad \text{and} \\ \Phi_B(n,m)(id-Q_s(m)) = \Phi_B(n,m)\hat{Q}_u(m)(id-Q_s(m)). \end{array} \tag{6.44}$$

Hence, taking into consideration (6.34) and (6.42), the estimates (6.23) are proved if $\|Q_s(m)\|$ is bounded - more precisely, if $\sup_{m\in\mathbb{N}} \|Q_s(m)\| < \infty$. For that we fix $m \in \mathbb{N}$. Consider $\xi \in \mathbb{R}^k$. Let $\xi_s := Q_s(m)\xi$ and $\xi_u := (id - Q_s(m))\xi$. Now

$$\begin{aligned} Q_s(m)\xi = \;& \xi_s = \hat{Q}_s(m)\xi_s \\ =\;& P_s\xi_s - \sum_{j=m+1}^{\infty} \Phi(m,j)(id-P_s)B(j-1)x_s(P_s\xi_s)(j-1,m) \\ =\;& P_s\xi - P_s\xi_u - \sum_{j=m+1}^{\infty} \Phi(m,j)(id-P_s)B(j-1)x_s(P_s\xi_s)(j-1,m). \end{aligned} \tag{6.45}$$

On the other hand

$$\begin{aligned} \xi_u =\;& \hat{Q}_u(m)\xi_u \\ =\;& (id-P_s)\xi_u - \sum_{j=1}^{m} \Phi(m,j)P_sB(j-1)x_u((id-P_s)\xi_u)(j-1,m). \end{aligned}$$

Therefore

$$P_s\xi_u = \sum_{j=1}^{m} \Phi(m,j)P_sB(j-1)x_u((id-P_s)\xi_u)(j-1,m).$$

Combining this with (6.45) yields:

$$\begin{aligned} Q_s(m)\xi = \; & P_s\xi - \sum_{j=1}^{m} \Phi(m,j)P_sB(j-1)x_u((id-P_s)\xi_u)(j-1,m) \\ & - \sum_{j=m+1}^{\infty} \Phi(m,j)(id-P_s)B(j-1)x_s(P_s\xi_s)(j-1,m). \end{aligned}$$

This equation can be written as

$$\begin{aligned} Q_s(m)\xi = \; & P_s\xi - \sum_{j=1}^{m} \Phi(m,j)P_sB(j-1)\Phi_B(j-1,m)\hat{Q}_u(m)(id-Q_s(m))\xi \\ & - \sum_{j=m+1}^{\infty} \Phi(m,j)(id-P_s)B(j-1)\Phi_B(j-1,m)\hat{Q}_s(m)Q_s(m)\xi. \end{aligned} \tag{6.46}$$

From that it follows

$$\begin{aligned} \|Q_s(m)\| \; & \leq \|P_s\| \\ & + \sum_{j=1}^{m} \|\Phi(m,j)P_s\|\|B(j-1)\|\|\Phi_B(j-1,m)\hat{Q}_u(m)\|(1+\|Q_s(m)\|) \\ & + \sum_{j=m+1}^{\infty} \|\Phi(m,j)(id-P_s)\|\|B(j-1)\|\|\Phi_B(j-1,m)\hat{Q}_s(m)\|\|Q_s(m)\| \\ \leq \; & \|P_s\| + \sum_{j=1}^{m} \tfrac{K_AK_BK}{\alpha_c}\left(\tfrac{\alpha_s}{\alpha_c}\right)^{m-j}\beta^{j-1} \\ & + \Big(\sum_{j=1}^{m} \tfrac{K_AK_BK}{\alpha_c}\left(\tfrac{\alpha_s}{\alpha_c}\right)^{m-j}\beta^{j-1} + \sum_{j=m+1}^{\infty} \tfrac{K_AK_BK}{\alpha_s}\left(\tfrac{\alpha_s}{\alpha_c}\right)^{j-m}\beta^{j-1}\Big)\|Q_s(m)\| \\ \leq \; & \|P_s\| + \tfrac{K_AK_BK}{\alpha_c}C + \tfrac{K_AK_BK}{\alpha_s}C\|Q_s(m)\|. \end{aligned} \tag{6.47}$$

Here $C := \sum_{j=0}^{\infty} \beta^j$. If besides (6.28) also

$$K_B < \frac{\alpha_s}{2K_AKC} \tag{6.48}$$

then (6.47) provides that $\|Q_s(m)\|$ is indeed bounded. (6.44) together with (6.34) and (6.42) gives an appropriate $\mathcal{K}$ according to the lemma.
Writing (6.20)as $\|B(n)\| < K_B\beta^{n_o}\beta^{n-n_o}$ and defining $K_{B,n_o} := K_B\beta^{n_o}$ we can make K_{B,n_o} as small as we wish. With this K_{B,n_o} we can do the same considerations and get the estimates (6.23) for all $n,m \geq n_o$. Then by enlarging $\mathcal{K}$ we get (6.23) for all $n,m \in \mathbb{N}$. ■

Remark 6.9 Similar to Lemma 6.8 one can prove that there exist projectors $Q_{cs}(n)$ such that

$$\begin{aligned} \|\Phi_B(n,m)Q_{cs}(m)\| &\leq \mathcal{K}(\alpha^c)^{n-m} \qquad &, n \geq m, \\ \|\Phi_B(n,m)(id-Q_{cs}(m))\| &\leq \mathcal{K}{\alpha_u}^{-(m-n)} \qquad &, m \geq n. \end{aligned}$$

Then we get projectors $Q_s(n)$, $Q_c(n)$ and $Q_u(n)$ defining an exponential trichotomy of (6.21): $Q_s(n)$ we get according to Lemma 6.8, $Q_c(n) := (id - Q_s(n))Q_{cs}(n)$ and $Q_u(n) := id - Q_{cs}(n)$. □

Remark 6.10 Let (6.21) be a variational equation associated with a nonlinear system $x(n+1) = f(x(n))$ having a hyperbolic fixed point p. Let $q \in W^{ss}(p)$ be a point in the strong stable manifold. Then we can apply Lemma 6.8 on $x(n+1) = Df(f^n(q))x(n)$. Similar to [6, Proposition 5.4] we can prove $\operatorname{im} Q_s(m) = T_{f^m(q)}W^{ss}(p)$. □

We will now consider the case that B in the equation (6.21) depends smoothly on a parameter $\lambda \in \mathbb{R}^l$:

$$x(n+1)(A + B(\lambda, n))x(n). \tag{6.49}$$

We will take on the assumptions concerning A made in Lemma 6.8. Moreover we assume that the estimate (6.20) of B holds true uniformely in λ and besides this also

$$\|D_\lambda B(\lambda, n)\| \leq K_B \beta^n,\ \beta \in (0,1). \tag{6.50}$$

Then, of course, the assertion of Lemma 6.8 remains true for all λ.

Remark 6.11 Recall that it was our goal to show that $\eta(x, \lambda) \neq 0$ iff x is not in the strong stable subspace. In accordance with our notations the strong stable subspaces are $\operatorname{im} Q_s(n)$. Now, combining Lemma 6.6 and Lemma 6.8 yields that $\eta(x, \lambda)$ has to be zero if and only if $x \in \operatorname{im} Q_s(0)$. □

Next we will look at the derivatives with respect to λ of the quantities under consideration:

Lemma 6.12 *With the above assumptions we get that there is a constant $K > 0$ such that*

$$\|D_\lambda(\Phi_B(\lambda, n, m)\hat{Q}_s(\lambda, m))\| \leq K\alpha_s^{n-m},\ n \geq m. \tag{6.51}$$

Proof As in the proof of Lemma 6.8 we have $\Phi_B(\lambda, n, m)\hat{Q}_s(\lambda, m)\xi = x_s(P_s\xi)(\lambda, n, m)$ and $\lambda \mapsto x_s(P_s\xi)(\lambda, \cdot, \cdot)$ maps smoothly from $\mathbb{R}^k$ into $\mathbb{S}_{\alpha_s}$. Hence

$$\begin{aligned} &\|D_\lambda \Phi_B(\lambda_o, n, m)\hat{Q}_s(\lambda, m)\xi)(\cdot)\| \\ &= \|D_\lambda x_s(P_s\xi)(\lambda_o, n, m))(\cdot)\| = \sup_{|\lambda|=1} \|D_\lambda x_s(P_s\xi)(\lambda_o, n, m))(\lambda)\|_{\mathbb{R}^k} \\ &\leq \|D_\lambda x_s(P_s\xi)(\lambda_o, \cdot, \cdot)\|_{\mathbb{L}(\mathbb{R}^k, \mathbb{S}_{\alpha_s})}\alpha_s^{n-m}. \end{aligned} \tag{6.52}$$

The latter inequality in (6.52) follows from the definition of $\|\cdot\|_{\mathbb{L}(\mathbb{R}^k, \mathbb{S}_{\alpha_s})}$. Finally the lemma follows by taking supremum for $\|\xi\| = 1$ in (6.52). ■

Remark 6.13 With similar arguments as in Lemma 6.12 we get also

$$\|D_\lambda(\Phi_B(\lambda, n, m)\hat{Q}_u(\lambda, m))\| \leq K\alpha_c^{n-m},\ m \geq n. \tag{6.53}$$

□

Remark 6.14 Let's consider the representation (6.46) of $Q_s(m)\xi$ with quantities depending on λ. Then estimates similar to that of $\|Q_s(m)\|$ given in (6.47) show that also $\|D_\lambda Q_s(\lambda,\cdot)\|$ is bounded. Then differentiating the equations corresponding to (6.44) with respect to λ leads to

$$\begin{aligned} \|D_\lambda(\Phi_B(\lambda,n,m)Q_s(\lambda,m))\| &\leq \mathcal{K}\alpha_s^{n-m},\ n\geq m, \\ \|D_\lambda(\Phi_B(\lambda,n,m)(id-Q_s(\lambda,m)))\| &\leq \mathcal{K}\alpha_c^{n-m},\ m\geq n. \end{aligned} \tag{6.54}$$

□

Looking closer at (6.46) one can prove that $\|D_\lambda Q_s(\lambda,\cdot)\|$ even decays exponentially -cf. [7, Lemma 1.1] for the differential equation case. However, for our purpose it is sufficient to know that $\|D_\lambda Q_s(\lambda,\cdot)\|$ remain bounded.

6.2 Estimates regarding $\tilde{P}$

Next we will make available necessary estimates concerning $\tilde{P}$. $\tilde{P}(u_i,u_{i+1},\lambda,N_{i+1})$ is defined by the direct sum decomposition (5.5). The existence of this decomposition was proved in [5, Lemma 4.4]. In the proof of this lemma $\tilde{P}$ is represented by

$$\tilde{P}(u_i,u_{i+1},\lambda,N_{i+1}) = S(u_i,u_{i+1},\lambda,N_{i+1})(id-P)S(u_i,u_{i+1},\lambda,N_{i+1})^{-1} \tag{6.55}$$

where

$$S(u_i,u_{i+1},\lambda,N_{i+1}) = (id-P^-(u_{i+1},\lambda,-N_{i+1}^-))P + (id-P^+(u_i,\lambda,N_{i+1}^+))(id-P). \tag{6.56}$$

P projects $\mathbb{R}^k$ on $T_pW^s(p)$ along $T_pW^u(p)$.

Lemma 6.15 *Suppose the eigenvalue condition (3.1). Then in the space* $\mathbb{L}(\mathbb{R}^k,\mathbb{R}^k)$ *of linear mappings it holds*

$$\lim_{N_{i+1}\to\infty} S(u_i,u_{i+1},\lambda,N_{i+1}) = id.$$

And moreover, there is a $K>0$ *such that*

$$\|S(u_i,u_{i+1},\lambda,N_{i+1})-id\| \leq K\Big(\max\{\alpha^s,(\alpha^u)^{-1}\}\Big)^{N_{i+1}^+}.$$

Proof Equation (6.56) provides

$$\begin{aligned} &\|S(u_i,u_{i+1},\lambda,N_{i+1})-id\| \\ &\quad\leq \|P^-(u_{i+1},\lambda,-N_{i+1}^-)P\| + \|P^+(u_i,\lambda,N_{i+1}^+)(id-P)\| \\ &\quad= \|(P^-(u_{i+1},\lambda,-N_{i+1}^-)-(id-P))P\| + \|(P^+(u_i,\lambda,N_{i+1}^+)-P)(id-P)\| \\ &\quad\leq \|P^-(u_{i+1},\lambda,-N_{i+1}^-)-(id-P)\|\|P\| + \|P^+(u_i,\lambda,N_{i+1}^+)-P\|\|id-P\|. \end{aligned}$$

Now the lemma results from [5, Corollary A.4] together with Lemma 6.2 and [5, Lemma A.5] - see also Lemma 6.6. More precisely: [5, Corollary A.4] states that both $\|P^-(u_{i+1},\lambda,-N_{i+1}^-)-(id-P)\|$ and $\|P^+(u_i,\lambda,N_{i+1}^+)-P\|$ tend exponentially fast to zero as N_{i+1} goes to infinity. The residual lemmata cited above ensure that the rate of decaying is as stated in the lemma.

■

Corollary 6.16 *Lemma 6.15 is valid also for* $S(u_i, u_{i+1}, \lambda, N_{i+1})^{-1}$.

Proof Let $GL(k)$ be the set of invertible linear maps $\mathbb{R}^k \to \mathbb{R}^k$. Then the mapping $GL(k) \to GL(k)$, $S \to S^{-1}$ is continuous (with respect to the topology induced by the norm of $\mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$. This proves (together with Lemma 6.15)

$$\lim_{N_{i+1}\to\infty} S(u_i, u_{i+1}, \lambda, N_{i+1})^{-1} = id.$$

The inequality

$$\|S(u_i, u_{i+1}, \lambda, N_{i+1})^{-1} - id\| \leq \tilde{K}\Big(\max\{\alpha^s, (\alpha^u)^{-1}\}\Big)^{N_{i+1}^+}$$

becomes clear by

$$\|S(u_i, u_{i+1}, \lambda, N_{i+1})^{-1} - id\| \leq \|S(u_i, u_{i+1}, \lambda, N_{i+1})^{-1}\|\|S(u_i, u_{i+1}, \lambda, N_{i+1}) - id\|$$

together with Lemma 6.15. ∎

Lemma 6.17 *Suppose the eigenvalue condition (3.1). Then in the space* $\mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$ *of linear mappings it holds*

$$\lim_{N_{i+1}\to\infty} \tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1}) = id - P.$$

And moreover, there is a $K > 0$ *such that*

$$\|\tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1}) - (id - P)\| \leq K\Big(\max\{\alpha^s, (\alpha^u)^{-1}\}\Big)^{N_{i+1}^+}.$$

Proof (6.55) provides

$$\begin{aligned}
&\|\tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1}) - (id - P)\| \\
&\quad = \|(S(u_i, u_{i+1}, \lambda, N_{i+1}) - id + id)(id - P)(S(u_i, u_{i+1}, \lambda, N_{i+1})^{-1} - id + id) - (id - P)\| \\
&\quad \leq \|(S(u_i, u_{i+1}, \lambda, N_{i+1}) - id)(id - P)(S(u_i, u_{i+1}, \lambda, N_{i+1})^{-1} - id)\| \\
&\qquad + \|(S(u_i, u_{i+1}, \lambda, N_{i+1}) - id)(id - P)\| + \|(S(u_i, u_{i+1}, \lambda, N_{i+1})^{-1} - id)(id - P)\|.
\end{aligned}$$

Now the lemma follows by Lemma 6.15 and Corollary 6.16. ∎

Lemma 6.18 *Let* $v^s \in T_pW^s(p) = \operatorname{im} P$. *Then*

$$\lim_{N_{i+1}\to\infty} \tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1})v^s = 0.$$

And moreover, there is a $C > 0$ *such that*

$$\|\tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1})v^s\| \leq C\Big(\max\{\alpha^s, (\alpha^u)^{-1}\}\Big)^{N_{i+1}^+} v^s.$$

Proof

$$\|\tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1})v^s\| = \|\tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1})v^s - (id - P)v^s\|$$
$$\leq \|\tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1}) - (id - P)\|\|v^s\|.$$

Now the lemma follows by Lemma 6.17. ■

Similar we get

Lemma 6.19 *Let $v^u \in T_pW^u(p) = \ker P$. Then*

$$\lim_{N_{i+1}\to\infty} \tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1})v^u = v^u.$$

And moreover, there is a $C > 0$ such that

$$\|\tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1})v^u - v^u\| \leq C\Big(\max\{\alpha^s, (\alpha^u)^{-1}\}\Big)^{N_{i+1}^+} v^u.$$

Proof

$$\tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1})v^u - v^u = (\tilde{P}(u_i, u_{i+1}, \lambda, N_{i+1}) - (id - P))v^u.$$

Again Lemma 6.17 provides the assertion. ■

At the end of this subsection we will make some remarks concerning the derivative of $\tilde{P}$. This projector is associated with variational equations of the form

$$\begin{aligned} v(n+1) &= D_1 f(\gamma^{+(-)}(u,\lambda)(n), \lambda)v(n) \\ &= \Big(D_1 f(p,\lambda) + \underbrace{(D_1 f(\gamma^{+(-)}(u,\lambda)(n),\lambda) - D_1 f(p,\lambda))}_{=:B^{+(-)}(u,\lambda,n)}\Big)v(n). \end{aligned} \tag{6.57}$$

Now it is immediately clear that $B^{+(-)}(u, \lambda, n)$ satisfies (6.20) uniformely in u and λ. To estimate $D_uB^{+(-)}(u, \lambda, n)$ we have to estimate $D_u\gamma^{+(-)}(u, \lambda)(n)$. We will do it exemplarily for "+". $\gamma^+(u, \lambda)(n + 1) = f(\gamma^+(u, \lambda)(n), \lambda)$. Hence

$$D_u\gamma^+(u,\lambda)(n+1) = D_1 f(\gamma^+(u,\lambda)(n),\lambda) D_u\gamma^+(u,\lambda)(n). \tag{6.58}$$

Furthermore $\gamma^+(u, \lambda)(0) \in W^s_\lambda(p)$ for fixed λ and all u. Hence

$$D_u\gamma^+(u,\lambda)(0) \in T_{\gamma^+(u,\lambda)(0)} W^s_\lambda(p). \tag{6.59}$$

Due to the exponential dichotomy of the equation $v(n+1) = D_1 f(\gamma^+(u, \lambda)(n), \lambda)v(n)$ equation (6.58) and formula (6.59) imply that there is a $K > 0$ such that

$$\|D_u\gamma^+(u,\lambda)(n)\| \leq K(\alpha^s)^n. \tag{6.60}$$

If we assume that $\mu^{s(u)}$ are simple -(EV) - then this estimate can be rendered more severe to

$$D_u\gamma^+(u,\lambda)(n) = \eta(D_u\gamma^+(u,\lambda)(0),\lambda)(\mu^s)^n + O\Big((\max\{\alpha^{ss}, (\alpha^s)^2\})^n\Big) \tag{6.61}$$

- cf. Lemma 6.6. Therefore $\|D_uB^{+(-)}(u, \lambda, n)\|$ decays exponentially. Hence, Remark 6.14 applied to this case reads

Corollary 6.20 *The derivatives of the projectors $P^{+(-)}$ with respect to u_i remain bounded as N_i tend to infinity.* ■

Then (6.56) and (6.55) - which describe the connection of P and $\tilde{P}$ - tell

Corollary 6.21 *The derivatives of $\tilde{P}^{+(-)}$ with respect to u_i remain bounded as N_i tend to infinity.* ■

7 Homoclinic Tangencies

We will apply our results to the case that q is a non-transversal homoclinic point. More precisely we assume that the principal eigenvalues are simple and real - (EV_R). The assumption $\dim U = 1$ we extend to

(HT) (i) $\dim U = 1$

(ii) $\operatorname{rank} \left.\frac{\partial(\xi^\infty, D_1\xi^\infty)}{\partial(u,\lambda)}\right|_{(u,\lambda)=(0,0)} = 2$

To exclude degeneracies we will also assume (SUB) and (HOM_A). According to [5, Section 3] the homoclinic tangencies condition (HT) implies that $l \geq 1$. For simplicity we put $l = 1$. Then $\xi^\infty(u,\lambda)$ can be transformed into $\lambda \pm u^2$. Because these two cases do not differ qualitatively from each other we will only deal with the "-" case. In this case we find for each $\lambda > 0$ u-values $u_1(\lambda) \neq u_2(\lambda)$ such that $\xi^\infty(u_i(\lambda),\lambda) = 0$. Assigned to these $u_i(\lambda)$ we have homoclinic points $q_i(\lambda)$, $i = 1, 2$, merging to the non-transversal homoclinic point q at $\lambda = 0$ and finally disappearing if λ becomes negative. See [5] for more details. Indeed for $\lambda > 0$ $q_i(\lambda)$, $i = 1, 2$, are transversal homoclinic points.
We will give an intiutive discussion of 1-periodic orbits.
The bifurcation equation $\Xi(N, \mathbf{u}, \lambda) = 0$ for detecting periodic orbits (hitting a small neighborhood of q exactlay once - so called 1-periodic orbits) only consits of a single equation

$$\xi^\infty(u,\lambda) + \xi(N,u,\lambda) = 0.$$

This equation can be written in normal form

$$\lambda - u^2 + \xi(N,u,\lambda) = 0 \tag{7.1}$$

or equivalently

$$\lambda = u^2 - \xi(N,u,\lambda).$$

Essentially, for fixed N the zeros of (7.1) form (a slightly deformed) parabola $\lambda = u^2$ which is shifted upwards or downwards depending on ξ is negative or positive. Our considerations in Section 5 show that

$$\begin{aligned} &\xi(N,u,\lambda) = c^u(u,\lambda)(\mu^u(\lambda))^{-N} + c^s(u,\lambda)(\mu^s(\lambda))^N + o\Big((\mu^u(\lambda))^{-N}\Big) + o\Big((\mu^s(\lambda))^N\Big) \\ &c^s(0,0), c^u(0,0) \neq 0. \end{aligned} \tag{7.2}$$

We will make a further assumption concerning the principal eigenvalues which allows to perceive the sign of ξ:

(EV_R$^+$) (EV_R), and $\mu^s(\lambda) > |\mu^u(\lambda)^{-1}|$.

Hence $\mu^s(\lambda) > 0$ and the sign of ξ coincides with the sign of $c^s(0,0)$.
If $c^s(0,0)$ is negative - the parabola will shifted upwards - then for sufficiently small $\hat{\lambda} \geq 0$ we have no intersection of $\{(u,\hat{\lambda}), u \in \mathbb{R}\}$ and the "parabola" $\lambda = u^2 - \xi$ and therefore no periodic orbits with period N.
While for positive $c^s(0,0)$ we find even for for small $\hat{\lambda} < 0$ periodic orbits with period N.
More precisely: There is a $\lambda^*(N)$, $\lambda^* > 0$ if $c^s < 0$ and $\lambda^* < 0$ if $c^s > 0$, such that for $\lambda > \lambda^*$ there are two 1-periodic orbits with period N which merge at $\lambda = \lambda^*$ and finally disappear if λ is less than λ^*.
Note that for fixed $\hat{\lambda} < 0$ (sufficiently small of course) we do not have periodic orbits of period N for arbitrary large N: If N is too large the entire "parabola" $\lambda = u^2 - \xi$ will lie above the line $\{(u,\hat{\lambda}), u \in \mathbb{R}\}$.

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