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Towards a Morse Theory for Random Dynamical Systems

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A generalization of the concepts of deterministic Morse theory to random dynamical systems is presented. Using the notions of attraction and repulsion in probability, the main building blocks of Morse theory such as attractor-repeller pairs, Morse sets, and the Morse decomposition are obtained for random dynamical systems.

Keywords: Attractor-Repeller pair, Morse theory, Morse set, Morse decomposition

2000 Mathematics Subject Classification: 34D25, 37B55, 37C70, 37H99

1 Introduction

One of the basic tasks of the theory of differential equations and dynamical systems is to study qualitative, asymptotic, long-term behavior of solutions/orbits. Of particular interest are compact sets which are invariant under the dynamics and attract or repel all nearby solutions. These attractors and repellers and their basins of attraction and repulsion form the dynamical skeleton of a system, describing its limit behavior in forward and backward time. Within an attractor there can be further (local) attractors and repellers the knowledge of which helps to understand the often complicated dynamics on the attractor. This is the topic of Morse theory. In a compact space X, Morse theory is a tool to construct a sequence of repellers $X = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_N = \emptyset$ corresponding to a sequence of attractors $\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_N = X$ and with it Morse sets $M_i = A_i \cap R_{i-1}$ forming the dynamical skeleton. Consider e.g.

$$\dot{x} = x^4 - x^2 \qquad \text{on the interval } [-1, 1]$$

with $A_1 = \{-1\}$, $A_2 = [-1, 0]$ and $R_2 = [0, 1]$, $R_1 = \{1\}$. The Morse sets M_1, M_2, M_3 describe the asymptotic behavior in contrast to the transient behavior in their complement (see Fig. 1)

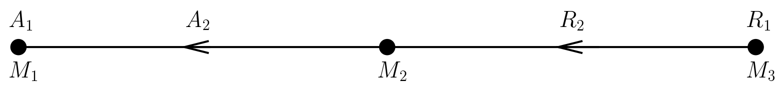

Figure 1: Morse sets for $\dot{x} = x^4 - x^2$ on $[-1, 1]$.

For an introduction to Morse theory for autonomous deterministic dynamical systems, see, e.g., Conley [5], Rybakowski [16] Chapter 3, Carbinatto and Rybakowski [3], Rybakowski [17] or Colonius and Kliemann [4] Appendix B2.

In this paper we undertake an approach towards a generalization of the notion of Morse decompositions to *random dynamical systems (RDS)*. Random dynamical systems are formally skew-product flows — but only in the measurable category. In particular, any topology and continuity is stripped-off from the driving or base flow. Random dynamical systems have attractors which typically are random sets (see e.g. Crauel and Flandoli [11], Crauel, Debussche and Flandoli [10], Crauel [6, 7, 8], Flandoli and Schmalfuß [12], Schmalfuß [19, 20], and for a brief survey Arnold [1] Chapter 1). However, there are different notions of attraction, e.g. pullback attraction, forward attraction or attraction in probability.

The main conceptual problem is to find definitions of attractor and repeller which are matching in the sense that they allow to prove that a random attractor has a corresponding repeller which then allows to define a Morse decomposition.

The structure of this paper is as follows: In Section 2 we will briefly review Morse decompositions for topological dynamical systems. In Section 3 we will recall some basic facts about random dynamical systems. Section 4 will be devoted to the concept of attractors and their basins of attraction as well as to the dual concept of repellers. The notion of a Morse decomposition for random dynamical systems will be presented in Section 5 which also contains our main theorem, stating the dynamical interpretation of a Morse decomposition. Section 6 contains a prototypical example: the flow on the projective space induced by a linear random dynamical system. We conclude in Section 7 with some remarks justifying our attractor notion.

2 Morse Decompositions for Dynamical Systems

We briefly recall the basic definitions of attractors, repellers, and of the Morse decomposition for topological dynamical systems.

Let X be a topological space. A continuous mapping $\varphi : \mathbb{R} \times X \to X$, $(t,x) \mapsto \varphi(t,x)$, is called a *topological dynamical system* if the family $\varphi(t,\cdot) = \varphi(t) : X \to X$ of self-mappings of X satisfies the flow properties $\varphi(0) = \mathrm{id}_X$, $\varphi(t+s) = \varphi(t) \circ \varphi(s)$ for all $t, s \in \mathbb{R}$, where "$\circ$" denotes composition of mappings. It follows that all mappings $\varphi(t)$ are homeomorphisms of X, and $\varphi(t)^{-1} = \varphi(-t)$.

2.1 Definition Let φ be a topological dynamical system on a compact metric space X.
(i) *Attractor:* Let A be a non-empty compact subset of X which is *invariant* under φ, i.e for which $\varphi(t)A = A$ for all $t \in \mathbb{R}$. Then A is called *attractor* of φ if there exists a neighborhood U of A (called *fundamental neighborhood*) with

$$\lim_{t\to\infty} d(\varphi(t)U, A) = 0. \tag{1}$$

(ii) *Repeller:* Let R be a non-empty compact subset of X which is *invariant* under φ. Then R is called *repeller* if there exists a neighborhood U of R with

$$\lim_{t\to-\infty} d(\varphi(t)U, R) = 0.$$

Note that (1) is equivalent to $\omega(U) = A$, i.e. A being the ω-limit set of a neighborhood of itself, similarly a repeller is the α-limit set of a neighborhood of itself.

2.2 Proposition *Let φ be a topological dynamical system on a compact metric space X, and let A be an attractor of φ. Then*

$$R := \{x \in X \,:\, \omega(x) \cap A = \emptyset\}$$

is a repeller and (A,R) is called attractor-repeller pair.

2.3 Definition Let φ be a topological dynamical system on a compact metric space X, and let (A_i, R_i) be attractor-repeller pairs with

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = X \qquad \text{and} \qquad X = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset.$$

Then the family $(M_i)_{i=1,\dots,n}$ of subsets of X defined by

$$M_i = A_i \cap R_{i-1}, \quad 1 \le i \le n$$

is called a *Morse decomposition* of X, each M_i is called *Morse set.*

We quote the following result on Morse decomposition for further use (see Conley [5] Chapter II or Rybakowski [16] Chapter III).

2.4 Theorem *Let φ be a topological dynamical system on a compact metric space X. Then the following two statements are equivalent.*

(i) $(M_i)_{i=1,\dots,n}$ *is a Morse decomposition.*

(ii) $(M_i)_{i=1,\dots,n}$ *attracts all orbits and cycles are not allowed, i.e.*

- $\omega(x), \alpha(x) \in \bigcup_{j=1}^{n} M_j$ *for each* $x \in X$.
- *If* $x_1, \dots, x_p$ *are points such that for some* $1 \leq j_0 \leq \dots \leq j_p \leq n$

$$\alpha(x_k) \subset M_{j_{k-1}} \quad \text{and} \quad \omega(x_k) \subset M_{j_k} \qquad \text{for } 1 \leq k \leq p,$$

then $j_0 \leq j_p$. *Furthermore,* $j_0 < j_p$ *if and only if* $x_k \notin \bigcup_{j=1}^{n} M_j$ *for some* k, *whereas otherwise* $j_0 = \dots = j_p$.

3 Random Dynamical Systems and Random Sets

A *random dynamical system* (RDS) consists of a continuous cocycle φ on a topological space over a measurable flow θ on a probability space. To be more precise, let $(\Omega, \mathscr{F}, P)$ be a probability space, and let $\theta : \mathbb{R}\times\Omega \to \Omega$ be a measurable map such that $\theta_{t+s} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$, where $\theta_t : \Omega \to \Omega$, $\theta_t(\omega) = \theta(t, \omega)$, and θ_t preserves P for every $t > 0$, and $(\theta_t)_{t\in\mathbb{R}}$ is ergodic. Given $(\Omega, \mathscr{F}, P; (\theta_t))$, let (X, d) be a metric space, and let $\varphi : \mathbb{R} \times \Omega \times X \to X$ be a measurable map such that the *cocycle property* holds, i.e.

$$\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega) \qquad \text{for all } t, s \in \mathbb{R} \text{ and } \omega \in \Omega, \tag{2}$$

where $\varphi(t, \omega) : X \to X$, $\varphi(t, \omega)x = \varphi(t, \omega, x)$, is assumed to be continuous for all $t \in \mathbb{R}$, $\omega \in \Omega$.

For a systematic discussion of RDS, comprising more general classes than those considered here, and also presenting several ways of their generation, see Arnold [1] Part I.

3.1 Remark Having assumed two-sided time for φ it is straightforward to verify that $\varphi(t, \omega)$ is a homeomorphism on X, and $\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t\omega)$.

We introduce some notation. Any set $M \subset X \times \Omega$ is determined by its *ω-sections* $M(\omega) = \{x \in X : (x, \omega) \in M\}$. We will sometimes identify M with $\omega \mapsto M(\omega)$. Recall that, for A and B non-empty subsets of X, the Hausdorff semi-metric $d(A, B)$ is defined by

$$d(A, B) := \sup_{a\in A} d(a, B) = \sup_{a\in A} \inf_{b\in B} d(b, a).$$

3.2 Definition (Random Set) (i) A function $\omega \mapsto M(\omega)$ taking values in the non-empty compact subsets of X is called a *compact random set* if $\omega \mapsto d(x, M(\omega))$ is measurable for each $x \in X$, where $d(x, M) := \inf_{y\in M} d(x, y)$.
(ii) A function $\omega \mapsto U(\omega)$ taking values in the open subsets of X is called an *open random set* if $\omega \mapsto U(\omega)^c$ is a closed random set, where U^c denotes the complement of U.

3.3 Definition (Invariance of Random Set) A random set M is said to be *forward invariant* under the RDS φ if $\varphi(t,\omega)M(\omega) \subset M(\theta_t\omega)$ for all $t \geq 0$. It is said to be *invariant* if $\varphi(t,\omega)M(\omega) = M(\theta_t\omega)$ for all $t \geq 0$.

3.4 Remark Suppose that, for some random set M, $\varphi(t,\omega)M(\omega) \subset M(\theta_t\omega)$ for all $t \in \mathbb{R}$. Then it is straightforward to verify that M is invariant, invoking Remark 3.1. Furthermore, a similar argument gives that $\varphi(t,\omega)M(\omega) = M(\theta_t\omega)$ for all $t \in \mathbb{R}$ for an invariant set M.

3.5 Definition (Isolated Invariant Random Set) An invariant random set M is called *isolated*, if there exists a random neighborhood U of M such that for each random variable x with the property that the orbit through x remains in U almost surely then the orbit already belongs to M almost surely, i.e. the inclusion

$$\varphi(t,\omega)x(\omega) \in U(\theta_t\omega) \quad \text{for all } t \in \mathbb{R} \text{ a.s.} \qquad \text{implies} \qquad x(\omega) \in M(\omega) \text{ a.s.}$$

4 Attractors and Repellers

4.1 Definition (Attractors and Repellers) Let φ be an RDS.
(i) A random compact set A which is invariant under φ is called *(local) attractor* of φ if there exists a random open neighborhood U of A (i.e. $A(\omega) \subset U(\omega)$ a.s.), which is forward invariant such that for each random closed set $V \subset U$ the distance $d(\varphi(t,\omega)V(\omega), A(\theta_t\omega))$ converges to 0 in probability, i.e.

$$\lim_{t\to\infty} P\big\{d(\varphi(t,\omega)V(\omega), A(\theta_t\omega)) > \varepsilon\big\} = 0 \qquad \text{for every } \varepsilon > 0. \tag{3}$$

The neighborhood U is said to be a *fundamental neighborhood of* A.
(ii) A random compact set R which is invariant under φ is called *(local) repeller* of φ if there exists a random open neighborhood U of R which is backward invariant such that for each random closed set $V \subset U$ the distance $d(\varphi(t,\omega)V(\omega), R(\theta_t\omega))$ converges to 0 in probability as $t \to -\infty$, i.e.

$$\lim_{t\to-\infty} P\big\{d(\varphi(t,\omega)V(\omega), R(\theta_t\omega)) > \varepsilon\big\} = 0 \qquad \text{for every } \varepsilon > 0. \tag{4}$$

The neighborhood U is said to be a *fundamental neighborhood of* R.

4.2 Remark (Attractors and Attracting Universes) Associated with an attractor, the notion of a *universe* is used in the literature (see e.g. Arnold [1] Definition 9.3.1). A universe denotes a family of subsets of $X \times \Omega$ such that with every element of this family also every of its subsets is an element of the family. An attractor for a universe is then a compact, invariant random set such that (3) holds for every element of the universe, and such, in addition, the attractor itself is an element of the universe.

Given a local attractor A with forward invariant neighborhood U satisfying (3) for each closed random $V \subset U$, we get a universe in this sense by taking

$$\mathfrak{U} = \big\{V \subset X \times \Omega : \overline{V}(\omega) \subset U(\omega)\big\} \; .$$

Clearly we have, for any $V \in \mathfrak{U}$,

$$\lim_{t\to\infty} d\big(\varphi(t,\omega)V(\omega), A(\theta_t\omega)\big) = 0 \qquad \text{in probability,} \tag{5}$$

and obviously $A \in \mathfrak{U}$.

4.3 Remark (i) Suppose that U_1, U_2 are fundamental neighborhoods of a local attractor A. Then also $U_1 \cap U_2$ is a fundamental neighborhood of A.

(ii) An RDS is a *nonautonomous* system in the way described by the cocycle property (2). It hence matters whether we define asymptotic properties like attractivity (a) by going from $-t$ to 0 or (b) by going from 0 to t, and then letting $t \to \infty$. The choice (a) offers itself as the mathematically natural one for the following reason: While t is moving, the quantity in question, $d(\varphi(t,\theta_{-t}\omega)V(\theta_{-t}\omega), A(\omega))$, is always studied at time 0, where typically ω-wise convergence can be expected. The choice (b), in contrast, seems to be physically natural, but considers quantities, namely $d(\varphi(t,\omega)V(\omega), A(\theta_t\omega))$, which are moving with t forever, being responsible for the fact that they often do not converge ω-wise.

The choice of the weaker mode of convergence in probability symmetrizes the situation and makes the two approaches equivalent since

$$P\big\{d(\varphi(t,\theta_{-t}\omega)V(\theta_{-t}\omega), A(\omega)) \geq \varepsilon\big\} = P\big\{d(\varphi(t,\omega)V(\omega), A(\theta_t\omega)) \geq \varepsilon\big\}$$

due to the fact that P is invariant under θ_t.

Random attractors defined by the choice (a) are know as "pullback attractors" and were introduced and studied by Crauel and Flandoli [11], Crauel, Debussche and Flandoli [10], Crauel [6, 7, 9], Flandoli and Schmalfuß [12] and Schmalfuß [19, 20] among others.

The concept of attractor given in Definition 4.1 (i) was introduced and studied by Ochs [14] under the name "weak attractor". Relations between these notions of attraction have been investigated by Scheutzow [18]. Note that an attractor as defined in Definition 4.1 attracts random sets which is in general stronger than the notion of *point attractors* attracting only random variables (see Crauel [7]).

We will make use of an auxiliary lemma.

4.4 Lemma *Suppose that U is a forward invariant random set for an RDS φ, i.e.,*

$$\varphi(t,\omega)U(\omega) \subset U(\theta_t\omega) \qquad \text{for every } t \geq 0,\ \omega \in \Omega.$$

Then for every $s \leq t$ we have

$$\varphi(t,\theta_{-t}\omega)U(\theta_{-t}\omega) \subset \varphi(s,\theta_{-s}\omega)U(\theta_{-s}\omega).$$

PROOF With $t = s + t - s$, and noting that $t - s \geq 0$, we have

$$\begin{aligned} \varphi(s+t-s, \theta_{-t}\omega)U(\theta_{-t}\omega) &= \varphi(s, \theta_{-s}\omega)\Big(\varphi(t-s, \theta_{-t}\omega)U(\theta_{-t}\omega)\Big) \\ &\subset \varphi(s, \theta_{-s}\omega)U\big(\theta_{t-s}(\theta_{-t}\omega)\big) \\ &= \varphi(s, \theta_{-s}\omega)U(\theta_{-s}\omega), \end{aligned}$$

where we used forward invariance to get $\varphi(t-s, \theta_{-t}\omega)U(\theta_{-t}\omega) \subset U(\theta_{t-s}\theta_{-t}\omega)$.

The following lemma is inspired by Proposition 2.5 in Ashwin and Ochs [2], where a similar result is proved for point attractors.

4.5 Lemma (Basin of Attraction and Repulsion) *Let φ be an RDS.*
(i) *Suppose that A is an attractor of φ with a forward invariant fundamental neighborhood U. Then*

$$C(\omega) := \{x \in X : \varphi(t, \omega)x \in U(\theta_t\omega) \text{ for some } t \geq 0\} \tag{6}$$

is a random set with the following properties

- *C is invariant,*
- *$C(\omega) = \bigcup_{t\geq 0} \varphi(-t, \theta_t\omega)U(\theta_t\omega) = \lim_{T\to\infty} \varphi(-T, \theta_T\omega)U(\theta_T\omega)$,*
- *A is attracting every closed random set $V \subset C$, i.e.* (3) *holds.*

C is called the basin of attraction of A.

(ii) *Suppose that R is an attractor of φ with a backward invariant fundamental neighborhood U. Then*

$$C(\omega) := \{x \in X : \varphi(t, \omega)x \in U(\theta_t\omega) \text{ for some } t \leq 0\}$$

is a random set with the following properties

- *C is invariant,*
- *$C(\omega) = \bigcup_{t\leq 0} \varphi(-t, \theta_t\omega)U(\theta_t\omega) = \lim_{T\to\infty} \varphi(T, \theta_{-T}\omega)U(\theta_{-T}\omega)$,*
- *R is repelling every closed random set $V \subset C$, i.e.* (4) *holds.*

C is called the basin of repulsion of R.

Note that the basin of attraction or repulsion, respectively, is independent of the fundamental neighborhood. This assertion follows from Proposition 5.1 about repellers below, and will be formulated in Corollary 5.2.

PROOF We prove only (i), the arguments for (ii) are similar.

To see that $C(\omega) = \bigcup_{t\geq 0} \varphi(-t, \theta_t\omega)U(\theta_t\omega)$, note that $x \in C(\omega)$ is equivalent to $\varphi(t, \omega)x \in U(\theta_t\omega)$ for some $t \geq 0$, or, equivalently, $x \in \varphi(-t, \theta_t\omega)U(\theta_t\omega)$, invoking $\varphi(t, \omega)^{-1} =$

$\varphi(-t,\theta_t\omega)$ again. But this inclusion holds if and only if $x \in \bigcup_{t\geq 0} \varphi(-t,\theta_t\omega)U(\theta_t\omega)$. From Lemma 4.4 (applied to $-T \leq -t$ for every $t \leq T$) we get

$$\bigcup_{0\leq t\leq T} \varphi(-t,\theta_t\omega)U(\theta_t\omega) = \varphi(-T,\theta_T\omega)U(\theta_T\omega) \tag{7}$$

for every $T \geq 0$. This implies, in particular, that $\varphi(-T,\theta_T\omega)U(\theta_T\omega)$ is increasing in T for every ω. Having assumed φ to have two-sided time, whence $\varphi(t,\omega)$ is a homeomorphism, this implies that C, given by $C(\omega) = \lim \varphi(-T,\theta_T\omega)U(\theta_T\omega)$, is an open random set.

Invariance of C follows from

$$\varphi(s,\omega)\big(\varphi(-T,\theta_T\omega)U(\theta_T\omega)\big) = \varphi(s-T,\theta_{T-s}\theta_s\omega)U(\theta_{T-s}\theta_s\omega)$$

for every $s \geq 0$, noting that the left hand side increases to $\varphi(s,\omega)C(\omega)$, while the right hand side increases to $C(\theta_s\omega)$, as $T \to \infty$.

To prove that A attracts every random closed set in C, choose a random closed set V with $V(\omega) \subset C(\omega) = \bigcup_{t\geq 0} \varphi(-t,\theta_t\omega)U(\theta_t\omega)$. By compactness of V together with (7) there exists $T = T(\omega) \geq 0$ such that $V(\omega) \subset \varphi(-T,\theta_T\omega)U(\theta_T\omega)$, and therefore $\varphi(t,\omega)V(\omega) \subset \varphi(t-T,\theta_T\omega)U(\theta_T\omega) \subset U(\theta_t\omega)$ for all $t > T$ by forward invariance of U. Define a random variable n by $n(\omega) = \inf\{n \in \mathbb{N} : \varphi(n,\omega)V(\omega) \subset U(\theta_n\omega)\}$, and put $\Omega_m = \{\omega : n(\omega) \leq m\}$. For any $\varepsilon > 0$ there is $N_0 \in \mathbb{N}$ such that $P(\Omega_{N_0}^c) < \frac{\varepsilon}{2}$. Define

$$Z(\omega) := \begin{cases} V(\omega) & \text{for } \omega \in \Omega_{N_0} \\ A(\omega) & \text{for } \omega \notin \Omega_{N_0}, \end{cases}$$

then Z is a compact random set with $\varphi(t,\omega)Z(\omega) \subset U(\theta_t\omega)$ for all ω and $t \geq 0$, and therefore $\lim d\big(\varphi(t,\omega)Z(\omega), A(\theta_t\omega)\big) = 0$ in probability for $t \to \infty$. Consequently, there exists $T(\varepsilon) > 0$ such that $P\big\{d\big(\varphi(t,\omega)Z(\omega), A(\theta_t\omega)\big) > \varepsilon\big\} \leq \frac{\varepsilon}{2}$ for all $t > T(\varepsilon)$. We therefore get

$$\begin{aligned} & P\big\{\omega : d\big(\varphi(t,\omega)V(\omega), A(\theta_t\omega)\big) > \varepsilon\big\} \\ \leq\ & P\Big(\big\{\omega : d\big(\varphi(t,\omega)V(\omega), A(\theta_t\omega)\big) > \varepsilon\big\} \cap \Omega_{N_0}\Big) + P(\Omega_{N_0}^c) \\ \leq\ & P\big\{d\big(\varphi(t,\omega)Z(\omega), A(\theta_t\omega)\big) > \varepsilon\big\} + P(\Omega_{N_0}^c) \\ <\ & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{for all } t > T(\varepsilon). \end{aligned}$$

which implies that V is attracted by A in probability. Since this holds for every closed $V \subset C$, A is an attractor with basin of attraction C. □

5 Morse Decompositions for Random Dynamical Systems

One of the building blocks of Morse theory is an attractor-repeller pair, i.e. a repeller corresponding to a given attractor. The proof of the following proposition follows Ochs [14].

5.1 Proposition (Repeller from Attractor) *Let φ be an RDS on a compact metric space X and let A be an attractor with a forward invariant fundamental neighborhood U, i.e., C as defined in* (6) *is the basin of attraction. Then R, given by*

$$R(\omega) = X \setminus C(\omega),$$

is a repeller with basin of repulsion $X \setminus A(\omega)$, and A and R are disjoint.

PROOF Lemma 4.5 implies that $R(\omega) = X \setminus C(\omega)$ is an invariant compact random set. Moreover, $A(\omega) \subset C(\omega)$, thus $A(\omega) \cap R(\omega) = \emptyset$.
Let $\varepsilon > 0$. Then $B_\varepsilon(R(\omega)) := \{x \in X : d(x, R(\omega)) < \varepsilon\}$ is an open random set. Let V be a random compact set with $V \subset A^c$. Using $\varphi(-t, \theta_t\omega)^{-1} = \varphi(t, \omega)$ and θ_t-invariance of P, we get for $t \geq 0$

$$\begin{aligned} P\{d(\varphi(-t,\theta_t\omega)V(\theta_t\omega), R(\omega)) \geq \varepsilon\} &= P\{\varphi(-t,\theta_t\omega)V(\theta_t\omega) \cap B_\varepsilon(R(\omega))^c \neq \emptyset\} \\ &= P\{\varphi(t,\omega)B_\varepsilon(R(\omega))^c \cap V(\theta_t\omega) \neq \emptyset\} \\ &= P\{\varphi(t,\theta_{-t}\omega)B_\varepsilon(R(\theta_{-t}\omega))^c \cap V(\omega) \neq \emptyset\}. \end{aligned}$$

Since $V \cap A = \emptyset$, we can find a $\delta > 0$ with $P\{V(\omega) \cap B_\delta(A(\omega)) \neq \emptyset\} < \frac{\varepsilon}{2}$. Since $B_\varepsilon(R(\omega))^c \subset C(\omega)$, the basin of attraction of A, there exists $T(\varepsilon, \delta) > 0$ such that

$$\begin{aligned} & P\big\{d\big(\varphi(t,\theta_{-t}\omega)B_\varepsilon(R(\theta_{-t}\omega))^c, A(\omega)\big) \geq \delta\big\} \\ &= P\big\{\varphi(t,\theta_{-t}\omega)B_\varepsilon(R(\theta_{-t}\omega))^c \cap B_\delta(A(\omega))^c \neq \emptyset\big\} < \frac{\varepsilon}{2} \end{aligned}$$

for all $t \geq T(\varepsilon, \delta)$. Now, we have

$$\begin{aligned} & P\big\{\varphi(t,\theta_{-t}\omega)B_\varepsilon(R(\theta_{-t}\omega))^c \cap V(\omega) \neq \emptyset\big\} \\ &= P\big\{\big(\varphi(t,\theta_{-t}\omega)B_\varepsilon(R(\theta_{-t}\omega))^c \cap V(\omega) \cap B_\delta(A(\omega))^c\big) \neq \emptyset\big\} \\ &\quad + P\big\{\big(\varphi(t,\theta_{-t}\omega)B_\varepsilon(R(\theta_{-t}\omega))^c \cap V(\omega) \cap B_\delta(A(\omega))\big) \neq \emptyset\big\} \\ &\leq P\big\{\varphi(t,\theta_{-t}\omega)B_\varepsilon(R(\theta_{-t}\omega))^c \cap B_\delta(A(\omega))^c \neq \emptyset\big\} + P\big\{V(\omega) \cap B_\delta(A(\omega)) \neq \emptyset\big\} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $t \geq T(\varepsilon, \delta)$, which implies that V is repelled by R. Consequently, R is a repeller with basin of repulsion $X \setminus A$. □

5.2 Corollary (Basin of Attraction is Well-Defined) *Let φ be an RDS on a compact metric space X and let A be an attractor with a forward invariant fundamental neighborhood U. Then the basin of attraction C, as defined in* (6)*, is independent of U almost surely.*

PROOF Suppose that A is an attractor with two basins of attraction C_1 and C_2. Then, by Proposition 5.3, $R_1(\omega) := X \setminus C_1(\omega)$ and $R_2(\omega) := X \setminus C_2(\omega)$ are two repellers with the same basin of repulsion $X \setminus A(\omega)$. Thus, R_1 is a random compact set lying in the basin of repulsion of R_2, so that

$$\lim_{t \to -\infty} d(\varphi(t, \theta_{-t}\omega)R_1(\theta_{-t}\omega), R_2(\omega)) = 0 \qquad \text{in probability.}$$

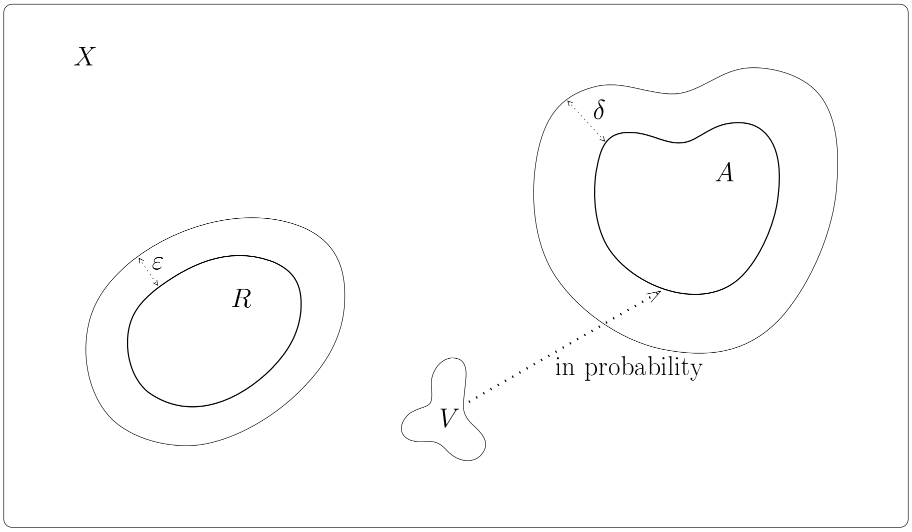

Figure 2: Construction of Repeller.

Invariance of R_1 implies

$$d(R_1(\omega), R_2(\omega)) = 0 \qquad \text{a.s.},$$

hence $R_1 \subset R_2$ a.s. Applying the same argument with the rôles of R_1 and R_2 interchanged we get $R_2 \subset R_1$ a.s., and thus $R_1 = R_2$ a.s. or, equivalently, $C_1 = C_2$ a.s. □

5.3 Definition (Attractor-Repeller Pair) Let φ be an RDS on a compact metric space X, and let A be an attractor with basin of attraction C. Then R, given by

$$R(\omega) = X \setminus C(\omega),$$

is called the *repeller corresponding to* A, and (A, R) is said to be an *attractor-repeller pair.*

5.4 Theorem *Suppose that $A_1 \subsetneq A_2$ a.s. are two attractors of φ with corresponding repellers R_1, R_2, respectively. Then $R_1 \supsetneq R_2$ a.s.*

Proof Since R_2 is a repeller with basin of repulsion A_2^c we have $R_2 \subset A_2^c$, and from $A_2^c \subset A_1^c$ we therefore get that R_2 is a compact random set in A_1^c, which is the basin of repulsion of R_1. Consequently, $\lim d\big(\varphi(-t, \theta_t\omega)R_2(\theta_t\omega), R_1(\omega)\big) = 0$ in probability for $t \to \infty$, which, in view of the invariance of R_2, implies $d(R_2(\omega), R_1(\omega)) = 0$ a.s., which means $R_2 \subset R_1$ a.s. It remains to prove that $R_1 \neq R_2$ a.s., which amounts to showing that $R_1 \setminus R_2 \neq \emptyset$ a.s.

Since $A_2 \setminus A_1 \neq \emptyset$ a.s., and A_1, A_2 are compact random sets, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $P(\Omega_\delta) \geq 1 - \frac{\varepsilon}{2}$, where $\Omega_\delta = \big\{\omega : d\big(A_2(\omega), A_1(\omega)\big) \geq \delta\big\}$. Put

$\delta(\omega) := \frac{1}{2}d(A_2(\omega), A_1(\omega))$, and define the random set

$$Z(\omega) := \begin{cases} B_{\delta(\omega)}(A_1^c(\omega)) \cap A_2(\omega) & \text{for } \omega \in \Omega_\delta \\ R_1(\omega) & \text{for } \omega \notin \Omega_\delta. \end{cases}$$

Then $Z(\omega) \subset A_1^c(\omega)$, so Z is repelled by R_1, hence for all $\delta_1 > 0$ there exists $T(\delta_1) > 0$ such that for all $t \geq T(\delta_1)$

$$P\big\{d\big(\varphi(-t, \theta_t\omega)Z(\theta_t\omega), R_1(\omega)\big) < \delta_1\big\} \geq 1 - \frac{\varepsilon}{2}$$

or, equivalently,

$$P\big\{\varphi(-t, \theta_t\omega)Z(\theta_t\omega) \subset B_{\delta_1}(R_1(\omega))\big\} \geq 1 - \frac{\varepsilon}{2}$$

for all $t \geq T(\delta_1)$. Now since $\varphi(-t, \theta_t\omega)Z(\theta_t\omega) \subset A_2(\omega)$ for all $\omega \in \Omega_\delta$, we have

$$\begin{aligned} & P\big(A_2 \cap B_{\delta_1}(R_1) \neq \emptyset\big) \\ \geq\ & P\big(\Omega_\delta \cap \{A_2 \cap B_{\delta_1}(R_1) \neq \emptyset\}\big) \\ \geq\ & P\big(\Omega_\delta \cap \big\{\varphi(-t, \theta_t\omega)Z(\theta_t\omega) \subset B_{\delta_1}(R_1(\omega))\big\}\big) \\ \geq\ & 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon \end{aligned}$$

for all $t \geq T(\delta_1)$, $\delta_1 > 0$ and $\varepsilon > 0$. That means, if we define $\Omega^*_{\delta_1} := \{\omega : A_2(\omega) \cap B_{\delta_1}(R_1(\omega)) \neq \emptyset\}$ then

$$P(\Omega^*_{\delta_1}) = 1 \text{ for every } \ \delta_1 > 0.$$

Put $\Omega^* := \cap_{n=1}^{\infty}\Omega^*_{\frac{1}{n}}$, then $P(\Omega^*) = 1$. We prove that $A_2(\omega) \cap R_1(\omega) \neq \emptyset$ for all $\omega \in \Omega^*$. Suppose that with some $\omega^* \in \Omega^*$ we have $A_2(\omega^*) \cap R_1(\omega^*) = \emptyset$. Then because of the compactness of $A_2(\omega^*)$ and $R_1(\omega^*)$, we have

$$\delta^* := \inf_{\substack{a \in A_2(\omega^*), \\ r \in R_1(\omega^*)}} d(a, r) > 0.$$

Thus $A_2(\omega^*) \cap B_{\delta^*}(R_1(\omega^*)) = \emptyset$, which is a contradiction to the definition of Ω^*. Therefore, $A_2 \cap R_1 \neq \emptyset$ a.s., and because $A_2 \cap R_2 = \emptyset$ a.s. we have $R_2 \neq R_1$ a.s., which yields $R_2 \subsetneq R_1$ a.s. □

5.5 Definition (Morse decomposition) Let φ be an RDS on a compact metric space X. Suppose that (A_i, R_i) are attractor-repeller pairs with

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = X \text{ a.s.} \qquad \text{and} \qquad X = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset \text{ a.s.}$$

Then the family $(M_i)_{i=1,\ldots,n}$ of subsets of X, defined by

$$M_i = A_i \cap R_{i-1}, \quad 1 \leq i \leq n$$

is called a *Morse decomposition* of X, and each M_i is called *Morse set.*

5.6 Example Consider the Stratonovich stochastic differential equation (SDE)

$$dX_t = (X_t - X_t^3)dt + (X_t - X_t^3) \circ dW_t.$$

in the interval $[-1,1]$. This SDE generates an RDS $\varphi : \mathbb{R} \times \Omega \times [-1,1] \to [-1,1]$ with

$$\varphi(t,\omega)x = \frac{x\mathrm{e}^{t+W_t(\omega)}}{(1 - x^2 + x^2\mathrm{e}^{2t+2W_t(\omega)})^{\frac{1}{2}}}$$

(see e.g. Kloeden [13], p. 123). It can be shown that $A_1 = \{-1\}$ is an attractor with basin of attraction $C = [-1,0)$; $A_2 = \{-1,1\}$ is an attractor with basin of attraction $C = [-1,0) \cup (0,1]$. Thus the corresponding repellers are $R_1 = [0,1]$ and $R_2 = \{0\}$. Therefore, the Morse sets are $M_1 = \{-1\}, M_2 = \{1\}$ and $M_3 = \{0\}$. Note that also $A_1 = \{1\}$, $R_1 = [-1,0]$ is an admissible choice yielding the Morse sets $M_1 = \{1\}, M_2 = \{-1\}$ and $M_3 = \{0\}$.

Example 5.6 is simply a modification of a deterministic Morse decomposition. For an example with random Morse sets, see Crauel [7] or the example in Section 6 below.

5.7 Lemma (Properties of Morse sets) *Morse sets are non-empty, invariant, pairwise disjoint and isolated.*

Proof Let $M_i = A_i \cap R_{i-1}$ be a Morse set. Then M_i is non-empty by Theorem 5.4. Furthermore, M_i is the intersection of two invariant sets and hence invariant. Let $M_j = A_j \cap R_{j-1}$ be another Morse set, assuming $i < j$ without loss of generality. Using the fact that $A_i \subset A_{j-1} \subset A_j$, $R_{j-1} \subset R_j$, we get

$$M_i \cap M_j = A_i \cap R_{i-1} \cap A_j \cap R_{j-1} = A_i \cap R_{j-1} \subset A_{j-1} \cap R_{j-1} = \emptyset.$$

To prove that M_i is isolated note that δ_A and δ_R, defined by

$$\delta_A(\omega) := \inf_{m \in M_i(\omega)} d(m, A_{i-1}(\omega)) \qquad \text{and} \qquad \delta_R(\omega) := \inf_{m \in M_i(\omega)} d(m, R_i(\omega)),$$

are random variables, and hence

$$U(\omega) := B_{\delta(\omega)}(M_i(\omega)) \qquad \text{with } \delta(\omega) = \tfrac{1}{2}\min\{\delta_A(\omega), \delta_R(\omega)\}$$

is a random set. Suppose that x is a random variable which satisfies

$$\varphi(t,\omega)x(\omega) \in U(\theta_t\omega) \qquad \text{for all } t \in \mathbb{R} \tag{8}$$

and $P\{x(\omega) \notin M_i(\omega)\} > 0$. Using the fact that $M_i^c = A_i^c \cup R_{i-1}^c$, A_i^c is the basin of repulsion of R_i and R_{i-1}^c is the basin of attraction of A_{i-1}, we can choose $T > 0$ such that $P\{d(\varphi(T,\omega)x(\omega), A_{i-1}(\omega)) < \delta\} > 0$ and $P\{d(\varphi(-T,\omega)x(\omega), R_i(\omega)) < \delta\} > 0$, contradicting (8). □

5.8 Remark (i) Using the theory of RDS on random subsets (see Chapter 1.9 in Arnold [1]) one can easily generalize the notion of a Morse decomposition to RDS which are restricted to compact invariant random subsets.

(ii) If $(M_i)_{i=1,\dots,n}$ is a Morse decomposition with defining attractor-repeller sequence (A_i, R_i) then (A_{i-1}, M_i) is an attractor-repeller pair in the compact metric space A_i.

(iii) Suppose that we are given a Morse decomposition $(M_i)_{i=1,\dots,n}$ with defining attractor-repeller sequence (A_i, R_i), $1 \leq i \leq n$. The question now is how to recover (A_i, R_i) based on what we know about $(M_i)_{i=1,\dots,n}$. To answer this question, first note that $M_1 = A_1$ and $M_n = R_{n-1}$. Since (A_{n-1}, R_{n-1}) is an attractor-repeller pair in the invariant compact metric space $X = A_n$, we can recover A_{n-1} by using the information we know about R_{n-1} (in fact, A_{n-1} is the complement in $X = A_n$ of the basin of repulsion of $M_n = R_{n-1}$). In general, suppose that we know A_i. Now, using (a), we can recover A_{i-1} by using the fact that A_{i-1} is the complement in A_i of the basin of repulsion of M_i. The same argument is used to recover the repeller R_{i-1} from the attractor A_{i-1}. Therefore by induction we can recover all attractor-repeller pairs based on the Morse decomposition.

We are now able to formulate our main result on a dynamical interpretation of a Morse decomposition.

5.9 Theorem (Dynamical Properties of Morse Decomposition) *Suppose that $(M_i)_{i=1,\dots,n}$ is a Morse decomposition for an RDS φ on a compact metric space X, given by attractor-repeller pairs (A_i, R_i), $1 \leq i \leq n$. Then the compact random set $\mathcal{M}$, given by*

$$\mathcal{M}(\omega) = \bigcup_{i=1}^{n} M_i(\omega),$$

determines the limiting behaviour of φ on X. Furthermore, cycles between the Morse sets are not allowed. More precisely:

(i) *For every X-valued random variable x the set $\mathcal{M}$ attracts both forward and backward in time, which means that $d\big(\varphi(t,\omega)x(\omega), \mathcal{M}(\theta_t\omega)\big)$ converges to 0 in probability for $t \to \pm\infty$.*

(ii) *If a random variable x is attracted by M_i and repelled by M_j for some $1 \leq i, j \leq n$, i.e., if*

$$\lim_{t\to\infty} d\big(\varphi(t,\omega)x(\omega), M_i(\theta_t\omega)\big) = 0 \textit{and} \lim_{t\to-\infty} d\big(\varphi(t,\omega)x(\omega), M_j(\theta_t\omega)\big) = 0$$

in probability, then $i \leq j$.

(iii) *If the conditions of (ii) are satisfied, then $i = j$ if and only if $x \in M_i$ a.s.*

(iv) *If $x_1, \dots, x_p$ are X-valued random variables such that for some $1 \leq j_0, \dots, j_p \leq n$, x_k is repelled by $M_{j_{k-1}}$ and attracted by M_{j_k}, then $j_0 \leq j_p$. Furthermore, $j_0 < j_p$ if and only if $P(x_k \notin \mathcal{M}) > 0$ for some k, whereas otherwise $j_0 = \dots = j_p$.*

Proof Invariance and compactness of $\mathcal{M}$ follow from invariance and compactness of the Morse sets M_i, $1 \leq i \leq n$.

In order to prove (i), partition Ω into

$$\Omega_i = \big\{x(\omega) \in R_i^c(\omega) \cap R_{i-1}(\omega)\big\}, \qquad 1 \le i \le n,$$

recalling that we have $X = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset$ for the repellers. Then

$$\begin{aligned} &P\{d\big(\varphi(t,\omega)x(\omega), \mathcal{M}(\omega)\big) > r\} \\ &= \sum_{i=0}^{n} P\Big(\big\{d\big(\varphi(t,\omega)x(\omega), M_i(\omega)\big) > r\big\} \cap \Omega_i\Big) \\ &= \sum_{i=0}^{n} P\Big(\big\{d\big(\varphi(t,\omega)x(\omega), A_i(\omega) \cap R_{i-1}(\omega)\big) > r\big\} \cap \Omega_i\Big), \end{aligned}$$

where each of the components of the last sum is small for t sufficiently big due to the fact that $x(\omega) \in R_i^c(\omega) \cap R_{i-1}(\omega)$ for $\omega \in \Omega_i$, whence it is attracted by $A_i \cap R_{i-1}$. Therefore x is attracted by $\mathcal{M}$ forward in time.

A completely analogous argument, using a partition of Ω according to repulsion by putting $\Omega_i = \{x(\omega) \in A_{i-1}^c(\omega) \cap A_i(\omega)\}$, $1 \le i \le n$, gives the assertion for attraction of $\mathcal{M}$ backward in time.

To prove assertion (ii), note that repulsion of $M_j = A_j \cap R_{j-1}$ implies that $P\big\{d\big(\varphi(-T,\theta_T\omega)x(\theta_T\omega), A_j(\omega)\big) \le \delta\big\}$ can be made arbitrarily close to one by choosing $T > 0$ large enough, which means that $\varphi(-T,\theta_T\omega)x(\theta_T\omega) \in \overline{B}_\delta(A_j(\omega))$ with probability arbitrarily close to one. Since $\overline{B}_\delta(A_j)$ is in the domain of attraction of A_j for δ sufficiently small with arbitrarily large probability, this implies that x is attracted by A_j with positive probability. Having assumed x to be attracted by $M_i = A_i \cap R_{i-1}$, we infer from $R_{i-1} \subset A_{i-1}^c$ that $A_j \cap A_{i-1}^c \ne \emptyset$ with positive probability (and therefore with probability one). Now $j < i$ would imply $A_j \subset A_{i-1}$, and therefore $A_j \cap A_{i-1}^c = \emptyset$, which would give a contradiction. Consequently, we must have $i \le j$.

In order to prove (iii), suppose that $i = j$. If $x \in M_i$ a.s. is not satisfied then $P\{x \notin A_i\} > 0$, or $P\{x \notin R_{i-1}\} > 0$. In case $x \notin A_i$ with positive probability, it is repelled by R_i with the same probability, which contradicts repulsion of x by M_i, since $R_i \cap M_i \subset R_i \cap A_i = \emptyset$ a.s., while in case $x \notin R_{i-1}$ with positive probability, then it is attracted by A_{i-1} with positive probability, whence attraction of x by M_i contradicts $A_{i-1} \cap M_i \subset A_{i-1} \cap R_{i-1} = \emptyset$ a.s. Consequently, attraction and repulsion of x by the same Morse set M_i implies $x \in M_i$ a.s. The other direction of the assertion is immediate from the invariance of the Morse sets.

Finally, to prove (iv) first note that (ii) implies $j_{k-1} \le j_k$ for $1 \le k \le p$, and (iii) gives $j_{k-1} < j_k$ in case $P(x_k \notin \mathcal{M}) > 0$ for some k. □

Note that the assertion of Theorem 5.9 does not translate literally to arbitrary (random) sets instead of points. In fact, already the deterministic example $\dot{x} = x(1-x)$ on $X = [0,1]$ has a Morse decomposition given by $A_1 = \{1\} \subsetneq A_2 = X$ and $R_1 = \{0\} \supsetneq R_2 = \emptyset$, giving $M_1 = A_1 = \{1\}$ and $M_2 = R_1 = \{0\}$. The compact set X is neither attracted nor repelled by $\mathcal{M} = M_1 \cup M_2 = \{1\} \cup \{0\}$. If, however, the random sets are assumed to have nonempty intersection with the Morse sets $\mathcal{M}$ a.s. then the assertions of Theorem 5.9 hold for those sets instead of points.

5.10 Remark (Characterization of Morse decomposition) Is the inverse of Theorem 5.9 true? Assume that M_i, $1 \leq i \leq n$, are compact invariant random sets such that conditions (i)–(iv) of Theorem 5.9 hold. Can we show that $(M_i)_{i=1,\dots,n}$ is a Morse decomposition? Actually property (ii) allows us to define an order relation

$$M_i \leq M_j \qquad :\Leftrightarrow$$

there are indices $j_0, \dots, j_p$ with $j_0 = i$ and $j_p = j$, and there are random variables $x_1, \dots, x_p \in X$, such that, for $k = 1, \dots, p$,

$$\lim_{t \to -\infty} d\big(\varphi(t,\omega)x_k(\omega), M_{j_{k-1}}(\theta_t\omega)\big) = 0 \quad \text{and} \quad \lim_{t \to \infty} d\big(\varphi(t,\omega)x_k(\omega), M_{j_k}(\theta_t\omega)\big) = 0.$$

Thus we can relabel the indices such that $M_j \leq M_k$ if and only if $j \leq k$.

To show that $(M_i)_{i=1,\dots,n}$ is a Morse decomposition we would have to show that $M_1 = A_1$ is an attractor in the sense of Definition 4.1, i.e. attracting closed sets in its basin. The crucial point is to find a forward invariant fundamental neighborhood. We are not aware of a technique to construct such a forward invariant neighborhood of M_1 in general. However, if we assume that $(M_i)_{i=1,\dots,n}$ is a Morse decomposition then we can show that

$$C_{j,i} := A_{j-1}^c \cap R_i^c \cap A_j \cap R_{i-1} \qquad \text{for } j > i$$

are invariant, open random sets with the property that closed random sets V with $V \subset C_{j,i}$ a.s. are repelled by M_j and attracted by M_i. Moreover, the $C_{j,i}$ and the M_i form a partition

$$X = \bigcup_{i=1}^{n} M_i \ \cup \ \bigcup_{j>i} C_{j,i}.$$

We see that the existence of those sets $C_{j,i}$ is a necessary condition for $(M_i)_{i=1,\dots,n}$ to be a Morse decomposition. On the other hand, once we have invariant, open random sets $C_{j,i}$ disjoint from $\mathcal{M}$ and with the property that closed random sets $V \subset C_{j,i}$ are repelled by M_j and attracted by M_i, we can use similar arguments as in Remark 5.8 to recover an attractor sequence A_i with corresponding repeller sequence R_i such that $M_i = A_i \cap R_{i-1}$, hence proving the inverse of Theorem 5.9.

6 A Prototypical Example

Suppose that $\Phi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a linear RDS over $(\Omega, \mathscr{F}, P; (\theta_t)_{t\in\mathbb{R}})$, satisfying the integrability condition of Oseledets' Multiplicative Ergodic Theorem (MET) (see, e.g., Arnold [1] Part II, Theorems 3.4.1 and 3.4.11 or, for the original formulation, Oseledets [15]). Then there are real numbers $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ for some n, $1 \le n \le d$, called the *Lyapunov exponents* of Φ, such that $\lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)x\| \in \{\lambda_1, \ldots, \lambda_d\}$ for all $x \in \mathbb{R}^d \setminus \{0\}$, for P-almost all ω. Furthermore, there are invariant random linear subspaces $E_1, \ldots, E_n$, called *Oseledets spaces*, such that

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)x\| = \lambda_i \quad \text{if and only if} \quad x \in E_i(\omega) \setminus \{0\}$$

where convergence is uniform with respect to $x \in E_i(\omega) \cap S^{d-1}$. In particular, $E_1(\omega) \oplus \cdots \oplus E_n(\omega) = \mathbb{R}^d$.

Note that equivalence of norms in finite dimensional linear spaces implies that all the assertions of the MET do not depend on the choice of a norm.

Linearity and invertibility of Φ imply that Φ induces RDS on each of the homogeneous spaces of the general linear group, amongst which there are the Graßmann manifolds and, as a special case of these, the projective space $\mathbb{P}^{d-1}$. For a systematic presentation see, e.g., Arnold [1] Chapter 6. We restrict ourselves here to the case of the projective space. Denote by $\mathbb{P}\Phi : \mathbb{R} \times \mathbb{P}^{d-1} \times \Omega \to \mathbb{P}^{d-1}$ the RDS induced by Φ on $\mathbb{P}^{d-1}$. Note that $\mathbb{P}^{d-1}$ is compact. Projecting the Oseledets spaces $E_1, \ldots, E_n$ of Φ to $\mathbb{P}^{d-1}$ defines compact random invariant sets $M_i = \mathbb{P}E_i$ for $\mathbb{P}\Phi$.

The proof of the next theorem is essentially a reformulation of well known results. We give a simplified argument for the case of Oseledets spaces which are orthogonal with respect to some scalar product, and we give the arguments for the unit sphere instead of the projective space. In the general case one may then use the fact that uniform attraction in probability is invariant under Lyapunov cohomology, and that general systems are Lyapunov cohomologous to systems with orthogonal Oseledets spaces. See, e.g., Arnold [1] Section 4.3 for details.

6.1 Theorem *Let Φ be a linear RDS satisfying the assumptions of the MET. Then the compact invariant random sets $M_i = \mathbb{P}E_i$ define a Morse decomposition of the induced RDS $\mathbb{P}\Phi$ on $\mathbb{P}^{d-1}$.*

PROOF It is sufficient to prove that $A_j = \mathbb{P}(E_1 \oplus \cdots \oplus E_j)$ defines a set attractor, the basin of attraction of which is $\mathbb{P}^{d-1} \setminus R_j$, where $R_j = \mathbb{P}(E_{j+1} \oplus \cdots \oplus E_n)$ is the repeller corresponding to A_j.

Suppose that the Oseledets spaces are orthogonal with respect to some scalar product, and denote by $\varphi(t,\omega)x = \frac{\Phi(t,\omega)x}{\|\Phi(t,\omega)x\|}$ the RDS induced by Φ on the unit sphere S^{d-1} with respect to the norm induced by the corresponding scalar product. We prove the stronger statement

$$\lim_{t\to\infty} d\big(\varphi(t,\omega)V(\omega), A_j(\theta_t\omega)\big) = 0 \qquad \text{a.s.}$$

for any random compact set V with $V(\omega) \subset S^{d-1} \setminus R_j(\omega)$ a.s., which implies convergence in probability.

Fixing ω, every $x \in V(\omega)$ can be written as $x = \sum_{i=1}^{j} \alpha_i x_i + \sum_{i=j+1}^{n} \alpha_i x_i$, where $\alpha_i x_i$ is the image of x under the projection to E_i, and $x_i \in E_i \cap S^{d-1}$. So we have $\|x\| = \|x_i\| = 1$ for $i = 1, \dots, n$. Since $x \notin R_i$, $\sum_{i=1}^{j} \alpha_i x_i \neq 0$. The map $P_j : V(\omega) \to E_1 \oplus \cdots \oplus E_j$, $x \mapsto \sum_{i=1}^{j} \alpha_i^2$, is continuous, hence $\delta^2 = \min_{x \in V(\omega)} \|P_j(x)\| > 0$. From the pairwise orthogonality of $E_i(\omega)$ and $\varphi(t,\omega)E_i(\omega) \cap S^{d-1} = E_i(\theta_t\omega) \cap S^{d-1}$, we get

$$\Big\| \sum_{i=1}^{j} \varphi(t,\omega)\alpha_i x_i \Big\|^2 = \sum_{i=1}^{j} \|\varphi(t,\omega)\alpha_i x_i\|^2 = \sum_{i=1}^{j} |\alpha_i|^2 \|\varphi(t,\omega)x_i\|^2 = \sum_{i=1}^{j} |\alpha_i|^2 \geq \delta^2.$$

Similary, $\big\| \sum_{i=j+1}^{n} \varphi(t,\omega)\alpha_i x_i \big\|^2 \leq \Delta^2$ for some $\Delta > 0$.
The MET implies $\lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)y\| = \lambda_i$ uniformly on $M_i, i = 1, \dots n$. Therefore for any $0 < \varepsilon < \frac{1}{2} \min_{i<j}(\lambda_i - \lambda_j)$, there exists $T(\varepsilon) > 0$ such that for all $t \geq T(\varepsilon) > 0$

$$\begin{aligned} \|\Phi(t,\omega)y\| &\geq \mathrm{e}^{(\lambda_i - \varepsilon)t}, \text{ for all } i = 1, \dots, j, \\ \|\Phi(t,\omega)y\| &\leq \mathrm{e}^{(\lambda_i + \varepsilon)t}, \text{ for all } i = j+1, \dots, n. \end{aligned}$$

Then we have

$$\begin{aligned} \Big\| \Phi(t,\omega) \sum_{i=1}^{j} \alpha_i x_i \Big\| &= \Big\| \sum_{i=1}^{j} \|\Phi(t,\omega)x_i\| \varphi(t,\omega)\alpha_i x_i \Big\| \\ &\geq \min_{1\leq i \leq j} \|\Phi(t,\omega)x_i\| \Big\| \sum_{i=1}^{j} \varphi(t,\omega)\alpha_i x_i \Big\| \geq \delta \mathrm{e}^{(\lambda_j - \varepsilon)t} \end{aligned}$$

and

$$\begin{aligned} \Big\| \Phi(t,\omega) \sum_{i=j+1}^{n} \alpha_i x_i \Big\| &= \Big\| \sum_{i=j+1}^{n} \|\Phi(t,\omega)x_i\| \varphi(t,\omega)\alpha_i x_i \Big\| \\ &\leq \max_{j+1\leq i \leq n} \|\Phi(t,\omega)x_i\| \Big\| \sum_{i=j+1}^{n} \varphi(t,\omega)\alpha_i x_i \Big\| \leq \Delta \mathrm{e}^{(\lambda_{j+1} + \varepsilon)t}, \end{aligned}$$

where we use $x_i \in E_i$, and that the Oseledets spaces are assumed to be orthogonal. Now put $u = \Phi(t,\omega) \sum_{i=1}^{j} \alpha_i x_i$ and $v = \Phi(t,\omega) \sum_{i=j+1}^{n} \alpha_i x_i$. Then for every $t \geq T(\varepsilon)$ we have $\|u\| \geq \delta \mathrm{e}^{(\lambda_j - \varepsilon)t}$ and $\|v\| \leq \Delta \mathrm{e}^{(\lambda_{j+1}+\varepsilon)t}$. Therefore

$$\begin{aligned} & d\big(\varphi(t,\omega)V(\omega), A_j(\theta_t\omega)\big) = \max_{x\in V(\omega)} \min_{a \in A(\theta_t\omega)} d(\varphi(t,\omega)x, a) \\ \leq\ & \max_{x\in V(\omega)} d\bigg(\varphi(t,\omega)x, \frac{\Phi(t,\omega)\sum_{i=1}^{j} \alpha_i x_i}{\|\Phi(t,\omega)\sum_{i=1}^{j} \alpha_i x_i\|} \bigg) = \max_{x\in V(\omega)} \Big\| \frac{u+v}{\|u+v\|} - \frac{u}{\|u\|} \Big\| \\ \leq\ & \max_{x\in V(\omega)} \Big\| \frac{u}{\|u\| - \|v\|} - \frac{u}{\|u\|} \Big\| + \max_{x\in V(\omega)} \frac{v}{\|u\| - \|v\|} \\ \leq\ & \max_{x\in V(\omega)} 2 \frac{\|v\|}{\|u\| - \|v\|} \leq 2 \frac{\frac{\Delta}{\delta} \mathrm{e}^{(2\varepsilon + \lambda_{j+1} - \lambda_j)t}}{1 - \frac{\Delta}{\delta} \mathrm{e}^{(2\varepsilon + \lambda_{j+1} - \lambda_j)t}} \simeq \mathrm{e}^{(2\varepsilon + \lambda_{j+1} - \lambda_j)t} \to 0 \text{ as } t \to \infty. \end{aligned}$$

Consequently, V is attracted by A_j. □

7 Discussion and Conclusions

In this paper Morse theory for RDS has been discussed with respect to attraction and repulsion in probability. We claim that pullback and/or forward attraction and repulsion will fail to serve our claim. In this section we briefly discuss some aspects.

Recall that we defined a repeller R associated to an attractor A to be the complement of its basin of attraction, see Proposition 5.1. The deterministic theory uses a different characterization, namely it defines

$$R = \{x \in X \ : \ (\text{Omega limit set of } x) \cap A = \emptyset\}.$$

The notion of Omega limit sets is available for RDS as well. For a random set B one defines

$$\Omega_B(\omega) = \bigcup_{T\geq 0} \overline{\bigcap_{t\geq T} \varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega)}. \tag{9}$$

It is well known that Omega limit sets are forward invariant, and they are invariant in case $\big(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega)\big)_{t\geq 0}$ is pre-compact. For an RDS φ consisting of homeomorphisms the proof is straightforward.

7.1 Lemma *Suppose that φ is an RDS on a compact metric space X such that $\varphi(t, \omega)$ is a homeomorphism for any $t \geq 0$ and $\omega \in \Omega$. Then the Omega limit set Ω_B of any random set B is invariant, i.e.*

$$\varphi(t, \omega)\Omega_B(\omega) = \Omega_B(\theta_s\omega) \qquad \textit{for all } t \geq 0, \ \omega \in \Omega.$$

Proof For any $t \geq 0$

$$\begin{aligned}
\varphi(t, \omega)\Omega_B(\omega) &= \varphi(t, \omega) \bigcap_{T\geq 0} \overline{\bigcup_{s\geq T} \varphi(s, \theta_{-s}\omega)B(\theta_{-s}\omega)} \\
&= \bigcap_{T\geq 0} \overline{\bigcup_{s\geq T} \varphi(t, \omega)\varphi(s, \theta_{-s}\omega)B(\theta_{-s}\omega)} \\
&= \bigcap_{T\geq 0} \overline{\bigcup_{s\geq T} \varphi(s + t, \theta_{-t-s}\theta_s\omega)B(\theta_{-t-s}\theta_s\omega)} \\
&= \bigcap_{T\geq 0} \overline{\bigcup_{s\geq T+t} \varphi(s, \theta_{-s}\theta_t\omega)B(\theta_{-s}\theta_t\omega)} = \Omega_B(\theta_s\omega).
\end{aligned}$$

□

If one wants to define the repeller associated to an attractor for an RDS φ by imitating the deterministic construction, i.e., by putting

$$R(\omega) = \{x \in X : \Omega_x(\omega) \cap A(\omega) = \emptyset\}, \tag{10}$$

one gets in view of Lemma 7.1

$$\begin{aligned} R(\theta_t\omega) &= \{x \in X : \Omega_x(\theta_t\omega) \cap A(\theta_t\omega) = \emptyset\} \\ &= \{x \in X : \varphi(t,\omega)\big(\Omega_x(\omega) \cap A(\omega)\big) = \emptyset\} \\ &= \{x \in X : \Omega_x(\omega) \cap A(\omega) = \emptyset\} = R(\omega) \end{aligned} \tag{11}$$

for every $t \geq 0$. Ergodicity of (θ_t) then implies that R is constant P-a.s., independent of ω, so (10) always gives a deterministic set. We have neither been able to prove nor to disprove that (10) defines an invariant set for φ. Note that this means to make an assertion about $\Omega_z(\omega)|_{z\,=\,\varphi(-t,\theta_t\omega)x}$, i.e., calculate Ω_z for deterministic z, and then evaluate it in $\varphi(-t,\theta_t\omega)x$. We give two examples to exhibit how (10) may look like.

7.2 Example Consider $X = S^1$ with a two-sided time RDS having a point attractor $A(\omega) = \{a(\omega)\}$, and a repeller $R(\omega) = \{r(\omega)\}$. Such an example is given, for instance, by the SDE

$$dx = \sin x \circ dW_1(t) + \cos x \circ dW_2(t)\,,$$

where W_1, W_2 are independent one-dimensional Wiener processes, see Crauel [7] Example 4.2. Using the same argument as in (11) one obtains that $\big\{x \in S^1 : \Omega_x(\omega) = \{a(\omega)\}\big\}$ as well as $\big\{x \in S^1 : \Omega_x(\omega) = \{r(\omega)\}\big\}$ are (θ_t)-invariant, and consequently they are both constant in ω outside a P-nullset. This implies that $\Omega_x(\omega) = \{a(\omega)\}$ for every $x \in S^1$, for P-almost all $\omega \in \Omega$, where the nullset does not depend on $x \in S^1$. Consequently,

$$R(\omega) = \{x \in S^1 : \Omega_x(\omega) \cap A(\omega) = \emptyset\} = \emptyset$$

for P-almost all $\omega \in \Omega$. Note that one should not confuse the Omega limit set $\Omega_r(\omega)$ of the random variable r, which is defined in (9), and satisfies $\Omega_r(\omega) = \{r(\omega)\}$, with $\Omega_z(\omega)|_{z\,=\,r(\omega)}$. The latter is the Omega limit set of a deterministic point, evaluated at the random variable r.

7.3 Example In Crauel [9] an example of a two-sided RDS on $X = S^1$ with a non-trivial forward attractor – hence also a weak attractor – is constructed, which does not have a non-trivial pullback attractor (just the compact state space S^1 itself). In particular, here one has $\Omega_x(\omega) = S^1$, hence the weak attractor is a proper subset of every $\Omega_x(\omega)$, for every $x \in S^1$. Again (10) gives $R(\omega) = \emptyset$.

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