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Ilchmann, Achim; Mehrmann, Volker

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Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69 3621
Fax: +49 3677 69 3270
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A behavioural approach to linear time-varying descriptor systems

Achim Ilchmann* and **Volker Mehrmann**†

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Abstract

We introduce a behavioural approach to linear, time-varying, differential algebraic (descriptor) systems. The analysis is "almost global" in the sense that the analysis is not restricted to an interval $\mathbb{I} \subset \mathbb{R}$ but is allowed for the "time axis" $\mathbb{R}\backslash\mathbb{T}$, where $\mathbb{T}$ is a discrete set of critical points, at which the solution may exhibit a finite escape time. Controllable, observable, autonomous, and adjoint behaviour for linear time-varying descriptor systems is introduced and characterized.

Keywords: Time-varying linear systems, descriptor systems, behavioural approach, controllability, observability, autonomous system, adjoint system

Nomenclature

$\mathcal{A}$	the ring of real analytic functions $f : \mathbb{R} \to \mathbb{R}$
$\mathcal{M}$	the field of real meromorphic functions
$\mathcal{A}[D]$, $\mathcal{M}[D]$	the skew polynomial ring of differential polynomials with coefficients in $\mathcal{A}, \mathcal{M}$ resp., indeterminate D, and multiplication rule $Df = fD + \dot{f}$
$\mathcal{C}^\infty(\mathbb{R}\backslash\mathbb{T}; \mathbb{R}^q)$	the real vector space of infinitely many times differentiable functions $f : \mathbb{R}\backslash\mathbb{T} \to \mathbb{R}^q$, $\mathbb{T} \subset \mathbb{R}$ a discrete set
$\mathcal{C}^\omega(\mathbb{I}, \mathbb{R}^q)$	the real vector space of real analytic functions $f : \mathbb{I} \to \mathbb{R}^q$, $\mathbb{I} \subset \mathbb{R}$ an open interval
I_d	$:= \text{diag}\{1, \ldots, 1\} \in \mathbb{R}^{d\times d}$
0_d	$:= (0, \ldots, 0)^T \in \mathbb{R}^d$

1 Introduction

We develop a behavioural approach to linear time-varying desriptor systems described by differential-algebraic equations of the form

$$\begin{aligned} E(t)\, \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t) + F(t)u(t), \end{aligned} \tag{1.1}$$

*Institut für Mathematik, Technische Universität Ilmenau, Weimarer Straße 25, 98693 Ilmenau, FRG; ilchmann@mathematik.tu-ilmenau.de

†Institut für Mathematik, MA 4-5, Technische Universität Berlin, Straße des 17. Juni 136, D-10623 Berlin, FRG; This research was supported by Deutsche Forschungsgemeinschaft through DFG Research Center *Mathematics for key technologies* in Berlin; mehrmann@math.tu-berlin.de

with real analytic matrices $A \in \mathcal{A}^{\ell\times n}$, $B \in \mathcal{A}^{\ell\times m}$, $C \in \mathcal{A}^{p\times n}$, $F \in \mathcal{A}^{p\times m}$, and $E \in \mathcal{A}^{\ell\times n}$ is allowed to be singular in the sense that $\operatorname{rk} E(t) < \min\{l, n\}$ for some $t \in \mathbb{R}$.

For notational convenience, we introduce the skew-polynomial rings $\mathcal{A}[D]$ and $\mathcal{M}[D]$ of differential polynomials with coefficients in $\mathcal{A}$, $\mathcal{M}$, respectively, indeterminate D, and multiplication rule

$$Df = fD + \tfrac{\mathrm{d}}{\mathrm{dt}} f. \tag{1.2}$$

This rule is a consequence of assuming the associative rule $(Df)g = D(fg)$, which yields $(Df)(g) = \frac{\mathrm{d}}{\mathrm{dt}} f \cdot g + f \cdot \frac{\mathrm{d}}{\mathrm{dt}} g = \left(\frac{\mathrm{d}}{\mathrm{dt}} f + fD\right)(g)$. Note that we distinguish between the algebraic indeterminate D and the differential operator $\frac{\mathrm{d}}{\mathrm{dt}}$; the algebraic object

$$R(D) = \sum_{i=0}^{n} R_i D^i \in \mathcal{M}[D]^{g\times q} \cong \mathcal{M}^{g\times q}[D],$$

acts on $\mathcal{C}^\infty$-functions w via

$$R(\tfrac{\mathrm{d}}{\mathrm{dt}})w(t) = \textstyle\sum_{i=0}^{n} R_i(t) w^{(i)}(t)\,.$$

In this notation, time-varying descriptor systems (1.1) may be rewritten as

$$R(\tfrac{\mathrm{d}}{\mathrm{dt}})w = 0, \quad \text{where } R(D) = \begin{bmatrix} ED - A & -B & 0 \\ -C & -F & I_p \end{bmatrix}, \quad \text{and} \quad w = \left(x^T, u^T, y^T\right)^T. \tag{1.3}$$

Skew polynomial rings are for example treated in the monograph [6], the ring $\mathcal{M}[D]$ has been introduced in [14] to study linear time-varying systems, and in [13] its algebraic properties have been exploited to achieve results on time-varying descriptor systems. In the present paper, we use $\mathcal{M}[D]$ only for notational convenience and if more general results in the context of [13] are used.

Systems of differential algebraic equations (often called descriptor systems) play an important role in modelling multi-body systems, electric circuits, or coupled systems of partial differential equations, see [1, 9]. In [11], for example, the model of a two-dimensional, three-link constrained mobile manipulator is studied which leads, after linearization along a trajectory, to a system of the form

$$\begin{aligned} M_0(t)\,\ddot{z}(t) + D_0(t)\,\dot{z}(t) + K_0(t)\,z(t) &= S_0\,u(t) + F_0^T\,\mu(t) \\ F_0\,z(t) &= 0\,, \end{aligned} \tag{1.4}$$

where $M_0, D_0, K_0 \in \mathcal{C}^\omega(\mathbb{I}, \mathbb{R}^{3\times 3})$ and $S_0, F_0^T \in \mathbb{R}^{3\times 2}$ with S_0 having full rank. We are interested in the behaviour, i.e. local solutions $t \mapsto (z(t), u(t))$ of (1.4). It can be shown that $\mu(\cdot)$ is a latent variable, see [21, Sec. 6.3] for its definition. Introducing the 8-dimensional variable $x(t) = [z(t), \dot{z}(t), \mu(t)]$ results in the equivalent descriptor system description (1.1) with $F \equiv 0$,

$$E(t) = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & M_0(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A(t) = \begin{bmatrix} 0 & I_3 & 0 \\ -K_0(t) & -D_0(t) & F_0^T \\ F_0 & 0 & 0 \end{bmatrix}, \quad B \equiv \begin{bmatrix} 0 \\ S_0 \\ 0 \end{bmatrix}, \tag{1.5}$$

and the specification of C is left open for the time being.

The analysis of the behaviour of (1.1) has to cope with three essential difficulties. First, the solutions of time-varying systems may exhibit critical points, i.e. a finite escape time. Secondly, descriptor systems behave quite differently than classical state space systems (i.e. $E = I_n$ in (1.1)). For state space systems, the function $u(\cdot)$ can be considered as an input function free to choose, and initial conditions can be arbitrary. This is in general not true for descriptor systems (1.1), since descriptor systems may contain algebraic constraints, which restrict the solutions, the set of possible inputs, and also the initial values to some manifold. Thirdly, some of the constraints that arise (the hidden constraints) are not explicit and thus it is not clear how to choose the underlying spaces for the descriptor variables x, u, y. Finally, the analytic property of the solution or behaviour is local, which is in contrast to the global algebraic properties of $R(D)$.
These difficulties are illustrated by the following example.

Example 1.1

(i) The scalar differential equations $t\dot{x} = -x$, $t^2\dot{x} = -x$, $t\dot{x} = x$, have local solutions $t \mapsto t^{-1}, e^{1/t}, t$, respectively. Hence at $t = 0$ the solution might be rational with a pole, or not even analytic, or does not have any pole, respectively.

(ii) The variables $x_1, \ldots, x_4, u_1, u_2$ of the descriptor system (1.1) with

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix},\ F = 0_{1\times 2}$$

satisfy the equivalent description

$$u_2 = 0,\ \dot{x}_2 = x_1,\ y = x_4,\ \dot{x}_3 = x_2 + u_1\,.$$

Thus, u_2 is constrained to be 0 and cannot be freely chosen, as it could in the case of state space systems. The variables x_1 and x_4 can be viewed as input or state variables, the system description does not determine this.
Note also that if we chose the input u_1 as a step function, then we would have to enlarge our solution space in order to allow that x_1 is a delta distribution. But even if we do so, then we have the problem that x_1 is not observable from the output y.
□

Hence the behavioural viewpoint, where state-, output-, and input-variables are not distinguished, seems the appropriate concept for the analysis of descriptor systems. The behavioural approach has been introduced by Willems [26, 27, 28, 29], see also the textbook [21] for a general presentation.

Motivated by Example 1.1, we consider local solutions of $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ belonging to

$$\mathcal{C}_t^\infty(\mathbb{R}^q) \ := \ \left\{ w \in \mathcal{C}^\infty(\mathbb{I}, \mathbb{R}^q) \,\middle|\, I \subset \mathbb{R} \text{ an open interval with } t \in I \right\}, \qquad t \in \mathbb{R}\,. \tag{1.6}$$

Definition 1.2 For $R(D) \in \mathcal{M}[D]^{g\times q}$, the *local behaviour* of the system $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ at $t \in \mathbb{R}$ is defined as

$$\mathfrak{B}_R^{\ker}(t) \ := \ \left\{ w \in \mathcal{C}_t^\infty(\mathbb{R}^q) \,\middle|\, R(\tfrac{\mathrm{d}}{\mathrm{d}\tau})w(\tau) = 0 \ \forall \tau \in \operatorname{dom} w \right\}. \tag{1.7}$$

The set $\mathfrak{B}_R^{\ker}(t)$ becomes a *real vector space* if endowed, for $w_1, w_2 \in \mathfrak{B}_R^{\ker}(t)$, with addition

$$(w_1 + w_2)(\tau) \;:=\; w_1(\tau) + w_2(\tau) \qquad \forall \tau \in \operatorname{dom} w_1 \cap \operatorname{dom} w_2\,,$$

and obvious scalar multiplication. The dimension of this vector space is defined as

$$\dim \mathfrak{B}_R^{\ker}(t) \;:=\; \sup \left\{ k \in \mathbb{N} \;\middle|\; \exists\, w_1, \ldots, w_k \in \mathfrak{B}_R^{\ker}(t) \text{ linearly independent on } \bigcap_{i=1}^{k} \operatorname{dom} w_i \right\}.$$

□

We are also interested in those points of the real axis, where the local solution is no longer extendable.

Definition 1.3 Consider the descriptor system (1.3). The set of *critical points*, where the solution is not defined, is given by

$$\mathbb{T}_R^{\text{crit}} := \left\{ t' \in \mathbb{R} \;\middle|\; \begin{array}{l} \text{there exists, for some } \varepsilon > 0, \text{ a } \mathcal{C}^\infty \text{ function} \\ w : (t' - \varepsilon, t') \to \mathbb{R}^p \text{ or } w : (t', t' + \varepsilon) \to \mathbb{R}^p \\ \text{which solves (1.3) and cannot be extended to} \\ (t' - \varepsilon, t'] \text{ or } [t', t' + \varepsilon)\,, \text{ respectively.} \end{array} \right\} \tag{1.8}$$

□

Note that for the three differential equations in Example 1.1(i) the sets of critical points are $\{0\}$, $\{0\}$, $\emptyset$, respectively.

Since E in (1.1) is real analytic, it follows that $\operatorname{rk} E(t) = \operatorname{rk}_{\mathcal{M}} E(\cdot)$ for almost all $t \in \mathbb{R}$, and hence the set of critical points is a discrete set. It is an open problem to characterize the set of critical points. However, we will determine discrete sets which include all critical points.

We define the appropriate behaviour, i.e. the solution space, of (1.3) on the time-axis $\mathbb{R}\backslash\mathbb{T}$, where $\mathbb{T}$ is discrete and includes the set of critical points of (1.3). Controllability and observability are defined in terms of trajectories (descriptor variables) which is a conceptual generalization of controllability and observability for state space systems.

For these systems in [5] controllability and observability has been studied in terms of derivative arrays. In [4] a first behaviour like approach for analytic coefficients has been discussed. A more general approach that allows for larger classes of coefficients and that can be implemented also numerically has been introduced in [17].

In [12] a first approach in the spirit of the present paper was presented for scalar systems. A completely different approach results from the study of differential-algebraic equations, see [1, 8]. A general solvability theory for nonsquare linear time-varying systems was first given in [16] and analysed for control problems in a behavioural context in [4, 18, 23], see also [17] for the general nonlinear case. In these papers, however, mainly the concept of regularization has been discussed, i.e., the problem of finding appropriate feedbacks that make the system regular and also decreases the index. Here we consider controllability and observability in the behavioural context.

This paper is organized as follows. In Section 2, we define critical points and follow the concepts of [16, 23] by deriving condensed forms for time-varying descriptor systems (1.3) to determine sets covering the critical points. In Section 3, controllability is defined, algebraically characterized, and related to the well known concepts of controllability. In

Section 4, we introduce the concept of autonomous behaviour for systems (1.3) and show how the behaviour of a descriptor system is a direct sum of an autonomous and controllable behaviour. In Section 5, observable behaviour is defined, it is related via the adjoint of the kernel representation to the controllable behaviour, and it is characterized algebraically.

2 Condensed forms

In this section, condensed forms with respect to state and input transformations are studied for time-varying descriptor systems (1.3). The condensed form allows to classify the solution sets and to identify the constraint manifolds for the variables. These forms are akin the forms derived in [4, 18]. If the system (1.3) is time-invariant, then the presented condensed forms are well known, see for example [3].

Theorem 2.1 Consider a time-varying descriptor system of the form (1.3).

(i) There exist orthogonal matrices $U_1 \in \mathcal{A}^{l\times l}, V_1 \in \mathcal{A}^{n\times n}$ so that (1.3) is transformed to

$$\begin{bmatrix} U_1 & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} E\,D-A & -B & 0 \\ -C & -F & I_p \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & I_{m+p} \end{bmatrix}, \tag{2.1}$$

corresponding to the descriptor system

$$\begin{array}{rcl} \Sigma_d\,\dot{x}_1 &=& A_{11}\,x_1 + A_{12}\,x_2 + A_{13}\,x_3 + B_1\,u \\ 0 &=& A_{21}\,x_1 + \Sigma_a\,x_2 + B_2\,u \\ 0 &=& A_{31}\,x_1 + B_3\,u \\ 0 &=& A_{41}\,x_1 \\ 0 &=& 0_{l-\nu} \\ y &=& C_1\,x_1 + C_2\,x_2 + C_3\,x_3 + F\,u, \end{array} \tag{2.2}$$

where $\Sigma_d \in \mathcal{A}^{d\times d}, \Sigma_a \in \mathcal{A}^{a\times a}$ are diagonal and invertible over $\mathcal{M}$ with $d = \mathrm{rk}_{\mathcal{M}}E$, and $B_3 \in \mathcal{A}^{\gamma\times m}$, $A_{41} \in \mathcal{A}^{f\times d}$ with full row rank, i.e. $\gamma = \mathrm{rk}_{\mathcal{M}}B_3$, $f = \mathrm{rk}_{\mathcal{M}}A_{41}$, and $\nu = d + a + \gamma + f$. All matrices are real analytic and of conforming formats.

(ii) There exist orthogonal matrices $U_2 \in \mathcal{A}^{l\times l}, V_2 \in \mathcal{A}^{n\times n}, W \in \mathcal{A}^{p\times p}, Z \in \mathcal{A}^{m\times m}$ so that (1.3) is transformed to

$$\begin{bmatrix} U_2 & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} E\,D-A & -B & 0 \\ -C & -F & I_p \end{bmatrix} \begin{bmatrix} V_2 & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & I_p \end{bmatrix}, \tag{2.3}$$

corresponding to the following descriptor system in *condensed form* (omitting the arguments t in the matrices and variables)

$$\begin{array}{rcl} \Sigma_d\,\dot{x}_1 &=& A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 + A_{15}x_5 + B_{11}u_1 + B_{12}u_2 \\ 0 &=& A_{21}x_1 + \Sigma_a x_2 + B_{21}u_1 + B_{22}u_2 \\ 0 &=& A_{31}x_1 + \Sigma_\gamma u_1 \\ 0 &=& \Sigma_f x_5 \\ 0 &=& 0 \\ y_1 &=& C_{11}x_1 + C_{12}x_2 + \Sigma_\omega x_3 + C_{15}x_5 + F_{11}u_1 + F_{12}u_2 \\ y_2 &=& C_{21}x_1 + C_{22}x_2 + C_{25}x_5 + F_{21}u_1 + F_{22}u_2 \end{array} \tag{2.4}$$

where $\Sigma_d, \Sigma_a, \Sigma_\gamma, \Sigma_f, \Sigma_\omega$ are diagonal matrices that are invertible over $\mathcal{M}$ and have sizes d, a, γ, f, ω, respectively. Furthermore, $\nu = d + a + \gamma + f$ and all matrices are real analytic and of conforming formats.

(iii) There exist matrices $U \in \mathcal{A}^{(l-p)\times(l-p)}$, $V \in \mathcal{M}^{n\times n}$ invertible over $\mathcal{M}$, $X \in \mathcal{M}^{p\times(l-p)}$, $W \in \mathcal{A}^{p\times p}$ orthogonal, $Z \in \mathcal{A}^{m\times m}$ orthogonal, a scalar function $\sigma \in \mathcal{A}$, and a permutation matrix $P \in \mathcal{A}^{(n+m)\times(n+m)}$ so that (1.3) is transformed to

$$\tilde{R}(D) := \begin{bmatrix} U & 0 \\ X & W \end{bmatrix} \begin{bmatrix} E\,D - A & -B & 0 \\ -C & -F & I_p \end{bmatrix} \begin{bmatrix} V & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & I_p \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I_p \end{bmatrix} \tag{2.5}$$

$$= \left[\begin{array}{c|ccc|ccc|cc} \sigma D I_d - \tilde{A}_{11} & -\tilde{A}_{13} & -\tilde{A}_{14} & -\tilde{B}_{12} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \Sigma_a^{-1}\tilde{B}_{22} & I_a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_f & 0 & 0 & 0 \\ \Sigma_\gamma^{-1}\tilde{A}_{31} & 0 & 0 & 0 & 0 & 0 & I_\gamma & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -\Sigma_\omega & 0 & -\sigma^{-1}\tilde{F}_{12} & 0 & 0 & 0 & I_\omega & 0 \\ -\sigma^{-1}\tilde{C}_{21} & 0 & 0 & -\sigma^{-1}\tilde{F}_{22} & 0 & 0 & 0 & 0 & I_p \end{array}\right]$$

corresponding to the meromorphic descriptor system in *standard condensed form*

$$\left.\begin{aligned} \sigma I_d \dot{x}_1 &= \tilde{A}_{11}x_1 + [\tilde{A}_{13},\ \tilde{A}_{14},\ \tilde{B}_{12}] \begin{bmatrix} x_3 \\ x_4 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} x_2 \\ x_5 \\ u_1 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & -\Sigma_a^{-1}\tilde{B}_{22} \\ 0 & 0 & 0 & 0 \\ -\Sigma_\gamma^{-1}\tilde{A}_{31} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_4 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ \sigma^{-1}\tilde{C}_{21} \end{bmatrix} x_1 + \begin{bmatrix} \Sigma_\omega & 0 & \sigma^{-1}\tilde{F}_{12} \\ 0 & 0 & \sigma^{-1}\tilde{F}_{22} \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ u_2 \end{bmatrix} \end{aligned}\right\} \tag{2.6}$$

where

$$\sigma(t) := \det \Sigma_d(t)\ \det \Sigma_a(t)\ \det \Sigma_\gamma(t)\ \det \Sigma_f(t)\,, \qquad \text{for all } t \in \mathbb{R}\,, \tag{2.7}$$

and all matrices are real analytic and of conforming formats. The integers d, a, γ, ω, f are invariants of (1.3).

Proof: The proof is constructive using a sequence of real analytic singular value decompositions. The singular value decomposition has been introduced in [2] for analytic matrices, and it is also valid for real analytic matrices. We will frequently use the multiplication rule (1.2) without saying so.

(i) Consider the first equation of (1.1) and choose orthogonal matrices $\tilde U \in \mathcal{A}^{l\times l}$, $\tilde V \in \mathcal{A}^{n\times n}$ so that

$$[\tilde R(D), -\tilde B] = \tilde U [ED - A, -B] \begin{bmatrix} \tilde V & 0 \\ 0 & I_m \end{bmatrix} = \left[\begin{bmatrix} \Sigma_d & 0 \\ 0 & 0 \end{bmatrix} D - \begin{bmatrix} \tilde A_{11} & \tilde A_{12} \\ \tilde A_{21} & \tilde A_{22} \end{bmatrix}, \begin{bmatrix} \tilde B_1 \\ \tilde B_2 \end{bmatrix} \right]$$

where $\Sigma_d \in \mathcal{A}^{d\times d}$ with $d = rk_{\mathcal{M}} E$ is diagonal.
Next, choose orthogonal matrices $\bar U \in \mathcal{A}^{(l-d)\times(l-d)}$, $\bar V \in \mathcal{A}^{(n-d)\times(n-d)}$ so that

$$\begin{aligned} [\bar R(D),\ -\bar B] &= \begin{bmatrix} I_{d_1} & 0 \\ 0 & \bar U \end{bmatrix} [\tilde R(D), -\tilde B] \begin{bmatrix} I_{d_1} & 0 & 0 \\ 0 & \bar V & 0 \\ 0 & 0 & I_m \end{bmatrix} \\ &= \left[\begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} D - \begin{bmatrix} \bar A_{11} & \bar A_{12} & \bar A_{13} \\ \bar A_{21} & \Sigma_a & 0 \\ \bar A_{31} & 0 & 0 \end{bmatrix}, \begin{bmatrix} \bar B_1 \\ \bar B_2 \\ \bar B_3 \end{bmatrix} \right], \end{aligned}$$

where $\Sigma_a \in \mathcal{A}^{a\times a}$ is diagonal and invertible over $\mathcal{M}$. Finally, choose an orthogonal $\hat U \in \mathcal{A}^{(l-d-a)\times(l-d-a)}$, so that $\begin{bmatrix} I_{d+a} & 0 \\ 0 & \hat U \end{bmatrix} [\bar R(D), -\bar B]$ has the form (2.2) with $B_3 \in \mathcal{A}^{\gamma\times m}$, $\gamma = \mathrm{rk}_{\mathcal{M}} B_3 = \mathrm{rk}_{\mathcal{M}} \bar B_3$ and $A_{41} \in \mathcal{A}^{f\times d}$, $f = \mathrm{rk}_{\mathcal{M}} A_{41}$. Performing all the transformations also on C and partitioning analogously shows (2.2).
(ii) We apply the so-called *index reduction process* as introduced in [18] to (2.4): Fix f variables of x_1, corresponding to some f linearly independent columns of A_{41}, i.e. choose a unitary matrix $Q \in \mathcal{A}^{d\times d}$ such that $A_{41}Q = [A_{41}^\alpha, A_{41}^\beta]$ with $A_{41}^\alpha \in \mathcal{A}^{f\times f}$ is invertible over $\mathcal{M}$. Then

$$0 = A_{41}x_1 = A_{41}^\alpha x_1^\alpha + A_{41}^\beta x_1^\beta, \qquad \begin{bmatrix} x_1^\alpha \\ x_1^\beta \end{bmatrix} := Qx_1,$$

and so

$$\dot x_1^\alpha = -(A_{41}^\alpha)^{-1} A_{41}^\beta \dot x_1^\beta - \frac{d}{dt}\left((A_{41}^\alpha)^{-1} A_{41}^\beta \right) x_1^\beta.$$

Inserting $\dot x_1^\alpha$ into the differential equation of (2.2) leaves $d-f$ differential equations. Note that we may have introduced meromorphic functions by the inverse of A_{41}^α and its derivative. A multiplication from the left with a real analytic function yields a description in the form (1.1), however the d differential equations have been reduced to $d-f$ differential equations and we may apply Part (i) again. This index reduction process stops after finitely many iterations, and we arrive at the following condensed form:

$$\begin{array}{rcl} \Sigma_d\, \dot x_1 &=& \hat A_{11}x_1 + \hat A_{12}x_2 + \hat A_{13}x_3 + \hat A_{14}x_4 + \hat B_1 u \\ 0 &=& \hat A_{21}x_1 + \Sigma_a x_2 + \hat B_2 u \\ 0 &=& \hat A_{31}x_1 + \hat B_3 u \\ 0 &=& \Sigma_f x_4 \\ 0 &=& 0 \\ y &=& \hat C_1 x_1 + \hat C_2 x_2 + \hat C_3 x_3 + \hat C_4 x_4 + Fu, \end{array} \tag{2.8}$$

where $\Sigma_d, \Sigma_a, \Sigma_f$ are diagonal matrices, invertible over $\mathcal{M}$, and of sizes d, a, f, respectively, and $\hat B_3 \in \mathcal{A}^{\gamma\times m}$ has full row rank over $\mathcal{A}$.
As a final step we perform an analytic singular value decomposition of $\hat C_3, \hat B_3$, respectively, and derive (2.4).

(iii) Using the fact that the fourth equation in (2.4) implies that $x_5 \equiv 0$, which can be extended even at points where Σ_f is singular, we can eliminate all terms invoking x_5 from all the other equations. This corresponds to multiplying (2.4) from the left by

$$\begin{bmatrix} \sigma\Sigma_d^{-1} & 0 \\ 0 & I_{l-d+p} \end{bmatrix} \begin{bmatrix} I_d & -A_{12}\Sigma_a^{-1} & -[B_{11} - A_{12}\Sigma_a^{-1}B_{21}]\Sigma_\gamma^{-1} & 0 & 0 & 0 & 0 \\ 0 & I_a & -B_{21}\Sigma_\gamma^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_\gamma & -A_{15}\Sigma_f^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{l-\nu} & 0 & 0 \\ 0 & -C_{12}\Sigma_a^{-1} & -F_{11}\Sigma_\gamma^{-1} & 0 & 0 & I_\omega & 0 \\ 0 & -C_{22}\Sigma_a^{-1} & -F_{21}\Sigma_\gamma^{-1} & 0 & 0 & 0 & I_{p-\omega} \end{bmatrix}$$

and from the right by

$$\begin{bmatrix} I_d & 0 & 0 & 0 \\ -[A_{21} - B_{21}\Sigma_\gamma^{-1}A_{31}]\Sigma_a^{-1} & I_a & 0 & 0 \\ -C_{11}\Sigma_\omega^{-1} & 0 & I_\gamma & 0 \\ 0 & 0 & 0 & I_{(n-\nu+f+m+p)\times(n-\nu+f+m+p)} \end{bmatrix}$$

yielding the transformed system

$$\begin{array}{rcl} \sigma\,\dot{x}_1 & = & \tilde{A}_{11}x_1 + \tilde{A}_{13}x_3 + \tilde{A}_{14}x_4 + \tilde{B}_{12}u_2 \\ 0 & = & \Sigma_a x_2 + \sigma^{-1}\tilde{B}_{22}u_2 \\ 0 & = & \tilde{A}_{31}x_1 + \Sigma_\gamma u_1 \\ 0 & = & \Sigma_f x_5 \\ 0 & = & 0_{l-\nu} \\ y_1 & = & \Sigma_\omega x_3 + \sigma^{-1}\tilde{F}_{12}u_2 \\ y_2 & = & \sigma^{-1}\tilde{C}_{21}x_1 + \sigma^{-1}\tilde{F}_{22}u_2 \end{array} \tag{2.9}$$

where all matrices are real analytic. This proves (2.6). □

Remark 2.2

(i) If the descriptor system (1.3) is time-invariant, then all transformations in Theorem 2.1 may be chosen as constant matrices and $\sigma = 1$.

(ii) For real analytic matrices as in (2.2), (2.4), (2.6), the analytic singular value decomposition is not uniquely defined; essentially, there is freedom to perform orthogonal transformations in the spaces associated with multiple singular values. However, this freedom can be removed by choosing minimal variation curves or by always choosing the analytic singular value decomposition to be closest to a reference point, [3, 20].

(iii) To derive (2.2), only an orthogonal transformation on the variables x in (2.1) has been applied. To derive (2.4), non-singular transformations on the variable x and orthogonal transformations on u have not been mixed.
To derive (2.6), we have used non-singular transformations on x, and orthogonal transformations on u. If we allow further linear combinations (which for classical systems where y, x, u are fixed a priori as outputs, states and controls, respectively, correspond to state feedback or output feedback), then we can simplify (2.6) further by removing blocks such as $\tilde{A}_{31}$ or by introducing almost everywhere invertible diagonal blocks in diagonal positions of the transformed matrices E or A. Note that the

transformation of derivative feedback is not an equivalence transformation, because under derivative feedback the characteristic quantities d, a, γ, f, w are not invariants and hence the properties of the system may be altered by this transformation completely, see [18] and Remark 2.3 below.

(iv) The description (2.6) is not quite of the form (1.1), since the coefficients of x_1 and u_2 in y_1 and y_2 may have poles at the zeros of σ.

(v) An immediate consequence of (2.6) is that the variables in x_1 represent couplings between algebraic equations and differential equations that are not influenced by u_1. Systems where such couplings between differential equations and algebraic equations occur are typically called *high index systems*. For a detailed discussion of different index concepts see [1, 8, 17, 18].

(vi) The transformation leading to (2.6) does not invoke any differentiation of u. Hence, if the variables denoted by u are classified as inputs a priori, then no extra differentiability conditions for these variables arise, see [4, 18]. □

The condensed forms (2.1), (2.4) and (2.6) allow to detect candidates for critical points.

Remark 2.3 Consider a descriptor system (1.3) and corresponding condensed forms (2.2), (2.4), (2.6). The set of its critical points as defined in Definition 1.3 is a subset of all zeros of σ in (2.7), however in general it is a proper subset, i.e.,

$$\mathbb{T}_R^{\text{crit}} \subset \mathbb{T}_R := \left\{ t' \in \mathbb{R} \,\middle|\, \sigma(t') = 0 \right\}. \tag{2.10}$$

To see this, consider the following descriptor system

$$\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} b_1(t) \\ 0 \end{bmatrix} u(t),$$

with b_1 real analytic, see e.g., [22]. Obviously, E has a rank drop at $t = 0$ and hence 0 is a potential candidate for a critical point, but since the solution is

$$x_1(t) = -b_1(t)u(t), \quad x_2(t) = 0,$$

the solution is defined everywhere. □

To characterize controllability we will also need the following staircase form which generalizes the staircase form of Van Dooren [25] to systems with variable coefficients.

Lemma 2.4 For any real analytic matrices $A \in \mathcal{A}^{n\times n}, B \in \mathcal{A}^{n\times m}$ there exist orthogonal matrices $P \in \mathcal{A}^{n\times n}$ and $Q \in \mathcal{A}^{m\times m}$ so that

$$P\left[DI_n - A, -B\right] \begin{bmatrix} P^T & 0 \\ 0 & Q \end{bmatrix}$$

$$= \left[\begin{array}{cccc|c|cc} DI_n - A_{11} & \cdots & \cdots & -A_{1,s-1} & -A_{1,s} & -B_1 & 0 \\ -[\hat{A}_{21}, 0] & \ddots & & \vdots & \vdots & 0 & 0 \\ & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & -[\hat{A}_{s-1,s-2}, 0] & DI_n - A_{s-1,s-1} & -A_{s-1,s} & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & DI_n - A_{ss} & 0 & 0 \end{array}\right] \begin{array}{c} n_1 \\ n_2 \\ \vdots \\ n_{s-1} \\ n_s \end{array} \tag{2.11}$$

where $n_1 \geq n_2 \geq \cdots \geq n_{s-1} \geq n_s \geq 0$, $n_{s-1} > 0$, $B_1 \in \mathcal{A}^{n_1 \times n_1}$ and $\hat{A}_{i,i-1} \in \mathcal{A}^{n_i \times n_i}$, are invertible over $\mathcal{M}$ for $i = 1, \ldots, s-1$.

Proof: A constructive proof is given by the so called *'Staircase' Algorithm* invoking the real analytic singular value decomposition (ASVD), see [2].
Whenever we use Σ in the following, it denotes a diagonal matrix.
Step 0: Choose orthogonal $U_B \in \mathcal{A}^{n \times n}$, $V_B \in \mathcal{A}^{m \times m}$ so that

$$B = U_B^T \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B \in \mathcal{A}^{n \times m} \quad \text{with invertible } \Sigma_B \in \mathcal{A}^{n_1 \times n_1},$$

and set

$$\begin{array}{rclcl} A_0 & := & U_B A U_B^T + \dot{U}_B U_B^T & = & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{with } A_{21} \in \mathcal{A}^{(n-n_1) \times n_1}, \\ B_0 & := & U_B B V_B & = & \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix}. \end{array}$$

Then by (1.2)

$$U_B \begin{bmatrix} DI_n - A, -B \end{bmatrix} \begin{bmatrix} U_B^T & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} DI_n - A_0, -B_0 \end{bmatrix}.$$

Step 1: If $n_1 < n$ and $A_{21} \neq 0$, then choose orthogonal $U_{21} \in \mathcal{A}^{(n-n_1) \times (n-n_1)}$, $V_{21} \in \mathcal{A}^{n_1 \times n_1}$ so that

$$A_{21} = U_{21} \begin{bmatrix} \Sigma_{21} & 0 \\ 0 & 0 \end{bmatrix} V_{21}^T \in \mathcal{A}^{(n-n_1) \times n_1} \quad \text{with invertible } \Sigma_{21} \in \mathcal{A}^{n_2 \times n_2},$$

and set

$$\begin{array}{rclcl} P_1 & := & \begin{bmatrix} V_{21}^T & 0 \\ 0 & U_{21}^T \end{bmatrix} & & \\ A_1 & := & P_1 A_0 P_1^T + \dot{P}_1 P_1^T & = & \left[\begin{array}{cc|c} * & * & * \\ \hline \Sigma_{21} & 0 & * \\ 0 & 0 & * \end{array}\right] + \left[\begin{array}{c|c} \dot{V}_{21}^T V_{21} & 0 \\ \hline 0 & \dot{U}_{21}^T U_{21} \end{array}\right] \\ B_1 & := & V_{21}^T \Sigma_B, & & \\ \widetilde{B}_1 & := & \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{A}^{n \times n}. & & \end{array}$$

This gives, again by (1.2) and for some $\tilde{A}_{32} \in \mathcal{A}^{(n-n_1-n_2) \times n_2}$,

$$P_1 \begin{bmatrix} DI_n - A_0, -B_0 \end{bmatrix} \begin{bmatrix} P_1^T & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} DI_n - A_1, -\widetilde{B}_1 \end{bmatrix}$$

$$= \left[DI_n - \left[\begin{array}{c|cc} * & * & * \\ \hline [\Sigma_{21}, 0] & * & * \\ 0 & \tilde{A}_{32} & * \end{array}\right], - \left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] \right]$$

Step 2: If $n_1 + n_2 < n$ and $\tilde{A}_{32} \neq 0$, then choose orthogonal $U_{32} \in \mathcal{A}^{(n-n_1-n_2)\times(n-n_1-n_2)}$, $V_{32}^T \in \mathcal{A}^{n_2 \times n_2}$ so that

$$\tilde{A}_{32} = U_{32} \begin{bmatrix} \Sigma_{32} & 0 \\ 0 & 0 \end{bmatrix} V_{32}^T \in \mathcal{A}^{(n-n_1-n_2)\times n_2} \quad \text{with invertible } \Sigma_{32} \in \mathcal{A}^{n_3 \times n_3},$$

and set

$$\begin{aligned} P_2 &:= \operatorname{diag}\left\{I_{n_1}, V_{32}^T, U_{32}^T\right\} \\ \hat{A}_{21} &:= V_{32}^T \Sigma_{21} \\ A_2 &:= P_2 A_1 P_2^T + \dot{P}_2 P_2^T \\ &= \left[\begin{array}{c|cc} * & * & * \\ \hline V_{32}^T[\Sigma_{21}, 0] & * & * \\ 0 & U_{32}^T \tilde{A}_{32} V_{32} & * \end{array}\right] + \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \dot{V}_{32}^T V_{32} & 0 \\ 0 & 0 & \dot{U}_{32}^T U_{32} \end{array}\right] \\ &= \left[\begin{array}{cc|cc|c} * & * & * & * & * \\ \hline \hat{A}_{21} & 0 & * & * & * \\ \hline 0 & 0 & \Sigma_{32} & 0 & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]. \end{aligned}$$

Then, for some $\tilde{A}_{43} \in \mathcal{A}^{(n-n_1-n_2-n_3)\times n_3}$,

$$P_2 P_1 \left[DI_n - A_0, -B_0\right] \begin{bmatrix} P_1^T P_2^T & 0 \\ 0 & I_m \end{bmatrix} = \left[DI_n - A_2, -\widetilde{B}_1\right]$$

$$= \left[DI_n - \left[\begin{array}{cc|cc|cc} * & * & * & * & * & * \\ \hline \hat{A}_{21} & 0 & * & * & * & * \\ \hline 0 & 0 & \Sigma_{32} & 0 & * & * \\ 0 & 0 & 0 & 0 & \tilde{A}_{43} & * \end{array}\right], - \left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] \right]$$

Step 3: In the remainder of the proof we proceed analogously as in Step 2 and terminate after finitely many steps with the from (2.11). This completes the proof of the lemma. □

Example 2.5 Consider the system (1.5). We see that critical points include those values of t where the mass matrix $M_0(t)$ changes rank. This happens for example when two arms of the manipulator are in one straight line.

Without loss of generality (by using an appropriate permutation of the basis), we may assume that the coordinate system for the Lagrange multipliers is such that $F_0 = [F_1\ 0]$ with non-singular $F_1 \in \mathbb{R}^{2\times 2}$ and if we partition

$$-K_0 = \begin{bmatrix} K_{11}(t) & K_{12}(t) \\ K_{21}(t) & K_{22}(t) \end{bmatrix}, \; M_0 = \begin{bmatrix} M_{11}(t) & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{bmatrix}, \; -D_0 = \begin{bmatrix} D_{11}(t) & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{bmatrix}, \; S_0 = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix},$$

with $K_{11}(t), M_{11}(t), D_{11}(t), S_1 \in \mathbb{R}^{2\times 2}$ and all other formats accordingly, then the system (1.5) may be written as

$$\begin{bmatrix} I_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & M_{11}(t) & M_{12}(t) & 0 \\ 0 & 0 & M_{21}(t) & M_{22}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ K_{11}(t) & K_{12}(t) & D_{11}(t) & D_{12}(t) & F_1^T \\ K_{21}(t) & K_{22}(t) & D_{21}(t) & D_{22}(t) & 0 \\ F_1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ S_1 \\ S_2 \\ 0 \end{bmatrix} u.$$

Since F_1 is constant and non-singular, we obtain $x_1 = 0$ and $\dot{x}_1 = 0$. Inserting this and changing the order of equations and blocks leads to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & M_{11}(t) & M_{12}(t) & 0 & 0 \\ 0 & M_{21}(t) & M_{22}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ K_{12}(t) & D_{11}(t) & D_{12}(t) & F_1^T & 0 \\ K_{22}(t) & D_{21}(t) & D_{22}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & F_1 \\ 0 & I_2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ S_1 \\ S_2 \\ 0 \\ 0 \end{bmatrix} u.$$

We can repeat the reduction process once more by using that $x_3 = 0$ and hence $\dot{x}_3 = 0$, which gives a system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & M_{22}(t) & 0 & 0 & 0 \\ 0 & M_{12}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_3 \\ \dot{x}_5 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ K_{22}(t) & D_{22}(t) & 0 & 0 & 0 \\ K_{12}(t) & D_{12}(t) & 0 & F_1^T & 0 \\ 0 & 0 & 0 & 0 & F_1 \\ 0 & 0 & I_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_3 \\ x_5 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ S_2 \\ S_1 \\ 0 \\ 0 \end{bmatrix} u.$$

Since the mass matrix M_0 is positive definite almost everywhere, we can eliminate the block M_{12} and obtain the system

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & M_{22}(t) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \dot{x}_2 \\ \dot{x}_4 \\ \hline \dot{x}_3 \\ \dot{x}_5 \\ \dot{x}_1 \end{array}\right]$$
$$= \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ K_{22}(t) & D_{22}(t) & 0 & 0 & 0 \\ \hline \tilde{K}_{12}(t) & \tilde{D}_{12}(t) & 0 & F_1^T & 0 \\ 0 & 0 & 0 & 0 & F_1 \\ 0 & 0 & I_2 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_2 \\ x_4 \\ \hline x_3 \\ x_5 \\ x_1 \end{array}\right] + \left[\begin{array}{c} 0 \\ S_2 \\ \hline \tilde{S}_1 \\ 0 \\ 0 \end{array}\right] u. \quad (2.12)$$

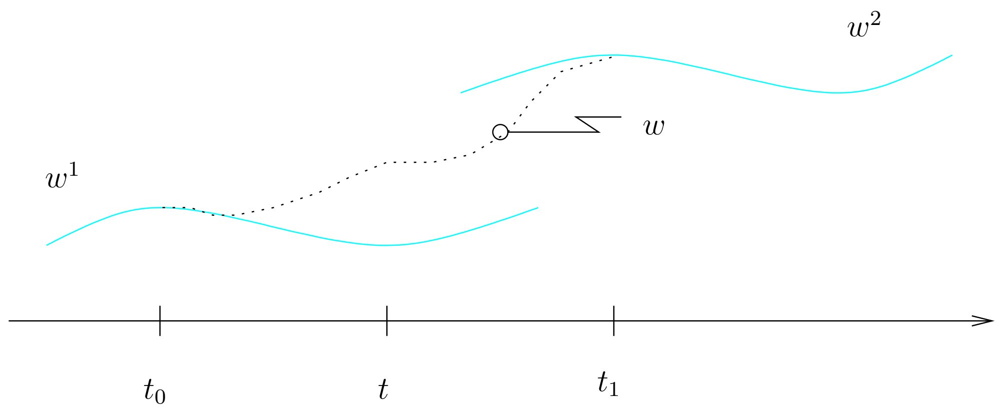

Figure 1: Local controllability at t

This is essentially (apart from diagonal matrices Σ) in the condensed form (2.2), with

$$\Sigma_d = \begin{bmatrix} 1 & 0 \\ 0 & M_{22}(t) \end{bmatrix}, \quad \Sigma_a = \begin{bmatrix} 0 & F_1^T & 0 \\ 0 & 0 & F_1 \\ I_2 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \tilde{S}_1 \\ 0 \\ 0 \end{bmatrix}.$$

It is then obvious how the more refined forms (2.4) and (2.6) can be determined. □

3 Controllability

In the present paper we are only interested in the concept of controllability for descriptor systems of the form (1.3). However, we repeat the definition for general systems of the form $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$, where $R(D) \in \mathcal{M}[D]^{g \times q}$, as introduced in [13].

Note that the family of linear sub-spaces of a linear space may be partially ordered by inclusion, and thus constitutes a lattice with respect to $+$ and $\cap$. Hence the following definition is well-defined.

Definition 3.1 A sub-vector space $\mathfrak{B}_R^{\mathrm{c}}(t)$ of $\mathfrak{B}_R^{\mathrm{ker}}(t)$ is called *locally controllable* at $t \in \mathbb{R}$ if, and only if, for every $w^1, w^2 \in \mathfrak{B}_R^{\mathrm{c}}(t)$ and every $t_0 \in (-\infty, t) \cap \mathrm{dom}\ w^1$ there exist $t_1 \in \mathrm{dom}\ w^2 \cap (t, \infty)$ and $w \in \mathfrak{B}_R^{\mathrm{c}}(t)$ such that

$$w(t) = \begin{cases} w^1(t), & t \in (-\infty, t_0] \cap \mathrm{dom}\ w^1 \\ w^2(t), & t \in [t_1, \infty) \cap \mathrm{dom}\ w^2. \end{cases}$$

$\mathfrak{B}_R^{\mathrm{contr}}(t) \subset \mathfrak{B}_R^{\mathrm{ker}}(t)$ is called the *largest controllable behaviour* of $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ at t if, and only if, every controllable behaviour $\mathfrak{B}_R^{\mathrm{c}}(t)$ of $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ satisfies $\mathfrak{B}_R^{\mathrm{c}}(t) \subset \mathfrak{B}_R^{\mathrm{contr}}(t)$ almost everywhere.

The system $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ is called *locally controllable at* $t \in \mathbb{R}$ if, and only if, the largest controllable behaviour is $\mathfrak{B}_R^{\mathrm{ker}}(t)$.

$R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ is called *controllable* if, and only if, it is locally controllable almost everywhere.

□

Remark 3.2

(i) Due to the linearity of the system $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$, the trajectory w^2 in Definition 3.1 may be replaced, without restriction of generality, by $w^2 = 0$.

(ii) Loosely speaking, controllability means that any two trajectories $w^1, w^2 \in \mathfrak{B}_R^{\ker}(t)$ can be connected by another trajectory $w \in \mathfrak{B}_R^{\ker}(t)$ so that in finite time w^1 moves via w into w^2, see Figure 1. A similar notion of controllability via trajectories was introduced in [10] for time-invariant Rosenbrock systems. For time-invariant state space systems, i.e. (1.3) with $E = I_n$, the concept of controllability coincides with the one introduced in [21, Sect. 5.2]. □

Remark 3.3 For descriptor systems with constant coefficients, several different concepts of controllability are known, see [3, 7, 19].

(i) System (1.1) with constant coefficients is called

completely controllable	iff	$\mathrm{rk}\,[\alpha E - \beta A, B] = l$ for all $(\alpha, \beta) \in \mathbb{C}^2/\{(0,0)\}$,
R-controllable	iff	$\mathrm{rk}\,[\lambda E - A, B]$ is full for all $\lambda \in \mathbb{C}$,
I-controllable	iff	$\mathrm{rk}\,[E, AS_\infty, B]$ is full, where S_∞ spans the kernel of E ,
strongly controllable	iff	the system is R-controllable and I-controllable.

It should be noted that these algebraic characterizations are sometimes misleading in the literature, since it is sometimes assumed that the rank of $[E, B]$ is full and sometimes not.

It follows that if system (1.1) is square and time-invariant, thus, in particular, $l = n$, then system (1.1) is *I-controllable* if, and only if, $n - (d + a + \gamma + f) = 0$, and *I-observable* if, and only if, $n - d - a - \omega - f = 0$. The constants a, d, f, γ are defined in Theorem 2.1(ii).

(ii) For time-invariant state-space systems, i.e. (1.1) with $E = I_n$, the algebraic conditions can be checked numerically via the staircase algorithm of [25]. In a similar fashion Lemma 2.4 will be used to check controllability for time-varying systems.

(iii) For time-invariant systems (1.3), Definition 3.1 corresponds to the concept of *R*-controllability. This follows from Theorem 3.4 below.

(iv) If the descriptor system (1.3) is time-varying, then Definition 3.1 is new, see [5, 23, 18] for a discussion of different controllability concepts for time-varying descriptor systems. □

Now Theorem 2.1 and Lemma 2.4 set us in a position to characterize controllability of time-varying descriptor systems (1.3).

Theorem 3.4 Consider a time-varying descriptor system (1.3) and assume that $R(D)$ has full row rank over $\mathcal{M}[D]$. Consider the condensed form (2.6) and σ as defined in (2.7). Set, for notational convenience,

$$G(t) := \tilde{A}_{11}(t)\,, \quad S(t) := [\tilde{A}_{13}(t),\ \tilde{A}_{14}(t),\ \tilde{B}_{12}(t)]\,, \quad v(t) := (x_3(t)^T,\ x_4(t)^T,\ u_2(t)^T)^T\,.$$

Then the following conditions are equivalent.

(i) (1.3) is locally controllable almost everywhere.

(ii) $R(D)$ is right invertible over $M[D]$.

(iii) (2.3) respectively (2.4) is locally controllable almost everywhere.

(iv) $\hat{R}(D) := [\sigma DI_d - G,\, S]$ is right invertible over $\mathcal{M}[D]$.

(v) In the staircase form (2.10) of the pair $[DI_d - G,\, S]$, the lower block is not present, i.e. $n_s = 0$.

(vi) There exist a discrete set $\mathbb{T} \subset \mathbb{R}$ such that for every

$$\begin{pmatrix} x_1^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} x_1^1 \\ v^1 \end{pmatrix} \in \mathfrak{B}_{\hat{R}}^{\ker}(t)$$

and for every open interval $\mathbb{I} \subset \mathbb{R} \setminus \mathbb{T}$ and all $t_0 \in \mathbb{I}$, there exists $t_1 > t_0,\ t_1 \in \mathbb{I}$, and $[x_1^T, v^T]^T \in \mathfrak{B}_{\hat{R}}^{\ker}(t)$, such that

$$(x_1(t), v(t)) \;=\; \begin{cases} (x_1^0(t), v^0(t))\,, & \text{if } t \in (-\infty, t_0] \cap \mathbb{R} \setminus \mathbb{T} \\ (x_1^1(t), v^1(t))\,, & \text{if } t \in [t_1, \infty] \cap \mathbb{R} \setminus \mathbb{T}. \end{cases}$$

Proof:
"(i) $\Leftrightarrow$ (ii)": This is proved in [13, Prop. 3.6].
"(ii) $\Leftrightarrow$ (iii)": The equivalence of local controllability almost everywhere of (1.3) and (2.4), respectively (1.1) and (2.6), follows from (2.3) by invoking orthogonality of U_2, V_2, W, Z.
"(ii) $\Leftrightarrow$ (iv)": By (2.5), there exist invertible matrices $\tilde{U} \in \mathcal{M}^{(l+p)\times(l+p)}$, $\tilde{V} \in \mathcal{M}^{(n+m+p)\times(n+m+p)}$ so that (1.1) is related to (2.6) in the form (1.3) by the transformation

$$\tilde{U}\begin{bmatrix} E\,D - A & -B & 0 \\ -C & -F & I_p \end{bmatrix}\tilde{V} \;=$$
$$\begin{bmatrix} \sigma\, DI_d - \tilde{A}_{11} & 0 & -\tilde{A}_{13} & -\tilde{A}_{14} & 0 & 0 & -\tilde{B}_{12} & 0 & 0 \\ 0 & -\Sigma_a & 0 & 0 & 0 & 0 & -\tilde{B}_{22} & 0 & 0 \\ -\tilde{A}_{31} & 0 & 0 & 0 & 0 & -\Sigma_\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Sigma_f & 0 & 0 & 0 & 0 \\ 0_{(l-\nu)\times d} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sigma\Sigma_\omega & 0 & 0 & 0 & -\tilde{F}_{12} & \sigma I_\omega & 0 \\ -\tilde{C}_{21} & 0 & 0 & 0 & 0 & 0 & -\tilde{F}_{22} & 0 & \sigma I_{p-\omega} \end{bmatrix}. \quad (3.1)$$

The above matrix is right invertible if, and only if, $l - \nu = l - d - a - \gamma - f = 0$ (which is a consequence of the full row rank assumption) and $[\sigma DI_d - G,\, S]$ is right invertible over $\mathcal{M}[D]$.
"(iv) $\Leftrightarrow$ (v)": By Lemma 2.4 there exist orthogonal matrices P and Q so that $P[\sigma DI_d - G, S]\begin{bmatrix} P^T & 0 \\ 0 & Q \end{bmatrix}$ is of the staircase form (2.10). Note that σ does not effect the staircase form. Now the equivalence "(iv) $\Leftrightarrow$ (v)" follows immediately since B_1 and $\hat{A}_{i,i-1}$ are invertible over $\mathcal{M}$ for $i = 1, \ldots, s-1$.
"(ii) $\Leftrightarrow$ (vi)": This equivalence follows readily from Definition 3.1 and from (3.1), since the set of zeros and poles of the coefficients of $\tilde{U}$ and $\tilde{V}$ is a discrete set. □

Note that the assumption that (1.3) has full row rank over $\mathcal{M}[D]$ is equivalent to $l - d - a - \gamma - f = 0$ in (2.6).

Note further that the characterization in Theorem 3.4 (ii) does not require a reinterpretation of variables. Moreover, in contrast to controllability of state space systems, here $u_1(\cdot)$ in (1.1) is not a "free input" variable.
For standard time-invariant state-space systems (i.e. $E = I_n$), the right invertibility of $R(D)$ in Theorem 3.4 is derived differently in [21, Th. 5.2.10].

Remark 3.5 For time-varying systems (1.1) with $E = I_n$, i.e. state space systems, it is well known that controllability of the system yields that it can be controlled in arbitrary short time. $\mathbb{I}$ in Definition 3.1 can be replaced by any arbitrary short open interval $\widehat{\mathbb{I}} \subset \mathbb{I}$. This holds also true for descriptor systems (1.3), since $\hat{R}(D)$ in Theorem 3.4 (iv) can be viewed locally as state space system, namely at those $t \in \mathbb{R}$ where $\sigma(t) \neq 0$; note that the zeros of σ are a discrete set. An alternative and constructive proof is given in [13, Th. 3.3] for general systems of the form $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$. $\square$

To illustrate these results consider the following examples.

Example 3.6

(i) $R(D) = [t^2 D + 1, 1]$ has right inverse $[0, 1]^T$ and hence, by Theorem 3.4, $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ is controllable.

(ii) Revisit the linearized model (1.5) of the three-link constrained mobile manipulator. In Example 2.5 it is shown that (1.5) is equivalent to (2.11). Rewriting (2.11) in the form (1.3) and invoking that M_{22} is invertible over $\mathcal{M}$ and F_1 is non-singular, it is easy to see that the corresponding $R(D)$ is right invertible. Therefore, by Theorem 3.4, the linearized model (1.5) is controllable. $\square$

4 Autonomous behaviour

In this section we show that the behaviour of a descriptor system (1.1) respectively (1.3) can be decomposed into the direct sum of a controllable and an autonomous behaviour. Loosely speaking, an autonomous behaviour consists of those solutions which are uniquely determined if they are known on an arbitrarily small open interval. For systems (1.3) we have to cope with the problem of finite escape time.

Definition 4.1 For $R(D) \in \mathcal{M}[D]^{g \times q}$, the system $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ is called *locally autonomous at* $t \in \mathbb{R}$ if, and only if, for any $w^1, w^2 \in \mathfrak{B}_R^{\mathrm{ker}}(t)$ with $w_1 \equiv w_2$ on some open interval $\mathbb{I} \subset \mathrm{dom}\, w^1 \cap \mathrm{dom}\, w^2$ with $t \in \mathbb{I}$ it follows that $w_1 \equiv w_2$ on $\mathrm{dom}\, w^1 \cap \mathrm{dom}\, w^2$.
The system $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ is called *autonomous* if, and only if, it is autonomous almost everywhere.
A real vector space $\mathfrak{B}_R^{\mathrm{aut}}(t) \subset \mathfrak{B}_R^{\mathrm{ker}}(t)$ is called *autonomous* if, and only if, for any $w^1, w^2 \in \mathfrak{B}_R^{\mathrm{aut}}(t)$ with $w_1 \equiv w_2$ on some open interval $\mathbb{I} \subset \mathrm{dom}\, w^1 \cap \mathrm{dom}\, w^2$ with $t \in \mathbb{I}$ it follows that $w_1 \equiv w_2$ on $\mathrm{dom}\, w^1 \cap \mathrm{dom}\, w^2$.

A parametrical family of sets $\mathfrak{B}_R^{\text{aut}}$, associated to $t \mapsto \mathfrak{B}_R^{\text{aut}}(t) \subset \mathfrak{B}_R^{\text{ker}}(t)$, is called *autonomous behaviour* if, and only if, $\mathfrak{B}_R^{\text{aut}}(t)$ is autonomous for almost all $t \in \mathbb{R}$. □

Remark 4.2

(i) Suppose that the descriptor system (1.3) is time-invariant, or even more general $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ for $R(D) \in \mathbb{R}[D]^{g \times q}$ but time-invariant, then Definition 4.1 corresponds to the concept introduced in [21, p. 67].

(ii) Let $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ be given as in (1.3). Since any non-trivial trajectory belonging to an autonomous subspace cannot be controlled, otherwise the definition of autonomy would be violated, it follows that

$$\mathfrak{B}_R^{\text{aut}}(t) \cap \mathfrak{B}_R^{\text{contr}}(t) \;=\; \{0\} \qquad \text{for almost all } t \in \mathbb{R}.$$

(iii) An autonomous behaviour $\mathfrak{B}_R^{\text{aut}}(t)$ of the system (1.3) is invariant under all transformations (2.1), (2.2), (2.4), (2.10). □

Proposition 4.3 Consider a descriptor system (1.3). Then there exists an autonomous behaviour $t \mapsto \mathfrak{B}_R^{\text{aut}}(t)$ such that

$$\mathfrak{B}_R^{\text{ker}}(t) = \mathfrak{B}_R^{\text{aut}}(t) \oplus \mathfrak{B}_R^{\text{contr}}(t) \qquad \text{for almost all } t \in \mathbb{R}.$$

Proof: Existence of this decomposition is proved in [13, Th. 4.3]. Non-uniqueness of $\mathfrak{B}_R^{\text{aut}}(t)$ is already known for time-invariant systems, see [21, Rem. 5.2.15]. However, the sum of an autonomous behaviour and the controllable behaviour is indeed uniquely defined. □

Example 4.4 Consider a time-varying state space system (1.3) with $E = I_n$. By [15] there exists $T \in \mathcal{A}^{n \times n}$ invertible over $\mathcal{A}$ so that the coordinate transformation $z := T^{-1}x$ converts (1.1) into

$$\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{dt}} z_1(t) &=& A_{11}(t)z_1(t) \;+\; A_{12}(t)z_2(t) \;+\; B_1(t)u(t) \\ \frac{\mathrm{d}}{\mathrm{dt}} z_2(t) &=& A_{22}(t)z_2(t) \\ y(t) &=& C_1(t)z_1(t) \;+\; C_2(t)z_2(t) \;+\; F(t)u(t), \end{array} \tag{4.1}$$

with all matrices real analytic of conforming formats, and controllable sub-system $\frac{\mathrm{d}}{\mathrm{dt}} z_1(t) = A_{11}(t)z_1(t) + B_1(t)u(t)$. Since (4.1) is a state space system, finite escape time does not occur and the controllable and autonomous subspaces can be described globally. Set

$$\hat{R}(D) \;:=\; \begin{bmatrix} DI - A_{11} & -A_{12} & -B_1 & 0 \\ 0 & DI - A_{22} & 0 & 0 \\ -C_1 & -C_2 & -F & -I_p \end{bmatrix}.$$

Then

$$\mathfrak{B}_{\hat R}^{\rm contr}(t) \;=\; \Big\{ w = (z_1^T, z_2^T, u^T, y^T)^T \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^{(n+m+p)}) \,\Big|\, \hat R(\tfrac{\rm d}{\rm dt}) w = 0 \;\wedge\; z_2 = 0 \Big\} \,\forall\, t \in \mathbb{R}$$

and

$$\mathfrak{B}_{\hat R}^{\rm aut}(t) \;=\; \left\{ w = (z_1^T, z_2^T, u^T, y^T)^T \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^{(n+m+p)}) \,\middle|\, \begin{array}{l} \hat R(\tfrac{\rm d}{\rm dt}) w = 0,\; z_1 = 0, \\ u = 0,\; \dot z_2 = A_{22}\, z_2 \end{array} \right\} \forall\, t \in \mathbb{R}$$

is an autonomous behaviour and hence, in the original coordinates, we have

$$\mathfrak{B}_R^{\rm contr}(t) \;=\; \begin{bmatrix} T(t) & 0 \\ 0 & I_{m+p} \end{bmatrix} \mathfrak{B}_{\hat R}^{\rm aut}(t) \oplus \begin{bmatrix} T(t) & 0 \\ 0 & I_{m+p} \end{bmatrix} \mathfrak{B}_{\hat R}^{\rm contr}(t) \qquad \forall\, t \in \mathbb{R}\,.$$

□

Remark 4.5 Consider a time-varying descriptor systems (1.1) in the condensed form (2.6). If (2.6) were controllable, then $\mathfrak{B}_R^{\rm aut} = \{0\}$ is the only autonomous behaviour of (2.6). To see this, note that (x_3, x_4, u_2) is free to choose and hence cannot be a non-zero component of an autonomous behaviour. Furthermore, since $[\sigma D I_d - G, S]$ is controllable by Theorem 3.4 (iii), it follows that x_1 is uniquely (modulo initial condition) determined by (x_3, x_4, u_2), and hence also not a non-trivial component of an autonomous behaviour. Finally, (2.6) yields that the remaining components x_2, x_5, u_1, y_1, y_2 are uniquely determined by x_3, x_4, u_2, x_1. This shows $\mathfrak{B}_R^{\rm aut} = \{0\}$.

If (2.6) is not controllable but has a non-trivial uncontrollable subspace, then there exists $\mathfrak{B}_R^{\rm aut} \neq \{0\}$ which is determined by the uncontrollable subspace as for state space systems, see Example 4.4. □

6 Observability

In this section, we study how one behaviour can be observed from another. Essential for this are the concepts of adjoints of matrices over $\mathcal{M}[D]$ and the adjoint of a kernel representation $\mathfrak{B}_R^{\ker}$.

Definition 6.1 The *adjoint* for matrices over $\mathcal{M}[D]$ is defined as

$$\cdot^{\rm ad} : \mathcal{M}^{n\times m}[D] \to \mathcal{M}^{m\times n}[D], \qquad \sum_{i=0}^k P_i D^i \mapsto \Big(\sum_{i=0}^k P_i D^i\Big)^{\rm ad} := \sum_{i=0}^k (-1)^i D^i P_i^T.$$

It is easy to show (see [13, Prop. 5.2]) that, for arbitrary matrices $P(D), Q(D)$ over $\mathcal{M}[D]$ with appropriate formats, we have

$$[P(D) + Q(D)]^{\rm ad} = P(D)^{\rm ad} + Q(D)^{\rm ad}, \qquad [P(D)\cdot Q(D)]^{\rm ad} = Q(D)^{\rm ad}\cdot P(D)^{\rm ad}. \quad (6.1)$$

Definition 6.2 Let $R(D) \in \mathcal{M}[D]^{g\times q}$ and $t \in \mathbb{R}$. The *local adjoint* of the kernel representation $\mathfrak{B}_R^{\ker}(t)$ of the system $R(\frac{\rm d}{\rm dt})w = 0$ is the image representation

$$\big\{ \tilde w \in \mathcal{C}_t^\infty(\mathbb{R}^q) \big| \;\; \exists\, l \in \mathcal{C}_t^\infty(\mathbb{R}^g)\ \forall\, \tau \in \text{ dom } w \cap \text{ dom } l \;:\; \tilde w(\tau) = R(\tfrac{\rm d}{\rm d\tau})^{\rm ad} l(\tau) \big\}\,. \quad (6.2)$$

□

Certainly, the projection onto the first component of the kernel representation

$$\left\{ (\tilde{w}, l) \in \mathcal{C}_t^\infty(\mathbb{R}^q) \times \mathcal{C}_t^\infty(\mathbb{R}^g) \;\middle|\; \forall \tau \in \text{ dom } \tilde{w} \cap \text{ dom } l \; : \; [I_q, R(\tfrac{\mathrm{d}}{\mathrm{d}\tau})^{\mathrm{ad}}] \begin{bmatrix} \tilde{w}(\tau) \\ l(\tau) \end{bmatrix} = 0 \right\}$$

yields the image representation (6.2).

The following definition is a straightforward generalization of observability for time-invariant systems in the behavioural set-up, see [21, Def. 5.3.2].

Definition 6.3 Let $[R_1(D), R_2(D)] \in \mathcal{M}[D]^{g\times(q_1+q_2)}$ and $t \in \mathbb{R}$. Then $w_2 \in \mathcal{C}_t^\infty(\mathbb{R}^{q_2})$ is called *locally observable at* $t \in \mathbb{R}$ *from* $w_1 \in \mathcal{C}_t^\infty(\mathbb{R}^{q_1})$ for $t \in \mathbb{R}$ if, and only if,

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ \tilde{w}_2 \end{bmatrix} \in \mathfrak{B}^{\ker}_{[R_1,R_2]}(t) \qquad \Longrightarrow \qquad \forall \tau \in \text{ dom } w_2 \cap \text{ dom } \tilde{w}_2 \; : \; w_2(\tau) = \tilde{w}_2(\tau)\,.$$

□

In [13, Prop. 5.7] it is shown that Definition 6.3 generalizes other well known concepts of observability, such as for time-varying state space systems (see for example [24]), time-varying Rosenbrock systems (see [14]). Furthermore, in [13, Th. 5.5, 5.6] the following is shown:

Proposition 6.4 For $[R_1(D), R_2(D)] \in \mathcal{M}[D]^{g\times(q_1+q_2)}$ the following two statements hold:

(i) Consider the system $[R_1(\tfrac{\mathrm{d}}{\mathrm{dt}}), R_2(\tfrac{\mathrm{d}}{\mathrm{dt}})] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$;
then w_2 is locally observable almost everywhere from w_1 if, and only if, $R_2(D)$ is left invertible over $M[D]$.

(ii) The system $[R_1(\tfrac{\mathrm{d}}{\mathrm{dt}}), R_2(\tfrac{\mathrm{d}}{\mathrm{dt}})] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$ is locally controllable almost everywhere if, and only if, l is locally observable almost everywhere from w with respect to the system

$$\begin{bmatrix} I_q \;, & \begin{matrix} R_1(\tfrac{\mathrm{d}}{\mathrm{dt}})^{\mathrm{ad}} \\ R_2(\tfrac{\mathrm{d}}{\mathrm{dt}})^{\mathrm{ad}} \end{matrix} \end{bmatrix} \begin{bmatrix} w \\ l \end{bmatrix} = 0.$$

□

An application of Proposition 6.4 to descriptor systems (1.3) yields the following result.

Theorem 6.5 Consider a descriptor system (1.3) with $R(D) = [R_1(D), R_2(D)]$ partitioned as

$$R_1(D) = \begin{bmatrix} ED - A \\ -C \end{bmatrix}, \qquad R_2(D) = \begin{bmatrix} -B & 0 \\ -F & I_p \end{bmatrix}.$$

Then the following are equivalent:

(i) x is locally observable from (u, y) almost everywhere,

(ii) $R_1(D)$ is left invertible over $\mathcal{M}[D]$,

(iii) the matrix

$$\begin{bmatrix} \sigma DI_d - \tilde{A}_{11} & -\tilde{A}_{13} & -\tilde{A}_{14} \\ -\tilde{A}_{31} & 0 & 0 \\ 0 & -\Sigma_\omega & 0 \\ -\sigma^{-1}\tilde{C}_{21} & 0 & 0 \end{bmatrix} \tag{6.3}$$

is left invertible over $\mathcal{M}[D]$, where the matrices in (6.3) are from the condensed form (2.4).

Proof: The equivalence "(i)$\leftrightarrow$ (ii)" follows Proposition 6.4 (i). To see "(ii)$\leftrightarrow$ (iii)", note that left invertibility of $R_2(D)$ is equivalent to

$$\begin{bmatrix} U & 0 \\ X & W \end{bmatrix} R_2(D)V = \begin{bmatrix} \sigma DI_d - \tilde{A}_{11} & 0 & -\tilde{A}_{13} & -\tilde{A}_{14} & 0 \\ 0 & -\Sigma_a & 0 & 0 & 0 \\ -\tilde{A}_{31}(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Sigma_f \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\Sigma_\omega & 0 & 0 \\ -\sigma^{-1}\tilde{C}_{21} & 0 & 0 & 0 & 0 \end{bmatrix}$$

being left invertible, where U, X, V, W are specified in Theorem 2.1 (iii). Since $\begin{bmatrix} U & 0 \\ X & W \end{bmatrix}$ and V are invertible over $\mathcal{M}$, the latter holds true if, and only if, (6.3) is left invertible. This completes the proof. □

Example 6.6 Consider again the linearized model (1.5) of the three-link constrained mobile manipulator. Suppose that the positions can be measured, corresponding to the additional equation

$$y = \begin{bmatrix} 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}. \tag{6.4}$$

In Example 2.5 we have shown that $x_1 = 0$, $x_3 = 0$ and thus $\dot{x}_1 = 0$ and $\dot{x}_3 = 0$ and permuting the variables accordingly to (2.12), we obtain

$$y = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} x_2 \\ x_4 \\ \hline x_3 \\ x_5 \\ x_1 \end{array}\right]. \tag{6.5}$$

Hence by Theorem 6.5, x is observable from (u, y) with respect to the system (1.5), (6.4) or equivalently system (2.11), (6.5) if, and only if,

$$\left[\begin{array}{cc|ccc} D-1 & 0 & 0 & 0 & 0 \\ -K_{22} & M_{22}D-D_{22} & 0 & 0 & 0 \\ \hline -\tilde{K}_{12} & -\tilde{D}_{12} & 0 & -F_1^T & 0 \\ 0 & 0 & 0 & 0 & F_1 \\ 0 & 0 & -I_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{6.6}$$

is left invertible over $\mathcal{M}[D]$. Since F_1 is invertible over $\mathcal{M}$, (6.6) is left invertible if, and only if, $\begin{bmatrix} D-1 & 0 \\ -K_{22} & M_{22}D-D_{22} \end{bmatrix}$ is invertible over $\mathcal{M}[D]$. Summarizing: x is observable from (u,v) almost everywhere if, and only if, K_{22} is invertible over $\mathcal{M}$. □

7 Conclusion

We have introduced a general behavioural approach to linear descriptor systems with real analytic coefficients. We have characterized autonomous, controllable and observable behaviour and have generalized results on time-varying ordinary differential equations and on time-invariant linear algebraic-differential equations. The results have been illustrated by several examples which demonstrates that the approach also helps in understanding practical problems such as constrained multibody systems.

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