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Continuous time-varying scalar systems - a behavioural approach

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Abstract

We introduce a behavioural approach to linear time-varying systems in kernel representation (1.1). The 1×2 matrix $[p(D), q(D)]$ is defined over $\mathcal{M}[D]$, i.e. the skew polynomial ring with real meromorphic coefficients $\mathcal{M}$, indeterminate D, and multiplication rule $Df = fD + \dot{f}$. Willem's behavioural approach to time-invariant systems and approaches to time-varying systems are generalized. We are aiming at a global approach in the sense that the time axis consists of the real numbers minus a discrete set of critical points where the system may exhibit finite escape time. Controllable and autonomous behaviours are introduced and characterized.

1 Introduction and preliminaries

In the present note, a behavioural approach to scalar linear time-varying systems is introduced. We consider systems respectively ordinary differential equations of the form

$$\left[p(\tfrac{\mathrm{d}}{\mathrm{d}t}), q(\tfrac{\mathrm{d}}{\mathrm{d}t})\right] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0, \qquad (1.1)$$

where $p(D), q(D) \in \mathcal{M}[D]$ are not both zero; $\mathcal{M}[D]$ denotes the skew polynomial ring with real meromorphic coefficients $\mathcal{M}$ (the quotient field of the ring of real analytic functions $\mathcal{A}$), indeterminate D, and - as D stands for the ordinary differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$ - skew multiplication rule

$$Df = fD + \tfrac{\mathrm{d}}{\mathrm{d}t} f.$$

Note that we distinguish between the algebraic indeterminate D and the differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$; if $r(D) = \sum_{i=0}^{n} r_i D^i \in \mathcal{M}[D]$, then

$$r(\tfrac{\mathrm{d}}{\mathrm{d}t})w = \textstyle\sum_{i=0}^{n} r_i(t) w^{(i)}(t).$$

The solution space for w_1, w_2 in (1.1) is crucial and its choice is not obvious. The following examples illustrate some of the difficulties when considering time-varying differential equations. The notation

$$\ker_{\mathcal{W}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \left\{ w \in \mathcal{W} \;\middle|\; r(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0 \right\}$$

is used, where $\mathcal{W}$ denotes some suitable solution space.

$\mathbf{r(D) = tD + 1}$: The function $t \mapsto w(t) = t^{-1}$ is a meromorphic solution of $r(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and has a singularity at $t = 0$. Therefore,

$$\ker_{\mathcal{A}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \ker_{\mathcal{C}^\infty} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \{0\},$$

but, for any interval $I \subseteq \mathbb{R}$ with $0 \notin I$,

$$\begin{aligned} \dim\ \ker_{\mathcal{M}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) &= \dim\ \ker_{\mathcal{A}_{|_I}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) \\ &= 1 \\ &= \deg r(D). \end{aligned}$$

If in the above example $\mathcal{W} = \mathcal{M}$, then the dimension of the solution space equals the degree of $r(D)$. That this is not a general rule is illustrated by the following example.

$\mathbf{r(D) = t^2D + 1}$: The function $t \mapsto w(t) = e^{\frac{1}{t}}$ solves $r(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. w is not meromorphic and has a singularity at $t = 0$. In this case we have that

$$\ker_{\mathcal{M}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \{0\}.$$

For any interval $I \subseteq \mathbb{R}$ with $0 \notin I$ we have

$$\dim\ \ker_{\mathcal{M}_{|_I}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = 1 = \deg r(D).$$

These two examples suggest that the zeros of the leading coefficient of $r(D)$ determine the critical points where the solutions might exhibit finite escape time, but this is not necessarily the case as shown by the following example.

$\mathbf{r(D) = tD - 1}$: The function $t \mapsto w(t) = t$ is an analytic solution of $r(\frac{d}{dt})w = 0$ and we have that

$$\dim \ \ker_{\mathcal{A}} r(\tfrac{d}{dt}) = 1 = \deg r(D).$$

However the solution is not as arbitrary as for time-varying systems, $w(0) = 0$ is the only value at $t = 0$.

The above examples show that studying the solution space of (1.1) requires a *local* investigation, excluding so-called "critical points" on the time-axis, i.e. points where solutions are not defined. On the other hand, the skew polynomial ring $\mathcal{M}[D]$ is a Euclidean domain which allows for a global algebraic investigation where critical points do not come into play. In this note we are aiming at an "almost global" theory of the solution space by exploiting the algebraic structure of matrices over $\mathcal{M}[D]$.

If $r(D) = \sum_{i=0}^{n} r_i D^i \in \mathcal{A}[D]$, $r_n \neq 0$, has analytic coefficients, then it is well known that the solutions of $r(\frac{d}{dt})w = 0$ are well defined at least on any open interval I not containing a zero of $r_n(\cdot)$. For this reason we define, for

$$\begin{aligned} &r(D) = \sum_{i=0}^{n} r_i D^i \in \mathcal{M}[D], \quad r_i = \alpha_i/\beta_i \\ &\alpha_i \in \mathcal{A},\ \beta_i \in \mathcal{A}\backslash\{0\},\ i = 1,\ldots,n,\ r_n \neq 0, \end{aligned}$$

the set

$$\hat{\mathbb{T}} := \{t \in \mathbb{R} \mid \alpha_n(t)\beta_0(t)\cdots\beta_n(t) = 0\} \qquad (1.2)$$

of *possibly critical points*.

To extend this definition to the matrix case $[p(D), q(D)] \in \mathcal{M}[D]^{1\times 2}$, we choose a unimodular matrix $V(D) \in \mathcal{M}[D]^{2\times 2}$ so that

$$[p(D), q(D)] = [r(D), 0]\ V(D)^{-1} \qquad (1.3)$$

for some $r(D) \in \mathcal{M}[D]$ which is unique up to similarity*. The existence of such $V(D)$ is due to the properties of $\mathcal{M}[D]$, which is a Euclidean domain, not containing any zero divisors and being simple, see Cohn (1991). $V(D)$ is not unique since for any $w_3(D) \in \mathcal{M}[D]$ and $w_4 \in \mathcal{M}\backslash\{0\}$ we have

$$[p(D), q(D)] = [r(D), 0] \begin{bmatrix} 1 & 0 \\ w_3(D) & w_4 \end{bmatrix} V(D)^{-1}.$$

From (1.3) we see that possibly critical points for a solution of (1.1) include the zeros and poles of the coefficients of the first row of $V(D)^{-1}$ and $\hat{\mathbb{T}}$. More precisely, let

$$V(D)^{-1} = \begin{bmatrix} \eta^1(D) \\ \eta^2(D) \end{bmatrix}, \quad \text{where}$$

$$\eta^j(D) = \sum_{i=0}^{n_j} \eta_i^j D^i \in \mathcal{M}^{1\times 2}[D].$$

Then the set of *possibly critical points* associated with (1.1) and (1.3) is

$$\mathbb{T} := \hat{\mathbb{T}} \cup \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} t \text{ is a pole or zero of some } \eta_i^1 \in \\ \mathcal{M}^{1\times 2} \text{ for } i = 1,\ldots,n_1 \end{array} \right\} \qquad (1.4)$$

where $\hat{\mathbb{T}}$ is defined in (1.2) for $r(D)$. Set

$$\mathbb{R}_{\mathbb{T}} := \mathbb{R} \setminus \mathbb{T}.$$

Note that $\mathbb{T}$ is discrete in $\mathbb{R}$.

We are now in a position to define the solution space of (1.1) as follows

$$C^\infty(\mathbb{R}_{\mathbb{T}}, \mathbb{R}^2) :=$$

$$\left\{ w : \mathbb{R}_{\mathbb{T}} \to \mathbb{R}^2 \;\middle|\; \begin{array}{l} w \text{ is infinitely many} \\ \text{times differentiable} \\ \text{on } \mathbb{R}_{\mathbb{T}} \end{array} \right\}.$$

Our results are related to early algebraic results on time-varying systems, see Silverman and Meadows (1967), Kamen (1976), Ylinen (1980), Fliess (1990), but generalize in particular the time-varying results by Ilchmann et al. (1984) and Fröhler and Oberst (1999), the time-invariant results by Hinrichsen and Prätzel-Wolters (1980) and the behavioural approach introduced by Willems (1981), see in particular the recent textbook by Polderman and Willems (1998). The novelty of this paper is a "global" approach.

The note is structured as follows. In Section 2 we introduce controllable behaviour and characterize it in terms of coprimeness. In Section 3 we derive the appropriate solution space to introduce autonomous behaviour, which then puts us in a position to define behaviour for (1.1).

2 Controllable behaviour

We are now in a position to generalize the notion of controllability as introduced by Hinrichsen and Prätzel-Wolters (1980) for time-invariant Rosenbrock systems and by Willems (1991) in the behavioural setup.

Definition 2.1 Consider (1.1) with $\mathbb{T}$ as in (1.4). A real vector space

$$\mathfrak{B}_c \subseteq \left\{ w \in C^\infty(\mathbb{R}_{\mathbb{T}}, \mathbb{R}^2) \;\middle|\; [p(\tfrac{d}{dt}), q(\tfrac{d}{dt})]w = 0 \right\}$$

is called a *controllable behaviour* of (1.1) if, and only if, for every trajectories $w_1, w_2 \in \mathfrak{B}_c$ and for every interval $I \subseteq \mathbb{R}_{\mathbb{T}}$, and each $t_0' \in I$, there exists $t_1' > t_0'$, $t_1' \in I$ and $w \in \mathfrak{B}_c$, such that

$$w(t) = \begin{cases} w_1(t), & \text{for all } t \leq t_0' \text{ and } t \in \mathbb{R}_{\mathbb{T}} \\ w_2(t), & \text{for all } t \geq t_1' \text{ and } t \in \mathbb{R}_{\mathbb{T}}. \end{cases}$$

*$r_1, r_2 \in \mathcal{M}[D]$ are *similar* if, and only if, $r_1 a = b r_2$ for some $a, b \in \mathcal{M}[D]$ for which r_1 and b (r_2 and a) are left (right) coprime, respectively.

Since the family of linear subspaces, partially ordered by inclusion, is a lattice with respect to $+$ and $\cap$, the *largest controllable behaviour* of (1.1)

$$\mathfrak{B}_{\text{contr}} \subseteq \left\{ w \in \mathcal{C}^\infty(\mathbb{R}_{\mathbb{T}}, \mathbb{R}^2) \middle|\ [p(\tfrac{\mathrm{d}}{\mathrm{dt}}), q(\tfrac{\mathrm{d}}{\mathrm{dt}})]w = 0 \right\},$$

that is, $\mathfrak{B}_{\text{contr}}$ is a controllable behaviour of (1.1) such that each controllable behaviour $\mathfrak{B}_c$ of (1.1) satisfies $\mathfrak{B}_c \subseteq \mathfrak{B}_{\text{contr}}$, is well defined.
The system (1.1) is called *controllable* if, and only if,

$$\mathfrak{B}_{\text{contr}} = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}_{\mathbb{T}}, \mathbb{R}^2) \middle|\ [p(\tfrac{\mathrm{d}}{\mathrm{dt}}), q(\tfrac{\mathrm{d}}{\mathrm{dt}})]w = 0 \right\}.$$

□

The following result generalizes the characterization of controllablility in terms of coprimeness as in Hinrichsen and Prätzel-Wolters (1980), Ilchmann et al. (1984), and Polderman and Willems (1998).

Theorem 2.2 The system (1.1) is controllable if, and only if, $p(D)$ and $q(D)$ are left coprime.

Proof: Consider (1.3) with corresponding $\mathbb{T}$ of possibly critical points as in (1.4) and let

$$\mathcal{S} := \left\{ w \in \mathcal{C}^\infty(\mathbb{R}_{\mathbb{T}}, \mathbb{R}^2) \middle|\ [p(\tfrac{\mathrm{d}}{\mathrm{dt}}), q(\tfrac{\mathrm{d}}{\mathrm{dt}})]w = 0 \right\}.$$

"$\Rightarrow$": Seeking a contradiction, suppose that $r(D)$ as defined in (1.3) has $\deg r(D) \geq 1$. Since $r(D)$ is a common left factor of $p(D)$ and $q(D)$, there exist $a(D), b(D) \in \mathcal{M}[D]$ such that $p(D)a(D) + q(D)b(D) = r(D)$. Let $I = (t_0, t_1) \subseteq \mathbb{R}_{\mathbb{T}}$, $t_0' \in I$ and the coefficients of $a(D), b(D)$ do neither have poles nor zeros in I. Choose some nonzero function $\varphi \in \mathcal{C}^\infty(I, \mathbb{R})$ such that $r(\frac{\mathrm{d}}{\mathrm{dt}})\varphi = 0$. If $\mathcal{S}$ was controllable, then there exists $t_1' > t_0'$, $t_1' \in I$, and some $w \in \mathcal{S}$ such that

$$w(t) = \begin{cases} \left(a(\tfrac{\mathrm{d}}{\mathrm{dt}}), b(\tfrac{\mathrm{d}}{\mathrm{dt}})\right)^T \varphi(t), & t \in (-\infty, t_0'] \cap I \\ 0, & t \in [t_1', \infty) \cap I. \end{cases}$$

Since I does not contain any critical points, φ and w are real analytic on I, and thus the identity property of analytic functions yields $a(\frac{\mathrm{d}}{\mathrm{dt}})\varphi(t) = b(\frac{\mathrm{d}}{\mathrm{dt}})\varphi(t) = 0$ for all $t \in [t_1', \infty) \cap I$. Applying again the fact that I does not contain any critical points, we conclude that $\varphi = 0$ on I, which contradicts the assumption that φ is nonzero.
"$\Leftarrow$": Suppose $r(D) = r \in \mathcal{M}\backslash\{0\}$. Since the system is linear, it is sufficient to show that for any $w_1 \in \mathcal{S}$ and $t_0' \in I$ there exist $t_1' > t_0'$, $t_1' \in I$ and some $w \in \mathcal{S}$ such that

$$w(t) = \begin{cases} w_1(t), & \text{for all } t \leq t_0' \text{ and } t \in \mathbb{R}_{\mathbb{T}} \\ 0, & \text{for all } t \geq t_1' \text{ and } t \in \mathbb{R}_{\mathbb{T}}. \end{cases} \tag{2.1}$$

It follows from (1.3) and $r \in \mathcal{M}$ that $w = V(\frac{\mathrm{d}}{\mathrm{dt}})\ (0, \varphi)^T$ for some $\varphi \in \mathcal{C}^\infty(\mathbb{R}_{\mathbb{T}}, \mathbb{R})$. Choose $\delta \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\delta(t) = \begin{cases} 1, & \text{for all } t \leq t_0' \\ 0, & \text{for all } t \geq t_1', \end{cases}$$

and define $w = V(\frac{\mathrm{d}}{\mathrm{dt}})\ \delta\ (0, \varphi)^T$. Then $w \in \mathcal{S}$ and (2.1) holds true. This completes the proof. □

It also can be shown that the largest controllable behaviour is uniquely defined in terms of $V(D)$ satisfying (1.3).

Proposition 2.3 For any $V(D) \in \mathcal{M}[D]^{2\times 2}$ satisfying (1.3) we have

$$\mathfrak{B}_{\text{contr}} = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}_{\mathbb{T}}, \mathbb{R}^2) \middle|\ [1, 0]\, V(\tfrac{\mathrm{d}}{\mathrm{dt}})^{-1} w = 0 \right\}.$$

3 Autonomous behaviour and behaviour

In this section we define an autonomous behaviour of systems (1.1). Due to the time-varying situation, this concept has to be defined slightly different than in Polderman and Willems (1998). Consider for example again $r(D) = tD + 1$. By Theorem 2.2 the system $r(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ is not controllable, and hence it should be autonomous. If $r(D)$ had real coefficients, then autonomous according to Polderman and Willems meant that any two solutions of $r(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$, which coincide on any interval, must coincide on the whole axis $\mathbb{R}$. However, for $r(D) = tD + 1$ the two solutions defined by

$$\varphi_1(t) = \begin{cases} 0, & t < 0 \\ t^{-1}, & t > 0 \end{cases},$$

$$\varphi_2(t) = \begin{cases} 0, & t < 0 \\ -t^{-1}, & t > 0 \end{cases}$$

do not satisfy this property. The reason is that the possible solution spaces $\mathcal{C}^\infty(\mathbb{R}_{\mathbb{T}}, \mathbb{R})$ or $\mathcal{A}(\mathbb{R}_{\mathbb{T}}, \mathbb{R})$ are too large. Since we do not want a local analysis, Schmale's (1985) concept of "cut neighbourhoods" is appropriate in our setup:
Consider $r(D) = \sum_{i=0}^{n} r_i D^i \in \mathcal{M}[D]$ with corresponding set of possibly critical points $\mathbb{T}$. Extend the domain of definition of the real meromorphic coefficients r_i to some open set $U \subseteq \mathbb{C}$ with $\mathbb{R} \subseteq U$ such that $\{\{t\} + i\mathbb{R}\} \cap U$ is connected for every $t \in \mathbb{R}$. Then

$$U_{\hat{\mathbb{T}}} := U \setminus \{\, t + i\mathbb{R}_{\geq 0} \mid t \in \hat{\mathbb{T}} \}$$

is called a *cut neighbourhood* of $\mathbb{R}$ relative to $\hat{\mathbb{T}}$. Note that we have $\mathbb{R}_{\hat{\mathbb{T}}} \subseteq U_{\hat{\mathbb{T}}}$. As the real meromorphic coefficients of $r(D)$ are now extended to $U_{\hat{\mathbb{T}}}$, we might also consider complex solutions $w : U_{\hat{\mathbb{T}}} \to \mathbb{C}$ of $r(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$. One advantage of this construction is, as a standard result of complex ordinary differential equations (see for example Walter [10, p. 213]), the "global" degree formula:

$$\dim\ \left\{ w : U_{\hat{\mathbb{T}}} \to \mathbb{C} \middle|\ r(\tfrac{\mathrm{d}}{\mathrm{dt}})w = 0 \right\} = \deg r(D).$$

The appropriate solution space for (1.1), factorized as in (1.3) with associated set of possibly critical points $\mathbb{T}$, is

$$\mathcal{E}^\infty(\mathbb{R}_\mathbb{T},\mathbb{R}^2) = \left\{ \begin{array}{l} w \in \mathcal{C}^\infty(\mathbb{R}_\mathbb{T},\mathbb{R}^2) \text{ is extendable to an} \\ \text{analytic function } w : U_\mathbb{T} \to \mathbb{C}^2 \text{ on} \\ \text{some open, simply connected set } U_\mathbb{T} \\ \text{including } \mathbb{R}_\mathbb{T} \end{array} \right\}.$$

Revisiting $r(D) = tD + 1$, we see that $\varphi_1, \varphi_2 \notin \mathcal{E}^\infty(\mathbb{R}_\mathbb{T},\mathbb{R})$ and the only solution of $r(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ in the extended space is $w(t) = t^{-1}$.
Now we are in a position to define autonomous behaviour of (1.1).

Definition 3.1 A real vector space

$$\mathfrak{B}_\mathrm{a} \subseteq \{w \in \mathcal{C}^\infty(\mathbb{R}_\mathbb{T},\mathbb{R}^2) \mid [p(\tfrac{\mathrm{d}}{\mathrm{d}t}), q(\tfrac{\mathrm{d}}{\mathrm{d}t})]w = 0\}$$

is called a *autonomous behaviour* of (1.1) if, and only if, for all trajectories $w_1 \in \mathfrak{B}_\mathrm{a}$ and $w_2 \in \mathcal{E}^\infty(\mathbb{R}_\mathbb{T},\mathbb{R}^2)$ with $[p(\frac{\mathrm{d}}{\mathrm{d}t}), q(\frac{\mathrm{d}}{\mathrm{d}t})]w_2 = 0$ we have:

$$w_1 = w_2 \quad \text{on some } I \subseteq \mathbb{R}_\mathbb{T} \quad \Longrightarrow \ w_1 = w_2 \text{ on } \mathbb{R}_\mathbb{T}.$$

□

It is obvious that for any autonomous behaviour $\mathfrak{B}_\mathrm{a}$ as in Definition 3.1 we have $\mathfrak{B}_\mathrm{a} \cap \mathfrak{B}_\mathrm{contr} = \{0\}$, and it is not difficult to see that for any $w \in \mathfrak{B}_\mathrm{a}$ there exists a unimodular $V(D)$ satisfying (1.3) such that

$$\begin{bmatrix} 1 & 0 \\ 0 & r(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w = 0;$$

the latter is the pendant to Proposition 2.3, and furthermore

$$\mathfrak{B}_\mathrm{a} \subseteq \{w \in \mathcal{E}^\infty(\mathbb{R}_\mathbb{T},\mathbb{R}^2) \mid [p(\tfrac{\mathrm{d}}{\mathrm{d}t}), q(\tfrac{\mathrm{d}}{\mathrm{d}t})]w = 0\}.$$

It can be proved that, for any $r(D)$ and $V(D)$ satisfying (1.3),

$$\mathfrak{B}_\mathrm{aut} := \left\{ w \in \mathcal{E}^\infty(\mathbb{R}_\mathbb{T},\mathbb{R}^2) \;\middle|\; \begin{bmatrix} 1 & 0 \\ 0 & r(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w = 0 \right\} \tag{3.1}$$

is an autonomous behaviour of (1.1) and has dimension

$$\dim \mathfrak{B}_\mathrm{aut} = \deg r(D).$$

Note that $\mathfrak{B}_\mathrm{aut}$ is not uniquely defined but depends on the factorization (1.3). However, it can be shown that, as in the time-invariant case, the direct sum $\mathfrak{B}_\mathrm{aut} \oplus \mathfrak{B}_\mathrm{contr}$ is uniquely defined.

Proposition 3.2 Consider (1.1), let $\mathfrak{B}_\mathrm{aut}$ be given as in (3.1), and let another autonomous behaviour of (1.1) be induced by a different factorization (1.3). Then

$$\mathfrak{B}_\mathrm{aut} \oplus \mathfrak{B}_\mathrm{contr} = \overline{\mathfrak{B}}_\mathrm{aut} \oplus \mathfrak{B}_\mathrm{contr}. \tag{3.2}$$

□

This sets us in a position to define the behaviour of (1.1) as follows.

Definition 3.3 Let $\mathfrak{B}_\mathrm{contr}$ be the controllable behaviour of (1.1) and let $\mathfrak{B}_\mathrm{aut}$ be some autonomous behaviour of (1.1) as in (3.1). Then

$$\mathfrak{B} := \mathfrak{B}_\mathrm{aut} \oplus \mathfrak{B}_\mathrm{contr}$$

is called the *behaviour* of (1.1).

□

Note the difference in the definition of behaviour compared to time-invariant systems as in Polderman and Willems (1998). For time-invariant systems, $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^2)$ is the right solution space to define the behaviour of (1.1) straight away. If the analogue $\mathcal{C}^\infty(\mathbb{R}_\mathbb{T},\mathbb{R}^2)$ is tried for time-invariant systems, one runs into the difficulties explained at the beginning of Section 3. One could think of taking the smaller solution space $\mathcal{E}^\infty(\mathbb{R}_\mathbb{T},\mathbb{R}^2)$ instead, but this one does not allow for inputs. For this reason we first define controllable and and autonomous behaviour and then, since (3.2) holds, behaviour as a sum.

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