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Proceedings of the 30th IEEE Conference on Decision and Control :
11 - 13 December 1991, Brighton, England / IEEE Control Systems
Society; Vol 1. – New York : IEEE, 1991, S. 83-84

DOI: [10.1109/CDC.1991.261260](https://doi.org/10.1109/CDC.1991.261260)

Exponential Stabilization using Piecewise Constant Gain Adaptation

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Abstract

An adaptive stabilizer for the class of SISO minimum phase systems with known sign is introduced. It extends the work of Willems and Byrnes but uses a 'threshold' based piecewise constant gain adaptation. This provides an exponential decaying state and moreover the set of thresholds which stabilize the terminal system exponentially are dense.

1 Introduction

In recent years attention has been focussed on non identifier based adaptive stabilization of SISO linear systems of the form

$$\dot{x}(t) = Ax(t) + bu(t), y(t) = cx(t), x(0) \in \mathbb{R}^n \quad (1.1)$$

where the state space dimension n is unknown, it is only assumed that the $\text{sign}(cb)$ is known and the system is *minimum phase*, i.e.

(i) The transfer function has no zeros in $\overline{\mathbb{C}_+}$.

(ii) (A, b, c) is stabilizable and detectable.

The simplest adaptive controller for this class of systems is due to Willems and Byrnes [5] and is based on the idea that the time-varying parameter $k(t)$ in the output feedback

$$u(t) = -\text{sgn}(cb)k(t)y(t) \quad (1.2)$$

finds the appropriate gain level by adapting $k(\cdot)$ via the evolution law

$$\dot{k}(t) = y(t)^2, \quad k(0) = k_0 \in \mathbb{R}. \quad (1.3)$$

For every initial data x_0, k_0 the closed loop system (1.1)-(1.3) guarantees that $\lim_{t \rightarrow \infty} x(t) = 0$ and $\lim_{t \rightarrow \infty} k(t) = k_\infty$ exists.

Based on these ideas, in Section 2 we present an adaptive stabilizer which uses a *piecewise constant* gain $k(\cdot)$ adjusted via a sequence of thresholds. This provides the bonus of exponential decaying of the state. However the terminal system $\dot{x} = [A - k_\infty bc]x$ is not necessarily stable. In Section 3 we show that for every fixed initial condition x_0 the sequences of thresholds which lead to an exponentially stable terminal system are dense.

2 Exponential decaying of the state

Proposition 2.1: Let $\{T_i\}_{i \in \mathbb{N}} \subset \mathbb{R}_+$ be a sequence of 'thresholds' which satisfies $\inf_{i \in \mathbb{N}} \{T_{i+1} - T_i\} > 0$. Then the adaptive controller

$$\left. \begin{aligned} u(t) &= -k(t)y(t) \\ \dot{s}(t) &= y(t)^2, \quad s(0) = T_1 \\ k(t) &= T_i \quad \text{for } s(t) \in [T_i, T_{i+1}) \end{aligned} \right\} \quad (2.1)$$

applied to any minimum phase system of the form (1.1) with $cb > 0$ yields a closed loop system with

- (i) The solution (x, s) of (1.1), (2.1) exists for all $t \geq 0$.
- (ii) $x(t)$ tends to zero exponentially.
- (iii) $\lim_{t \rightarrow \infty} s(t) = s_\infty$ exists.
- (iv) For some $j \in \mathbb{N}$ and $t^* > 0$, $k(t) = T_j$ for all $t \geq t^*$.

Proof: (a) Since the right hand side of the closed loop system is discontinuous in x existence and uniqueness of the solution as well as finite escape time has to be clarified. This can be done similar to [1] and considering the following steps on finite intervals first. We omit it for brevity and assume that (x, s) exists on $\mathbb{R}_+$.

(b) It has been proved in [2] that the minimum phase property implies the inequality

$$\frac{1}{2} y(t)^2 \leq M + M \int_{t_0}^t y(s)^2 ds + cb \int_{t_0}^t u(s)y(s) ds \quad (2.2)$$

for some $M = M(x_0) > 0$ and all $t \geq t_0$, $t_0 \geq 0$. Inserting the feedback into (2.2) yields

$$\frac{1}{2} y(t)^2 \leq M + \int_{t_0}^t [M - cbk(s)]y(s)^2 ds. \quad (2.3)$$

If $k(\cdot) \notin L_\infty$ then $y(\cdot) \notin L_2$ and the right hand side of (2.3) goes to $-\infty$ as t tends to ∞ . This contradicts the positivity of the left hand side and proves boundedness of $k(t)$.

(c) $k(\cdot) \in L_\infty$ yields $y(\cdot) \in L_2$ and a well known argument gives $\lim_{t \rightarrow \infty} x(t) = 0$, see [5]. This proves (iii) and (iv).

(d) Using the notation of (iv) it follows that $x(t)$ satisfies

$$\dot{x}(t) = [A - T_j bc]x(t), \quad \text{for all } t \geq t^*.$$

Since the terminal system is linear time-invariant and detectable, every mode which is not exponentially stable would contradict the boundedness of $s(\cdot)$. Therefore (ii) is proved. $\square$

Remark 2.2:

(i) In case of unknown sign of the high frequency gain cb the same result holds true if the control strategy is modified by using the concept of switching functions due to Nussbaum [4], see also [5].

(ii) Proposition 2.1 can be extended to the class of adaptive controllers which use piecewise constant gain adaptation. The essential ingredient is that after a finite time the closed loop system $\dot{x} = [A - kbc]x$ is a time-invariant linear system.

3 Exponential stability of the terminal system

In Section 2 we have shown that the adaptive stabilizer (2.1) ensures an exponential decaying state trajectory corresponding to the initial condition $x_0 \in \mathbb{R}^n$. However the terminal system $\dot{x} = [A - T_j bc]x$ must not be exponentially stable, the trajectory may lie in a stable subspace.

In this section we shall show that for each fixed initial condition $x_0 \in \mathbb{R}^n$ the set of thresholds which achieve an exponentially stable terminal system is dense in the set of all admissible thresholds.

Proposition 3.1: Under the assumptions of Proposition 2.1 for every fixed system (1.1) and every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ and $T'_1, \dots, T'_N$ such that $|T_i - T'_i| < \varepsilon$ for all $i \in \underline{N}$ and the switching sequence $T'_1, \dots, T'_N, T_{N+1}, \dots$ used for the adaptive stabilizer yields an exponentially stable terminal system, i. e. $\sigma(A - T'_N bc) \subset \mathbb{C}_-$. $x_c \neq 0$, (c, A) observable.

Proof: Using the notation

$$A_\varepsilon := A - (k + \varepsilon)bc$$

$$p_\varepsilon(s) := \det(sI_n - A_\varepsilon), \quad z(s) := c \cdot \text{adj}(sI_n - A) \cdot b$$

it follows that

$$p_\varepsilon(s) = p_0(s) + \varepsilon z(s) \quad \text{for all } \varepsilon \in \mathbb{R}. \quad (3.1)$$

We proceed in several steps.

(a) All eigenvalues of A_ε in $\overline{\mathbb{C}_+}$ are moving if ε moves. Suppose the contrary, i. e. for some $s_0 \in \overline{\mathbb{C}_+}$ $\text{const. } 0 \equiv p_0(s_0) + \varepsilon z(s_0)$ locally in ε around 0. This is not possible since $z(s_0) \neq 0$ by the minimum phase assumption.

(b) For every $\varepsilon^* > 0$ there exists $\varepsilon \in [0, \varepsilon^*)$ such that A_ε has distinct eigenvalues in $\overline{\mathbb{C}_+}$. Suppose this is not the case. Then $p_\varepsilon(s)$ has eigenvalues of multiplicity $l \geq 2$ for all $\varepsilon \in [0, \varepsilon_1), \varepsilon_1 > 0$. Therefore

$$p_\varepsilon(s) = (s - \lambda(\varepsilon))^l q_\varepsilon(s) \quad \text{for all } \varepsilon \in [0, \varepsilon_1)$$

and in particular it follows from (3.1) that for $s = \lambda(\varepsilon)$

$$p_\varepsilon(s) = (s - \lambda(0))^l q_0(s) + \varepsilon z(s) = 0$$

where $q_0(s) \neq 0$ in $[0, \varepsilon_1)$. Differentiation w.r.t. s yields also

$$p'_\varepsilon(s) = l(s - \lambda(0))^{l-1} q_0(s) + (s - \lambda(0))^l q'_0(s) + \varepsilon z'(s) = 0$$

Using $z(\lambda(\varepsilon)) \neq 0$ an algebraic manipulation gives for $s = \lambda(\varepsilon)$

$$(s - \lambda(0))^{(l-1)} \cdot \psi(s) \equiv 0 \quad \text{for } \varepsilon \in [0, \varepsilon_1),$$

where

$\psi(s) := l q_0(s) z(s) + (s - \lambda(0)) [q'_0(s) z(s) - q_0(s) z'(s)]$
Continuity of $\lambda(\varepsilon)$ yields that $\psi(s)$ is uniformly bounded away from 0 for ε_1 sufficiently small. Therefore $\lambda(\varepsilon) = \lambda(0)$, which is impossible by (a).

(c) Suppose A_0 has distinct eigenvalues and for some $\varepsilon_1 > 0$, $v(\varepsilon) A_\varepsilon = \lambda(\varepsilon) v(\varepsilon)$ for λ continuous in $\varepsilon \in [0, \varepsilon_1)$. Then there exists an $\varepsilon_2 \in (0, \varepsilon_1)$ such that

$$v(\varepsilon)x \neq 0 \quad \text{for all } \varepsilon \in (0, \varepsilon_2).$$

To prove this, it is sufficient to prove the statement for a single x since the unstable subspace is finite dimensional. Note that the eigenvectors depend locally analytically on ε . Therefore it remains to consider the case

$$v(\varepsilon)x \neq 0 \quad \text{for all } \varepsilon \in [0, \varepsilon_3) \text{ for some } \varepsilon_3 > 0$$

This implies that $\begin{bmatrix} \lambda(\varepsilon) - A_\varepsilon & x \\ c & 0 \end{bmatrix}$ is singular whence

$$\det \begin{bmatrix} \lambda(\varepsilon) - A & x \\ c & 0 \end{bmatrix} = 0 \quad \text{for all } \varepsilon \in [0, \varepsilon_3). \text{ By (a), } \lambda(\varepsilon)$$

is not constant. So we have $\det \begin{bmatrix} sI_n - A & x \\ c & 0 \end{bmatrix} \equiv 0$ on $\mathbb{C}$. But this contradicts detectability as the initial state $x(0) = x$ for the system $\dot{x} = Ax, y = Cx$ has zero output.

(d) Now we are able to prove the proposition. Suppose we apply the controller (2.1) and the last switch occurs for T_M at time $t = t_M$. If the terminal system is not stable replace T_M by $T_M + \varepsilon$ according to (b) so that the eigenvalues are separate. Next (c) can be applied to replace T_M by $T_M + \varepsilon + \varepsilon'$ to ensure non-zero projection of $x(t_M)$ into the unstable eigensubspace of $A - (T_M + \varepsilon + \varepsilon')bc$. Repeating this change finitely many times yields exponential stability of the terminal system. This proves the proposition. $\square$

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Acknowledgement

This work was supported, in part, by the German Research Association (DFG) and the EEC SCIENCE programme under grant number ERBSC1 - CT000433.