**Rolf Steyer**

# *How to get it all: Average and individual causal effects, and why individuals differ in their effects. Design and Data analysis*

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# Abstract

A design and a method of data analysis is presented which yield

(a) estimates of the *average causal effect* of a treatment variable on a response variable in the sense of Rubins approach to causality

(b) estimates of the *variance of the individual causal effects* and

(c) of the *covariance between pretest and individual causal effects*.

(d) It is shown how to include variables in the analysis that *explain the interindividual differences in the individual causal effects* of the treatment variable on the response variable.

All this is based on a specific design with random assignment of units to the treatment conditions, assessing a pretest and introducing some additional assumptions which, however, can be tested in the analysis as well.

# My view of the world in an experimental study

**Table 1.** *Individual causal effects, equal and unequal treatment probabilities.*

$$E(Y \mid X = x) := \sum_u E(Y \mid X = x, U = u) \cdot P(U = u \mid X = x)$$

| Person | $P(U=u)$ sampling probability | $E(Y \mid X = 1, U = u)$ Expected outcome | $E(Y \mid X = 0, U = u)$ Expected outcome | $E(Y \mid X = 1, U = u) - E(Y \mid X = 0, U = u)$ Individual causal effec | $P(X = 1 \mid U=u)$ treatment probability in experiment 1 | $P(X = 1 \mid U=u)$ treatment probability in experiment 2 |
|---|---|---|---|---|---|---|
| $u_1$ | 1/8 | 82 | 68 | 14 | 1/2 | 8/9 |
| $u_2$ | 1/8 | 89 | 81 | 8 | 1/2 | 7/9 |
| $u_3$ | 1/8 | 101 | 89 | 12 | 1/2 | 6/9 |
| $u_4$ | 1/8 | 108 | 102 | 6 | 1/2 | 5/9 |
| $u_5$ | 1/8 | 118 | 112 | 6 | 1/2 | 4/9 |
| $u_6$ | 1/8 | 131 | 119 | 12 | 1/2 | 3/9 |
| $u_7$ | 1/8 | 139 | 131 | 8 | 1/2 | 2/9 |
| $u_8$ | 1/8 | 152 | 138 | 14 | 1/2 | 1/9 |
| | | Individual causal laws | | | Good design | Bad design |

Average causal laws:

$$CUE(Y \mid X = x) = \sum_u E(Y \mid X = x, U = u) \cdot P(U = u)$$

Good design implies: $E(Y \mid X = x)$ reflect average causal laws

Bad design implies: $E(Y \mid X = x)$ do not reflect average causal laws

# 1. The Single-Unit Trial

We consider the *following single-unit trial*: sample a unit (or person) from a given set of units (the population), observe its assignment (or assign it) to one of two treatment conditions and register the outcome.

The set of possible outcomes of the single-unit trial described above might be of the form:

$$\Omega = \Omega_U \times \Omega_X \times \mathrm{IR} \quad .$$

Notation:

- $U: \Omega \rightarrow \Omega_U$ *Person variable* or *unit variable*
- $X: \Omega \rightarrow \Omega_X$ *treatment variable*, with values 0 and 1
- $Y: \Omega \rightarrow \mathrm{IR}$ *Outcome variable*

# 2. Individual and Average Causal Effects I

Define the random variables $f_0(U)\colon \Omega \to \mathrm{IR}$ and $f_1(U)\colon \Omega \to \mathrm{IR}$ by:

$$f_0(u) = E(Y \mid X = 0, U = u)$$

for all values $u$ of $U$

and

$$f_1(u) = E(Y \mid X = 1, U = u) - E(Y \mid X = 0, U = u)$$

for all values $u$ of $U$

Then:

$$E(Y \mid X, U) = f_0(U) + f_1(U) \cdot X$$

# 2. Individual and Average Causal Effects II

**Definition 1**

- $f_1(u) =:$ *Individual causal effect of unit u*
- $E\,[\,f_1(U)] =:$ *Average causal effect*

- $f_0(U) =: \eta_0$ *Expected outcome variable under "control"*
- $f_0(U) + f_1(U) =: \eta_1$ *Expected outcome variable under "treatment"*

These two variables, $\eta_0$ and $\eta_1$ replace Rubins "potential outcome variables".

# Causal Unbiasedness

**Definition 2.** The regression $E(Y \mid X)$ as well as its values $E(Y \mid X = 0)$ and $E(Y \mid X = 1)$ are called *causally unbiased* if the following equations hold:

$$E(Y \mid X = 0) = E\,[f_0(U)]$$

$$E(Y \mid X = 1) = E\,[f_0(U)] + E\,[f_1(U)]$$

**Corollary 1.** If $E(Y \mid X)$ is causally unbiased, then:

$$ACE = E(Y \mid X = 1) - E(Y \mid X = 0)$$

# Identification of the Average Causal Effect I

The following equations are always true ($X$ with values 0 and 1):

$$E(Y \mid X) = \alpha_0 + \alpha_1 \cdot X$$

and

$$E(Y \mid X) = E\,[E(Y \mid X, U) \mid X\,]$$
$$= E\,[\,f_0(U) \mid X\,] + E\,[\,f_1(U) \mid X\,] \cdot X\,.$$

These equations show that the slope $\alpha_1$ of the linear regression $E(Y \mid X) = \alpha_0 + \alpha_1 \cdot X$ is the average causal effect if

$$E\,[f_0(U) \mid X\,] = E\,[f_0(U)]$$ Weak

and

$$E\,[f_1(U) \mid X\,] = E\,[f_1(U)]$$ Ignorability

# Identification of the Average Causal Effect II

**Theorem 1.** Weak ignorability implies *causal unbiasedness* of the regression $E(Y \mid X)$ and, therefore,

$$ACE = E(Y \mid X = 1) - E(Y \mid X = 0) = \alpha_1$$

**Theorem 2.** Sufficient conditions for weak ignorability and, therefore, for causal unbiasedness are:

- Independence of $U$ and $X$
- Unit-treatment homogeneity: $E(Y \mid X, U) = \alpha_0 + \alpha_1 \cdot X$
- Independence of $X$ and $(\eta_0, \eta_1)$ (= Rubins Ignorability)
- Unconfoundedness of the regression $E(Y \mid X)$ (see Def. 3)

# Identification of the Average Causal Effect III

**Definition 3.** The regression $E(Y \mid X)$ is called unconfounded if for each value $x$ of $X$:

$$P(X = x \mid U = u) = P(X = x), \quad \text{for all values } u \text{ of } U,$$

or

$$E(Y \mid X = x, U = u) = E(Y \mid X = x), \quad \text{for all values } u \text{ of } U.$$

# 4. The Single-Unit Trial With a Covariate

We consider the *following single-unit trial*: sample a unit from a given set of units (the population), observe a covariate (a pretreatment variable), observe the assignment of the unit to one of two treatment conditions and register the outcome.

The set of possible outcomes of the single-unit trial described above might be of the form:

$$\Omega = \Omega_U \times \Omega_Z \times \Omega_X \times \mathrm{IR} \quad .$$

Notation:

- $U: \Omega \rightarrow \Omega_U$ *Person variable or unit variable*
- $Z: \Omega \rightarrow \Omega_Z$ *covariate* or *pretreatment variable*
- $X: \Omega \rightarrow \Omega_X$ *treatment variable*, with values 0 and 1
- $Y: \Omega \rightarrow \mathrm{IR}$ *Outcome variable*

# 5. Conditional Average Causal Effects

Aside from

$$E(Y \mid X, U) = f_0(U) + f_1(U) \cdot X$$

we can now also consider

$$E(Y \mid X, Z) = g_0(Z) + g_1(Z) \cdot X \ .$$

We can define:

$E[\, f_1(U) \mid Z = z] =:$ *CACE Conditional average causal effect*

of $X$ on $Y$ given a value $z$ of the covariate $Z$

# Identification of the Conditional Average Causal Effect I

The following equations are always true:

$$E(Y \mid X, Z) = g_0(Z) + g_1(Z) \cdot X \qquad (1)$$

and

$$E(Y \mid X, Z) = E\,[E(Y \mid X, Z, U) \mid X, Z\,] \qquad (2)$$

If we assume

- $E(Y \mid X, Z, U) = E(Y \mid X, U)$ ???
- $E\,[f_0(U) \mid X, Z\,] = E\,[f_0(U) \mid Z\,]$ Conditional
- $E\,[f_1(U) \mid X, Z\,] = E\,[f_1(U) \mid Z\,]$ weak ignorability

then defining $g_0(Z) := E\,[f_0(U) \mid Z\,]$ and $g_1(Z) := E\,[f_1(U) \mid Z\,]$

shows that the slopes $g_1(z)$ of the conditional linear regression (1) are the conditional average causal effects of $X$ on $Y$ given $Z = z$.

# Identification of the Conditional Average Causal Effect II

**Theorem 3.** Conditional weak ignorability and ??? imply *conditional causal unbiasedness* of the regression $E(Y \mid X, Z)$ and, therefore,

$$CACE = E(Y \mid X = 1, Z = z) - E(Y \mid X = 0, Z = z) = g_1(z)$$

for each value $z$ of $Z$

*and* $E[g_1(Z)] = E[\, f_1(U)\, ] = ACE.$

**Theorem 4.** Sufficient conditions for conditional weak ignorability and, therefore, for conditional causal unbiasedness are:

- Conditional independence of $U$ and $X$ given $Z$
- Conditional unit-treatment homogeneity: $E(Y \mid X, Z, U) = g_0(Z) + g_1(Z) \cdot X$
- Conditional independence of $X$ and $(\eta_0, \eta_1)$ given $Z$ (= Rubins Ignorability)
- Conditional unconfoundedness of the regression $E(Y \mid X, Z)$ (see Def. 5)

# Identification of the Conditional Average Causal Effect III

**Definition 5.** The regression $E(Y \mid X, Z)$ is called *conditionally unconfounded* given $Z$ if for all values $z$ of $Z$ the following proposition holds:

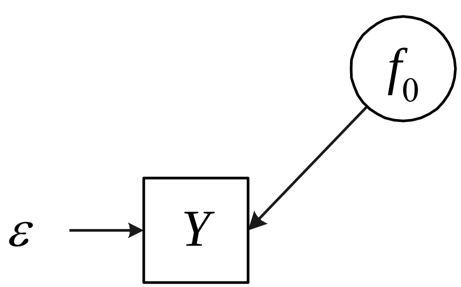
For each value $x$ of $X$:

(a) $P(X = x \mid Z = z, U = u) = P(X = x \mid Z = z)$, for all values $u$ of $U$,

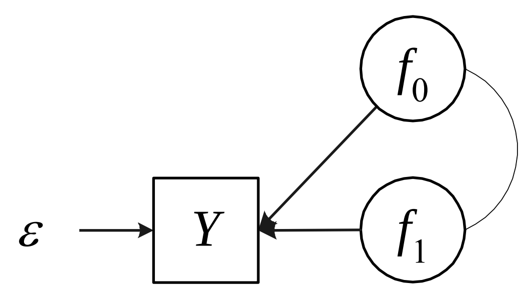
or

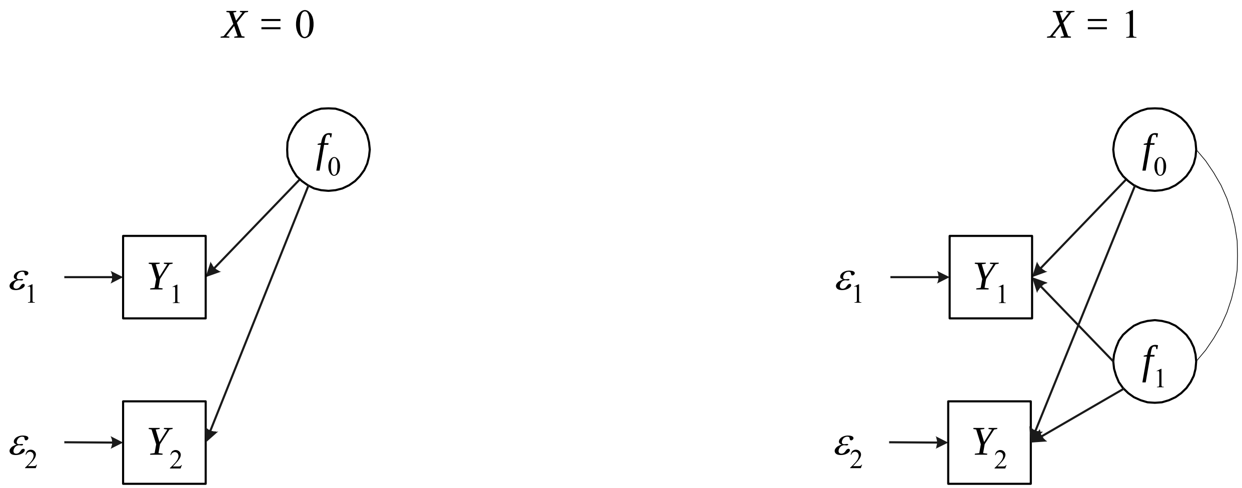
(b) $E(Y \mid X = x, Z = z, U = u) = E(Y \mid Z = z, X = x)$, for all values $u$ of $U$.

$X = 0$



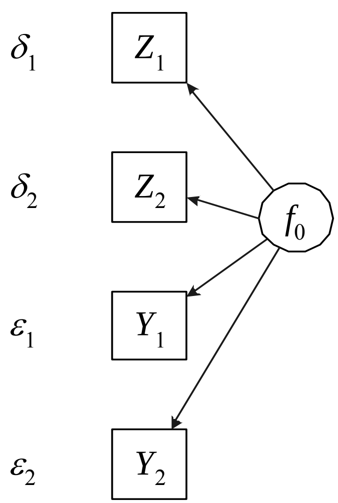

$X = 1$

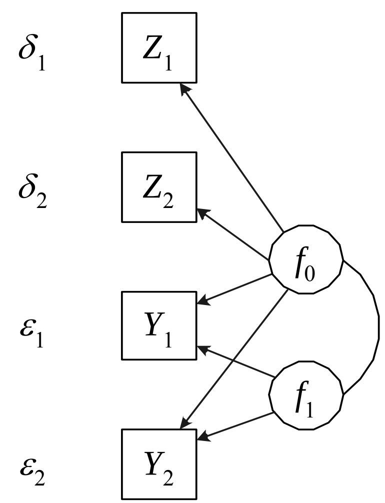



X = 0
f0
ε1
Y1
ε2
Y2
X = 1
f0
ε1
Y1
f1
ε2
Y2

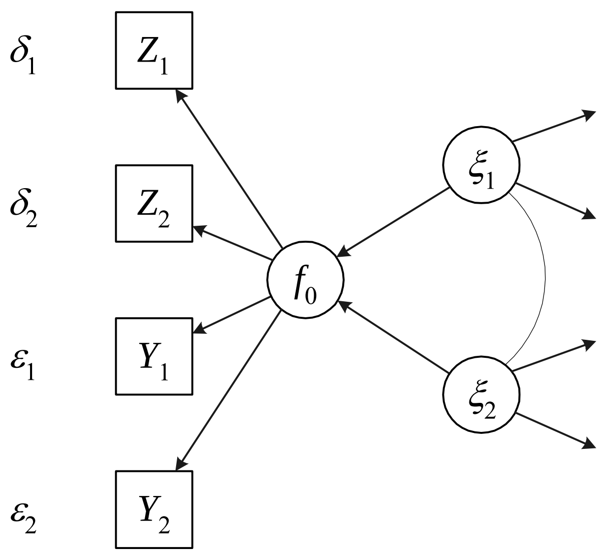
$X = 0$ $\longrightarrow$

$X = 1$ $\longrightarrow$

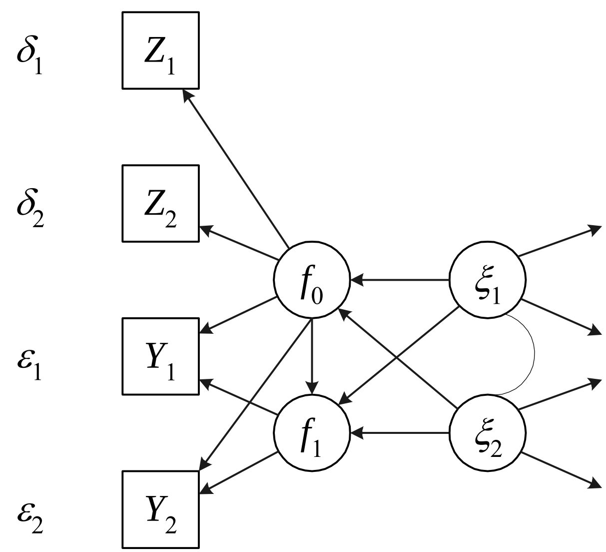


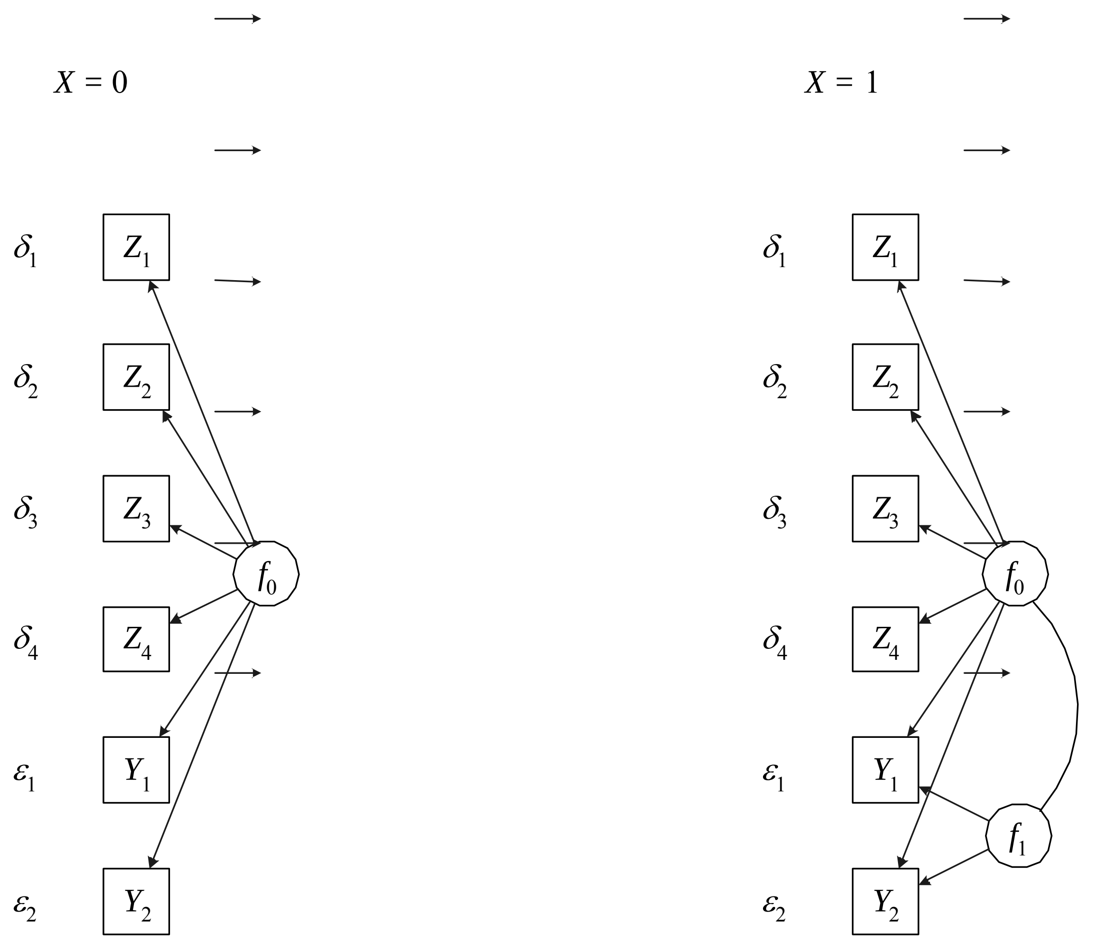



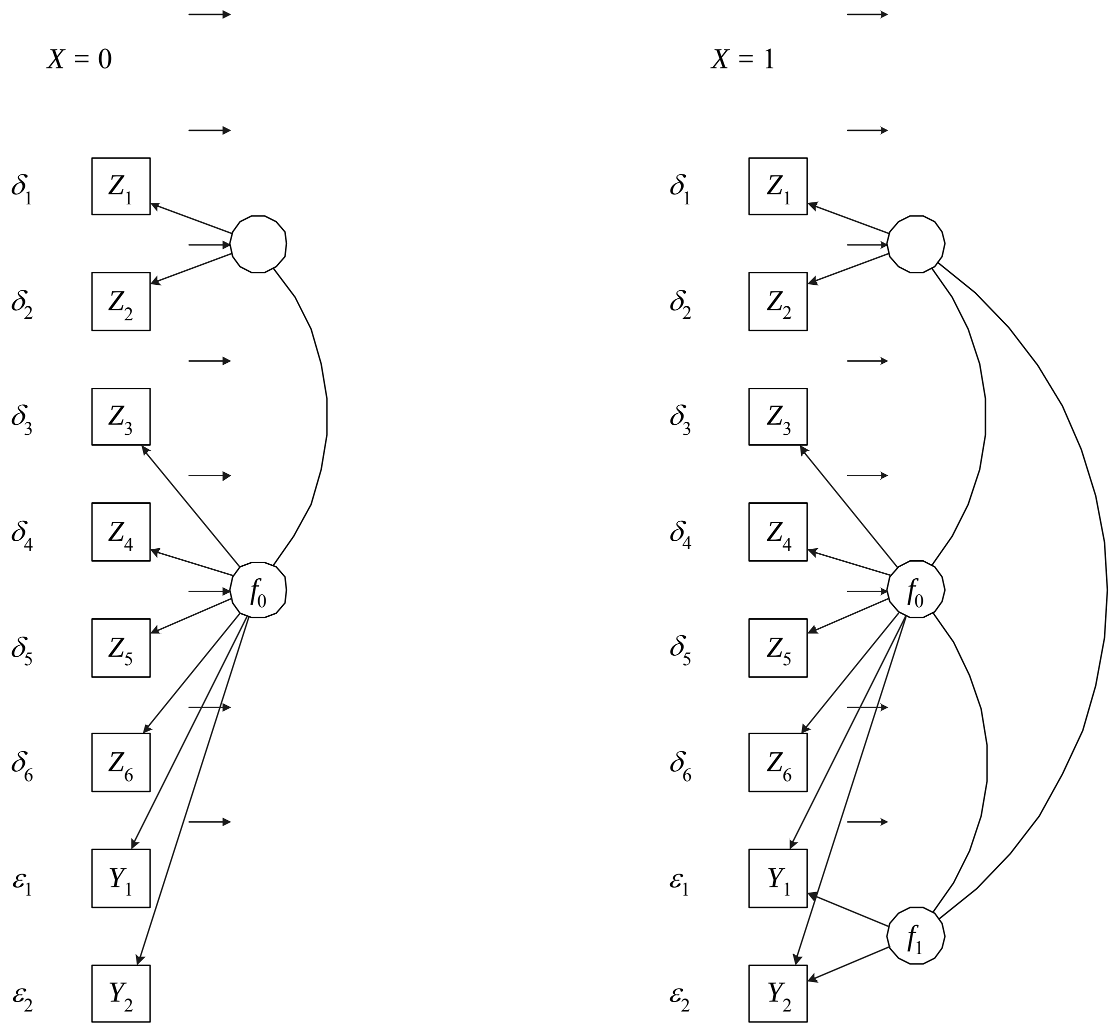
$X = 0$




$X = 1$




X = 0
δ1
δ2
δ3
δ4
ε1
ε2
Z1
Z2
Z3
Z4
Y1
Y2
f0
X = 1
δ1
δ2
δ3
δ4
ε1
ε2
Z1
Z2
Z3
Z4
Y1
Y2
f0
f1


X = 0
X = 1
δ1
δ2
δ3
δ4
δ5
δ6
ε1
ε2
Z1
Z2
Z3
Z4
Z5
Z6
Y1
Y2
f0
f1

# Rubin: Unit homogeneity

**Table 1.** *Illustration of a homogeneous population*

| $U$ Person variable | $Y_1$ Potential outcome under treatment | $Y_0$ Potential outcome under control | $Y_0 - Y_1$ Individual causal effect |
|---|---|---|---|
| $u_1$ | 110 | 100 | 10 |
| $u_2$ | 110 | 100 | 10 |
| $u_3$ | 110 | 100 | 10 |
| $u_4$ | 110 | 100 | 10 |
| $u_5$ | 110 | 100 | 10 |
| $u_6$ | 110 | 100 | 10 |
| $u_7$ | 110 | 100 | 10 |
| $u_8$ | 110 | 100 | 10 |
| Mean | **110** | **100** | **10** |

# *Rubin: Sampling with heterogenous units 1*

**Table 1.** *Example illustrating individual and average causal effects*

(a) Experiment with non-comparable groups

| $U$ Person variable | $Y_1$ Potential outcome under treatment | $Y_0$ Potential outcome under control | $Y_1 - Y_0$ Individual causal effect |
|---|---|---|---|
| $u_1$ | **82** | 68 | 14 |
| $u_2$ | 89 | **81** | 8 |
| $u_3$ | **101** | 89 | 12 |
| $u_4$ | **108** | 102 | 6 |
| $u_5$ | 118 | **112** | 6 |
| $u_6$ | **131** | 119 | 12 |
| $u_7$ | 139 | **131** | 8 |
| $u_8$ | 152 | **138** | 14 |
| Mean | **105.5** | **115.5** | 10 |

*Note*. The red numbers are selected.

# *Rubin: Sampling with heterogenous units 2*

(b) Experiment with comparable groups

| *U* Person variable | $Y_1$ Potential outcome under treatment | $Y_1$ Potential outcome under control | $Y_1 - Y_0$ Individual causal effect |
|---|---|---|---|
| $u_1$ | 82 | **68** | 14 |
| $u_2$ | **89** | 81 | 8 |
| $u_3$ | **101** | 89 | 12 |
| $u_4$ | 108 | **102** | 6 |
| $u_5$ | **118** | 112 | 6 |
| $u_6$ | 131 | **119** | 12 |
| $u_7$ | 139 | **131** | 8 |
| $u_8$ | **152** | 138 | 14 |
| Mean | **115** | **105** | 10 |

*Note.* The red numbers are selected.

# Rubin: Ignorability

| $U$ Person Variable | $Y_1$ Potential outcome under treatment | $Y_0$ Potential outcome under control | $Y_1 - Y_0$ Individual causal effect | $P(X = 1 \mid Y_0, Y_1)$ treatment probability in experiment 1 | $P(X = 1 \mid Y_0, Y_1)$ treatment probability in experiment 2 |
|---|---|---|---|---|---|
| $u_1$ | 85 | 74 | 11 | 3/4 | 8/9 |
| $u_2$ | 85 | 74 | 11 | 1/4 | 7/9 |
| $u_3$ | 101 | 89 | 12 | 1/2 | 6/9 |
| $u_4$ | 108 | 102 | 6 | 1/2 | 5/9 |
| $u_5$ | 118 | 112 | 6 | 1/2 | 4/9 |
| $u_6$ | 131 | 119 | 12 | 1/2 | 3/9 |
| $u_7$ | 139 | 131 | 8 | 1/2 | 2/9 |
| $u_8$ | 152 | 138 | 14 | 1/2 | 1/9 |
| Mean | 115 | 105 | 10 | | |

# Rubin: Propensity

| $U$ Person Variable | $Y_1$ Potential outcome under treatment | $Y_0$ Potential outcome under control | $Y_1 - Y_0$ Individual causal effect | $P(X=1 \mid U)$ treatment probability in experiment 1 | $P(X=1 \mid U)$ treatment probability in experiment 2 |
|---|---|---|---|---|---|
| $u_1$ | 85 | 74 | 11 | 1/2 | 8/9 |
| $u_2$ | 85 | 74 | 11 | 1/2 | 7/9 |
| $u_3$ | 101 | 89 | 12 | 1/2 | 6/9 |
| $u_4$ | 108 | 102 | 6 | 1/2 | 5/9 |
| $u_5$ | 118 | 112 | 6 | 1/2 | 4/9 |
| $u_6$ | 131 | 119 | 12 | 1/2 | 3/9 |
| $u_7$ | 139 | 131 | 8 | 1/2 | 2/9 |
| $u_8$ | 152 | 138 | 14 | 1/2 | 1/9 |
| Mean | 115 | 105 | 10 | | |

# Beyond Rubin 1: Expected outcomes

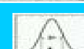
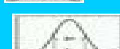
**Table 1.** *Individual causal effects.*

| Person variable $U$ | $E(Y \mid X = 1, U = u)$ Expected outcome | $E(Y \mid X = 0, U = u)$ Expected outcome | $E(Y \mid X = 1, U = u) - E(Y \mid X = 0, U = u)$ Individual causal effect |
|---|---|---|---|
| $u_1$ | 82 | 68 | 14 |
| $u_2$ | 89 | 81 | 8 |
| $u_3$ | 101 | 89 | 12 |
| $u_4$ | 108 | 102 | 6 |
| $u_5$ | 118 | 112 | 6 |
| $u_6$ | 131 | 119 | 12 |
| $u_7$ | 139 | 131 | 8 |
| $u_8$ | 152 | 138 | 14 |

Individual causal laws

# Beyond Rubin 2 : Individual distributions

**Table 1.** *Individual causal effects and treatment probabilities.*

| Person variable $U$ | $E(Y \mid X = 1, U = u)$ Expected outcome | | $E(Y \mid X = 0, U = u)$ Expected outcome | | $E(Y \mid X = 1, U = u) - E(Y \mid X = 0, U = u)$ Individual causal effect |
|---|---|---|---|---|---|
| $u_1$ | 82 |  | 68 |  | 14 |
| $u_2$ | 89 |  | 81 |  | 8 |
| $u_3$ | 101 |  | 89 |  | 12 |
| $u_4$ | 108 |  | 102 |  | 6 |
| $u_5$ | 118 |  | 112 |  | 6 |
| $u_6$ | 131 |  | 119 |  | 12 |
| $u_7$ | 139 |  | 131 |  | 8 |
| $u_8$ | 152 |  | 138 |  | 14 |

Individual causal laws

# Beyond Rubin 3: Individual and average causal laws

**Table 1.** *Individual and average causal effects.*

| Person variable $U$ | $P(U=u)$ sampling probability | $E(Y \mid X = 1, U = u)$ Expected outcome | $E(Y \mid X = 0, U = u)$ Expected outcome | $E(Y \mid X = 1, U = u) - E(Y \mid X = 0, U = u)$ Individual causal effect |
|---|---|---|---|---|
| $u_1$ | 1/8 | 82 | 68 | 14 |
| $u_2$ | 1/8 | 89 | 81 | 8 |
| $u_3$ | 1/8 | 101 | 89 | 12 |
| $u_4$ | 1/8 | 108 | 102 | 6 |
| $u_5$ | 1/8 | 118 | 112 | 6 |
| $u_6$ | 1/8 | 131 | 119 | 12 |
| $u_7$ | 1/8 | 139 | 131 | 8 |
| $u_8$ | 1/8 | 152 | 138 | 14 |

Individual causal laws

Average causal laws:

$$CUE(Y \mid X = x) = \sum_u E(Y \mid X = x, U = u) \cdot P(U = u)$$

# Beyond Rubin 4 : A good design

**Table 1.** *Individual causal effects and equal treatment probabilities.*

| Person variable $U$ | $P(U=u)$ sampling probability | $E(Y \mid X=1, U=u)$ Expected outcome | $E(Y \mid X=0, U=u)$ Expected outcome | $E(Y \mid X=1, U=u) - E(Y \mid X=0, U=u)$ Individual causal effect | $P(X=1 \mid U=u)$ treatment probability in experiment 1 |
|---|---|---|---|---|---|
| $u_1$ | 1/8 | 82 | 68 | 14 | 1/2 |
| $u_2$ | 1/8 | 89 | 81 | 8 | 1/2 |
| $u_3$ | 1/8 | 101 | 89 | 12 | 1/2 |
| $u_4$ | 1/8 | 108 | 102 | 6 | 1/2 |
| $u_5$ | 1/8 | 118 | 112 | 6 | 1/2 |
| $u_6$ | 1/8 | 131 | 119 | 12 | 1/2 |
| $u_7$ | 1/8 | 139 | 131 | 8 | 1/2 |
| $u_8$ | 1/8 | 152 | 138 | 14 | 1/2 |

Individual causal laws

Good design

Average causal laws:

$$CUE(Y \mid X=x) = \sum_u E(Y \mid X=x, U=u) \cdot P(U=u)$$

Good design implies: $E(Y \mid X=x)$ reflect average causal laws

# Beyond Rubin 5 : CUE and $E(Y \mid X = x)$

**Table 1.** *Individual causal effects, equal and unequal treatment probabilities.*

$$E(Y \mid X = x) := \sum_u E(Y \mid X = x, U = u) \cdot P(U = u \mid X = x)$$

| Person | $P(U=u)$ sampling probability | $E(Y \mid X = 1, U = u)$ Expected outcome | $E(Y \mid X = 0, U = u)$ Expected outcome | $E(Y \mid X = 1, U = u) - E(Y \mid X = 0, U = u)$ Individual causal effec | $P(X = 1 \mid U=u)$ treatment probability in experiment 1 | $P(X = 1 \mid U=u)$ treatment probability in experiment 2 |
|---|---|---|---|---|---|---|
| $u_1$ | 1/8 | 82 | 68 | 14 | 1/2 | 8/9 |
| $u_2$ | 1/8 | 89 | 81 | 8 | 1/2 | 7/9 |
| $u_3$ | 1/8 | 101 | 89 | 12 | 1/2 | 6/9 |
| $u_4$ | 1/8 | 108 | 102 | 6 | 1/2 | 5/9 |
| $u_5$ | 1/8 | 118 | 112 | 6 | 1/2 | 4/9 |
| $u_6$ | 1/8 | 131 | 119 | 12 | 1/2 | 3/9 |
| $u_7$ | 1/8 | 139 | 131 | 8 | 1/2 | 2/9 |
| $u_8$ | 1/8 | 152 | 138 | 14 | 1/2 | 1/9 |
| | | Individual causal laws | | | Good design | Bad design |

Average causal laws:

$$CUE(Y \mid X = x) = \sum_u E(Y \mid X = x, U = u) \cdot P(U = u)$$

Good design implies: $E(Y \mid X = x)$ reflect average causal laws

Bad design implies: $E(Y \mid X = x)$ do not reflect average causal laws

# Beyond Rubin 6 : Individual and average causal effects

$U$: Person or unit variable
$X$: treatment variable, dichotomous with values 0 and 1
$Y$: Outcome variable

$$E(Y \mid X, U) = g_0(U) + g_1(U) \cdot X$$

$g_1(u)$ =: Individual causal effect of unit $u$
$E[g_1(U)]$ =: Average causal effect

The next equation is always true if $X$ is dichotomous:

$$\begin{aligned} E(Y \mid X) &= E[E(Y \mid X, U) \mid X] \\ &= E[g_0(U) \mid X] + E[g_1(U) \mid X] \cdot X . \end{aligned}$$

This equation shows that the slope $\alpha_1$ of the linear regression $E(Y \mid X) = \alpha_0 + \alpha_1 \cdot X$ is the average causal effect if

$$E[f_0(U) \mid X] = E[f_0(U)] \quad \text{and} \quad E[f_1(U) \mid X] = E[f_1(U)]. \qquad (0.1)$$

Sufficient conditions for (0.1) are:

(a) Independence of $U$ and $X$.
(b) Independence of $X$ and $(Y_0, Y_1)$, (= Rubins Ignorability)
where $Y_0 := f_0(U)$ and $Y_1 := f_0(U) + f_1(U)$

# Pearl 1: His analysis of Rubins Causal Model

If

(a) *X* and *U* are independent (randomization) or

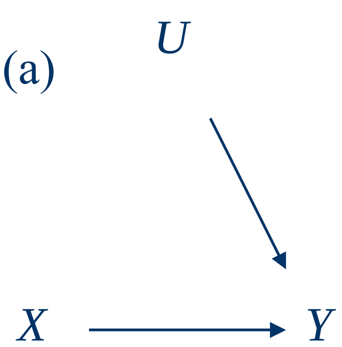
(b) $E(Y \mid X, U) = E(Y \mid X)$ (unit homogeneity),

then $E(Y \mid X = 1) - E(Y \mid X = 0)$ is the causal effect of *X* on *Y*, *i.e. PFE = ACE.*

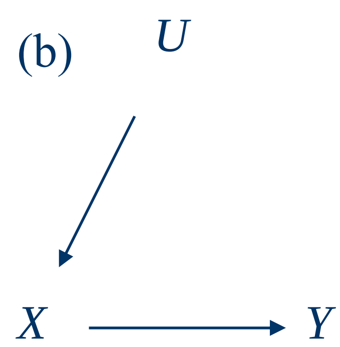
Note that

$$E(Y \mid X = x) = \sum_u E(Y \mid X = x, U = u) \cdot P(U = u \mid X = x)$$

(a)




(b)




*U*: Person or unit variable

*X*: treatment variable

*Y*: Outcome

# Pearl 2: His analysis of Rubins Causal Model

The causal effect of *X* on *Y* can be computed by:

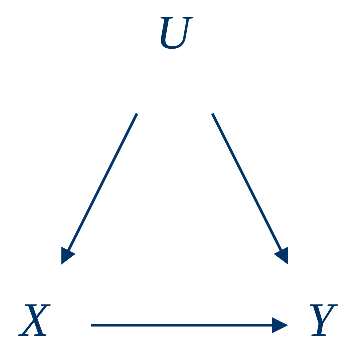
$CUE(Y \mid X = 1) - CUE(Y \mid X = 0)$, where

$$CUE(Y \mid X = x) = \sum_u E(Y \mid X = x, U = u) \cdot P(U = u)$$

*U*: Person or unit variable

*X*: treatment variable

*Y*: Outcome



# Pearl 1 + 2: His analysis of Rubins Causal Model

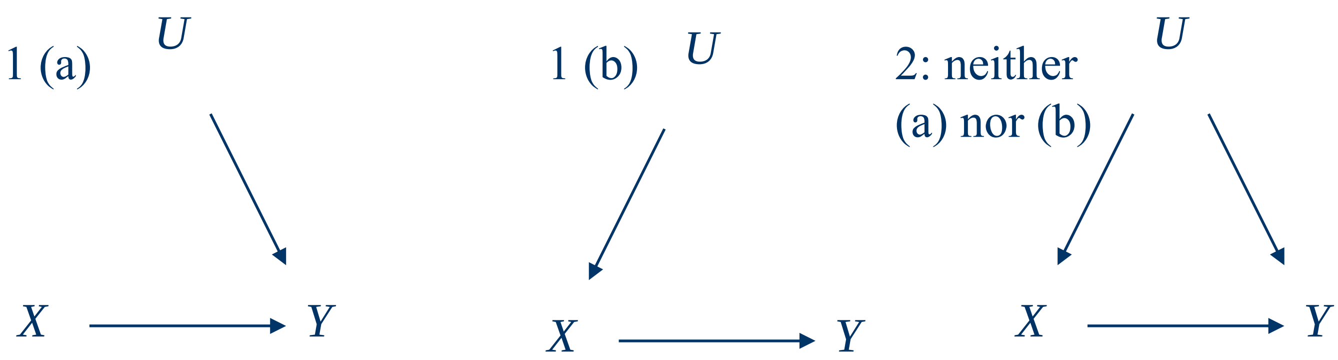
(1) If (a) $X$ and $U$ are independent or (b) $E(Y \mid X, U) = E(Y \mid X)$, then $E(Y \mid X = 1)$ -

$$E(Y \mid X = x) = \sum_u E(Y \mid X = x, U = u) \cdot P(U = u \mid X = x)$$

(2) The causal effect of $X$ on $Y$ can be computed by:

$CUE(Y \mid X = 1)$ - $CUE(Y \mid X = 0)$, where

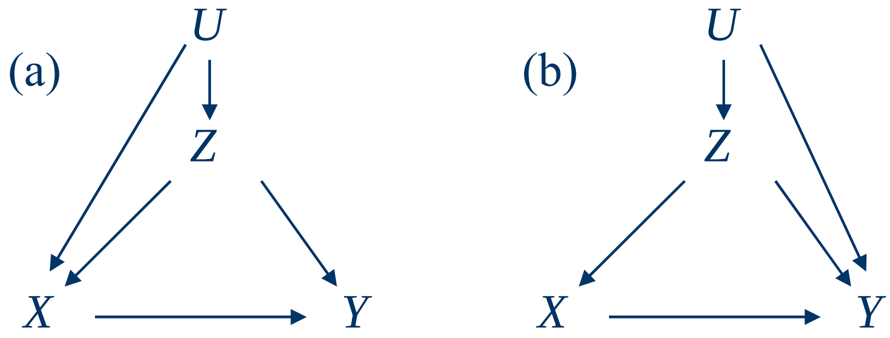
$$CUE(Y \mid X = x) = \sum_u E(Y \mid X = x, U = u) \cdot P(U = u)$$



# Pearl 3: His analysis of Rubins Causal Model

If (a) $E(Y \mid X, U, Z) = E(Y \mid X, Z)$ or (b) $P(X \mid U, Z) = P(X \mid Z)$
and
$E(Y \mid X, U, Z) = E(Y \mid X, U)$,
then the causal effect of $X$ on $Y$ can be computed by:
$CUE(Y \mid X = 1) - CUE(Y \mid X = 0)$, where

$$CUE(Y \mid X = x) = \sum_z E(Y \mid X = x, Z = z) \cdot P(Z = z)$$



# Beyond Rubin 6 : Unconfoundedness, Definition

*There are conditions under which, for each value x of X:*

$$E(Y\,|\,X=x) := \sum_u E(Y\,|\,X=x, U=u) \cdot P(U=u\,|\,X=x)$$

*is equal to*

$$\sum_u E(Y\,|\,X=x, U=u) \cdot P(U=u).$$

For each value $x$ of $X$:

(1) $P(U=u\,|\,X=x) = P(U=u) \quad \forall\, u$

**or**

(2) $E(Y\,|\,X=x, U=u) = E(Y\,|\,X=x) \quad \forall\, u$

} *Unconfoundedness of the regression* $E(Y\,|\,X)$

# Beyond Rubin 7: Unconfoundedness - Example

**Table 2.** An example in which the treatment regression $E(Y \mid X)$ is unconfounded and causally unbiased but none of the other causality criteria discussed in this paper hold

| Observational-unit variable $U$ | $P(U=u)$ | $W$ (gender) | $P(X=x_1 \mid U=u)$ | $E(Y \mid X=x_1, U=u)$ | $P(X=x_2 \mid U=u)$ | $E(Y \mid X=x_2, U=u)$ | $P(X=x_3 \mid U=u)$ | $E(Y \mid X=x_3, U=u)$ |
|---|---|---|---|---|---|---|---|---|
| $u_1$ | 1/8 | $m$ | 1/2 | 82 | 1/10 | 105 | 4/10 | 110 |
| $u_2$ | 1/8 | $m$ | 1/2 | 89 | 1/10 | 105 | 4/10 | 110 |
| $u_3$ | 1/8 | $m$ | 1/2 | 101 | 2/10 | 105 | 3/10 | 110 |
| $u_4$ | 1/8 | $m$ | 1/2 | 108 | 2/10 | 105 | 3/10 | 110 |
| $u_5$ | 1/8 | $f$ | 1/2 | 118 | 3/10 | 105 | 2/10 | 110 |
| $u_6$ | 1/8 | $f$ | 1/2 | 131 | 3/10 | 105 | 2/10 | 110 |
| $u_7$ | 1/8 | $f$ | 1/2 | 139 | 4/10 | 105 | 1/10 | 110 |
| $u_8$ | 1/8 | $f$ | 1/2 | 152 | 4/10 | 105 | 1/10 | 110 |

*Note:* The (unconditional) probabilities for the three treatments are $P(X=x_1) = 1/2$, $P(X=x_2) = P(X=x_3) = 1/4$.

# Beyond Rubin 8: Unconfoundedness – Equivalent Definition

For each value $x$ of $X$:

(1) $P(U = u \mid X = x) = P(U = u)$,

***or***

(2) $E(Y \mid X = x, U = u) = E(Y \mid X = x)$

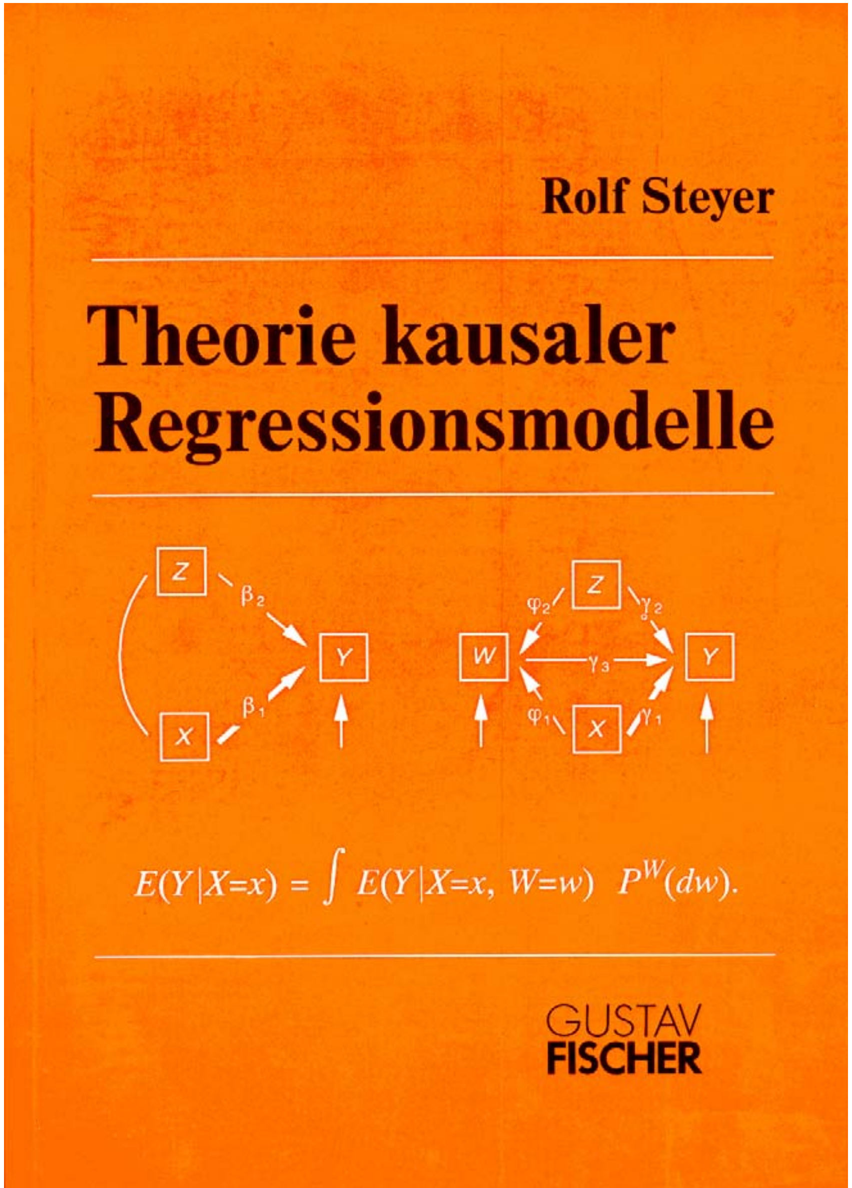
$\Bigg\}$ *Unconfoundedness of the regression* $E(Y \mid X)$

*Unconfoundedness is equivalent to:*

*For each* $W := f(U)$:

$$E(Y \mid X = x) := \sum_w E(Y \mid X = x, W = w) \cdot P(W = w)$$

for each value $x$ of $X$.


Rolf Steyer
Theorie kausaler Regressionsmodelle
Z
X
Y
β2
β1
W
Z
X
Y
φ2
φ1
γ2
γ3
γ1
E(Y|X=x) = ∫ E(Y|X=x, W=w) P^W(dw).
GUSTAV FISCHER

TRENDS IN MATHEMATICAL PSYCHOLOGY
E. Degreef and J. Van Buggenhaut (editors)



# CAUSAL LINEAR STOCHASTIC DEPENDENCIES: THE FORMAL THEORY

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The formal background of the theory of causal linear stochastic dependence is provided, which was introduced by Steyer (1984). The theory presented is concerned with those kinds of dependencies which can be described by specifying the functional form of a conditional expectation $E(Y|X)$. This includes also those situations in which X is a multidimensional random variable. The main concepts of the theory are <u>causal</u> and <u>weak causal linear stochastic dependencies</u>, the definition of which is based on the pre- and equiorderedness relations of sigma-fields and stochastic variables, on the notion of potential

# *A formal theory of causality 0*
# *(Steyer 1984, 1992)*

$$[\ (\Omega, \mathcal{A}, P),\quad E(Y \mid X)\ ]$$

Probability space

*Random variables* on $(\Omega, \mathcal{A}, P)$
$Y: \Omega \rightarrow \Re$ (real-valued)
$X: \Omega \rightarrow \Omega_X$
must be „measurable“

i. e., all events associated with $X$ and $Y$ are elements in $\mathcal{A}$

*Regression* or *conditional expectation*, i. e. that function of $X$, the values of which are the conditional expected values $E(Y \mid X = x)$

Regressive dependence of $Y$ on $X$
$E(Y \mid X) \neq E(Y)$

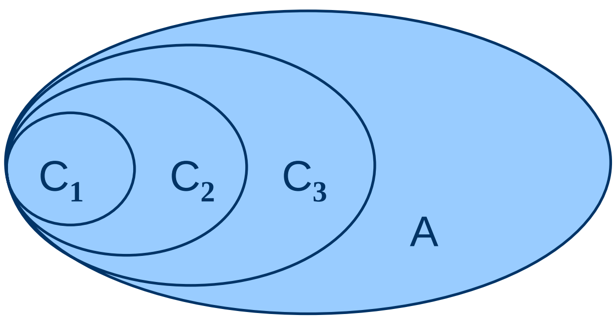
# *A formal theory of causality 2 (Steyer, 1984, 1992)*

$$[(\Omega, \mathsf{A}, P), E(Y \mid X), (\mathsf{C}_t, t \in T), \mathsf{D}]$$

same as before

Monotonically nondecreasing family of σ-algebras $\mathsf{C}_t \subset \mathsf{A}$

$\mathsf{D} \subset \mathsf{A}$, a sub-σ-algebra of $\mathsf{A}$.



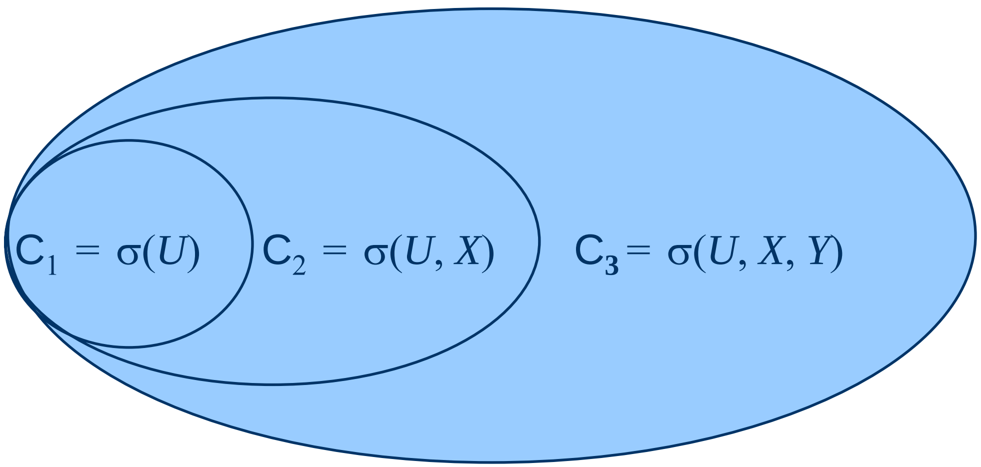

used to define preorderedness relation between events and random variables.
[Random variables generate σ-algebras $\subset \mathsf{A}$.]

used to define „potential confounders“ $W$ (random variables). Their generated σ-algebra is a subset of $\mathsf{D}$.

Pre-orderedness
$W \rightarrow X \rightarrow Y$

# A formal theory of causality 3 (Steyer, 1984, 1992)

$$[\ (\Omega, \mathcal{A}, P),\ E(Y \mid X),\ (\mathcal{C}_t, t \in T),\ \mathcal{D}\ ]$$




$\mathcal{D} = \mathcal{C}_1 = \sigma(U)$
Potential confounders $W$:
measurable with respect to $\mathcal{D}$

Pre-orderedness
$U \rightarrow X \rightarrow Y$

# *A formal theory of causality 4 (Steyer, 1984, 1992)*

## Causality conditions

*Strict Causality*

$E(Y \mid X, W) = E(Y \mid X)$ for each potential confounder $W$

*Strong Causality*

$E(Y \mid X, W) = E(Y \mid X) + f(W)$ for each potential confounder $W$

*Weak Causality (= Unconfoundedness)*

If $W$ is a potential confounder, then, for $P^X$-almost every value $x$ of $X$:

$$E(Y \mid X = x) = \int E(Y \mid X = x, W = w)\, P^W(dw)$$

i.e., if $W$ is discrete:

$$E(Y \mid X = x) = \sum_w E(Y \mid X = x, W = w)\; P(W = w)$$

# Sufficient conditions for Weak Causality (Steyer, 1992)

1. Stochastic independence of *X* and D implies Weak Causality. [If D is defined to be generated by *U*, the random variable, the values of which are the observational units drawn from the population, then this independence can be deliberately created via random assignment of units to treatment conditions.]

2. Both, Strict and Strong Causality Conditions imply Weak Causality.

# Applications

- Experimental design techniques such as randomization, conditional randomization etc.
- Data analysis techniques such as
  - ➢ Nonorthogonal Analysis of Variance
  - ➢ Analysis of Covariance
  - ➢ Computation of causal effects in structural equation models
  - ➢ Tests of confounding
  - ➢ Data mining for causal dependencies

    ….

# Nonorthogonal Analysis of Variance

**Table 1.** Example for a nonorthogonal analysis of variance design

| | | Need for therapy | | | | | |
|---|---|---|---|---|---|---|---|
| Treatment | strong $Z = z_1$ | | medium $Z = z_2$ | | weak $Z = z_3$ | | total |
| 1 $X = x_1$ | 120 | (40) | 110 | (20) | 60 | (6) | (66) |
| 2 $X = x_2$ | 100 | (14) | 100 | (80) | 100 | (14) | (108) |
| 3 $X = x_3$ | 80 | (6) | 90 | (20) | 140 | (40) | (66) |
| total | | (60) | | (120) | | (60) | (240) |

*Note*. True cell means and, in parentheses, cell frequencies.

# Conclusions

- The mathematical structure of causal stochastic dependencies is now well-known
- The theory of stochastic causality helps in deciding between competing strategies for data analysis
- The theory also leads to new ways of data analysis
- Many statistical problems in these data analyses are not yet solved

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