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Time-Varying Linear Control Systems: A Geometric Approach

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The concept of $[A, B]$-invariance for time-varying (not necessarily continuous) linear systems is introduced. A natural approach via time-varying subspaces is given. A disturbance decoupling and a noninteracting problem is studied.

1. Introduction

THE concept of $[A, B]$-invariance has been introduced by Basile & Marro (1969) and Wonham & Morse (1970) to solve various decoupling and pole-assignment problems for linear time-invariant multivariable systems. This concept was generalized to nonlinear systems (see e.g. Hirschhorn (1981), Isidori, Krener, Gori-Giori, & Monaco (1981), and Isidori (1985)) and to infinite-dimensional linear systems (see e.g. Curtain (1985, 1986)). Here I introduce a geometric approach for time-varying systems of the form

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t) + S(t)\boldsymbol{q}(t), \\ \boldsymbol{y}(t) &= C(t)\boldsymbol{x}(t), \end{aligned} \tag{1.1}$$

where A, B, C, and S are piecewise analytic matrix functions (see Section 2 for details); $\boldsymbol{q}$ is viewed as a disturbance entering the system via S. The main problem is as follows: When it is possible to determine a feedback matrix function F such that, in the closed-loop system

$$\begin{aligned} \boldsymbol{x}(t) &= (A + BF)(t)\boldsymbol{x}(t) + S(t)\boldsymbol{q}(t), \\ \boldsymbol{y}(t) &= C(t)\boldsymbol{x}(t), \end{aligned} \tag{1.2}$$

the disturbance $\boldsymbol{q}$ has no influence on the output $\boldsymbol{y}$ on a given open time interval $\mathcal{J}$?

The following example will illustrate an important difference between time-invariant and time-varying systems with respect to disturbance decoupling. Let

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \begin{bmatrix} 0 & t & 0 \\ a_1(t) & a_2(t) & a_3(t) \\ a_4(t) & a_5(t) & a_6(t) \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ b(t) \\ 0 \end{bmatrix} \boldsymbol{u}(t) + \begin{bmatrix} 0 \\ 0 \\ s(t) \end{bmatrix} \boldsymbol{q}(t), \\ y(t) &= [c(t), 0, 0]\boldsymbol{x}(t), \end{aligned} \tag{1.3}$$

where a_i $(i = 1,\dots,6)$, b, s, and c are real analytic functions, with a_3, b, s, and c not identically zero.

If the feedback matrix is denoted by $F(t) = [f_1(t), f_2(t), f_3(t)]$, then

$$(A+BF)(t) = \begin{bmatrix} 0 & t & 0 \\ (a_1+bf_1)(t) & (a_2+bf_2)(t) & (a_3+bf_3)(t) \\ a_4(t) & a_5(t) & a_6(t) \end{bmatrix}.$$

By a simple calculation, it is seen that $\boldsymbol{q}$ has no influence on y if and only if $a_3 + bf_3 \equiv 0$. If $b(t') = 0$ and $a_3(t') \neq 0$ for some $t' \in \mathbb{R}$, then $|f_3(t)| = |a_3(t)/b(t)|$ tends to infinity as $t \to t'$. This shows that disturbance decoupling might only be possible within certain intervals. One result of the present paper is to determine these intervals.

If the entries of the matrices A, B, S, and C consist of real analytic functions, then the present set up is a specialization of the nonlinear approach. However, there are several reasons to introduce a self-contained geometric approach for time-varying systems of the form (1.1): (a) the class of piecewise real analytic systems is much richer than the class of time-varying systems covered by the nonlinear approach. (b) The mathematical approach using time-varying subspaces is a natural one for the analysis of time-varying linear disturbance-decoupling problems. There is no need to use differential geometry. (c) The concept of $[A, B]$-invariance has a nice geometric interpretation, not given in the nonlinear case (see Theorem 4.5(iv)). It also can be dualized in a canonical way. (d) The maximal intervals where disturbance decoupling is possible are determined by the zeros of certain functions of time. (e) A sufficient condition when disturbance decoupling is possible on I is given. This condition can be checked on a computer if, for instance, the matrices in (1.1) are defined over $\mathbb{R}[t]$. (f) If disturbance decoupling is possible, then a constructive algorithm is given to determine the feedback matrix function F.

The organization of the paper is as follows. In Section 2, I present basic results on families of linear subspaces $\mathcal{V}(t)$ in $\mathbb{R}^n$ which depend on time $t \in \mathbb{R}$. In Section 3, a new description of $\mathcal{R}(t)$, the vector space of all states that can be controlled to zero at time t, is given. This leads to a time-varying family of subspaces which is A-invariant. The concept of $[A, B]$-invariances for piecewise real analytic systems is introduced and characterized in Section 4. In particular, the concept of A-invariance is discussed. In Section 5 it is shown that the concept of $[A, B]$-invariance is dual to (C, A)-invariance. An algorithm is presented which determines in a finite number of steps the smallest (C, A)-invariant family of subspaces containing a family $\mathcal{L}(t)$. Finally, in Section 6, the disturbance-decoupling problem for piecewise analytic state-space systems is introduced. We give a characterization of this problem. For analytic systems, one can check by means of the largest $[A, B]$-invariant family of subspaces included in $\ker C(t)$ if and where the disturbance-decoupling problem is solvable. In Section 7, controllability subspace families are defined and characterized. They become important for the noninteracting problem which is tackled in Section 8. In the Appendix, a greatest common left divisor and a least common right multiple for matrices with real analytic entries is determined.

2. Time-varying families of subspaces

In this section, families of subvector spaces which depend piecewise analytically on time t are analysed. These families become an important tool in the present geometric approach.

A function $f:\mathbb{R}\to\mathbb{R}$ is called *piecewise real analytic* if there exists a disjoint partition $\{[a_j, a_{j+1}) : j\in\mathbb{Z}\}$ of $\mathbb{R}$, with $\{a_j\}_{j\in\mathbb{Z}}$ a discrete set so that each restriction $f\restriction_{(a_j, a_{j+1})}$ is real analytic and has a real analytic extension on some $(a_j^{\mathrm{L}}, a_{j+1}^{\mathrm{R}})$, where $a_j^{\mathrm{L}} < a_j < a_{j+1} < a_{j+1}^{\mathrm{R}}$. The ring of piecewise real analytic functions is denoted by

$$\mathbb{A}_{\mathrm{p}} = \{f:\mathbb{R}\to\mathbb{R} : f \text{ piecewise real analytic}\},$$

and a differentiation on $\mathbb{A}_{\mathrm{p}}$ is defined in a canonical way:

$$\mathsf{D} : \mathbb{A}_{\mathrm{p}} \to \mathbb{A}_{\mathrm{p}} : f \mapsto \dot{f},$$

where $\dot{f}(t) = \dot{g}_j(t)$ for $t\in[a_j, a_{j+1})$ and g_j is any real analytic left continuation of $f\restriction_{(a_j, a_{j+1})}$.

Put

$$\mathbb{A} = \{f:\mathbb{R}\to\mathbb{R} : f \text{ real analytic}\}, \qquad \mathbb{M} = \{f:\mathbb{R}\to\bar{\mathbb{R}} : f \text{ real meromorphic}\},$$

and define the ring $\mathbb{M}_{\mathrm{p}}$ of piecewise real meromorphic functions analogously to $\mathbb{A}_{\mathrm{p}}$.

We call $\mathcal{V} = (\mathcal{V}(t))_{t\in\mathbb{R}}$ a *time-varying subspace* if $\mathcal{V}(t)$ is a subspace of $\mathbb{R}^n$ for every $t\in\mathbb{R}$. So $\mathcal{V}$ is a family of subspaces parametrized by $t\in\mathbb{R}$. Let $\mathbf{W}_n$ denote the set of all time-varying subspaces $\mathcal{V} = (\mathcal{V}(t))_{t\in\mathbb{R}}$, where $\mathcal{V}(t)$ is a subspace of $\mathbb{R}^n$ for every $t\in\mathbb{R}$. If $\mathcal{V}(t)$ is given by

$$\mathcal{V}(t) = V(t)\mathbb{R}^k \quad (t\in\mathbb{R}),$$

where $V\in\mathbb{A}_{\mathrm{p}}^{n\times k}$, then $\mathcal{V}$ is called the time-varying subspace *generated* by V.

A problem arises: If $\mathcal{V}\in\mathbf{W}_n$ has a generator $V\in\mathbb{A}_{\mathrm{p}}^{n\times k}$, then $\mathcal{V}^\perp := (\mathcal{V}(t)^\perp)_{t\in\mathbb{R}} \in \mathbf{W}_n$ does not generally have some piecewise analytic generator $W\in\mathbb{A}_{\mathrm{p}}^{n\times k'}$. Consider, for instance, $\mathcal{V}(t) = t\mathbb{R}$; then

$$\mathcal{V}(t)^\perp = \begin{cases} 0 & \text{if } t\neq 0,\\ \mathbb{R} & \text{if } t=0,\end{cases}$$

so that $\mathcal{V}$ belongs to $\mathbf{W}_1$ but does not have a piecewise analytic generator. To cope with this, equivalence classes are introduced: Two families $\mathcal{V}_1, \mathcal{V}_2\in\mathbf{W}_n$ are called *equal almost everywhere* (a.e.) on an interval $\mathcal{J}\subset\mathbb{R}$, denoted by

$$\mathcal{V}_1(t) \overset{\text{ae}}{=} \mathcal{V}_2(t) \quad \text{on } \mathcal{J},$$

if $\mathcal{V}_1(t) = \mathcal{V}_2(t)$ for all $t\in\mathcal{J}\setminus\mathcal{N}$, for some discrete set $\mathcal{N}$. In this sense one obtains for the preceeding example $\mathcal{V}(t)^\perp \overset{\text{ae}}{=} \{0\}$. Analogously, one defines $\mathcal{V}_1$ is *included a.e.* in $\mathcal{V}_2$ on $\mathcal{J}$. The reference to $\mathcal{J}$ is omitted if $\mathcal{J}=\mathbb{R}$. The notation

$$\mathcal{V}_1(t) \overset{\subset}{\underset{}{\overset{\text{ae}}{=}}} \mathcal{V}_2(t) \quad \text{on } \mathcal{J}$$

is used if $\mathcal{V}_1(t)\subset\mathcal{V}_2(t)$ for all $t\in\mathcal{J}$ and $\mathcal{V}_1(t)\overset{\text{ae}}{=}\mathcal{V}_2(t)$ on $\mathcal{J}$. Clearly 'equal almost everywhere' is an equivalence relation on $\mathbf{W}_n$, and the equivalence class of $\mathcal{V} \in \mathbf{W}_n$ is denoted by

$$\bar{\mathcal{V}} = \{\mathcal{W} \in \mathbf{W}_n : \mathcal{V}(t) \overset{\text{ae}}{=} \mathcal{W}(t)\}.$$

In the Appendix it is shown that a greatest common left divisor and a least common right multiple exist respectively for matrices with analytic and piecewise analytic entries. These algebraic results set us in a position to determine generators for time-varying families.

LEMMA 2.1 Let $C \in \mathbb{A}^{p\times n}$, $V \in \mathbb{A}^{n\times k}$, and $\mathcal{V}(t) = V(t)\mathbb{R}^k$. Then there exist real analytic matrices $\hat{V}$, $\hat{U}$, $\hat{C}$, and $\hat{W}$, of formats $n \times k$, $n \times (n-l)$, $n \times s$, and $n \times s'$ respectively, which have constant ranks and satisfy

(i) $$\mathcal{V}(t) \overset{\subset}{\underset{\text{ae}}{=}} \hat{V}(t)\mathbb{R}^l, \qquad l = \text{rk}_{\mathbb{M}} V,$$

(ii) $$\mathcal{V}(t)^{\perp} \overset{\supset}{\underset{\text{ae}}{=}} \hat{U}(t)\mathbb{R}^{n-l},$$

(iii) $$\ker C(t) \overset{\supset}{\underset{\text{ae}}{=}} \hat{C}(t)\mathbb{R}^s, \qquad s = n - \text{rk}_{\mathbb{M}}\, C = \text{rk}_{\mathbb{M}}\, \hat{C},$$

(iv) $$\mathcal{V}(t) \cap \ker C(t) \overset{\text{ae}}{=} \hat{W}(t)\mathbb{R}^{s'}, \qquad s' = \text{rk}_{\mathbb{M}}\, \hat{C} + \text{rk}_{\mathbb{M}}\, V - \text{rk}_{\mathbb{M}}\, [\hat{C}, V].$$

Proof.

(i) By Silverman & Bucy (1970), there exists $S \in \text{GL}_n(\mathbb{A})$ so that

$$V^{\mathsf{T}}S = [V_1, 0], \qquad V_1 \in \mathbb{A}^{k\times l}, \qquad \text{rk}_{\mathbb{M}}\, V_1 = l.$$

Therefore $[(V(t)\mathbb{R}^k]^{\perp} \overset{\supset}{\underset{\text{ae}}{=}} S(t)\begin{bmatrix} 0 \\ \mathrm{I}_{n-l} \end{bmatrix}\mathbb{R}^{n-l}$, and $\hat{V}(t) := S^{-\mathsf{T}}(t)\begin{bmatrix} \mathrm{I}_l \\ 0 \end{bmatrix}$ has constant rank and satisfies (i).

Condition (ii) is valid for $\hat{U}(t) := S(t)\begin{bmatrix} 0 \\ \mathrm{I}_{n-l} \end{bmatrix}$.

(iii) Let $R \in \text{GL}_n(\mathbb{A})$ so that

$$C(t)R(t) = [C_1(t), 0], \qquad C_1 \in \mathbb{A}^{p\times(n-s)}, \qquad \text{rk}_{\mathbb{M}}\, C_1 = n - s.$$

Clearly, $\hat{C}(t) := R(t)\begin{bmatrix} 0 \\ \mathrm{I}_s \end{bmatrix}$ satisfies (ii).

(iv) Use Lemma 9.1(ii) to determine some $W \in \text{lcrm}\,(\hat{C}, V)$ (see the beginning of Section 9 for a definition of lcrm) with $\text{rk}_{\mathbb{M}}\, W = s'$. Now, by (i), one can choose $\hat{W} \in \mathbb{A}^{n\times s}$ so that (iv) holds true. □

To characterize when the rank of $V(t)$ is constant in t, for $V \in \mathbb{A}^{n\times k}$, the following definition is needed.

DEFINITION 2.2 For a family $\mathcal{V} \in \mathbf{W}_n$, let

$$P(t) : \mathbb{R}^n \to \mathcal{V}(t)$$

be the orthogonal projection on $\mathcal{V}(t)$ along $\mathcal{V}(t)^{\perp}$. We call $\mathcal{V}$ an *analytic family* if $P \in \mathbb{A}^{n\times n}$, and a *piecewise analytic* (p.a.) *family* if $P \in \mathbb{A}_{\text{p}}^{n\times n}$.

Note that analyticity of $V \in \mathbb{A}^{n\times k}$ does not ensure that the family $\mathcal{V}$ generated by V is an analytic family; consider for instance $V(t) = t$.

PROPOSITION 2.3 If $\mathcal{V} \in \mathbf{W}_n$ is generated by $V \in \mathbb{A}^{n\times k}$, then $\mathcal{V}$ is an analytic family if and only if the function $t \mapsto \mathrm{rk}_{\mathbb{R}} V(t)$ is constant on $\mathbb{R}$.

Proof. If the orthogonal projector $P(t)$ on $\mathcal{V}(t)$ is real analytic in $t \in \mathbb{R}$, then Corollary A.5 in Gohberg, Lancaster, & Rodman (1983) gives that the function $t \mapsto \mathrm{rk}_{\mathbb{R}} V(t)$ is constant (continuity of P is already sufficient). Conversely, if $t \mapsto \mathrm{rk}_{\mathbb{R}} V(t)$ is constant on $\mathbb{R}$, then Proposition A.11 in Gohberg, Lancaster, & Rodman (1983) gives that $\mathcal{V}$ is an analytic family. □

Proposition 2.3 will be extended to the piecewise analytic situation. For this, a definition is necessary.

DEFINITION 2.4 $V \in \mathbb{A}_{\mathrm{p}}^{n\times k}$ is said to have *piecewise constant* (p.c.) rank if there exists a disjoint partition $\{[a_j, a_{j+1}) : j \in \mathbb{Z}\}$ of $\mathbb{R}$ so that each restriction $V\restriction_{(a_j, a_j+1)}$ is real analytic and has a real analytic extension V_j on some $(a_j^{\mathrm{L}}, a_{j+1}^{\mathrm{R}})$, where $a_j^{\mathrm{L}} < a_j < a_{j+1} < a_{j+1}^{\mathrm{R}}$, such that $\mathrm{rk}_{\mathbb{R}} V_j(t)$ is constant for all $t \in (a_j^{\mathrm{L}}, a_{j+1}^{\mathrm{R}})$.

PROPOSITION 2.5 If $\mathcal{V} \in \mathbf{W}_n$ is generated by $V \in \mathbb{A}_{\mathrm{p}}^{n\times k}$, then $\mathcal{V}$ is a p.a. family if and only if V has p.c. rank.

Proof. If $\mathcal{V}$ is a p.a. family, then there exists a partition $\{[a_j, a_{j+1}) : j \in \mathbb{Z}\}$ of $\mathbb{R}$ so that each restriction $V\restriction_{[a_j, a_{j+1})}$ and $P\restriction_{[a_j, a_{j+1})}$ is real analytic and has a real analytic extension V and P respectively on some $(a_j^{\mathrm{L}}, a_{j+1}^{\mathrm{R}})$, where $a_j^{\mathrm{L}} < a_j < a_{j+1} < a_{j+1}^{\mathrm{R}}$. Now it follows from Proposition 2.3 that $\mathrm{rk}_{\mathbb{R}} V(t)$ is constant in $t \in (a_j^{\mathrm{L}}, a_{j+1}^{\mathrm{R}})$ for each $j \in \mathbb{Z}$. This proves that V has p.c. rank. The opposite direction follows by reversing the foregoing arguments. □

PROPOSITION 2.6 Let $C \in \mathbb{A}_{\mathrm{p}}^{p\times n}$ and $V \in \mathbb{A}_{\mathrm{p}}^{n\times k}$. Then there exist $\hat{V} \in \mathbb{A}_{\mathrm{p}}^{n\times k}$ and $\hat{U}, \hat{C}, \hat{W} \in \mathbb{A}_{\mathrm{p}}^{n\times n}$ with p.c. ranks so that

(i) $$V(t)\mathbb{R}^k \overset{\subset}{\underset{\text{ae}}{=}} \hat{V}(t)\mathbb{R}^n,$$

(ii) $$\left(V(t)\mathbb{R}^k\right)^{\perp} \overset{\supset}{\underset{\text{ae}}{=}} \hat{U}(t)\mathbb{R}^n,$$

(iii) $$\ker C(t) \overset{\supset}{\underset{\text{ae}}{=}} \hat{C}(t)\mathbb{R}^n,$$

(iv) $$V(t)\mathbb{R}^k \cap \ker C(t) \overset{\text{ae}}{=} \hat{W}(t)\mathbb{R}^{s'}.$$

Proof. To prove (i), choose an interval $[a_j, a_{j+1})$ so that $V\restriction_{[a_j, a_{j+1})}$ is real analytic and has a real analytic extension V_j on $(a_j^{\mathrm{L}}, a_{j+1}^{\mathrm{R}})$. Then, by Lemma 2.1(i), there exists $\hat{V}_j \in \mathbb{A}\restriction^{n\times n}_{(a_j^{\mathrm{L}}, a_{j+1}^{\mathrm{R}})}$ with constant rank so that

$$V_j(t)\mathbb{R}^k \overset{\subset}{\underset{\text{ae}}{=}} \hat{V}_j(t)\mathbb{R}^n \quad \text{on } (a_j^{\mathrm{L}}, a_{j+1}^{\mathrm{R}}).$$

Since this can be done for every interval of the partition corresponding to $V \in \mathbb{A}_{\mathrm{p}}^{n\times k}$, statement (i) is proved. For the proof of (ii)–(iv), use the similar arguments. □

PROPOSITION 2.7 (i) Suppose $\mathcal{V} \in \mathbf{W}_n$ is generated by $V \in \mathbb{A}^{n\times k}$ and $\mathrm{rk}_{\mathbb{R}} V(t)$ is constant in $t \in \mathbb{R}$. If $\boldsymbol{v} \in \mathbb{A}^n$ satisfies

$$\boldsymbol{v}(t) \in \mathcal{V}(t) \quad \text{for all } t \in \mathbb{R}\setminus\mathcal{N}, \tag{2.1}$$

where $\mathcal{N}$ is a discrete set, then there exists $\boldsymbol{r} \in \mathbb{A}^k$ so that

$$\boldsymbol{v}(t) = V(t)\boldsymbol{r}(t) \quad \text{for all } t \in \mathbb{R} \tag{2.2}$$

and thus $\boldsymbol{v}(t) \in \mathcal{V}(t)$ for all $t \in \mathbb{R}$.

(ii) Suppose $\mathcal{V} \in \mathbf{W}_n$ is generated by $V \in \mathbb{A}_p^{n\times k}$ and V has p.c. rank. If $\boldsymbol{v} \in \mathbb{A}_p^n$ satisfies (2.1), then (2.2) is valid for some $\boldsymbol{r} \in \mathbb{A}_p^k$.

Proof. (i) Let $l \in \mathbb{N}$ so that $\operatorname{rk}_{\mathbb{R}} V(t) = l$ for all $t \in \mathbb{R}$. Then, by Silverman & Bucy (1970), there exists $S \in \mathrm{GL}_k(\mathbb{A})$ such that $VS^{-1} = [W, 0]$ for some $W \in \mathbb{A}^{n\times l}$ with $\operatorname{rk}_{\mathbb{R}} W(t) = l$ for all $t \in \mathbb{R}$. Put

$$\boldsymbol{r} = S^{-1}\begin{bmatrix} \boldsymbol{r}' \\ \mathbf{0}_{k-l} \end{bmatrix},$$

where $\boldsymbol{r}' := W^{\mathrm{T}}(WW^{\mathrm{T}})^{-1}\boldsymbol{v}$; then $\boldsymbol{r}$ satisfies (2.2).

(ii) Use the notation of Definition 2.4. It is sufficient to prove the assertion on some interval $(a_j^{\mathrm{L}}, a_{j+1}^{\mathrm{R}})$, where $\operatorname{rk}_{\mathbb{R}} V_j(\bullet)$ is constant. Thus (ii) follows from (i). □

Time-varying subspaces arise when controllability subspaces of time-varying systems are considered. This will be described in the following section.

3. The controllable family

In this section, we present a new description of the controllable and the unreconstructible vector spaces, at time $t_0 \in \mathbb{R}$, of the system

$$\left.\begin{aligned} \dot{\boldsymbol{x}}(t) &= A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= C(t)\boldsymbol{x}(t) \end{aligned}\right\} \quad (t \in \mathbb{R}), \tag{3.1}$$

where $\boldsymbol{u}(t) \in \mathbb{R}^m$, $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{y}(t) \in \mathbb{R}^p$, and A, B, C are $n \times n$, $n \times m$, $p \times n$ matrix functions with entries in a ring $\mathcal{R}$ of real functions, so that (3.1) has a unique solution.

From a system-theoretic point of view, this section is interesting for itself. On the other hand, it will be seen that time-varying families of subspace introduced in Section 2 arise in a natural way. Moreover, the controllable family of a system is an example of an A-invariant family which will be defined in Section 4. Finally, the controllable family is used for a characterization of the soution of the disturbance-decoupling problem in Section 6.

Throughout the following, a *fundamental matrix* of

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t)$$

is denoted by $X(\bullet)$, and the *transition matrix* by

$$\Phi(t, t_0) = X(t)X(t_0)^{-1}.$$

Suppose that

$$T = [t_{ij}] \in \mathrm{GL}_n(\mathcal{R}) = \{T \in \mathcal{R}^{n\times n} : (\exists\, T^{-1} \in \mathcal{R}^{n\times n})(\forall\, t \in \mathbb{R})\ T(t)T(t)^{-1} = \mathrm{I}_n\}$$

and $\dot{T} = [\dot{t}_{ij}] \in \mathcal{R}^{n\times n}$; then the *coordinate transformation*

$$z(t) := T(t)^{-1}x(t)$$

converts the system (3.1) into

$$\left.\begin{aligned} \dot{z}(t) &= A'(t)z(t) + B'(t)u(t) \\ y(t) &= C'(t)z(t) \end{aligned}\right\} \quad (t \in \mathbb{R}), \tag{3.2}$$

where

$$A' = T^{-1}AT - T^{-1}\dot{T} \in \mathcal{R}^{n\times n}, \qquad B' = T^{-1}B \in \mathcal{R}^{n\times m}, \qquad C = CT \in \mathcal{R}^{p\times n}; \tag{3.3}$$

and the transition matrix $\Phi'(t, t_0)$ of $\dot{z}(t) = A'(t)z(t)$ is given by

$$\Phi'(t, t_0) = T(t)^{-1}\Phi(t, t_0)T(t_0).$$

In this case, (3.1) and (3.2) are called *similar*. For the sake of brevity, the matrix pair $(A, B) \in \mathcal{R}^{n\times n} \times \mathcal{R}^{n\times m}$ and the triple $(A, B, C) \in \mathcal{R}^{n\times n} \times \mathcal{R}^{n\times m} \times \mathcal{R}^{p\times n}$ are associated with the system (3.1).

As opposed to time-invariant systems, due to the much richer class of coordinate transformations, a system (A, B) is always similar to a system with constant free motion. More precisely, the coordinate transformation $T(\bullet) = X(\bullet)$, where X is a fundamental matrix of $\dot{x}(t) = A(t)x(t)$, converts (A, B) into $(0, X^{-1}B) \in \mathcal{R}^{n\times n} \times \mathcal{R}^{n\times m}$.

For computations, the following operational setup will be helpful. By identifying each $f \in \mathcal{R}$ with the multiplication operator $g \mapsto fg$, we may consider $\mathcal{R}$ as a subring of $\mathrm{end}_{\mathbb{R}}(\mathcal{R})$, the ring of $\mathbb{R}$-endomorphisms of $\mathcal{R}$. If $\mathcal{R}$ *is differentially closed*, i.e. $\dot{f} \in \mathcal{R}$ for all $f \in \mathcal{R}$, then the differential operator

$$\mathsf{D} : \mathcal{R} \to \mathcal{R} : f \mapsto \mathsf{D}f = \dot{f}$$

is an element of $\mathrm{end}_{\mathbb{R}}(\mathcal{R})$ as well. The composition of D and f in $\mathrm{end}_{\mathbb{R}}(\mathbb{M})$ is given by

$$(\mathsf{D}f)(g) = \mathsf{D}(fg) = f\dot{g} + \dot{f}g = (f\mathsf{D} + \dot{f})(g) \quad \text{for all } f,g \in \mathcal{R},$$

and one has the multiplication rule

$$\mathsf{D}f = f\mathsf{D} + \dot{f} \quad \text{for all } f \in \mathcal{R}. \tag{3.4}$$

LEMMA 3.1 If $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$ and $T \in \mathrm{GL}_n(\mathbb{A})$ then, for $i \in \mathbb{N}$ and $(A', B') \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$ satisfying (3.3), we have

$$T^{-1}(\mathsf{D}I_n - A)T = \mathsf{D}I_n - A', \tag{3.5}$$

$$T^{-1}(\mathsf{D}I_n - A)^i(B) = (\mathsf{D}I_n - A')^i(B'), \tag{3.6}$$

$$(\mathsf{D}I_n - A)^i(B) = X(X^{-1}B)^i, \tag{3.7}$$

where $(\mathsf{D}I_n - A)^i(B) := (\mathsf{D}I_n - A)[(\mathsf{D}I_n - A)^{i-1}(B)]$.

Proof. (3.5) is an immediate consequence of (3.3) and (3.4). We prove (3.6) by induction on i. For $i = 0$, it holds true by (3.3). If (3.6) is true for i, we conclude

$$\begin{aligned} T^{-1}(\mathsf{D}I_n - A)^{i+1}(B) &= T^{-1}(\mathsf{D}I_n - A)T[T^{-1}(\mathsf{D}I_n - A)^i(B)] \\ &= (\mathsf{D}I_n - A')(\mathsf{D}I_n - A')^i(B). \end{aligned}$$

Equation (3.7) is also easily shown by induction. □

We will now describe the vector space $\mathcal{R}(t_0)$ of all states which can be controlled to zero at time t_0. For this, consider the *controllability Gramian*

$$W(t_0, t_1) := \int_{t_0}^{t_1} \Phi(t_0, s)B(s)B^{\mathrm{T}}(s)\Phi^{\mathrm{T}}(t_0, s)\,\mathrm{d}s$$

associated with (3.1) for $\mathcal{R} = \mathbb{C}_{\mathrm{p}}$, the space of piecewise continuous functions. Then there exists a control $\boldsymbol{u} \in \mathbb{C}_{\mathrm{p}}^m$ which forces the state $\boldsymbol{x}_0 \in \mathbb{R}^n$ at time t_0 to zero in time $t_1 - t_0 > 0$, i.e.

$$\Phi(t_1, t_0)\boldsymbol{x}_0 + \int_{t_0}^{t_1} \Phi(t_1, s)B(s)\boldsymbol{u}(s)\,\mathrm{d}s = 0,$$

if and only if $\boldsymbol{x}_0 \in \operatorname{im} W(t_0, t_1)$ (cf. Kalman 1960). Thus

$$\mathcal{R}(t_0) := \bigcup_{t_1 > t_0} \operatorname{im} W(t_0, t_1)$$

is the vector space of all states which can be controlled at time t_0 to zero in finite time. We call $\mathcal{R} = (\mathcal{R}(t))_{t \in \mathbb{R}}$ the *controllable family* of the system $(A, B) \in \mathbb{C}_{\mathrm{p}}^{n\times n} \times \mathbb{C}_{\mathrm{p}}^{n\times m}$. Clearly,

$$\mathcal{R}(t_0) \subset \Phi(t_0, t_{-1})\mathcal{R}(t_{-1}) \quad \text{for } t_{-1} \leq t_0.$$

If $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$ is a real analytic system, then $\operatorname{rk}_{\mathbb{R}} W(t_0, t)$ is constant for all $t \in \mathbb{R} \setminus \{t_0\}$. Thus

$$\mathcal{R}(t) = \Phi(t, t_0)\mathcal{R}(t_0) \quad \text{for all } t, t_0 \in \mathbb{R}. \tag{3.8}$$

Moreover it can be shown that $\mathcal{R}(t)$ can be computed without knowledge of the transition matrix.

PROPOSITION 3.2 Suppose $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$ and $t_0 < t_1$. Then

$$\operatorname{im} W(t_0, t_1) = \Phi(t_0, t) \sum_{i \geq 0} \operatorname{im} [\mathrm{DI}_n - A(t)]^i(B)(t) \quad \text{for all } t \in [t_0, t_1]. \tag{3.9}$$

Suppose that $[A, B] \in \mathbb{A}_{\mathrm{p}}^{n\times n} \times \mathbb{A}_{\mathrm{p}}^{n\times m}$ and that $\{[a_j, a_{j+1}) : j \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$ such that A and B are analytic on every (a_j, a_{j+1}). Then, for $t_0 \in [a_0, a_1)$ and $t_1 \in [a_N, a_{N+1})$, one obtains

$$\operatorname{im} W(t_0, t_1) = \sum_{i \geq 0} \operatorname{im} [\mathrm{DI}_n - A(t_0)]^i(B)(t_0) + \sum_{j=1}^{N} \sum_{i \geq 0} \operatorname{im} [\mathrm{DI}_n - A(a_j)]^i(B)(a_j). \tag{3.10}$$

Proof. The main idea of the proof is due to Schmale (1981). Using Lemma 3.1, it is easily seen that, without loss of generality, one may assume $A = 0$ and $X = \mathrm{I}_n$. So it remains to prove

$$\operatorname{im} W_2(t_0, t_1) = \sum_{i \geq 0} \operatorname{im} B(t)^i \quad \text{for all } t \in [t_0, t_1],$$

which is equivalent to

$$\ker W_2(t_0, t_1) = \Big(\sum_{i \geq 0} \operatorname{im} B(t)^i\Big)^{\perp} \quad \text{for all } t \in [t_0, t_1].$$

Due to the properties of analytic functions, it is easily seen that

$$\begin{aligned}
& \quad q \in \ker W_2(t_0, t_1) \\
\Leftrightarrow & \quad B(t)^{\mathrm{T}} q = 0 \quad \text{for all } t \in [t_0, t_1] \\
\Leftrightarrow & \quad B(t)^{\mathrm{T}i} q = 0 \quad \text{for some } t \in [t_0, t_1] \quad \text{for all } i \geq 0 \\
\Leftrightarrow & \quad q \in \bigcap_{i \geq 0} \ker B(t)^{\mathrm{T}i} = \Big(\sum_{i \geq 0} \operatorname{im} B(t)^i\Big)^{\perp} \quad \text{for some } t \in [t_0, t_1].
\end{aligned}$$

This proves (3.9). Equation (3.10) follows from (3.9) and the fact that

$$\operatorname{im} W_2(t_0, t_1) = \operatorname{im} W_2(t_0, a_1) + \cdots + \operatorname{im} W_2(a_N, t_1). \quad \square$$

COROLLARY 3.3 If $(A, B) \in \mathbb{A}^{n \times n} \times \mathbb{A}^{n \times m}$, then

$$\mathcal{R}(t) = \sum_{i \geq 0} \operatorname{im}[\mathrm{DI}_n - A(t)]^i (B)(t) \quad \text{for all } t \in \mathbb{R}, \tag{3.11}$$

and the controllable family is an analytic family.

Proof. Equation (3.11) follows from (3.9), and analyticity follows from Proposition 2.3 and (3.8).

Remark 3.4. (i) Set $t = t_0$ in (3.9). Then, for time-invariant systems, an application of the Cayley–Hamilton-Theorem reduces (3.9) to the well-known fact that the *controllable space* is given by

$$\operatorname{im} B + \operatorname{im} AB + \cdots + \operatorname{im} A^{n-1}B.$$

(ii) In general, it is not possible to restrict the sum in (3.9) independently of t_0 to only finitely many summands. See an example in Kamen (1979; p. 871).

(iii) If the entries of A and B are polynomial functions, it can be shown that the sum in (3.9) can be restricted to finitely many summands; compare the subclass of constant-rank systems considered in Silverman (1971) and Kamen (1979).

It is also possible to define an *unreconstructible family* and to prove its dual relationships to the controllable family; see Ilchmann (1989).

4. [A, B]-invariant time-varying subspaces

Throughout this section, piecewise analytic systems $(A, B) \in \mathbb{A}_{\mathrm{p}}^{n \times n} \times \mathbb{A}_{\mathrm{p}}^{n \times m}$ are considered. The concept and notation of time-varying subspaces introduced in Section 2 will be used.

DEFINITION 4.1 Suppose that $(A, B) \in \mathbb{A}_{\mathrm{p}}^{n \times n} \times \mathbb{A}_{\mathrm{p}}^{n \times m}$ and that $\mathcal{V} \in \mathbf{W}_n$ is generated by $V \in \mathbb{A}_{\mathrm{p}}^{n \times k}$. Then $\mathcal{V}$ is called *meromorphically* [A, B]-*invariant* if there exist $N \in \mathbb{M}_{\mathrm{p}}^{k \times k}$ and $M \in \mathbb{M}_{\mathrm{p}}^{n \times k}$ such that

$$(\mathrm{DI}_n - A)(V) = VN + BM. \tag{4.1}$$

$\mathcal{V}$ is called [A, B]-*invariant* if (4.1) holds true for some N and M with entries in $\mathbb{A}_{\mathrm{p}}$ instead of $\mathbb{M}_{\mathrm{p}}$. If $B = 0$, we speak of (meromorphic) A-invariance.

This is an extension of the concept of $[A, B]$-invariance introduced by Basile & Marro (1969) for time-invariant systems $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, see also Wonham (1974). In this case a constant vector subspace $\mathcal{V}$ of $\mathbb{R}^n$ is called *$[A, B]$-invariant* if

$$A\mathcal{V} \subset \mathcal{V} + \operatorname{im} B.$$

Clearly, $\mathcal{V}$ viewed as a constant family belongs to $\mathbf{W}_n$ and $\mathcal{V}$ is $[A, B]$-invariant in the sense of Definition 4.1.

A simple example illustrates the difference between $[A, B]$-invariance and meromorphic $[A, B]$-invariance. Put

$$\mathcal{V}(t) = \boldsymbol{v}(t)\mathbb{R}, \qquad \boldsymbol{v}(t) = \begin{bmatrix} 0 \\ t \end{bmatrix}, \qquad A = 0_{2\times 2}, \qquad B(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{4.2}$$

Then

$$(\mathrm{DI}_n - A)(\boldsymbol{v})(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} t^{-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 0$$

and thus $\mathcal{V}$ is meromorphically $[A, B]$-invariant, not $[A, B]$-invariant.

PROPOSITION 4.2 Suppose that either

(i) $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$, $V \in \mathbb{A}^{n\times k}$, and $\operatorname{rk}_{\mathbb{R}} [V(t), B(t)]$ is constant for $t \in \mathbb{R}$

or

(ii) $(A, B) \in \mathbb{A}_{\mathrm{p}}^{n\times n} \times \mathbb{A}_{\mathrm{p}}^{n\times m}$, $V \in \mathbb{A}_{\mathrm{p}}^{n\times k}$, and $[V, B]$ has p.c. rank.

Then $\mathcal{V}$ generated by V is meromorphically $[A, B]$-invariant if and only if $\mathcal{V}$ is $[A, B]$-invariant.

Proof. (i) It has to be shown that meromorphic $[A, B]$-invariance implies $[A, B]$-invariance. Meromorphic $[A, B]$-invariance yields that

$$[\mathrm{DI}_n - A(t)](V)(t) \in [V(t), B(t)]\mathbb{R}^{m+k} \quad \text{for almost all } t \in \mathbb{R}.$$

Now the result follows from Proposition 2.7(i).

(ii) Since $[V, B]$ has p.c. rank, it is sufficient to prove the assertion on an interval $(a_j^{\mathrm{L}}, a_{j+1}^{\mathrm{R}})$ where $[V, B](t)$ has constant rank (the notation of Definition 2.4 is used). Now (ii) follows from (i). □

Remark 4.3. Suppose (i) or (ii) of Proposition 4.2 is satisfied. Then it follows from the proof of Proposition 4.2 that $\mathcal{V}$ is $[A, B]$-invariant if and only if

$$\operatorname{im}[\mathrm{DI}_n - A(t)](V)(t) \subset \mathcal{V}(t) + \operatorname{im} B(t) \quad \text{for all } t \in \mathcal{N},$$

where $\mathcal{N}$ is a discrete set.

The following basic properties of (meromorphic) $[A, B]$-invariance are immediate.

Remark 4.4. For $\mathcal{V} \in \mathbf{W}_n$ with generator $V = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k] \in \mathbb{A}_{\mathrm{p}}^{n\times k}$, the following statements hold true.

(i) Suppose that $(A, B) \in \mathbb{A}_{\mathrm{p}}^{n\times n} \times \mathbb{A}_{\mathrm{p}}^{n\times m}$ is similar to $(A', B') \in \mathbb{A}_{\mathrm{p}}^{n\times n} \times \mathbb{A}_{\mathrm{p}}^{n\times m}$ via $T \in GL_n(\mathbb{A}_{\mathrm{p}})$. Then $\mathcal{V}$ is (meromorphically) $[A, B]$-invariant iff $T^{-1}\mathcal{V}$ is (meromorphically) $[A', B']$-invariant.

(ii) $\mathcal{V}$ is (meromorphically) $[A, B]$-invariant iff, for every $\boldsymbol{v} = \sum_{i=1}^{k} \alpha_i \boldsymbol{v}_i$, with

$\alpha_i \in \mathbb{A}_p$ (resp. $\alpha_i \in \mathbb{M}_p$), there exist $\boldsymbol{r} \in \mathbb{A}_p^k$ and $\boldsymbol{s} \in \mathbb{A}_p^m$ (resp. $\boldsymbol{r} \in \mathbb{M}_p^k$ and $\boldsymbol{s} \in \mathbb{M}_p^m$) such that

$$(\mathrm{DI}_n - A)(\boldsymbol{v}) = V\boldsymbol{r} + B\boldsymbol{s}.$$

(iii) The sum of two (meromorphically) $[A, B]$-invariant families is (meromorphically) $[A, B]$-invariant as well.

The concept of $[A, B]$-invariance becomes clearer by the following theorem. Furthermore this result is important for the solvability of the disturbance-decoupling problem tackled in Section 6.

THEOREM 4.5 Suppose that $(A, B) \in \mathbb{A}_p^{n\times n} \times \mathbb{A}_p^{n\times m}$ and that $\mathcal{V} \in \mathbf{W}_n$ is generated by $V \in \mathbb{A}_p^{n\times k}$ with $\mathrm{rk}_{\mathbb{R}}\, V(t) = k$ for all $t \in \mathbb{R}$. Let $P(t): \mathbb{R}^n \to \mathcal{V}(t)$ denote the orthogonal projector on $\mathcal{V}(t)$ along $\mathcal{V}^{\perp}(t)$. Then the following are equivalent:

(i) $\mathcal{V}$ is $[A, B]$-invariant, i.e. there exist $N \in \mathbb{A}_p^{k\times k}$ and $M \in \mathbb{A}_p^{m\times k}$ such that

$$[\mathrm{DI}_n - A(t)](V)(t) = V(t)N(t) + B(t)M(t) \quad \text{for all } t \in \mathbb{R}.$$

(ii) There exists an $F \in \mathbb{A}_p^{m\times n}$ such that $\mathcal{V}$ is $(A + BF)$-invariant.

(iii) There exist $\tilde{N} \in \mathbb{A}_p^{n\times n}$ and $\tilde{M} \in \mathbb{A}_p^{m\times n}$ such that

$$[\mathrm{DI}_n - A(t)](P)(t) = P(t)\tilde{N}(t) + B(t)\tilde{M}(t) \quad \text{for all } t \in \mathbb{R}.$$

(iv) There exist $N \in \mathbb{A}_p^{k\times k}$ and $M \in \mathbb{A}_p^{m\times k}$ such that

$$V(t)\Psi(t_0, t)^{\mathrm{T}} = \Phi(t, t_0) + \int_{t_0}^{t} \Phi(t, s)B(s)M(s)\Psi(t_0, s)^{\mathrm{T}}\, ds \quad \text{for all } t \in \mathbb{R},$$

where Φ and Ψ denote respectively the transition matrices of

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t), \qquad \dot{\boldsymbol{x}}(t) = N(t)^{\mathrm{T}}\boldsymbol{x}(t).$$

Proof. (i)$\Rightarrow$(ii): Define $F = M(V^{\mathrm{T}}V)^{-1}(V)^{\mathrm{T}}$. Then

$$[\mathrm{DI}_n - (A + BF)](V) = (\mathrm{DI}_n - A)(V) - BFV = VN,$$

which proves (ii). The proof of (ii)$\Rightarrow$(i) is trivial.

(i)$\Rightarrow$(iii): Put $Q = V^{\mathrm{T}}(VV^{\mathrm{T}})^{-1}P$. Then $VQ = P$ and

$$(\mathrm{DI}_n - A)(P) = V(NQ - Q) + BMQ.$$

(iii)$\Rightarrow$(i) If $Q = V^{\mathrm{T}}(VV^{\mathrm{T}})^{-1}P$, then

$$[(\mathrm{DI}_n - A)(V)]Q = (\mathrm{DI}_n - A)(P) + V\dot{Q} = P\tilde{N} + B\tilde{M} + V\dot{Q}.$$

Since $\mathrm{rk}_{\mathbb{R}}\, P(t) = k$ for all $t \in \mathbb{R}$, there exists $Q_r \in \mathbb{A}_p^{n\times k}$ so that $QQ_r = \mathrm{I}_k$. Thus

$$(\mathrm{DI}_n - A)(V) = P\tilde{N}Q_r + B\tilde{M}Q_r + PQ_r\dot{Q}Q_r,$$

which proves (i).

(i)$\Rightarrow$(iv): Multiplying the equation in (i) from the left by $T^{-1} = \Phi(\bullet, t_0)^{-1}$ yields

$$\dot{V}' = V'N + B'M,$$

where $V' = T^{-1}V$ and $B' = T^{-1}B$. This may be integrated, after transposition, to

$$V'^{\mathrm{T}}(t) = \Psi(t, t_0)V'^{\mathrm{T}}(t_0) + \int_{t_0}^{t} \Psi(t, s)M^{\mathrm{T}}(s)B'^{\mathrm{T}}(s)\, \mathrm{d}s.$$

By transposition and multiplication it follows that

$$V'(t)\Psi(t_0, t)^{\mathrm{T}} = V'(t_0) + \int_{t_0}^{t} B'(s)M(s)\Psi(t_0, s)^{\mathrm{T}}\, \mathrm{d}s$$

Multiplication from the left by $T(t)$ gives

$$V(t)\Psi(t_0, t)^{\mathrm{T}} = \Phi(t, t_0)V(t_0) + \int_{t_0}^{t} \Phi(t, s)B(s)M(s)\Psi(t_0, t)^{\mathrm{T}}\, \mathrm{d}s.$$

This proves (iv).

To prove (iv)$\Rightarrow$(i), reverse the arguments in the proof of (i)$\Rightarrow$(iv). □

As an immediate consequence of Theorem 4.5, one obtains the following result.

COROLLARY 4.6 Suppose that $A \in \mathbb{A}_{\mathrm{p}}^{n\times n}$ and that $\mathcal{V} \in \mathbf{W}_n$ is generated by $V \in \mathbb{A}^{n\times k}$ with $\mathrm{rk}_{\mathbb{R}}\, V(t) = k$ for all $t \in \mathbb{R}$. Then the following are equivalent.

(i) $\mathcal{V}$ is A-invariant.

(ii) There exists $\tilde{N} \in \mathbb{A}_{\mathrm{p}}^{n\times n}$ such that $[\mathrm{D}I_n - A(t)](P)(t) = P(t)\tilde{N}(t)$ for all $t \in \mathbb{R}$.

(iii) $\mathcal{V}(t) = \Phi(t, t_0)\mathcal{V}(t_0)$ for all $t, t_0 \in \mathbb{R}$.

Remark 4.7. If a real analytic system $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$ is considered and V is real analytic, then in Theorem 4.5 and Corollary 4.6 all matrices are also real analytic. The proofs carry over completely.

Remark 4.8. Condition (iii) in Corollary 4.6 implies that, for every $\boldsymbol{x}_0 \in \mathcal{V}(t_0)$, the free trajectory $\Phi(t, t_0)\boldsymbol{x}_0$ remains in $\mathcal{V}(t)$ for all $t \in \mathbb{R}$. Condition (iv) in Theorem 4.5 says that, if $\boldsymbol{x}_0 \in \mathcal{V}(t_0)$, then there exists a control $\boldsymbol{u} \in \mathbb{A}_{\mathrm{p}}^{m}$ such that the forced motion

$$\boldsymbol{x}_{\boldsymbol{u}}(t; t_0, \boldsymbol{x}_0) = \Phi(t, t_0)\boldsymbol{x}_0 + \int_{t_0}^{t} \Phi(t, s)B(s)\boldsymbol{u}(s)\, \mathrm{d}s$$

can be hold in $\mathcal{V}(t)$ for every $t \in \mathbb{R}$. For time-invariant systems, the latter condition is also sufficient for $[A, B]$-invariance. It is an open problem whether this is also valid for time-varying systems.

EXAMPLE 4.9 For time-invariant systems $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, it is well known that the controllable subspace $\sum_{i=0}^{n-1} A^i \operatorname{im} B$ is the smallest A-invariant subspace that contains $\operatorname{im} B$; see e.g. Wonham (1985: §1.2). This is extended to the analytic situation as follows: The controllable family $\mathcal{R}$ (see Section 3) of an analytic system $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$ is the smallest A-invariant family that contains $\operatorname{im} B(\bullet)$. In fact, $\mathcal{R}$ is an analytic family (see Corollary 3.3), and thus A-invariance follows from Corollary 4.6. Use of the representation (3.11) of $\mathcal{R}(t)$ yields $\operatorname{im} B(t) \subset \mathcal{R}(t)$. If $\mathcal{V} = (\mathcal{V}(t))_{t\in\mathbb{R}} \in \mathbf{W}_n$ is another A-invariant family with $\operatorname{im} B(t) \subset \mathcal{V}(t)$, then $[\mathrm{D}I_n - A(t)]^i(B)(t) \subset \mathcal{V}(t)$ for all $i \in \mathbb{N}$. Thus $\mathcal{R}(t) \subset \mathcal{V}(t)$, by (3.11), and therefore $\mathcal{R}$ is the smallest A-invariant family that contains $\operatorname{im} B(\bullet)$.

If we do not assume that the rank of V is constant, then the feedback constructed in Theorem 4.5(ii) may have poles. For disturbance-decoupling problems it is important to locate these poles.

PROPOSITION 4.10 Suppose that $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$, with $\mathrm{rk}_{\mathbb{M}}\, B = m$, and that $\mathcal{V} \in \mathbf{W}_n$ is generated by $V \in \mathbb{A}^{n\times k}$ with $\mathrm{rk}_{\mathbb{M}}\, V = k$.

If $\mathcal{V}$ is meromorphically $[A, B]$-invariant, then there exists analytic matrices U_1, U_2, T, and W, of formats $k\times s$, $m\times s$, $s\times s$, and $s\times k$ respectively, and $T^{-1} \in \mathbb{M}^{s\times s}$ so that

$$(\mathrm{DI}_n - A)V = VU_1T^{-1}W + BU_2T^{-1}W. \tag{4.3}$$

T^{-1} has poles at t' if and only if

$$\dim\,[V(t')\mathbb{R}^k + B(t')\mathbb{R}^m] < \max_{t\in\mathbb{R}} \dim\,[V(t)\mathbb{R}^k + B(t)\mathbb{R}^m]. \tag{4.4}$$

Proof. By Lemma 9.1(i), any $G \in \mathrm{gcld}\,(V, B) \subset \mathbb{A}^{n\times s}$ satisfies

$$G\mathbb{A}^s = V\mathbb{A}^k + B\mathbb{A}^m, \qquad G = VU_1 + BU_3,$$

for some $U_1 \in \mathbb{A}^{k\times s}$ and $U_3 \in \mathbb{A}^{m\times s}$. Let $\hat{G} \in \mathbb{A}^{n\times s}$ with $\mathrm{rk}_{\mathbb{R}}\, \hat{G}(t) = s$ for all $t \in \mathbb{R}$ (see Lemma 2.1) such that

$$G(t)\mathbb{R}^s \overset{\subset}{\overset{\text{ae}}{=}} \hat{G}(t)\mathbb{R}^s.$$

Then $G = \hat{G}T$ for some $T \in \mathbb{A}^{s\times s}$ with $T^{-1} \in \mathbb{M}^{s\times s}$. Since $\mathcal{V}$ is meromorphically $[A, B]$-invariant and $\hat{G}$ is left invergible over $\mathbb{A}$ there exists some $W \in \mathbb{A}^{s\times k}$ so that $(\mathrm{DI}_n - A)(V) = \hat{G}W$. This proves (4.3). Clearly, T^{-1} has poles at t' if and only if $\mathrm{rk}_{\mathbb{R}}\, G(t') < \mathrm{rk}_{\mathbb{M}}\, G$, which proves (4.4). □

EXAMPLE 4.11 Let $(A, B) \in \mathbb{A}^{3\times 3} \times \mathbb{A}^{3\times 1}$ be given by

$$A(t) = \begin{bmatrix} t^2 & \sin t & -t(t^2+2) \\ a_4(t) & a_5(t) & a_6(t) \\ -1 & 0 & t \end{bmatrix}, \qquad B(t) = \begin{bmatrix} -t^2(t-2) \\ t^2 \\ t-2 \end{bmatrix},$$

and let $\mathcal{V}(t) = V(t)\mathbb{R}$, where $V(t) = [t^2, 0, -1]^{\mathsf{T}}$. In the notation of the proof of Proposition 4.10, Lemma 2.1 gives

$$\mathrm{gcld}\,(V, B) \ni G(t) = \begin{bmatrix} t^2 & -t^2(t-2) \\ 0 & t^2 \\ -1 & t-2 \end{bmatrix}$$

$$= VU_1 + BU_3 = \begin{bmatrix} t^2 \\ 0 \\ -1 \end{bmatrix}[1, \quad 0] + \begin{bmatrix} -t^2(t-2) \\ t^2 \\ t-2 \end{bmatrix}[0, \quad 1]$$

and $G(t) = \hat{G}(t)T(t) = \begin{bmatrix} t^2 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -(t-2) \\ 0 & t^2 \end{bmatrix}$. Also, $\mathcal{V}$ is meromorphically

$[A, B]$-invariant since, for

$$W(t) = \begin{bmatrix} -t(t+1) \\ -a_4(t)t^2 + a_6(t) \end{bmatrix} \quad \text{and} \quad T^{-1}(t) = \begin{bmatrix} 1 & -t^{-2}(t-2) \\ 0 & t^{-2} \end{bmatrix},$$

we have

$$\begin{aligned}[] [\mathrm{DI}_n - A(t)](V)(t) &= \begin{bmatrix} -t^3(t+1) \\ -a_4(t)t^2 + a_6(t) \\ t(t+1) \end{bmatrix} = \hat{G}(t)W(t) = G(t)T(t)^{-1}W(t) \\ &= V(t)[t^{-2}(t-2)[-a_4(t)t^2 + a_6(t)] - t(t+1)] + B(t)[t^{-2}(-a_4(t)t^2 + a_6(t))] \\ &= V(t)N(t) + B(t)M(t). \end{aligned}$$

Since $\mathrm{rk}_{\mathbb{R}}\,[V(t), B(t)] = \{1 \text{ if } t = 0;\ 2 \text{ if } t \neq 0\}$, it follows that $t' = 0$ is the only pole of $T^{-1}(t')$, or equivalently

$$\dim\,[V(t')\mathbb{R} + B(t')\mathbb{R}] < 2.$$

In the proof of Theorem 4.5, an analytic feedback matrix F was determined so that

$$[\mathrm{DI}_n - (A + BF)](V) \subset \mathcal{V}.$$

In the present example, V does not have constant rank. Therefore, in general, F is a meromorphic matrix:

$$F(t) = \frac{-a_4(t)t^2 + a_6(t)}{t^2(t^4+1)}\,[t^2, 0, -1] = M(V^{\mathrm{T}}V)^{-1}V^{\mathrm{T}},$$

and

$$B(t)F(t) = \frac{-a_4(t)t^2 + a_6(t)}{t^4+1} \begin{bmatrix} -t^2(t-2) & 0 & t-2 \\ t^2 & 0 & -1 \\ t-2 & 0 & -(t-2)/t^2 \end{bmatrix}.$$

Whether F has poles depends on the zeros of a_4 and a_6.

Remark 4.12. For time-invariant systems, Hautus (1980) introduces a 'frequency-domain characterization' of $[A, B]$-invariant subspaces. In the case of time-varying systems, a polynomial characterization of $[A, B]$-invariant families of subspaces is also possible via a certain skew polynomial approach, see Ilchmann (1989).

5. Duality between $[A, B]$-invariance and (C, A)-invariance

For time-invariant systems $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$, a constant subspace $\mathcal{V}$ of $\mathbb{R}^n$ is called *(C, A)-invariant* if $A(\mathcal{V} \cap \ker C) \subset \mathcal{V}$. It is well-known (see e.g. Schumacher (1979)) that $\mathcal{V}$ is $[A, B]$-invariant if and only if $\mathcal{V}^{\perp}$ is $(B^{\mathrm{T}}, -A^{\mathrm{T}})$-invariant. For time-varying systems $(A, B, C) \in \mathbb{A}_p^{n\times n} \times \mathbb{A}_p^{n\times m} \times \mathbb{A}_p^{p\times n}$ it has already been mentioned in Section 2 that, in general, the time-varying subspace $\ker C(\cdot)$ does not have a generator $W \in \mathbb{A}_p^{n\times k'}$. Even if $\mathcal{V} \in \mathbf{W}_n$ has a generator $V \in \mathbb{A}_p^{n\times k}$, then the orthogonal complement $\mathcal{V}^{\perp} = (\mathcal{V}(t)^{\perp})_{t\in\mathbb{R}}$ does, in general, not have a piecewise analytic generator. Therefore equivalence classes were introduced:

$$\bar{\mathcal{V}} := \{W \in \mathbf{W}_n : W(t) \stackrel{\text{ae}}{=} \mathcal{V}(t)\} \quad \text{for } \mathcal{V} = (\mathcal{V}(t))_{t \in \mathbb{R}} \in \mathbf{W}_n.$$

By Proposition 2.6, for every $\mathcal{V}$ generated by $V \in \mathbb{A}_{\mathrm{p}}^{n \times k}$ and $B \in \mathbb{A}_{\mathrm{p}}^{n \times m}$, one can find $W \in \mathbb{A}_{\mathrm{p}}^{n \times (n-k)}$ and $W' \in \mathbb{A}_{\mathrm{p}}^{n \times k'}$ with p.c. ranks so that

$$(W(t)\mathbb{R}^{n-k})_{t \in \mathbb{R}} \in \overline{\mathcal{V}^{\perp}} \quad \text{and} \quad (W'(t)\mathbb{R}^{k'})_{t \in \mathbb{R}} \in \overline{(\mathcal{V}(t) \cap \ker B^{\mathrm{T}}(t))_{t \in \mathbb{R}}}.$$

Let $\bar{\mathbf{W}}_n := \{\bar{\mathcal{V}} : \mathcal{V} \in \mathbf{W}_n\}$; then the concept of (meromorphic) $[A, B]$-invariance is extended as follows.

Definition 5.1 Suppose $(A, B) \in \mathbb{A}_{\mathrm{p}}^{n \times n} \times \mathbb{A}_{\mathrm{p}}^{n \times m}$. Then $\bar{\mathcal{V}} \in \bar{\mathbf{W}}_n$ is called (meromorphically) $[A, B]$*-invariant* if there exists a $\mathcal{V} \in \bar{\mathcal{V}}$ so that $\mathcal{V}$ is (meromorphically) $[A, B]$-invariant.

Now (C, A)-invariance, as defined above for constant systems, can be extended to the time-varying situation as follows.

Definition 5.2 Suppose that $A \in \mathbb{A}_{\mathrm{p}}^{n \times n}$ and $C \in \mathbb{A}_{\mathrm{p}}^{p \times n}$, and that $\mathcal{V} \in \mathbf{W}_n$ is generated by $V \in \mathbb{A}_{\mathrm{p}}^{n \times k}$. Choose $\hat{W} \in \mathbb{A}_{\mathrm{p}}^{n \times s}$ so that

$$\hat{W}(t)\mathbb{R}^s \stackrel{\text{ae}}{=} \mathcal{V}(t) \cap \ker C(t).$$

Then $\bar{\mathcal{V}}$ is called (C, A)*-invariant* if

$$([\mathrm{DI}_n - A(t)](\hat{W})(t)\mathbb{R}^s)_{t \in \mathbb{R}} \in \bar{\mathcal{V}}. \tag{5.1}$$

Remark 5.3. (i) In Definition 5.2, one has some freedom in choosing $\hat{W}$. By Proposition 2.6, $\hat{W}$ may be chosen with p.c. rank. Also by Proposition 2.6, choose $\hat{V} \in \mathbb{A}_{\mathrm{p}}^{n \times k}$ with p.c. rank such that $(\hat{V}(t)\mathbb{R}^k)_{t \in \mathbb{R}} \in \bar{\mathcal{V}}$. Now it follows from Proposition 2.7(ii) that $\bar{\mathcal{V}}$ is (C, A)-invariant if and only if

$$(\mathrm{DI}_n - A)(\hat{W}) = \hat{V}R \quad \text{for some } R \in \mathbb{A}_{\mathrm{p}}^{k \times s} \tag{5.2}$$

(ii) Since there always exists $\hat{V} \in \mathbb{A}_{\mathrm{p}}^{n \times k}$ with p.c. rank so that $(\hat{V}(t)\mathbb{R}^k)_{t \in \mathbb{R}} \in \bar{\mathcal{V}}$, it makes no sense to introduce *meromorphic* (C, A)-invariance similar to meromorphic $[A, B]$-invariance.

(iii) It is easily verified that statements analogous to Remark 4.4 hold true for (C, A)-invariance.

Proposition 5.4 Suppose that $A \in \mathbb{A}_{\mathrm{p}}^{n \times n}$ and $C \in \mathbb{A}_{\mathrm{p}}^{p \times n}$, and that $\mathcal{V} \in \mathbf{W}_n$ is generated by $V \in \mathbb{A}_{\mathrm{p}}^{n \times k}$. Then $\bar{\mathcal{V}}$ is (C, A)-invariant if and only if $\overline{\mathcal{V}^{\perp}}$ is meromorphically $[-A^{\mathrm{T}}, C^{\mathrm{T}}]$-invariant.

Proof. By Remark 5.3(i), one may assume, without restriction of generality, that V has p.c. rank. Choose, by Lemma 2.1(ii), $\hat{U} \in \mathbb{A}_{\mathrm{p}}^{n \times l}$ with p.c. rank so that

$$\mathcal{V}^{\perp}(t) = \hat{U}(t)\mathbb{R}^l.$$

Since $\hat{W} \in \mathbb{A}_{\mathrm{p}}^{n \times s}$ satisfies

$$\hat{W}(t)\mathbb{R}^s \overset{\subset}{\underset{}{\stackrel{\text{ae}}{=}}} V(t)\mathbb{R}^k \cap \ker C(t),$$

one obtains, for arbitrary columns $\hat{\boldsymbol{u}}$ of $\hat{U}$ and $\hat{\boldsymbol{w}}$ of $\hat{W}$, the identity $\langle \hat{\boldsymbol{u}}(t), \hat{\boldsymbol{w}}(t)\rangle = 0$ for all $t \in \mathbb{R}$. Thus

$$[(\mathrm{D}I_n - A)(\hat{\boldsymbol{w}})]^{\mathrm{T}}\hat{\boldsymbol{u}} = -\hat{\boldsymbol{w}}^{\mathrm{T}}\dot{\hat{\boldsymbol{u}}} - \hat{\boldsymbol{w}}^{\mathrm{T}}A^{\mathrm{T}}\hat{\boldsymbol{u}} = -\hat{\boldsymbol{w}}^{\mathrm{T}}[(\mathrm{D}I_n - A^{\mathrm{T}})(\hat{\boldsymbol{u}})]. \tag{5.3}$$

If $\bar{V}$ is (C, A)-invariant, then (5.2) yields $(\mathrm{D}I_n - A)(\hat{\boldsymbol{w}}) = V\boldsymbol{r}$ for some $\boldsymbol{r} \in \mathbb{A}_{\mathrm{p}}^k$, and (5.3) gives $\langle (\mathrm{D}I_n - A^{\mathrm{T}})(\hat{\boldsymbol{u}}), \hat{\boldsymbol{w}}\rangle = 0$. Therefore

$$\begin{aligned}[\mathrm{D}I_n + A(t)^{\mathrm{T}}](\hat{\boldsymbol{u}})(t) &\in [V(t) \cap \ker C(t)]^{\perp} + \operatorname{im} C(t)^{\mathrm{T}} \\ &= [\hat{U}(t), C(t)^{\mathrm{T}}]\mathbb{R}^{l+p} \quad \text{for all } t \in \mathbb{R}.\end{aligned} \tag{5.4}$$

Consider an interval $[a_j, a_{j+1})$ so that $[\hat{U}, C^{\mathrm{T}}]\restriction_{[a_j, a_{j+1})}$ is real analytic and can be real-analytically extended to $[\hat{U}, C^{\mathrm{T}}]_j$ on some $\mathcal{J}_j := (a_j^{\mathrm{L}}, a_{j+1}^{\mathrm{R}})$, where $a_j^{\mathrm{L}} < a_j < a_{j+1} < a_{j+1}^{\mathrm{R}}$. Then, by Lemma 2.1(i), there exists $\hat{G}_j \in \mathbb{A}\restriction_{\mathcal{J}_j}^{n\times(l+p)}$ with constant rank so that

$$[\hat{U}, C^{\mathrm{T}}]_j(t)\mathbb{R}^{l+p} \overset{\mathrm{ae}}{\subseteq} \hat{G}_j(t)\mathbb{R}^{l+p} \quad \text{on } \mathcal{J}_j.$$

Since $\mathrm{rk}_{\mathbb{M}\restriction_{\mathcal{J}_j}}[\hat{U}, C^{\mathrm{T}}] = \mathrm{rk}_{\mathbb{M}\restriction_{\mathcal{J}_j}} \hat{G}_j$, there exists $T_j \in \mathbb{A}\restriction_{\mathcal{J}_j}^{(l+p)\times(l+p)}$ so that $T_j \in GL_{l+p}(\mathbb{M}\restriction_{\mathcal{J}_j})$ and $[\hat{U}, C^{\mathrm{T}}]_j T_j^{-1} = \hat{G}_j$. Now it follows from (3.4) and Proposition 2.7(ii) that $\overline{V^{\perp}}$ is meromorphically $[-A^{\mathrm{T}}, C^{\mathrm{T}}]$-invariant.

Conversely, if $V^{\perp}$ is meromorphically $[-A^{\mathrm{T}}, C^{\mathrm{T}}]$-invariant, then

$$(\mathrm{D}I_n + A^{\mathrm{T}})(\hat{\boldsymbol{u}}) = \hat{U}\boldsymbol{m} + C^{\mathrm{T}}\boldsymbol{n}$$

for some $\boldsymbol{m} \in \mathbb{M}_{\mathrm{p}}^l$ and $\boldsymbol{n} \in \mathbb{M}_{\mathrm{p}}^p$. Thus (5.3) yields

$$\langle [\mathrm{D}I_n - A(t)](\hat{\boldsymbol{w}})(t), \hat{\boldsymbol{u}}(t)\rangle = 0 \quad \text{for all } t \in \mathbb{R};$$

whence $[\mathrm{D}I_n - A(t)](\hat{\boldsymbol{w}})(t) \in V(t)$ for all $t \in \mathbb{R}$. This completes the proof. □

For time-invariant systems $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, it is well known (see e.g. Wonham (1985; p. 91)) that the *maximal* $[A, B]$*-invariant subspace* V^* included in a subspace $L \subset \mathbb{R}^n$ can be determined as follows:

$$V^0 := L, \qquad V^i := L \cap A^{-1}(\operatorname{im} B + V^{i-1}) \quad (i \in \mathbb{N}).$$

This sequence is decreasing and stops after at most $k = \dim L$ steps, with $V^k = V^*$. It is not clear how to generalize this algorithm to time-varying systems and subspaces. Instead one can determine the smallest (C, A)-invariant family that contains a given family L, and use duality to obtain the largest meromorphic $[A, B]$-invariant family that is included in a given family $L^{\perp}$. For $\bar{V}_1, \bar{V}_2 \in \bar{\mathbf{W}}_n$, define

$$\bar{V}_1 < \bar{V}_2 \quad \text{if} \quad V_1(t) \overset{\mathrm{ae}}{\subseteq} V_2(t).$$

DEFINITION 5.5 Suppose $A \in \mathbb{A}_{\mathrm{p}}^{n\times n}$, $B \in \mathbb{A}_{\mathrm{p}}^{n\times m}$, and $C \in \mathbb{A}_{\mathrm{p}}^{p\times n}$. Then $\overline{V_*(L)} \in \bar{\mathbf{W}}_n$ is called the *smallest* (C, A)*-invariant family* that contains $\bar{L} \in \bar{\mathbf{W}}_n$ if the following conditions (i)–(iii) all hold:

(i) $\overline{V_*(L)}$ is (C, A)-invariant;

(ii) $\bar{L} < \overline{V_*(L)}$;

(iii) if $\bar{W} \in \bar{\mathbf{W}}_n$ is (C, A)-invariant and $\bar{L} < \bar{W}$, then $\overline{V_*(L)} < \bar{W}$.

$\overline{V^*(L')} \in \bar{\mathbf{W}}_n$ is called the *largest meromorphically* $[A, B]$*-invariant family* contained in $\bar{L}' \in \bar{\mathbf{W}}_n$ if the following conditions (i)′–(iii)′ all hold.

(i)′ $\overline{\mathcal{V}^*}(\mathcal{L}')$ is meromorphically $[A, B]$-invariant;
(ii)′ $\overline{\mathcal{V}^*}(\mathcal{L}) < \bar{\mathcal{L}}'$;
(iii)′ if $\bar{\mathcal{W}} \in \bar{\mathbf{W}}_n$ is meromorphically $[A, B]$-invariant and $\bar{\mathcal{W}} < \bar{\mathcal{L}}'$, then

$$\bar{\mathcal{W}} < \overline{\mathcal{V}^*}(\mathcal{L}').$$

To present an algorithm that determines $\overline{\mathcal{V}_*}(\mathcal{L})$, some notation is needed. Suppose that

$$\mathcal{W}(t) \overset{\text{ae}}{=} \hat{W}(t)\mathbb{R}^q \quad \text{for } \hat{W} \in \mathbb{A}_{\text{p}}^{n\times q} \text{ with p.c. rank},$$

$$\ker C(t) \overset{\text{ae}}{=} \hat{C}(t)\mathbb{R}^s \quad \text{for } \hat{C} \in \mathbb{A}_{\text{p}}^{n\times s} \text{ with p.c. rank}.$$

Then, by Lemma 9.3(ii), there exists $R \in \mathbb{A}_{\text{p}}^{q\times r_1}$ so that

$$\tilde{W} \overset{\text{ae}}{=} \hat{W}R \in \text{lcrm}_{\mathbb{A}_{\text{p}}}(\hat{W}, \hat{C}) \quad \text{for } \tilde{W} \in \mathbb{A}_{\text{p}}^{n\times r} \text{ with p.c. rank}.$$

Now, by Proposition 2.7(ii), for $w \in \mathbb{A}_{\text{p}}^n$ with

$$w(t) \in \mathcal{W}(t) \cap \ker C(t) \quad \text{for all } t \in \mathbb{R},$$

there exists $q \in \mathbb{A}_{\text{p}}^r$ such that

$$w(t) \overset{\text{ae}}{=} \tilde{W}(t)q(t) \overset{\text{ae}}{=} \hat{W}(t)R(t)q(t).$$

Thus it makes sense to define

$$(\mathrm{DI}_n - A)(\mathcal{W} \cap \ker C) := (\mathrm{DI}_n - A)(\hat{W}R)\mathbb{R}^{r_1}.$$

Proposition 5.6 Suppose that $A \in \mathbb{A}_{\text{p}}^{n\times n}$ and $C \in \mathbb{A}_{\text{p}}^{p\times n}$, and that $\mathcal{L} \in \mathbf{W}_n$ is generated by $L \in \mathbb{A}_{\text{p}}^{n\times q}$. Then the sequence $(\mathcal{W}_i : i \in \mathbb{N}_0)$, given by

$$\mathcal{L}_0 := \mathcal{L}, \qquad \mathcal{W}_{i+1} := \mathcal{W}_i + (\mathrm{DI}_n - A)(\mathcal{W}_i \cap \ker C) \quad (i \in \mathbb{N}_0), \tag{5.5}$$

is *increasing* in the sense that $\bar{\mathcal{W}}_i < \bar{\mathcal{W}}_{i+1}$ for $i \in \mathbb{N}_0$, and there exists $k \leq n$ so that

$$\overline{\mathcal{V}_*}(\mathcal{L}) = \bar{\mathcal{W}}_k = \bar{\mathcal{W}}_{k+l} \quad \text{for every } l \in \mathbb{N}.$$

Proof. Let $R_1 \in \mathbb{A}_{\text{p}}^{q\times r_1}$ be such that $LR_1 \in \text{lcrm}_{\mathbb{A}_{\text{p}}}(L, \hat{C})$; then

$$\mathcal{W}_1(t) \overset{\text{ae}}{=} L(t)\mathbb{R}^q + [\mathrm{DI}_n - A(t)](LR_1)(t)\mathbb{R}^{r_1} = W_1(t)\mathbb{R}^{q+r_1},$$

where $W_1 := [L, (\dot{L} - AL)R_1]$. Proceeding in this way, one obtains

$$\mathcal{W}_i(t) \overset{\text{ae}}{=} W_i(t)\mathbb{R}^{q+r_1+\cdots+r_i},$$

where $W_i := [W_{i-1}, (\dot{W}_{i-1} - AW_{i-1})R_i]$ and $W_{i-1}R_i \in \text{lcrm}_{\mathbb{A}_{\text{p}}}(W_{i-1}, \hat{C})$. Therefore $\bar{\mathcal{W}}_i < \bar{\mathcal{W}}_{i+1}$ for $i \in \mathbb{N}_0$. Then, if $\text{rk}_{\mathbb{M}} W_k = \text{rk}_{\mathbb{M}} W_{k+1}$ for some $k \in \mathbb{N}_0$, one gets $\bar{\mathcal{W}}_k = \bar{\mathcal{W}}_{k+l}$ for all $l \in \mathbb{N}$. By construction, $\bar{\mathcal{W}}_k$ is (C, A)-invariant and $\bar{\mathcal{L}} < \bar{\mathcal{W}}_k$. So it remains to prove that, if $\bar{\mathcal{V}}$ is (C, A)-invariant and $\bar{\mathcal{L}} < \bar{\mathcal{V}}$, then $\bar{\mathcal{W}}_k < \bar{\mathcal{V}}$. By assumption,

$$(\mathrm{DI}_n - A)(\mathcal{V} \cap \ker C) \overset{\text{ae}}{\subseteq} \mathcal{V},$$

and induction on i gives

$$\mathcal{W}_i = \mathcal{W}_{i-1} + (\mathrm{D}I_n - A)(\mathcal{W}_{i-1} \cap \ker C) < \mathcal{V} + (\mathrm{D}I_n - A)\mathcal{V}.$$

This completes the proof. □

The duality between the smallest (C, A)-invariant family and largest meromorphically $[A, B]$-invariant family is given as follows.

PROPOSITION 5.7 Suppose $A \in \mathbb{A}_{\mathrm{p}}^{n\times n}$ and $C \in \mathbb{A}_{\mathrm{p}}^{p\times n}$. If $\bar{\mathcal{L}} \subset \bar{\mathcal{W}}_n$ and some $\mathcal{L} \in \bar{\mathcal{L}}$ is generated by $L \in \mathbb{A}_{\mathrm{p}}^{n\times k}$, then the following are equivalent.

(i) $\bar{\mathcal{V}}$ is the smallest (C, A)-invariant family containing $\bar{\mathcal{L}}$.

(ii) $\overline{\mathcal{V}^\perp}$ is the largest meromorphically $[-A^{\mathsf{T}}, C^{\mathsf{T}}]$-invariant family included in $\overline{\mathcal{L}^\perp}$.

Proof. (i)⇒(ii): If $\mathcal{W}$ is a representative of the largest meromorphically $[-A^{\mathsf{T}}, C^{\mathsf{T}}]$-invariant family that is included in $\mathcal{L}^\perp$, then

$$\mathcal{V}^\perp(t) \overset{\text{ae}}{\subset} \mathcal{W}(t) \overset{\text{ae}}{\subset} \mathcal{L}^\perp(t)$$

and thus

$$\mathcal{L}(t) \overset{\text{ae}}{\subset} \mathcal{W}^\perp(t) \overset{\text{ae}}{\subset} \mathcal{V}(t).$$

Since $\bar{\mathcal{V}}$ is the smallest (C, A)-invariant family, it follows that $\bar{\mathcal{W}} = \bar{\mathcal{V}}$. The reverse direction is proved analogously. □

Remark 5.8. For a real analytic system $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$, it is demonstrated how to determine $\overline{\mathcal{V}^*}(\ker C(t))$: By Proposition 5.7 this problem is equivalent to determine $\overline{\mathcal{V}_*}$, the smallest $(B^{\mathsf{T}}, -A^{\mathsf{T}})$-invariant family containing $\overline{[\ker C(t)]^\perp}$. Let

$$[\ker C(t)]^\perp \overset{\subset}{\underset{\text{ae}}{=}} L(t)\mathbb{R}^q \quad \text{for some } \hat{C} \in \mathbb{A}^{n\times q} \text{ with } \mathrm{rk}_{\mathbb{R}}\, L(t) = q \;\forall\, t \in \mathbb{R}, \tag{5.6}$$

$$\ker B^{\mathsf{T}}(t) \overset{\supset}{\underset{\text{ae}}{=}} \hat{C}(t)\mathbb{R}^s \quad \text{for some } \hat{C} \in \mathbb{A}^{n\times s} \text{ with } \mathrm{rk}_{\mathbb{R}}\, \hat{C}(t) = s \;\forall\, t \in \mathbb{R}. \tag{5.7}$$

Applying algorithm (5.5) yields

$$[\ker C(t)]^\perp \subset L(t)\mathbb{R}^q \subset \mathcal{W}_{i-1}(t) \subset \mathcal{W}_i(t) \subset \hat{V}(t)\mathbb{R}^{n-k}$$

for some $\hat{V} \in \mathbb{A}^{n\times(n-k)}$, with $\mathrm{rk}_{\mathbb{R}}\, \hat{V}(t) = n-k$ for all $t \in \mathbb{R}$, satisfying

$$\overline{(\hat{V}(t)\mathbb{R}^{n-k})_{t\in\mathbb{R}}} = \bar{\mathcal{W}}_n.$$

Thus

$$\ker C(t) \supset [L(t)\mathbb{R}^q]^\perp \supset \mathcal{W}_{i-1}^\perp \supset \mathcal{W}_i^\perp(t) \supset V(t)\mathbb{R}^k$$

for $V \in \mathbb{A}^{n\times k}$ such that $[\hat{V}(t)\mathbb{R}^{n-k}]^\perp = V(t)\mathbb{R}^k$ for all $t \in \mathbb{R}$. By Proposition 5.7, it follows that $\bar{\mathcal{V}} = \overline{\mathcal{V}^*}(\ker C(t))$.

EXAMPLE 5.9 $\overline{\mathcal{V}^*}(\ker C)$ will be calculated for a system (A, B, C), where $A \in \mathbb{A}^{3\times 3}$ and $B \in \mathbb{A}^{3\times 1}$ are as in Example 4.11 and $C(t) := [1, 0, t^2]$. Using the notation of Remark 5.8, one has

$$L(t) = \begin{bmatrix} 1 \\ 0 \\ t^2 \end{bmatrix}, \qquad \hat{C}(t) = \begin{bmatrix} 1 & 1 \\ 0 & t-2 \\ t^2 & 0 \end{bmatrix},$$

which satisfy (5.6) and (5.7). For this situation, the algorithm

$$\mathcal{W}_0 = L\mathbb{R}, \qquad \mathcal{W}_i = \mathcal{W}_{i-1} + (\mathbf{D}\mathbf{I}_n + A^{\mathrm{T}})(\mathcal{W}_{i-1} \cap \hat{C}\mathbb{R}^2) \quad (i \in \mathbb{N}),$$

is as follows:

$$\mathcal{W}_1(t) = \begin{bmatrix} 1 \\ 0 \\ t^2 \end{bmatrix}\mathbb{R} + \left(\frac{\mathrm{d}}{\mathrm{d}t} + A^{\mathrm{T}}(t)\right)\left(\begin{bmatrix} 1 \\ 0 \\ t^2 \end{bmatrix}\right)\mathbb{R} = \begin{bmatrix} 1 & 0 \\ 0 & \sin t \\ t^2 & 0 \end{bmatrix}\mathbb{R}^2,$$

$$\mathcal{W}_2(t) = \mathcal{W}_1(t) + \left(\frac{\mathrm{d}}{\mathrm{d}t} + A^{\mathrm{T}}(t)\right)([1, 0, t^2]^{\mathrm{T}})\mathbb{R} = \mathcal{W}_i(t) \quad \text{for all } i \geq 1.$$

Therefore $\overline{\mathcal{V}_*}$, given by

$$\mathcal{V}_*(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ t^2 & 0 \end{bmatrix}\mathbb{R}^2,$$

is the smallest $(B^{\mathrm{T}}, -A^{\mathrm{T}})$-invariant family containing $\overline{[\ker C(t)]^{\perp}}$. Thus $\overline{\mathcal{V}^*}$ is given by

$$\mathcal{V}^*(t) = [-t^2, 0, 1]^{\mathrm{T}}\mathbb{R}.$$

Now we are in a position to prove the main result of this section, which is a summary of the foregoing.

PROPOSITION 5.10 (see Fig. 1) The set $\mathbf{S}_{(C, A)}$, consisting of the (C, A)-invariant families of $\overline{\mathbf{W}}_n$, is a lattice with respect to the operations

$$\overline{\mathcal{V}_1} \wedge \overline{\mathcal{V}_2} = \overline{\mathcal{V}_1 \cap \mathcal{V}_2}, \qquad \overline{\mathcal{V}_1} \vee \overline{\mathcal{V}_2} = \overline{\mathcal{V}_*}(\mathcal{V}_1 + \mathcal{V}_2),$$

where $\overline{\mathcal{V}_*}(\mathcal{V}_1 + \mathcal{V}_2)$ denotes the smallest (C, A)-invariant family that contains $\overline{\mathcal{V}_1 + \mathcal{V}_2}$. Furthermore the set $\mathbf{S}_{[A, B]}$, consisting of the meromorphically $[A, B]$-

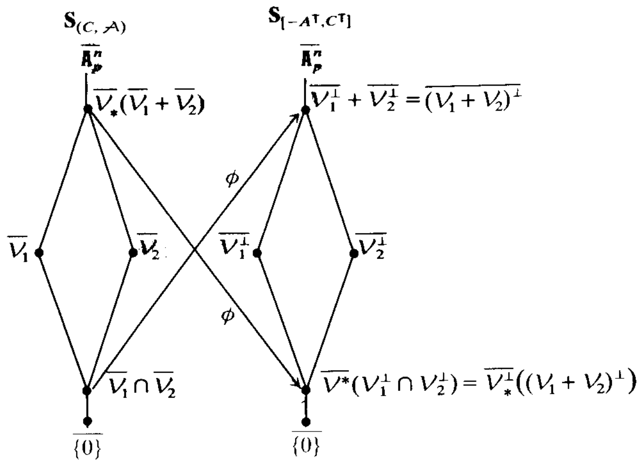

FIG. 1

invariant families of $\bar{\mathbf{W}}_n$, is a lattice with respect to the operations

$$\overline{V_1} \wedge \overline{V_2} = \overline{V^*(V_1 \cap V_2)}, \qquad \overline{V_1} \vee \overline{V_2} = \overline{V_1 + V_2},$$

where $\overline{V^*(V_1 \cap V_2)}$ denotes the largest meromorphically $[A, B]$-invariant family which is included in $V_1 \cap V_2$. The map

$$\phi : \mathbf{S}_{(C,A)} \to \mathbf{S}_{[-A^\top, C^\top]} : \bar{V} \mapsto \overline{V^\perp}$$

is a lattice anti-isomorphism, where 'anti' means

$$\phi(\overline{V_1} \wedge \overline{V_2}) = \phi(\overline{V_1}) \vee \phi(\overline{V_2}), \qquad \phi(\overline{V_1} \vee \overline{V_2}) = \phi(\overline{V_1}) \wedge \phi(\overline{V_2}).$$

Proof. It is easily seen that the definition of the lattice operations does not depend on the representatives. It remains to prove that ϕ is an anti-isomorphism. Using the fact that $[V_1(t) \cap V_2(t)]^\perp = V_1(t)^\perp + V_2(t)^\perp$ holds for finite dimensional vector spaces, we obtain

$$\begin{aligned}\phi(\overline{V_1} \wedge \overline{V_2}) &= \phi(\overline{V_1 \cap V_2}) = \overline{(V_1 \cap V_2)^\perp} = \overline{V_1^\perp + V_2^\perp} \\ &= \overline{\phi(V_1) + \phi(V_2)} = \phi(\overline{V_1}) \vee \phi(\overline{V_2}).\end{aligned}$$

This proves the first equation of the anti-isomorphism. To prove the second one, use Proposition 5.7 to conclude

$$\begin{aligned}\phi(\overline{V_1} \vee \overline{V_2}) &= \phi(\overline{V_*(V_1 + V_2)}) = \overline{V_*^\perp(V_1 + V_2)} \\ &= \overline{V^*((V_1 + V_2)^\perp)} = \overline{V^*(V_1^\perp \cap V_2^\perp)} = \phi(\overline{V_1}) \wedge \phi(\overline{V_2}).\end{aligned}$$

6. Disturbance-decoupling problem

In this section, we consider a system $(A, B, C) \in \mathbb{A}_p^{n\times n} \times \mathbb{A}_p^{n\times m} \times \mathbb{A}_p^{p\times n}$ with an additional disturbance $\boldsymbol{q} \in \mathbb{C}_p^s$ entering the system via $S \in \mathbb{A}_p^{n\times s}$:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t) + S(t)\boldsymbol{q}(t), \\ \boldsymbol{y}(t) &= C(t)\boldsymbol{x}(t).\end{aligned} \tag{6.1}$$

The *disturbance-decoupling problem* (DDP) on (t_0, t_1) is to find a state-feedback matrix $F \in \mathbb{A}_p^{m\times n}$ such that arbitrary $\boldsymbol{q}$ has no influence on the output $\boldsymbol{y}$, given by

$$\boldsymbol{y}(t) = C(t)\left(\Phi_F(t, , t_0)\boldsymbol{x}_0 + \int_{t_0}^{t} \Phi_F(t, s)S(s)\boldsymbol{q}(s)\,\mathrm{d}s\right) \quad \text{for all } t \in (t_0, t_1),$$

of the closed-loop system

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= A_F(t)\boldsymbol{x}(t) + S(t)\boldsymbol{q}(t), \\ \boldsymbol{y}(t) &= C(t)\boldsymbol{x}(t),\end{aligned} \tag{6.2}$$

where $A_F(t) := A(t) + B(t)F(t)$ and $\Phi_F(\bullet, \bullet)$ denotes the transition matrix of

$$\dot{\boldsymbol{x}}(t) = A_F(t)\boldsymbol{x}(t).$$

This notation will be used throughout the remaining sections. The following definition is an extension of the time-invariant case; see Wonham (1985).

DEFINITION 6.1 The system (6.1) is called *disturbance-decoupled on* (t_0, t_1) if, for some $F \in \mathbb{A}_p^{m\times n}$, we have

$$y(t) = C(t)\int_{t_0}^{t} \Phi_F(t, s)S(s)q(s)\,ds = 0 \quad \text{for all } t \in (t_0, t_1) \text{ and arbitrary } q \in \mathbb{C}_p^s. \tag{6.3}$$

Using the controllability Gramian of the closed-loop system (6.2), namely

$$W_F(t_0, t_1) := \int_{t_0}^{t} \Phi_F(t_0, s)S(s)S^{\mathrm{T}}(s)\Phi_F^{\mathrm{T}}(t_0, s)\,ds,$$

the DDP for piecewise analytic systems can be characterized as follows.

PROPOSITION 6.2 Let $\mathcal{J} = (t_0, t_1)$ and $F \in \mathbb{A}_p^{m\times n}$. Let $\{[a_j, a_{j+1}) : j \in \mathbb{Z}\}$ be a partition of $\mathbb{R}$ so that A_F and S are real analytic on $(a_j, a_{j+1}) \cap \mathcal{J}$. Then the following are equivalent.

(i) (6.1) is disturbance-decoupled on $\mathcal{J}$ by F.

(ii) $\Phi_F(t, t_0)\,\mathrm{im}\,W_F(t_0, t) \subset \ker C(t)$ for all $t \in \mathcal{J}$.

(iii)

$$\Phi_F(t, t_0)\Big(\sum_{i\geq 0} \mathrm{im}\,[\mathrm{DI}_n - A_F(t_0)]^i(S)(t_0) + \sum_{j=1}^{N}\sum_{i\geq 0} \mathrm{im}\,[\mathrm{DI}_n - A_F(a_j)]^i(S)(a_j)\Big) \subset \ker C(t)$$

for all $t \in \mathcal{J}$, where $t_0 \in [a_0, a_1)$ and $t \in (a_N, a_{N+1})$.

Proof. Consider the map

$$L_{t_0,t} : \mathbb{C}_p^s \to \mathbb{R}^n : q \mapsto \int_{t_0}^{t} \Phi_F(t, s)S(s)q(s)\,ds$$

By Knobloch & Kappel (1974: p. 103), it follows that

$$\begin{aligned}\mathrm{im}\,L_{t_0,t} &= \mathrm{im}\int_{t_0}^{t} \Phi_F(t, s)S(s)S^{\mathrm{T}}(t)\Phi_F^{\mathrm{T}}(t, s)\,ds\\ &= \Phi_F(t, t_0)\,\mathrm{im}\,W_F(t_0, t).\end{aligned}$$

This proves (i)$\Leftrightarrow$(ii). The equivalence of (ii) and (iii) is a consequence of Proposition 3.2. □

If the system (6.1) is analytic, then Proposition 3.2 yields the following simple result.

COROLLARY 6.3 An analytic system (6.1) is disturbance-decoupled on $\mathcal{J}$ by $F \in \mathbb{A}\!\restriction_{\mathcal{J}}^{m\times n}$ iff

$$\sum_{i\geq 0} \mathrm{im}\,[\mathrm{DI}_n - A_F(t)]^i(S)(t) \subset \ker C(t) \quad \text{for all } t \in \mathcal{J}. \quad \square$$

Because of the identity theorem for analytic functions, we need to check condition (6.3), for an analytic system (6.1), only on an arbitrary small interval $(t_0, t_0 + \varepsilon)$. More precisely, we have the following result.

PROPOSITION 6.4 Suppose that (6.1) is analytic and that $F \in \mathbb{A}\lceil_{\mathcal{J}}^{m\times n}$, where $\mathcal{J} = (t_0, t_1)$; then the following are equivalent.

(i) (6.1) is disturbance-decoupled on $\mathcal{J}$ by F.

(ii) 6.1 is disturbance-decoupled on $(t_0, t_0 + \varepsilon)$ by F, for arbitrary $\varepsilon \in (0, t_1 - t_0)$.

Proof. Ilchmann (1989: Proposition 1.2.2), shows that (6.3) does not depend on whether we admit a piecewise continuous or an analytic disturbance. Since the vector function

$$t \mapsto \boldsymbol{\varphi}(t, \boldsymbol{q}) := \int_{t_0}^{t} \Phi_F(t, s) S(s) \boldsymbol{q}(s)\, \mathrm{d}s$$

is real analytic on $\mathcal{J}$ for every $\boldsymbol{q} \in \mathbb{A}\lceil_{\mathcal{J}}^{s}$, the identity theorem for analytic functions yields that $C(t)\varphi(t, \boldsymbol{q}) = 0$ for all $t \in \mathcal{J}$ if and only if $C(t)\varphi(t, \boldsymbol{q}) = 0$ for all $t \in (t_0, t_0 + \varepsilon)$, for some $\varepsilon \in (0, t_1 - t_0)$. This proves the proposition. □

For an analytic system (6.1), the largest meromorphically $[A, B]$-invariant subspace $\mathcal{V}^*(\ker C)$ included in $\ker C(t)$ with generator $V \in \mathbb{A}^{n\times k}$ of constant rank k was constructed in Remark 5.8. By Proposition 4.10, one obtains

$$(\mathrm{DI}_n - A_F)(V) = V U_1 T^{-1} W,$$

where $F = U_2 T^{-1} W (V^{\mathrm{T}} V)^{-1} V^{\mathrm{T}}$. Thus the set of *critical points* for the feedback F is given by

$$\mathcal{P} = \{t' \in \mathbb{R} : \text{an entry of } U_2 T^{-1} W \text{ has a pole at } t'\}.$$

Let $\mathcal{J} \subset \mathbb{R}$ be an open interval. Then F is analytic on $\mathcal{J}$ if $\mathcal{P} \cap \mathcal{J} = \varnothing$; furthermore, by Proposition 4.10, $\mathcal{P} \cap \mathcal{J} = \varnothing$ if $\mathrm{rk}_{\mathbb{R}}\, [V(t), B(t)]$ is constant for all $t \in \mathcal{J}$. Now, for every $\mathcal{J} \subset \mathbb{R} \setminus \mathcal{P}$, the differential equation

$$\dot{\boldsymbol{x}}(t) = A_F(t)\boldsymbol{x}(t) \quad (t \in \mathcal{J})$$

in solvable on $\mathcal{J}$. This sets us in a position to state the main result of this section, which is a generalization of the constant case; see Wonham (1985: Thm 4.2).

THEOREM 6.5 Suppose that the system (6.1) is analytic and that $V \in \mathbb{A}^{n\times k}$ with $\mathrm{rk}_{\mathbb{R}}\, V(t) = k$ for all $t \in \mathbb{R}$ generates $\mathcal{V}^*(\ker C)$ constructed in Remark 5.8. Then, for $\mathcal{J} = (t_0, t_1)$, the following statements hold.

(i) If the DDP is solvable on $\mathcal{J}$ by $F \in \mathbb{A}\lceil_{\mathcal{J}}^{m\times n}$, then $S(t)\mathbb{R}^s \subset V(t)\mathbb{R}^k$ for all $t \in \mathcal{J}$.

(ii) If $\bar{\mathcal{J}} \subset \mathbb{R} \setminus \mathcal{P}$ and $S(t)\mathbb{R}^s \subset V(t)\mathbb{R}^k$ for all $t \in \mathcal{J}$, then the DDP is solvable on $\mathcal{J}$ by $F \in \mathbb{A}\lceil_{\mathcal{J}}^{m\times n}$ given in (6.4).

Proof. (i) By Corollary 6.3,

$$\mathrm{im}\, S(t) \subset \sum_{i\geq 0} \mathrm{im}\, [\mathrm{DI}_n - A_F(t)]^i(S)(t) \subset \ker C(t) \quad \text{for all } t \in \mathcal{J}.$$

By Corollary 3.3, there exists $\tilde{V} \in \mathbb{A}\lceil_{\mathcal{J}}^{n\times s}$ with constant rank on $\mathcal{J}$ so that

$$\tilde{\mathcal{V}}(t) := \tilde{V}(t)\mathbb{R}^s = \sum_{i\geq 0} \mathrm{im}\, [\mathrm{DI}_n - A_F(t)]^i S(t) \quad \text{for all } t \in \mathcal{J}.$$

Thus Theorem 4.5 yields that $\tilde{V}$ is $[A, B]$-invariant on $\mathcal{J}$. This, together with $\tilde{V}(t) \subset \ker C(t)$ for all $t \in \mathbb{R}$, gives

$$\tilde{V}(t)\mathbb{R}^s \overset{\text{ae}}{\subset} V(t)\mathbb{R}^k \quad \text{for all } t \in \mathcal{J}.$$

Since $V(t)$ has constant rank on $\mathcal{J}$, one gets

$$S(t)\mathbb{R}^s \subset \tilde{V}(t)\mathbb{R}^s \subset V(t)\mathbb{R}^k \quad \text{for all } t \in \mathcal{J},$$

which proves (i).

(ii) Since $S(t)\mathbb{R}^s \subset V(t)\mathbb{R}^k$ for all $t \in \mathcal{J}$ and $\left(V(t)\mathbb{R}^k\right)_{t\in\mathbb{R}}$ is A_F-invariant, one obtains

$$\begin{aligned} \operatorname{im} S(t) &\subset \sum_{i\geq 0} \operatorname{im} [\mathrm{D}I_n - A_F(t)]^i(S)(t) \\ &\subset \sum_{i\geq 0} \operatorname{im} [\mathrm{D}I_n - A_F(t)]^i(V)(t) \\ &= V(t)\mathbb{R}^k \subset \ker C(t) \end{aligned}$$

for all $t \in \mathcal{J}$. Now (ii) follows from Corollary 6.3. □

Remark 6.6. If the entries of the matrices of (6.1) belong to the ring $\mathbb{R}[t]$, then it is computationally not too expensive to check the assumptions of Theorem 6.5(ii). The main tool is to transform a matrix into an upper triangular form.This algorithm is described in detail in, for instance, Wolovich (1979). It can be implemented via the algebraic programming system REDUCE (see Hearn (1985)).

EXAMPLE 6.7 Consider a system of the form (6.1) specified by

$$A(t) = \begin{bmatrix} t_2 & \sin t & -t(t^2+2) \\ a_4(t) & a_5(t) & a_6(t) \\ -1 & 0 & t \end{bmatrix}, \qquad B(t) = \begin{bmatrix} -t^2(t-2) \\ t^2 \\ t-2 \end{bmatrix}, \qquad S(t) = \begin{bmatrix} t^3 \\ 0 \\ -t \end{bmatrix},$$

$C(t) = [1, 0, t^2]$.

By Example 5.9, $V(t) = [-t^2, 0, 1]^{\mathrm{T}}$ is a generator of $\mathcal{V}^*(\ker C)$. By Example 4.11, the set of critical points is $\mathcal{P} = \{0\}$. Since $\operatorname{im} S(t) \subset \mathcal{V}^*(t)$ for all $t \in \mathbb{R}$, Theorem 6.5(ii) says that the distrubance-decoupling problem is solvable on every open interval $\mathcal{J} \subset \mathbb{R}$ with $0 \notin \bar{\mathcal{J}}$.

7. Controllability-subspace families

In this section, the concept of controllability subspaces (see Wonham (1985)) is extended to analytic time-varying system $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$.

DEFINITION 7.1 A family of subspaces $\mathcal{V} \in \mathbf{W}_n$ generated by $V \in \mathbb{A}^{n\times k}$ is called a *controllability subspace family* (c.s.f.) of $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$ if (i) $\mathcal{V}$ is $[A, B]$-invariant and (ii) for every $\boldsymbol{x}_0 \in \mathcal{V}(t_0)$ and $\boldsymbol{x}_1 \in \mathcal{V}(t_1)$, with $t_0 < t_1$, there exists a control $\boldsymbol{u} \in \mathbb{C}_{\mathrm{p}}^m$ such that the forced trajectory of $\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t)$ satisfies

$$\boldsymbol{x}(t) \in \mathcal{V}(t) \quad \text{for all } t \in (t_0, t_1), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \qquad \boldsymbol{x}(t_1) = \boldsymbol{x}_1. \tag{7.1}$$

In the case of time-invariant systems, (ii) implies (i); see Wonham (1985: Section 5.1).

EXAMPLE 7.2 The controllable family $\mathscr{R}$ of $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$ is a c.s.f. In Example 4.9, it is shown that $\mathscr{R}$ is A-invariant, whence it is $[A, B]$-invariant. For $\boldsymbol{x}_0 \in \mathscr{R}(t_0)$ and $\boldsymbol{x}_1 \in \mathscr{R}(t_1)$, a control $\boldsymbol{u} \in \mathbb{C}_p^m$ satisfying (7.1) can be constructed as follows. Set

$$\hat{\boldsymbol{x}}_1 := \Phi(t_0, t_1)\boldsymbol{x}_1 \in \mathscr{R}(t_0), \qquad \boldsymbol{x}_\Delta := \boldsymbol{x}_0 - \hat{\boldsymbol{x}}_1 \in \mathscr{R}(t_0).$$

Choose $\boldsymbol{u} \in \mathbb{C}_p^m$ such that

$$\hat{\boldsymbol{x}}(t) := \Phi(t, t_0)x_\Delta + \int_{t_0}^{t} \Phi(t, s)B(s)\boldsymbol{u}(s)\,\mathrm{d}s$$

fulfills $\hat{\boldsymbol{x}}(t_1) = \mathbf{0}$ and $\hat{\boldsymbol{x}}(t) \in \mathscr{R}(t)$ for all $t \in [t_0, t_1]$. Thus

$$\boldsymbol{x}(t) := \Phi(t, t_0)\boldsymbol{x}_0 + \int_{t_0}^{t} \Phi(t, s)B(s)\boldsymbol{u}(s)\,\mathrm{d}s = \Phi(t, t_1)\boldsymbol{x}_1 + \hat{\boldsymbol{x}}(t),$$

which is in $\mathscr{R}(t)$, with $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ and $\boldsymbol{x}(t_1) = \boldsymbol{x}_1$.

PROPOSITION 7.3 Suppose that $\mathcal{V}(t) = V(t)\mathbb{R}^k$ for some $V \in \mathbb{A}^{n\times k}$ with $\mathrm{rk}_{\mathbb{R}}\, V(t) = k$ for all $t \in \mathbb{R}$. Then $\mathcal{V}$ is a c.s.f. of $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$ if and only if

$$\mathcal{V}(t) = \sum_{i\geq 0} \mathrm{im}\,[\mathrm{D}I_n - A_F(t)]^i BG(t) \quad \text{for all } t \in \mathbb{R} \tag{7.2}$$

for some $F \in \mathbb{A}^{m\times n}$ and $G \in \mathbb{A}^{m\times m}$.

Proof. Assume (7.2). Then, for given $\boldsymbol{x}_0 \in \mathcal{V}(t_0)$ and $\boldsymbol{x}_1 \in \mathcal{V}(t_1)$, Example 7.2 shows there is some $\hat{\boldsymbol{u}} \in \mathbb{C}_p^m$ such that

$$\begin{gathered} \dot{\boldsymbol{x}}(t) = A_F(t)\boldsymbol{x}(t) + BG(t)\hat{\boldsymbol{u}}(t), \\ \boldsymbol{x}(t) \in \mathcal{V}(t) \quad \text{for } t \in [t_0, t_1], \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \quad \boldsymbol{x}(t_1) = \boldsymbol{x}_1. \end{gathered} \tag{7.3}$$

Thus condition (ii) of Definition 7.1 is satisfied. $\mathcal{V}$ is $[A, B]$-invariant since it is the controllable family of the system (7.3); see Example 4.9.

To prove the converse, let $F \in \mathbb{A}^{m\times n}$ be such that $(\mathrm{D}I_n - A_F)(V) = VN$ for some $N \in \mathbb{A}^{k\times n}$. Then $\mathcal{V}$ is also a c.s.f. of the system

$$\dot{\boldsymbol{x}} = A_F(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t).$$

By Lemma 9.1(i), choose $G \in \mathbb{A}^{m\times m}$ and $L \in \mathbb{A}^{k\times m}$ such that

$$BG = VL \in \mathrm{lclm}\,(B, V).$$

This proves '$\subset$' in (7.2). For the opposite inclusion, let $\mathscr{R}(t)$ denote the controllable family of (7.3). Clearly $\mathcal{V}(t) \subset \mathscr{R}(t)$ and, since $\mathscr{R}(t)$ can be represented by the right-hand side of (7.2) (see (3.11)), the proof is complete. □

Proposition 7.3 and the following proposition are generalizations of the constant case; see Wonham (1985; p. 104).

PROPOSITION 7.4 Suppose that $\mathcal{V} \in \mathbf{W}_n$ is generated by $V \in \mathbb{A}^{n\times k}$ with constant rank k. If $\mathcal{V}$ is a c.s.f. of $(A, B) \in \mathbb{A}^{n\times n} \times \mathbb{A}^{n\times m}$ and $\operatorname{im} B(t) \cap \mathcal{V}(t) = \operatorname{im} BG(t)$ for some $G \in \mathbb{A}^{m\times n}$, then

$$\mathcal{V}(t) = \sum_{i\geq 0} \operatorname{im}\,[\mathrm{DI}_n - A_F(t)]^i (BG)(t) \quad \text{for all } t \in \mathbb{R},$$

for any $F \in \mathbb{A}^{m\times n}$ that satisfies

$$(\mathrm{DI}_n - A_F)V = VN \quad \text{for some } N \in \mathbb{A}^{k\times k}. \tag{7.4}$$

Proof. By Proposition 7.3, there exists an $F_0 \in \mathbb{A}^{m\times n}$ so that

$$\mathcal{V}(t) = \sum_{i\geq 0} \operatorname{im}\,[\mathrm{DI}_n - A_{F_0}(t)]^i (BG)(t).$$

Put

$$\mathcal{V}'(t) := \sum_{i\geq 0} \operatorname{im}\,[\mathrm{DI}_n - A_F(t)]^i (BG)(t)$$

for some F that satisfies (7.4), then $\mathcal{V}'(t) \subset \mathcal{V}(t)$. For the reverse inclusion it is sufficient to show that $\mathcal{V}'$ is $(A + BF)$-invariant. This is proved completely analogously to the constant case; see Wonham (1985: p. 105.) □

8. Noninteracting control

Consider a state-space system with several outputs:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t), \\ \boldsymbol{z}_i(t) &= C_i(t)\boldsymbol{x}(t) \quad (i \in \underline{k}), \end{aligned} \tag{8.1}$$

where A, B, and C_i are respectively $n \times n$, $n \times m$, and $p_i \times n$ matrix functions defined over $\mathbb{A}$. The *restricted decoupling problem* (RDP) for (8.1) is to find a $F \in \mathbb{A}^{m\times n}$ and c.s.f.'s $\mathcal{V} \in \mathbf{W}_n$ $(i \in \underline{k})$ such that the following conditions are satisfied for all $t \in \mathbb{R}$ and $i \in \underline{k}$:

$$\mathcal{V}_i(t) = \sum_{\lambda\geq 0} \operatorname{im}\,[\mathrm{DI}_n - A_F(t)]^\lambda (BG_i)(t), \tag{8.2}$$

where $G_i \in \mathbb{A}^{m\times n}$ is such that $\operatorname{im} BG_i(t) = \operatorname{im} B(t) \cap \mathcal{V}_i(t)$,

$$C_j(t)\mathcal{V}_i(t) = 0 \quad \text{for } i \neq j, \qquad C_i(t)\mathcal{V}_i(t) = \operatorname{im} C_i(t). \tag{8.3a,b}$$

(8.3a) is called the *noninteraction* condition and is equivalent to

$$\mathcal{V}_i(t) \subset \bigcap_{j\neq i} \ker C_j(t). \tag{8.4}$$

(8.3b) is called the *output controllability* condition and is equivalent to

$$\mathcal{V}_i(t) + \ker C_i(t) = \mathbb{R}^n. \tag{8.5}$$

(8.2) is referred to as the *compatibility* condition of the families $\mathcal{V}_i$.

DEFINITION 8.1 Some families $\mathcal{V}_i \in \mathbf{W}_n$ with respective generators $V_i \in \mathbb{A}^{n\times r_i}$ $(i \in \underline{k})$ are called *compatible* relative to (8.1) if there exist $F \in \mathbb{A}^{m\times n}$ and $N_i \in \mathbb{A}^{r_i\times r_i}$ so that

$$(\mathrm{DI}_n - A_F)(\mathcal{V}_i) = V_i N_i \quad \text{for } i \in \underline{k}. \tag{8.6}$$

LEMMA 8.2 Suppose that $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_1 \cap \mathcal{V}_2$ in $\mathbf{W}_n$ are generated respectively by $V_i \in \mathbb{A}^{n\times r_i}$, with $\mathrm{rk}_{\mathbb{R}}\, V_i(t)$ constant for all $t \in \mathbb{R}$, for $i \in \underline{3}$. If there exist $F_i \in \mathbb{A}^{m\times n}$ and $N_i \in \mathbb{A}^{r_i\times r_i}$ so that

$$(\mathrm{DI}_n - A_{F_i})(\mathcal{V}_i) = V_i N_i \quad \text{for } i \in \underline{3},$$

then $\mathcal{V}_1$ and $\mathcal{V}_2$ are compatible.

Proof. Let

$$P_i(t) : \mathbb{R} \to \mathcal{V}_i(t) \backslash [\mathcal{V}_1(t) \cap \mathcal{V}_2(t)] \quad (i = 1, 2) \quad \text{and} \quad P_3(t) : \mathbb{R} \to \mathcal{V}_1(t) \cap \mathcal{V}_2(t)$$

denote the orthogonal projections on $\mathcal{V}_i(t) \backslash [\mathcal{V}_1(t) \cap \mathcal{V}_2(t)]$ and $\mathcal{V}_1(t) \cap \mathcal{V}_2(t)$ respectively. Then, by the assumptions and Proposition 2.3, it follows that $P_1 \in \mathbb{A}^{m\times n}$ for $i \in \underline{3}$. Thus, for

$$F := F_1 P_1 + F_2 P_2 + F_3 P_3 \in \mathbb{A}^{m\times n},$$

(8.6) is satisfied. □

DEFINITION 8.3 Some families $\mathcal{V}_i \in \mathbf{W}_n$ $(i \in \underline{k})$ are called *independent* if

$$\mathcal{V}_i(t) \cap \sum_{j\neq i}^{k} \mathcal{V}_j = \{0\} \quad \text{for all } i \in \underline{k}.$$

LEMMA 8.4 For $i \in \underline{k}$, let $\mathcal{V}_i \in \mathbf{W}_n$ be generated by $V_i \in \mathbb{A}^{n\times r_i}$ with $\mathrm{rk}_{\mathbb{R}}\, V_i(t)$ constant for all $t \in \mathbb{R}$. If the families $\mathcal{V}_i$ are independent and

$$(\mathrm{DI}_n - A_{F_i})(\mathcal{V}_i) = V_i N_i$$

for some $F_i \in \mathbb{A}^{m\times n}$ and $N_i \in \mathbb{A}^{r_i\times r_i}$, for $i \in \underline{k}$, then $\mathcal{V}_i$ are compatible.

Proof. Since $\mathcal{V}_i$ are independent, there exists a $\mathcal{Y} \in \mathbf{W}_n$ such that

$$\mathbb{R}^n = \mathcal{V}_1(t) \oplus \cdots \oplus \mathcal{V}_k(t) \oplus \mathcal{Y}(t) \quad \text{for all } t \in \mathbb{R}.$$

According to this decomposition, we define

$$F(t) : \mathbb{R}^n \to \mathbb{R}^n : \sum_{i=1}^{k} \boldsymbol{v}_i(t) + \boldsymbol{y}(t) \mapsto \sum_{i=1}^{k} F_i(t)\boldsymbol{v}_i(t).$$

Since $\mathcal{V}_i$ have constant dimensions, $F \in \mathbb{A}^{m\times n}$. Thus F satisfies (8.6). □

Using the previous lemmas, we are now in a position to prove the main result of this section, i.e. a characterization of the RDP which is a generalization of the constant case given in Wonham (1985: §9.3).

PROPOSITION 8.5 Suppose

$$\bigcap_{i=1}^{k} \ker C_i(t) = \{0\}.$$

Then the RDP is solvable iff there exist c.s.f.'s $\mathcal{V}_i$, generated respectively by $V_i \in \mathbb{A}^{n\times r_i}$ $(i \in \underline{k})$ of constant ranks, such that

$$\mathcal{V}_i(t) \subset \bigcap_{j\neq i} \ker C_j(t), \qquad \mathcal{V}_i + \ker C_i(t) = \mathbb{R}^n, \tag{8.7a,b}$$

for all $t \in \mathbb{R}$ and $i \in \underline{k}$.

Proof. The 'only if' part is immediate from the definition. To prove the 'if' part, note that, by $[A, B]$-invariant of $\mathcal{V}_i$, there exist $F_i \in \mathbb{A}^{m\times n}$ and $N_i \in \mathbb{A}^{r_i\times r_i}$ such that

$$(\mathrm{D}I_n - A_{F_i})(V_i) = V_i N_i \quad (i \in \underline{k}).$$

The families $\hat{\mathcal{K}}_i$ defined by

$$\hat{\mathcal{K}}_i(t) := \bigcap_{j\neq i} \ker C_j(t) \quad (i \in \underline{k})$$

are independent. This is proved analogously to the time-invariant case; see Wonham (1985: p. 225). Since $\mathcal{V}_i(t) \subset \hat{\mathcal{K}}_i(t)$, it follows that the $\mathcal{V}_i$'s are also independent. By Lemma 8.4, they are compatible. Application of Proposition 7.3 yields (8.2) and the proof is complete. □

9. Appendix: Algebraic properties of matrices with piecewise analytic entries

Here the existence of greatest common left divisors and least common right multiples of matrices over $\mathbb{A}_p$ is shown.

Suppose $P \in \mathbb{A}^{n\times k}$ and $Q \in \mathbb{A}^{n\times l}$. Then $G \in \mathbb{A}^{n\times r}$ is called a *greatest common left divisor* of P and Q—written $G \in \mathrm{glcd}\,(P, Q)$ for short—if, for every common left divisor G' of P and Q, there exists an analytic matrix R of appropriate size such that $G'R = G$. The matrix function $K \in \mathbb{A}^{n\times s}$ is called a *least common right multiple* of P and Q—written $K \in \mathrm{lcrm}\,(P, Q)$ for short—if, for every common right multiple K' of P and Q, there exists an analytic matrix S of appropriate size such that $K' = KS$.

Greatest common left divisors and least common right multiples of matrices over certain rings have been examined by several authors (see, for example, Mac Duffee (1956)). Unfortunately their results are only valid for Euclidean domains or principal-ideal domains; the set of real analytic functions is not a principal-ideal domain, however it is a *Bezout ring*, i.e. if $f,g \in \mathbb{A}$ have no common zeros, then there exists $a,b \in \mathbb{A}$ so that $af + bg = 1$; see Narasimhan (1985: §6.4). Nevertheless, the proof of the following lemma is partially based on Mac Duffee's ideas.

LEMMA 9.1 Suppose $P \in \mathbb{A}^{n\times k}$ and $Q \in \mathbb{A}^{n\times l}$, with $\mathrm{rk}_{\mathbb{M}}\, P = k$, $\mathrm{rk}_{\mathbb{M}}\, Q = l$, and $\mathrm{rk}_{\mathbb{M}}\, [P, Q] = r$. Then:

(i) There exists $G \in \mathrm{glcd}\,(P, Q)$, with $\mathrm{rk}_{\mathbb{M}}\, G = r$, which is unique up to multiplication by an invertible matrix from the right. Furthermore there exist analytic matrices U_1 and U_3 of appropriate sizes such that

$$G = PU_1 + QU_3, \qquad G\mathbb{A}^r = P\mathbb{A}^k + Q\mathbb{A}^l.$$

(ii) There exists $K \in \mathrm{lcrm}\,(P, Q)$, with $\mathrm{rk}_{\mathbb{M}}\, K = s$, which is unique up to multiplication by an invertible matrix from the right, and

$$K\mathbb{A}^{k+l-r} = P\mathbb{A}^k \cap Q\mathbb{A}^l.$$

Proof. (i) By Silverman & Bucy (1970), there exists $U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \in \mathrm{GL}_{k+l}(\mathbb{A})$ such that

$$[P, Q]U = [G, 0_{n\times s}], \qquad \mathrm{rk}_{\mathbb{M}}\, G = r, \tag{9.1}$$

where $s := k + l - r$. Let $V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$ be the inverse of U partitioned in such a form that

$$\begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}\begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix} = \begin{bmatrix} \mathrm{I}_r & 0 \\ 0 & \mathrm{I}_s \end{bmatrix}.$$

Then $P = GV_1$, $Q = GV_2$, and G is a common left divisor of P and Q. All matrices used in the following are defined over $\mathbb{A}$ and are of appropriate formats. Now it is proved that G is a *greatest* common left divisor. Assume $P = \hat{G}W$, $Q = \hat{G}W'$, and $\hat{G} = GR$. Since $\mathrm{rk}_{\mathbb{M}}\, \hat{G} \leq \mathrm{rk}_{\mathbb{M}}\, G$, it is assumed without restriction of generality that $\hat{G}$ is an $n \times r$ matrix. By (9.1),

$$G = \hat{G}S,$$

where $S := WU_1 + W'U_3$. Thus $\mathrm{rk}_{\mathbb{R}}\, G(t) = \mathrm{rk}_{\mathbb{R}}\, \hat{G}(t)$ for all $t \in \mathbb{R}$. Let $\mathcal{J} \subset \mathbb{R}$ be an open interval such that G is left-invertible over $\mathbb{A}\restriction_{\mathcal{J}}$. Then $G(t) = G(t)R(t)S(t)$ for all $t \in \mathcal{J}$ implies $\mathrm{I}_r = R(t)S(t)$ for all $t \in \mathcal{J}$. Since R and S are analytic, $\mathrm{I}_r = R(t)S(t)$ holds on $\mathbb{R}$. Therefore $G \in \mathrm{gcld}\,(P, Q)$, and the uniqueness statement is proved as well. (We note here that Wedderburn (1915) proves that a matrix over the ring of holomorphic functions can be transformed into a diagonal matrix by unimodular matrix operations; cf. Narasimhan (1985). This result is also valid for matrices over the ring of real analytic functions. However, the weaker result of Silverman & Bucy (1970) is sufficient for our purposes.)

(ii) $K := PU = -QU_4$ is a common right multiple of P and Q. First it will be proved that $\mathrm{rk}_{\mathbb{M}}\, K = s$. Assume $\mathrm{rk}_{\mathbb{M}}\, K < s$. Then there exists a $Z \in \mathrm{GL}_n\,(\mathbb{A})$ such that

$$KZ = [\hat{K}, 0] = PU_2Z = -QU_4Z.$$

Now P and Q are left-invertible on an open interval $\mathcal{J} \subset \mathbb{R}$, so U_2Z and $-U_4Z$ are of the form $[*, 0]$ on $\mathcal{J}$. Therefore

$$U\begin{bmatrix} \mathrm{I}_k & 0 \\ 0 & Z \end{bmatrix} = \begin{bmatrix} U_1 & [*, 0] \\ U_2 & [*, 0] \end{bmatrix},$$

which contradicts the invertibility of U on $\mathcal{J}$. Secondly it is proved that $K \in \mathrm{lcrm}\,(P, Q)$. Let

$$K' = PY = QY',$$

and choose $\tilde{G} \in \mathrm{gcld}\,(K, K')$ with $\tilde{G}H = K$ and $\tilde{G}H' = K'$. Clearly $\mathrm{rk}_{\mathbb{M}}\, \tilde{G} \geq s$. By (i), there exist N and N' such that $\tilde{G} = KN + K'N'$, and thus

$$\tilde{G}\mathbb{A}^{s'} \subset P\mathbb{A}^k \cap Q\mathbb{A}^l.$$

Since (by (i)) $\max_{t\in\mathbb{R}} \dim_{\mathbb{R}} [P(t)\mathbb{R}^k \cap Q(t)\mathbb{R}^l] = s$, we have $\operatorname{rk}_{\mathbb{M}} \tilde{G} \leq s$. Therefore $\operatorname{rk}_{\mathbb{M}} \tilde{G} = s$, and so we may take $\tilde{G}$ to be an $n \times s$ matrix without restricting generality. From the equations above, we compute

$$P(U_2N + YN') = KN + K'N' = \tilde{G} = Q(-U_4N + Y'N').$$

Let $E := U_2N + YN'$ and $F := -U_4N + Y'N'$; then

$$-QU_4 = PU_2 = K = \tilde{G}H = PEH = QFH.$$

Since Q and P are left-invertible on some open interval $\mathcal{J} \subset \mathbb{R}$, we have $-U_4 = FH$ and $U_2 = EH$ on $\mathcal{J}$. Therefore

$$\mathrm{I}_s = V_3U_2 + V_4U_4 = V_3EH - V_4FH = (V_3E - V_4F)H \quad \text{on } \mathcal{J}$$

and, since all the matrices involved are analytic, H is invertible over $\mathbb{A}$. Thus $K' = \tilde{G}H' = KH^{-1}H'$. Using similar arguments, one can prove that H' is invertible over $\mathbb{A}$ also; whence $K = \tilde{G}H = K'H'^{-1}H$. This completes the proof. □

Remark 9.2. It is also possible to define, and to show nonempty, sets $\operatorname{gcld}_{\mathbb{A}_p}(P, Q)$ and $\operatorname{lcrm}_{\mathbb{A}_p}(P, Q)$ for matrices P and Q defined over $\mathbb{A}_p$ instead of $\mathbb{A}$. This is demonstrated for $\operatorname{gcld}_{\mathbb{A}_p}(p, q)$, where $p, q \in \mathbb{A}_p$. Suppose that $\{[a_j, a_{j+1}) : j \in \mathbb{Z}\}$ is a disjoint partition of $\mathbb{R}$ such that $p_j, q_j \in \mathbb{A}\restriction_{[a_j, a_{j+1})}$ have analytic extension on both sides of (a_j, a_{j+1}). For short, put

$$f_j := f\restriction_{[a_j, a_{j+1})} \quad \text{for } f = g \text{ or } f = h.$$

Choose $g_j \in \operatorname{gcrd}(p_j, q_j) \subset \mathbb{A}\restriction_{[a_j, a_{j+1})}$, with $g_j = p_jc_j + q_jd_j$ and $c_j, d_j \in \mathbb{A}\restriction_{[a_j, a_{j+1})}$, for $j \in \mathbb{Z}$. Now it is straightforward to prove that g, defined by

$$g\restriction_{[a_j, a_{j+1})} := g_j,$$

satisfies $g \in \operatorname{gcld}_{\mathbb{A}_p}(p, q) \subset \mathbb{A}_p$.

Using Remark 9.2 it is immediate that the statements of Lemma 9.1 can be extended to piecewise analytic matrices as follows.

LEMMA 9.3 Suppose that $P \in \mathbb{A}_p^{n\times k}$ and $Q \in \mathbb{A}_p^{n\times l}$, and that $\{[a_j, a_{j+1}) : j \in \mathbb{Z}\}$ is a disjoint partition of $\mathbb{R}$ so that $P\restriction_{[a_j, a_{j+1})}$ and $Q\restriction_{[a_j, a_{j+1})}$ have real analytic extensions on some $(a_j^{\mathrm{L}}, a_{j+1}^{\mathrm{R}})$, where $a_j^{\mathrm{L}} < a_j < a_{j+1} < a_{j+1}^{\mathrm{R}}$. Then:

(i) There exists $G \in \operatorname{gcld}_{\mathbb{A}_p}(P, Q) \subset \mathbb{A}_p^{n\times n}$ so that

$$G\restriction_{[a_j, a_{j+1})} \text{ is of the form } [*, 0_{n\times s}],$$

where $\operatorname{rk}_{\mathbb{M}} P\restriction_{[a_j, a_{j+1})} = k_j$, $\operatorname{rk}_{\mathbb{M}} Q\restriction_{[a_j, a_{j+1})} = l_j$, $\operatorname{rk}_{\mathbb{M}} [P, Q]\restriction_{[a_j, a_{j+1})} = r_j$, with $s_j := k_j + l_j - r_j$. Furthermore there exist $U_1 \in \mathbb{A}_p^{k\times n}$ and $U_3 \in \mathbb{A}_p^{l\times n}$ so that

$$G = PU_1 + QU_3.$$

(ii) There exists $K \in \operatorname{lcrm}_{\mathbb{A}_p}(P, Q) \subset \mathbb{A}_p^{n\times n}$ with $\operatorname{rk}_{\mathbb{M}} K\restriction_{[a_j, a_{j+1})} = s_j$ and

$$K\mathbb{A}_p^n = P\mathbb{A}_p^k + Q\mathbb{A}_p^l.$$

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