Ilchmann, Achim ; Townley, Stuart:

Adaptive Sampling Control of High-Gain Stabilizable Systems

Zuerst erschienen in:
IEEE Trans. on Autom. Control 44 (1999), S.1961 - 1966
DOI: 10.1109/9.793786

Adaptive Sampling Control of High-Gain Stabilizable Systems

Achim Ilchmann and Stuart Townley

***Abstract*—It is well known that proportional output feedback control can stabilize any relative-degree one, minimum-phase system if the sign of the feedback is correct and the proportional gain is high enough. Moreover, there exist simple adaptation laws for tuning the proportional gain (so-called high-gain adaptive controllers) which do not need to know the system and do not attempt to identify system parameters.**

In this paper the authors consider sampled versions of the high-gain adaptive controller. The motivation for sampling arises from the possibility that the output of a system may not be available continuously, but only at sampled times. The main point of interest is the need to develop techniques for adapting the sampling rate, since the stiffness of the system increases as the proportional gain is increased. Our main result shows that adaptive sampling stabilization is possible if the product hk of the decreasing sampling interval h and the increasing proportional gain k decreases at a rate proportional to $1/\log k$.

***Index Terms*—Adaptive stabilization, minimum-phase systems, proportional control, sampled-data control.**

I. Introduction

In this paper, we will show that the ideas and techniques of high-gain adaptive output feedback stabilization carry over when the output of the system is not available continuously but is only available at sampled instants of time. This situation arises naturally when digital computations of control inputs are used.

It is well known (see, [13]) that

$$u(t) = -k(t)y(t), \qquad \dot{k}(t) = y^2(t)$$

is a continuous-time, high-gain adaptive stabilizer for the class of minimum-phase systems with positive high-frequency gain. This controller arose from the work of [6] and has been developed by [5], [4], [2], [11], [3], [1], and [12], to name but a few. While not all of these papers deal with adaptive control of minimum-phase systems, they are all similar in spirit in the sense that the adaptation of the controller gain is not based on any attempt to identify the parameters of the system. We continue in this spirit but focus on developing a mechanism to deal with the restriction that the output is only available at sampled time instants. The main novelty, which distinguishes this problem from either continuous or discrete-time adaptive control, is the need to develop suitable mechanisms for adjusting a variable

Manuscript received June 17, 1998. Recommended by Associate Editor, G. Tao. This work was supported by the Human Capital and Mobility Programme under Project CHRX-CT93-0402 and the University of Exeter, Small Grants Committee.

The authors are with the School of Mathematical Sciences, University of Exeter, Exeter EX4 4QE, U.K. and the Centre for Systems and Control Engineering, School of Engineering and Computer Science, University of Exeter, Exeter EX4 4QF, U.K.

Publisher Item Identifier S 0018-9286(99)07139-1.

sampling rate. Reference [9] is the only paper we are aware of which deals with this issue.

We focus on adaptive stabilization of minimum-phase multi-input/multi-output systems with the spectrum of the high-frequency gain unmixed. These systems are high-gain stabilizable. While this might be considered a restriction, it is precisely in this high-gain case that the conflict between gain adaptation and sampling is most apparent. Variable sampling could of course be considered in many other situations in adaptive control.

More precisely, we consider systems to be stabilized to be of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \qquad x(0) = x_0 \quad (1)$$

where $A \in \mathbb{R}^{n\times n}, B, C^T \in \mathbb{R}^{n\times m}, x_0 \in \mathbb{R}^n$ and n are all unknown. The assumption that (1) is *minimum phase* means that

$$\det\begin{bmatrix} sI_n - A & B \\ C & 0\end{bmatrix} \neq 0, \qquad \text{for all} \quad s \in \overline{\mathbb{C}_+}. \quad (2)$$

The assumption that the high-frequency gain CB is *unmixed* means that

$$\sigma(sCB) \subset \mathbb{C}_+ \quad \text{for some unknown} \quad s \in \{-1, 1\}. \quad (3)$$

The sign of the high-frequency gain is called positive if and only if

$$\sigma(CB) \subset \mathbb{C}_+. \quad (4)$$

The *control objectives* are described as follows. Design a simple scalar adaptation law

$$k_{j+1} = f(k_j, y_j), \qquad t_{j+1} = g(t_j, k_j) \quad (5)$$

so that the proportional sampled-data output feedback

$$u(t) = -k_j y_j, \qquad t \in [t_j, t_{j+1}) \quad (6)$$

which uses only sampled output information $y_j := y(t_j)$, when applied to a system (1) satisfying (2) and either (3) or (4), yields a closed-loop system (1), (5), (6) with convergent gain adaptation, positive sampling interval length, and stabilized sampled output, i.e.,

$$\lim_{j\to\infty} k_j = k_\infty \in \mathbb{R}, \qquad \lim_{j\to\infty} t_{j+1} - t_j = h_\infty > 0$$
$$\lim_{j\to\infty} y_j = 0.$$

The paper is organized as follows. Section II is devoted to sampled-data adaptive stabilization of multivariable systems satisfying (2) and either (3) or (4), while in Section III we study the intersampling behavior and prove that under additional mild assumptions we can guarantee that the continuous-time state $x(t)$ tends to zero.

II. Sampling Stabilization of Multivariable Systems

The following theorem is the main result of this section. An adaptive gain and sampling time mechanism is presented which stabilizes the output at the sampling instants and guarantees convergent gain and sampling period adaptation.

Theorem 2.1: Suppose the system (1) satisfies (2) and (4), i.e., (1) is minimum phase with positive high-frequency gain. Define the adaptive-sampling output feedback law by

$$u(t) = -k_i y_i, \qquad t \in [t_i, t_{i+1}) \quad (7)$$

where $y_i := y(t_i)$, and $\{k_j\}_{j\in\mathbb{N}_0}$ and $\{t_j\}_{j\in\mathbb{N}_0}$ are generated by the gain and sampling-time adaptation mechanism

$$h_i = \frac{1}{k_i \log k_i}, \qquad t_{i+1} = t_i + h_i$$
$$k_{i+1} = k_i + k_i h_i \|y_i\|^2, \qquad \text{for all} \quad i \in \mathbb{N}_0 \quad (8)$$

with $t_0 = 0$ and $k_0 > 1$. Then the closed-loop system (1), (7), and (8) admits a unique solution $x(\cdot)$ defined on the whole half-axis $[0, \infty)$. Furthermore:

1) $\lim_{i\to\infty} k_i = k_\infty \in \mathbb{R}$;
2) $\lim_{i\to\infty} h_i = h_\infty > 0$;
3) $\{y_i\}_{i\in\mathbb{N}_0} \in l^2$.

Before proving this result we discuss the underlying adaptation, especially the adaptation of the sampling rate.

Remark 2.2—1): The basic ideas underlying Theorem 2.1 can be motivated by considering the simplest situation of scalar systems

$$\dot{x}(t) = ax(t) + bu(t), \quad y(t) = cx(t), \qquad x(0) = x_0 \quad (9)$$

where $a, b, c, x_0 \in \mathbb{R}$ all unknown. It is well known (see, e.g., [13]) that the continuous-time adaptive control law

$$u(t) = -k(t)y(t), \qquad \dot{k}(t) = y^2(t) \quad (10)$$

will stabilize any system given by (9) with $cb > 0$. The reason, loosely speaking, is that in the resulting closed-loop system

$$\dot{x}(t) = [a - k(t)cb]x(t) \quad (11)$$

$k(t)$ must increase until $a - k(t)cb$ is negative, after which $x(t)$ tends to zero exponentially and $k(t)$ converges to a finite limit.

A Euler discretization of the k dynamics in (10), with a step length η_j, is given by

$$\frac{k_{j+1} - k_j}{\eta_j} = y_j^2. \quad (12)$$

On the other hand, sampling (11) on a sampling interval of length h_j gives $x(t_j)$ determined approximately by a Euler discretization of (11) with step length h_j. Since the "stiffness" of (11) increases affinely with $k(t)$, one would to need to sample (11) at a rate faster than $1/k(t)$. It is also natural to sample the x-dynamics (which are responding to changes in k) more rapidly than the numerical integration of the k-dynamics. With these observations in mind we choose

$$\eta_j = \frac{1}{\log k_j} \quad (13)$$

and

$$h_j = t_{j+1} - t_j = [k_j \log k_j]^{-1} = o(\eta_j). \quad (14)$$

Note that (12) and (13) coincide with (8), k_j is monotonically increasing and h_j given by (14) is monotonically decreasing. If k is large enough, so that $a - kcb$ is negative and the continuous-time system (2.5) is stable, and the sampling period h_j is small enough, then exponential decay of $x(t)$ can be expected.

2): The idea of using a variable sampling rate has been considered by [9]. Our approach differs from this work in two crucial aspects. In Owens the functional dependence between h and k $(k = k(h))$ is such that

$$\lim_{h\to 0} hk(h) = \hat{k}_\infty > 0 \quad (15)$$

whereas in our approach $\lim_{h\to 0} hk(h) = \lim_{k\to\infty} (1/k\log k)k = 0$. More significantly, in the context of adaptive control without identification, we require neither the extra assumptions (15) nor that $0 < N\hat{k}_\infty CB < 2$ holds for $N = +1$ or -1, imposed in [9].

3): We stress that, in general, we cannot expect $x(t) \to 0$ as $t \to \infty$. However, in the case $n = 1$, boundedness and monotonicity of k_j and h_j gives

$$|x(t)| \le [e^{|a|h_0} + cb\, e^{|a|h_0} k_\infty h_0]|x_j|, \qquad \text{for} \quad t \in [t_j, t_{j+1}).$$

Since $\{y_j\}_{j\in\mathbb{N}_0} \in l^2$ and $c \neq 0$ it follows that x_j tends to zero, so that $\lim_{t\to\infty} x(t) = 0$.

The crucial step in proving Theorem 2.1 is the investigation of fixed (nonadaptive) high-gain feedback control. Indeed, if a feedback of the form

$$u(t) = -ky_i, \qquad t \in [hi, h(i+1))$$

with fixed gain k and sampling length $h = (k \ \log \ k)^{-1}$, is applied to (1), with sufficiently high gain k, then every solution of the closed-loop system tends to zero exponentially. The following lemma, which is of interest in its own right, clarifies this "high-gain" idea.

Lemma 2.3: Consider a system (1) satisfying (2) and (4), and let $h = (k \ \log k)^{-1}$. Then there exists $\overline{k} > 1$ sufficiently large such that for all $k \geq \overline{k}$, the feedback

$$u(t) = -ky_i, \qquad t \in [hi, h(i+1)) \tag{16}$$

applied to (1) yields an exponentially stable closed-loop system

$$\dot{x}(t) = Ax(t) - kBCx_i, \qquad t \in [hi, h(i+1)). \tag{17}$$

Here $x_i = x(t_i)$. Moreover, the associated discrete-time system

$$x_{i+1} = [I_n + h(A - kBC) + h^2 U_{h,k}]x_i \tag{18}$$

where $U_{h,k} := T_h(A)[A - kBC]$ and $T_h(A) := (1/2!)A + (1/3!)hA^2 + \cdots$, is power stable, i.e., there exists some $M > 0$ and $\zeta_h \in (0,1)$, independent of $\overline{k}$, so that

$$\|x_{i+1}\| \leq M\zeta_h^{i+1-i_0}\|x_{i_0}\|, \qquad \text{for all} \quad i \geq i_0. \tag{19}$$

Proof: Applying variation-of-constants to (17) yields

$$x_{i+1} = \left[e^{Ah} - k \int_0^h e^{As} \, ds \, BC\right]x_i.$$

Equation (18) follows from the uniform power series expansion

$$\begin{aligned} e^{Ah} - k\int_0^h e^{As}\,ds\,BC &= \left[I_n + hA + \frac{1}{2!}\,h^2A^2 + \cdots\right] \\ &\quad - k\left[hI_n + \frac{h^2}{2!}\,A + \cdots\right] BC. \end{aligned}$$

Since (1) is minimum phase and $\det CB \neq 0$, the state space can be decomposed into $\operatorname{im} B \oplus \ker C$ so that without loss of generality (see, e.g., [1, p. 11]), we may assume A, B, C, and x are of the form

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \qquad B = \begin{bmatrix} CB \\ 0 \end{bmatrix}$$
$$C = [I_m, 0], \qquad x = \begin{pmatrix} y \\ z \end{pmatrix}$$

with $\sigma(A_4) \subset C_-$ and blocks structured according to $y \in \mathbb{R}^m, z \in \mathbb{R}^{n-m}$. Setting

$$\Psi_{h,k} := I_n - kh\begin{bmatrix} CB & 0 \\ 0 & 0 \end{bmatrix} + h\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{20}$$

(18) becomes

$$x_{i+1} = [\Psi_{h,k} + h^2 U_{h,k}]x_i. \tag{21}$$

To prove that there exists $\overline{k}$ sufficiently large so that (19) holds for all $k > \overline{k}$, we consider

$$V(y,z) := \begin{pmatrix} y \\ z \end{pmatrix}^T R \begin{pmatrix} y \\ z \end{pmatrix}, \qquad \text{with} \quad R = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

as a Lyapunov-function candidate. Here $P = P^T \in \mathbb{R}^{m\times m}$, $Q = Q^T \in \mathbb{R}^{(n-m)\times(n-m)}$ denote the positive-definite solutions of

$$(CB)^T P + P(CB) = I_m \quad \text{and} \quad A_4^T Q + QA_4 = -I_{n-m}. \tag{22}$$

Computing $\Delta V_i := V(y_{i+1}, z_{i+1}) - V(y_i, z_i)$ along the solution of (21) gives

$$\Delta V_i = [x_i^T(\Psi_{h,k}^T + h^2 U_{h,k}^T)]R[(\Psi_{h,k} + h^2 U_{h,k})x_i] - x_i^T R x_i.$$

Let $\overline{k}_1 > 1$. Then there exists $M_1 > 0$, so that

$$\|\Psi_{h,k}\| + \frac{1}{k\sqrt{\log k}}\,\|U_{h,k}\| \leq M_1, \qquad \text{for all} \quad k \geq \overline{k}_1.$$

Hence, there exists $M_2 > 0$ so that

$$\begin{aligned} \Delta V_i &= x_i^T \Psi_{h,k}^T R \Psi_{h,k} x_i - x_i^T R x_i \\ &\quad + 2h^2 x_i^T U_{h,k}^T R \Psi_{h,k} x_i + h^4 x_i^T U_{h,k}^T R U_{h,k} x_i \\ &\leq x_i^T[\Psi_{h,k}^T R \Psi_{h,k} - R]x_i \\ &\quad + M_2\left(\frac{h}{\sqrt{\log k}}\|x_i\|^2 + \frac{h^2}{\log k}\|x_i\|^2\right) \end{aligned}$$

for all $k \geq \overline{k}_1$. Now

$$\begin{aligned} \Psi_{h,k}^T R \Psi_{h,k} &= \left(I + h\begin{bmatrix} -kCB + A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^T\right) \\ &\quad \cdot \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}\left(I + h\begin{bmatrix} -kCB + A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right) \end{aligned}$$

so that by using (22)

$$\begin{aligned} &\Psi_{h,k}^T R \Psi_{h,k} - R \\ &\quad = -h\begin{bmatrix} kI_m & 0 \\ 0 & I_{n-m} \end{bmatrix} \\ &\qquad + h\begin{bmatrix} A_1^T P + PA_1 & A_3^T Q + PA_2 \\ A_2^T P + QA_3 & 0 \end{bmatrix} \\ &\qquad + h^2\begin{bmatrix} -kCB + A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^T R \begin{bmatrix} -kCB + A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}. \end{aligned}$$

Therefore, there exists $M_3 > 0$ so that

$$\begin{aligned} \Delta V_i \leq &-hk\|y_i\|^2 - h\|z_i\|^2 + hM_3(\|y_i\|^2 + \|y_i\|\,\|z_i\|) \\ &+ h^2 M_3[k^2\|y_i\|^2 + k\|y_i\|\,\|z_i\| + \|z_i\|^2] \\ &+ M_2\left[\frac{h}{\sqrt{\log k}} + \frac{h^2}{\log k}\right][\|y_i\|^2 + \|z_i\|^2] \end{aligned}$$

and hence, by using $\|y\| \cdot \|z\| \leq 2M_3\|y\|^2 + (2M_3)^{-1}\|z\|^2$

$$\begin{aligned} \Delta V_i \leq &-hk\left[1 - M_3\left(\frac{1}{k} + \frac{2M_3}{k} + hk + 2M_3 h\right)\right. \\ &\left. - \frac{M_2}{k\sqrt{\log k}}\left(1 + \frac{h}{\sqrt{\log k}}\right)\right]\|y_i\|^2 \\ &- h\left[\frac{1}{2} - \frac{hk}{2} - M_3 h - \frac{M_2 h}{\sqrt{\log k}} - \frac{M_2 h}{\log k}\right]\|z_i\|^2 \end{aligned} \tag{23}$$

for all $k \geq \overline{k}_1$. By choosing $\overline{k} > \overline{k}_1$ sufficiently large, we obtain for all $k > \overline{k}$

$$\Delta V_i \leq -\frac{h}{4}k\|y_i\|^2 - \frac{h}{4}\|z_i\|^2. \tag{24}$$

Hence

$$V(x_{i+1}) - V(x_i) \leq -\frac{h}{4}\|x_i\|^2 \leq -\frac{h}{4\|R\|}V(x_i)$$

for all $i \geq i_0$. Therefore

$$V(x_{i+1}) \leq (1 - h(4\|R\|)^{-1})^{i+1-i_0}V(x_{i_0}).$$

Then (19) follows, with $\zeta_h := \sqrt{1 - h(4\|R\|)^{-1}}$, by using the standard inequalities $\sigma_{\min}(R)\|x\|^2 \leq x^T R x \leq \|R\|\,\|x\|^2$.

It remains to prove exponential stability of (17). Again by variation-of-constants applied to (17) for $t \in h[i, (i+1))$ we have, for some suitable $M_4 > 0$, that

$$\begin{aligned}\|x(t)\| &\le \left[e^{\|A\|h} + k\ \frac{e^{\|A\|h}-1}{\|A\|}\|BC\|\right]\|x_i\| \\ &\le \frac{M_4\|R\|}{\sigma_{\min}(R)}\zeta_h^{i+1}\|x_0\|\end{aligned}$$

and, for $-\alpha := (1/h)\ \log \zeta_h < 0$, we conclude that

$$\|x(t)\| \le \frac{M_4\|R\|}{\sigma_{\min}(R)}e^{-\alpha h(i+1)}\|x_0\| \le \frac{M_4\|R\|}{\sigma_{\min}(R)}e^{-\alpha t}\|x_0\|. \qquad \square$$

Proof of Theorem 2.1: Existence and uniqueness of the solution to the closed-loop system (1), (7), and (8) is obvious. Clearly $\{k_i\}_{i\in\mathbb{N}_0}$ is nondecreasing. To prove convergence of $\{k_i\}_{i\in\mathbb{N}_0}$, suppose to the contrary that $\lim_{i\to\infty}\ k_i = \infty$. Consider the associated discrete-time system

$$x_{i+1} = [I_n + h_i(A - k_iBC) + h_i^2U_{h_i,k_i}]x_i \tag{25}$$

which we derive from (17) as in the proof of Lemma 3.3; see (20) and (21). Analogously to (24) we obtain, for sufficiently large $i_0 \in \mathbb{N}_0$ (recall that $\lim_{i\to\infty}\ k_i = \infty$) and all $i \ge i_0$

$$\Delta V_i \le -\frac{h_i}{4}\ k_i\|y_i\|^2 = -\frac{1}{4}\ (k_{i+1} - k_i).$$

Hence

$$\begin{aligned}&V(y_N, z_N) - V(y_{i_0}, z_{i_0}) \\ &\quad = \sum_{i=i_0}^{N-1}\ \Delta V(y_i, z_i) \le -\frac{1}{4}\ \sum_{i=i_0}^{N-1}\ (k_{i+1} - k_i) \\ &\quad = -\frac{1}{4}[k_N - k_{i_0}]\end{aligned}$$

so that

$$\tfrac{1}{4}k_N \le V(y_{i_0}, z_{i_0}) + \tfrac{1}{4}\ k_{i_0}, \qquad \text{for all } N \ge i_0.$$

This contradicts the unboundedness of $\{k_i\}_{i\in\mathbb{N}_0}$. Hence $\{k_i\}_{i\in\mathbb{N}_0}$ converges.

Now the statements 1) and 2) are immediate from boundedness of $\{k_i\}_{i\in\mathbb{N}_0}$. The proof is completed by noting that 3) follows from the inequality

$$\frac{1}{\log k_\infty}\ \sum_{i=0}^{N}\ \|y_i\|^2 \le \sum_{i=0}^{N}\ \frac{1}{\log k_i}\ \|y_i\|^2 = k_{N+1} - k_0 \le k_\infty.$$

$\square$

If the sign of the high-frequency gain CB is unknown, so that we only know that (3) holds, then the adaptation law has to additionally find the sign for the feedback. For continuous-time feedback this problem was solved by the famous contribution of [7]. Nussbaum's idea was to introduce sign-switching feedbacks of the form $u(t) = -k(t)\sin\sqrt{k(t)}y(t)$. The analogue for adaptive-sampling feedback control is obtained by replacing the continuous sign changing function $\sin\sqrt{k}$ by a piecewise continuous sign changing function as follows.

Algorithm 2.4: For a monotone nondecreasing sequence $1 < k_0 \le k_1 \le \cdots$ define a *switching sequence* $\{S_i\}_{i\in\mathbb{N}_0} \subset \{-1, 1\}$ by the flow chart shown in Fig. 1, initialized with $i = L = 0$, $S_0 = 1$ and where

$$\chi_i(k, S) := \begin{cases} 1, & \text{if } k_0 = \cdots = k_i \\ \dfrac{1}{k_i - k_0}\displaystyle\sum_{j=0}^{i-1}(k_{j+1} - k_j)S_j, & \\ \quad\text{otherwise.} & \end{cases} \tag{26}$$

If $\{k_i\}_{i\in\mathbb{N}_0}$ diverges to infinity, then the Algorithm 2.4 ensures that $\chi_i \in (-1, 1)$ for all $i \ge i_0$ (for i_0 sufficiently large) and χ_i has the two accumulation points $+1$ and -1. Thus S_i will stay at $+1$, respectively -1, for longer and longer intervals and it is then natural to choose the feedback

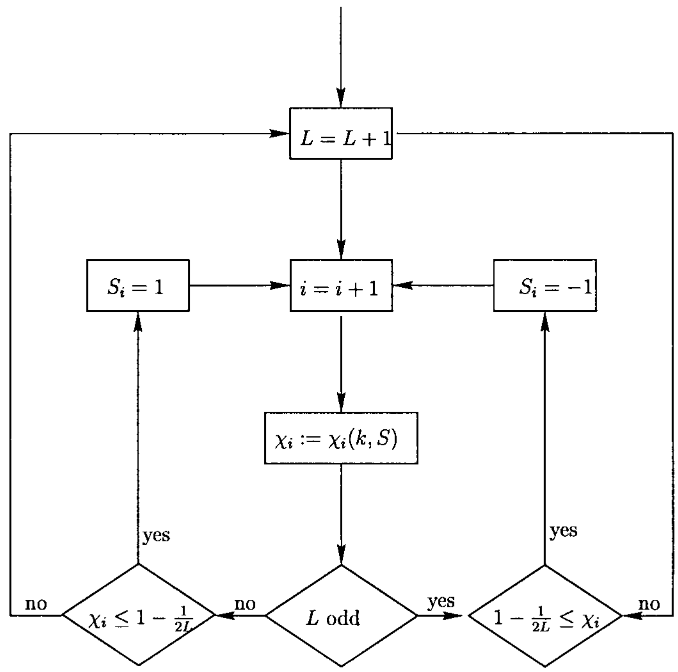

Fig. 1. The switching procedure.

$$u(t) = -k_iS_iy_i, \qquad t \in [t_i, t_{i+1}). \tag{27}$$

This switching procedure is similar to the one used in [9]. However, our implementation is direct and does not use a "switching activation sequence."

Theorem 2.5: Suppose the system (1) satisfies (2) and (3). Let $S_0 = 1$ and $k_0 > 1$. Then the adaptive-sampling output feedback law (27), where $y_i := y(t_i)$, and the gain and sampling-time are adapted according to (8) in Theorem 2.1, with switching sequence $\{S_i\}_{i\in\mathbb{N}_0}$ defined by Algorithm 2.4, applied to (1) yields a closed-loop system which admits a unique solution $x(\cdot)$ defined on the whole half-axis $[0, \infty)$. Furthermore:

1) $\lim_{i\to\infty}\ k_i = k_\infty \in \mathbb{R}$;
2) $\lim_{i\to\infty}\ h_i = h_\infty > 0$;
3) $\lim_{i\to\infty}\ \chi_i = \chi_\infty \in (-1, 1]$;
4) there exists some i_0 such that $S_i = S_{i_0}$ for all $i \ge i_0$;
5) $\{y_i\}_{i\in\mathbb{N}_0} \in l^2$.

Proof: This proof is very similar to the proof of Theorem 2.1. The main difference is in proving convergence of $\{k_i\}_{i\in\mathbb{N}_0}$, and the remainder is straightforward and is omitted.

Suppose that $\{k_i\}_{i\in\mathbb{N}_0}$ is not bounded. For suitable $s = +1$ or -1 we have

$$(CB)^TP + P(CB) = sI_n.$$

If k in (20) is replaced by kS, then we can deduce, as in the proof of Lemma 2.3 [see in particular (23)] that

$$\begin{aligned}\Delta V_i \le &-sh_ik_iS_i\|y_i\|^2 + \Bigg[M_3(h_i + 2M_3h_i + h_i^2k_i^2 + 2M_3h_i^2k_i) \\ &- \frac{M_2h_i}{\sqrt{\log k_i}}\left(1 + \frac{h_i}{\sqrt{\log k_i}}\right)\Bigg]\|y_i\|^2 \\ &- h_i\left[\frac{1}{2} - \frac{h_ik_i}{2} - M_3h_i - \frac{M_2h_i}{\sqrt{\log k_i}} - \frac{M_2h_i}{\log k_i}\right]\|z_i\|^2.\end{aligned}$$

Therefore, there exists i_0 sufficiently large and $M_5 > 0$, so that for all $i \ge i_0$

$$\begin{aligned}\Delta V_i &\le \left[\frac{M_5}{\log k_i} - s h_i k_i S_i\right]\|y_i\|^2 \\ &= [M_5 - sS_i]\,[k_{i+1} - k_i].\end{aligned}$$

Hence, for all $N \ge i_0$

$$\begin{aligned}&V(y_N, z_N) - V(y_{i_0}, z_{i_0}) \\ &\quad\le M_5[k_N - k_{i_0}] - s\sum_{i=i_0}^{N-1} S_i[k_{i+1} - k_i] \\ &\quad= (k_N - k_{i_0})\left[M_5 - \frac{s}{k_N - k_{i_0}}\sum_{i=i_0}^{N-1} S_i[k_{i+1} - k_i]\right].\end{aligned}$$

Now it is easy to show, by construction of the switching sequence, that the right-hand side above tends to $-\infty$, whilst the left-hand side is bounded. Hence we have a contradiction and therefore $\{k_i\}_{i\in\mathbb{N}_0}$ is bounded. This completes the proof. □

III. Stabilization of the State by Sampling Output Feedback

In Remark 2.2-3), we indicated that we would not expect the adaptive controller in Section II to stabilize the whole state, not even the state at sampling instants. In fact, while the sequence $y(t_i)$ converges to zero, the following example (see [9] for the details) shows that the continuous-time output $y(t)$ need not converge to zero. Consider the minimum phase system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \quad \text{with} \\ A &= \begin{bmatrix} 0 & 1 \\ -4\pi^2 & 0\end{bmatrix}, \qquad B = \begin{bmatrix}1\\1\end{bmatrix}, \qquad C = (1, 0) \qquad (28)\end{aligned}$$

for which the high-frequency gain $CB = 1$. It is easy to see that with initial data $x(0) = (0\, 2\pi)^T, t_0 = 0$, and $k_0 > 1$ such that $h_0 = (k_0 \log k_0)^{-1} = 1$, the adaptive feedback law (7), (8) applied to (28) yields $y_j = 0, h_j = 1, k_j = k_0$ for all $j \in \mathbb{N}_0$ and $u(\cdot) \equiv 0$, but $x(t) = (\sin 2\pi t,\, 2\pi\cos 2\pi t)$.

Note that this example is pathological since the sampling times occur exactly where the continuous-time output vanishes. Since the continuous-time system (1) is detectable (this is a consequence of the minimum phase assumption; see, [1]), we could overcome this problem by choosing the sampling periods in such a way that sampling preserves detectability. Now it is well known that the sampled system (with constant sampling period $h > 0$) is detectable if and only if

$$\frac{\lambda - \mu}{2\pi i}\, h \notin \mathbb{Z} \quad \text{for any} \quad \lambda \ne \mu, \qquad \lambda, \mu \in \sigma(A) \cup \{0\}. \qquad (29)$$

We shall modify the adaptive sampling time algorithm (8) under the additional assumption that (29) holds for some known h. This detectability of the sampled system at some known sampling time h is used in [8] in constructing sampled-data identification-based adaptive controllers. The main benefit of the extra assumption is that (29) then holds for a sampling period h/q, for any $q \in \mathbb{N}$. We exploit this in the following result.

Theorem 3.1: Suppose the system (1) satisfies (2) and (4). Let h be such that (29) holds. Then the adaptive sampling output feedback law (7) where $y_i := y(t_i)$, and $\{k_i\}_{i\in\mathbb{N}_0}$ and $\{t_i\}_{i\in\mathbb{N}_0}$ are generated by the gain and sampling-time adaptation mechanism

$$\hat{h}_i = \frac{1}{j}\, h$$

where

$$\begin{aligned} &j \text{ is such that } h_i = \frac{1}{k_i \log k_i} \in \left[\frac{1}{j+1}, \frac{1}{j}\right) \\ &t_{i+1} = t_i + \hat{h}_i \quad \text{and} \quad k_{i+1} = k_i + k_i\hat{h}_i\|y_i\|^2 \qquad (30)\end{aligned}$$

with $t_0 = 0$, $k_0 > 1$, applied to (1) yields a closed-loop system which admits a unique solution $x(\cdot)$ defined on the whole axis $[0, \infty)$. Moreover:

1) $\lim_{i\to\infty} k_i = k_\infty \in \mathbb{R}$;
2) there exists some $i_0, j_\infty \in \mathbb{N}$ such that $\hat{h}_i = (1/j_\infty)h$ for all $i \ge i_0$;
3) $\{y_i\}_{i\in\mathbb{N}_0} \in l^2$;
4) $\lim_{t\to\infty} x(t) = 0$.

Remark 3.2—1): If an upper bound of A in the spectral norm is known, i.e., for some $M > 0$ we have $\|A\| \le M$, then we can choose $h \in (0, (3/M))$. This is an immediate consequence of

$$0 < \left|h\frac{\lambda - \mu}{2\pi i}\right| \le \frac{3}{M}\frac{2M}{2\pi} < 1$$

from which (29) follows.

2): If A is rational, then (29) holds for any $h \in \mathbb{Q}$. To see this, note that since $\det(\lambda I_n - A)$ is a polynomial with rational coefficients, the real and imaginary parts of the eigenvalues of A are algebraic numbers. And since the difference of any two algebraic numbers is algebraic (see, e.g., [10]), we have for any $h \in \mathbb{Q}$ that $h(\lambda - \mu) \notin 2\pi i\mathbb{Z}$ and therefore the claim follows.

Proof of Theorem 3.1: This proof is similar to the proof of Theorem 2.1 but with h_i replaced by $\hat{h}_i$. Existence and uniqueness of the solution is again straightforward.

Step 1: We will prove boundedness of $\{k_i\}_{i\in\mathbb{N}_0}$. Suppose to the contrary that $\lim_{i\to\infty} k_i = \infty$. Analogously to (24) we can find $i_0 \in \mathbb{N}$ such that for all $i \ge i_0$

$$\Delta V_i \le -\frac{\hat{h}_i}{2}\, k_i\|y_i\|^2 = -\frac{1}{2}(k_{i+1} - k_i)$$

and 1) follows as in part 1) of the proof of Theorem 3.2.

Step 2: Since k_i converges to k_∞, there exists some $j_\infty \in \mathbb{N}$ and $i_0 \in \mathbb{N}$ such that

$$h_i = \frac{1}{k_i \log k_i} \in \left[\frac{1}{j_\infty + 1}, \frac{1}{j_\infty}\right), \qquad \text{for all} \quad i \ge i_0.$$

This proves 2), and 3) follows from

$$\frac{h}{\log k_\infty}\sum_{i=0}^{N}\|y_i\|^2 \le \sum_{i=0}^{N} k_i\hat{h}_i\|y_i\|^2 = k_{N+1} - k_0 \le k_\infty.$$

Step 3: It remains to prove 4). To this end consider

$$x_{i+1} = e^{A\hat{h}_i}x_i + \left[\int_0^{\hat{h}_i} e^{sA}B\, ds\right]u_i, \qquad y_i = Cx_i$$

which is the sampled version of (1) on a sampling interval of length $\hat{h}_i$. Since $h_i \in [(1/j_\infty + 1), (1/j_\infty))$ for all $i \ge i_0$ it follows that $\hat{h}_i = (h/j_\infty) := \hat{h}$ for all $i \ge i_0$. Hence

$$x_{i+1} = e^{A\hat{h}}x_i + \left[\int_0^{\hat{h}} e^{sA}B\, ds\right]u_i, \qquad y_i = Cx_i.$$

By the minimum phase assumption (A, C) is detectable. Using (29) it follows that $(e^{A\hat{h}}, C)$ is detectable. Since by 3) we have

$\lim_{i\to\infty} y_i = 0$, it follows that $\lim_{i\to\infty} x_i = 0$. Now for all $t \in [t_i, t_{i+1})$ we have

$$\begin{aligned}\|x(t)\| &= \left\| e^{A(t-t_i)} x_i - k_i \int_{t_i}^{t} e^{A(t-s)} BCx_i ds \right\| \\ &\le \left[e^{\|A\|\hat{h}} + k_\infty \frac{e^{\|A\|\hat{h}} - 1}{\|A\|} \|BC\| \right] \|x_i\|.\end{aligned}$$

This shows $\lim_{t\to\infty} x(t) = 0$ and completes the proof. □

IV. Conclusions

In this paper we have considered the problem of how to adapt a variable sampling rate in the high-gain adaptive stabilization of minimum phase, relative degree one systems. The adaptive sampling rate is used to counter the increasing stiffness of the closed-loop system caused by an increased gain which is needed to exploit the stability of the zero dynamics. The ideas explored in this paper are prototypical of many problems in adaptive sampled-data control where the possible conflict between variable stiffness, caused by adaptive gains, and adaptive sampling rates has to be resolved.

References

[1] A. Ilchmann, *Non-Identifier-Based High-Gain Adaptive Control*. London, U.K.: Springer-Verlag, 1993.

[2] H. K. Khalil and A. Saberi, “Adaptive stabilization of a class of nonlinear systems using high-gain feedback,” *IEEE Trans. Automat. Contr.*, vol. 32, pp. 1031–1035, 1987.

[3] H. Logemann and B. Mårtensson, “Adaptive stabilization of infinite-dimensional systems,” *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1869–1883, 1992.

[4] B. Mårtensson, “Adaptive stabilization,” Ph.D dissertation, Lund Inst. Technol., Lund, Sweden, 1986.

[5] I. Mareels, “A simple selftuning controller for stably invertible systems,” *Syst. Contr. Lett.*, vol. 4, pp. 5–16, 1984.

[6] A. S. Morse, “Recent problems in parameter adaptive control,” in *Outils et Modèles Mathématiques pour l’Automatique, l’Analyze de Systèmes et le Traitment du Signal* I. D. Landau, Ed.____1983, pp. 733–740.

[7] R. D. Nussbaum, “Some remarks on a conjecture in parameter adaptive control,” *Syst. Contr. Lett.* vol. 3, pp. 243–246, 1983.

[8] R. Ortega and G. Kreisselmeier, “Discrete-time, model reference adaptive control for continuous-time systems using generalized sampled-data hold functions,” *IEEE Trans. Automat. Contr.*, vol. 35, pp. 334–338, 1990.

[9] D. H. Owens, “Adaptive stabilization using a variable sampling rate,” *Int. J. Contr.*, vol. 63, no. 1, pp. 107–119, 1996.

[10] H. E. Rose, *A Course in Number Theory*. New York: Oxford Univ. Press, 1988.

[11] E. P. Ryan, “Adaptive stabilization of a class of uncertain nonlinear systems: A differential inclusion approach,” *Syst. Contr. Lett.*, vol. 10, pp. 95–101, 1988.

[12] S. Townley, “Topological aspects of universal adaptive stabilization,” *SIAM J. Contr. Optim.*, vol. 34, no. 3, pp. 1044–1070, 1996.

[13] J. C. Willems and C. I. Byrnes, “Global adaptive stabilization in the absence of information on the sign of the high frequency gain,” in *Lect. Notes in Control and Inf. Sciences*, vol. 62. Berlin, Germany: Springer-Verlag, 1984.