On Output Feedback Control of Infinite-dimensional Systems

Dissertation zur Erlangung des akademischen Grades Doctor rerum naturalium

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betreut durch

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30 Juli 2015

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urn:nbn:de:gbv:ilm1-2015000297

Acknowledgments

I am very grateful to my advisors Achim Ilchmann, Timo Reis and Carsten Trunk for making this research project possible and guiding me through the world of mathematical research. I am much obliged to the Klaus Tschira Stiftung for supporting this project financially over four years. Furthermore, I would like to thank Mark Opmeer and Olof Staffans for their valuable hints on the subject of this thesis.

I am thankful to all of my colleagues, in particular Leslie Leben, Hannes Gernandt, Florian Kelma, and Stefan Brechtken for his technical support. I would like to thank my sister Ulrike for constantly sending me motivation and my parents Marie-Luise and Manfred for making this life possible in the first place. Last but not least I am grateful for Sabina Kim who supported me unconditionally in the stressful final stage of writing this thesis.

Contents

1. Introduction

Linear systems theory is concerned with physical processes that convert an input signal u into an output signal y in a linear, time-invariant and causal manner. Mathematically, this is captured in the simple equation $y = \mathfrak{D}u$, where $\mathfrak{D}$ is a linear time-invariant causal operator that maps one signal space into another. The system that processes the signal usually comprises an internal state x which is, as opposed to y, not directly measurable. In the theory of compatible well-posed linear systems, these processes are described by a differential equation of the type

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t),$$

where A, B, C and D are linear operators between Hilbert spaces. A notably challenging situation arises when these Hilbert spaces need to be infinite-dimensional and the operators may be unbounded. Such infinite-dimensional systems have received much attention in the past 30 years, see e.g. the monographs [BDDM07, CZ95, JZ12, Sta05, TW09] for an overview. Typical examples are diffusion processes such as heat conduction, wave propagation, and many others that are for example mentioned in [CZ95, JZ12, TW09]. Typical control goals are for instance to make the output follow a desired reference trajectory [BGHS13, LT97, Pau11], or to make the output insensitive to disturbances [vK93, Mik02, MG90]. The present thesis is motivated by the desire to achieve each of these two goals in a practically feasible way for infinite-dimensional systems.

1.1. Summary of the thesis

In order to complete the task of trajectory tracking we employ funnel control, a very simple control strategy that makes the output follow the reference trajectory in a strict way. Namely, it evolves in a funnel around the reference trajectory that can be specified by the user. The second output control strategy that we employ aims at minimizing noise amplification of the closed-loop system. It is a special version of the famous $\mathcal{H}^\infty$-control problem [vK93].

Both methods require considerable preparation in terms of structural systems analysis. For both control schemes we construct state space transformation to obtain realizations in terms of matrix-like representations. While state space transformations are standard tools in the finite-dimensional theory, they have been neglected in the infinite-dimensional theory so far. This is mostly due to the lack of matrix representations for unbounded linear operators. In fact, the transformations we use may become unbounded themselves which leads to considerable technical difficulties. Nevertheless, the present work shows that it is sometimes possible and beneficial to make use of such transformations.

Funnel control for finite-dimensional systems requires two properties: A relative degree condition and stable zero dynamics. The concept of relative degree for infinite-dimensional system appears only in [MR07, LT97], where the relative-degree of a state linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t),$$

is defined as a natural number determined by the behavior of the transfer function at infinity. To the definition of [MR07], we add a smoothness condition on the control and observation operator that guarantees the existence of certain invariant subspaces [MR07, Zwa89]. This condition allows us to develop two special realizations: The zero dynamics form and the Byrnes-Isidori form. The latter is a generalization of the popular Byrnes-Isidori form in [BI91], and both are suitable for determining the zero dynamics, which is roughly speaking, the set of all trajectories (x, u) that satisfy

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad Cx(t) = 0 \qquad \forall\, t \geqslant 0.$$

While the importance of the zero dynamics is undisputed [BG09, BGH94, BGHS13, BGIS06, Isi95, MR07, MR10], there is no proper definition nor characterization for infinite-dimensional systems so far. We fill this gap by giving a universal definition of zero dynamics for well-posed linear systems, and we show that the zero dynamics are, at least in some cases, characterized by a strongly continuous semigroup on some subspace of the state space. For systems with relative degree, the zero dynamics form shows that this subspace is precisely the largest feedback invariant subspace in $\ker C$, which was extensively studied in [Cur84, Cur86, MR07, Zwa88, Zwa89]. The aforementioned semigroup allows us to characterize the stability of the zero dynamics.

To systems with exponentially stable zero dynamics and relative degree one we subsequently apply the funnel controller. This control law is an easily implemented algebraic calculation of the form

$$u(t) = k(t, y(t) - y_{\mathrm{ref}}(t)) \cdot (y(t) - y_{\mathrm{ref}}(t)),$$

where k is a special nonlinear function that determines the performance funnel, and $y_{\mathrm{ref}}(t)$ is the reference signal to be tracked. The challenge is to identify the systems for which this works. It has been established for finite-dimensional linear, non-linear and differential-algebraic systems as well as some functional differential equations in [BIR12b, BIW14, IRS02, IRT05], respectively. In this thesis we first show that the funnel control is successfully applicable to systems of relative degree one with exponentially stable zero dynamics. Thereafter, we prove the same result for transfer functions that have a series expansion of the form

$$\mathbf{G}(s) = \sum_{k=0}^{\infty} \frac{c_k}{s + \lambda_k}, \qquad s \in \mathbb{C}_{>0},$$

where $c_k, \lambda_k \in \mathbb{R}_{\geqslant 0}$. Such a transfer function is for instance realized by the following boundary control system with "collocated" control and observation:

$$\begin{aligned} \tfrac{\partial}{\partial t} x(\xi, t) &= \Delta x(\xi, t), && (\xi, t) \in \Omega \times \mathbb{R}_{>0}, \\ u(t) &\equiv \partial_\nu x(\xi, t), && (\xi, t) \in \partial\Omega \times \mathbb{R}_{>0}, \\ y(t) &= \int_{\partial\Omega} x(\xi, t)\, \mathrm{d}\sigma_\xi, && (\xi, t) \in \partial\Omega \times \mathbb{R}_{>0}. \end{aligned}$$

Here, Ω is a bounded, smooth domain and $u(t)$ and $y(t)$ are scalar. This non-trivial example is thoroughly analyzed within this thesis, including a discussion of the zero dynamics and the evolution of the state under funnel control.

The second output control strategy that we develop is a special version of the famous $\mathcal{H}^\infty$-control problem, and closely related to the linear quadratic optimal control problem. The infinite-dimensional versions of these problems have received extensive study, amongst others in [vK93, Mik02, Sta98b] and [BDDM07, CZ95, Mik02, PS87, Sta98c], respectively. The solutions always comprise an infinite-dimensional observer that estimates the internal states of the system and is impossible to implement because of its infinite-dimensional nature. Therefore, some efforts to design finite-dimensional controllers with similar performance have been made in [Cur03, Cur06]. The method described therein is to approximate the transfer function by a finite-dimensional-realization in order to construct a finite-dimensional controller that is of practical use and achieves the control task with a possible decline in performance. We develop this approach further by applying the approximation method of balanced truncation to a certain $\mathcal{H}^\infty$-balanced realization. This method was proposed in [MG91] for finite-dimensional systems and has been outlined in [Cur03, FSS13] for infinite-dimensional state linear systems. We convert these ideas into rigorous proofs, encompassing the more general class of Pritchard-Salamon systems.

Let us explain this approach: The method of balancing and truncation requires an output normalized or a balanced realization. Such realizations were given in [CG86, GCP88, GLP90, Gui12, GO14, Obe86, Obe87, Obe91, OMS91, Sta05]. For systems with compact Hankel operator we generalize certain balancing transformation that were first introduced in [TP87] for finite-dimensional systems. While all of the above works considering infinite-dimensional systems construct the balanced realization from a Hankel operator that is often not known explicitly, our transformations can be applied to any existing realization provided that a factorization of each Gramian is given. These factorizations can for example be obtained from an algorithm that is described in [ORW13] for infinite-dimensional systems. Moreover, in order to calculate the approximation, it suffices to evaluate the factors on a finite number of vectors.

With this approximation method at hand, the next obstacle is that it is only works for stable systems. The remedy is to stabilize a given system first by a state feedback and then perform balanced truncation on the closed-loop system. In the spirit of [MG91] we use a state feedback that solves a certain linear quadratic optimization problem. This has the advantage that the solution of this optimization problem can be obtained by solving an algebraic Riccati equation, which also yields the observability Gramian of the closed-loop system. In the process of balancing and truncation both Gramians are diagonalized, and we will show that the truncated Gramians solve two analogous algebraic Riccati equations. With this information we may construct a finite-dimensional robust $\mathcal{H}^\infty$-controller by the method described in e.g. [TSH01, MG90].

1.2. Outline of the thesis

In Chapter 2 basic definitions and elementary results about well-posed linear systems are recapped. Almost everything in this chapter is known, except for Section 2.6, in which a non-trivial, recurring example of the heat with boundary control is introduced.

In Chapter 3 we consider state linear systems that have a well-defined relative degree within the natural numbers. After giving our definition of relative degree in Section 3.1, we derive the zero dynamics form in Section 3.2. Subsequently, the Byrnes-Isidori form can easily be derived from the zero dynamics form of the dual system in Section 3.3.

For well-posed linear systems we define zero dynamics and their stability concepts in Chapter 4. The zero dynamics of systems with relative degree are characterized in Section 4.1 with the aid of the zero dynamics form and the Byrnes-Isidori form. In Section 4.2 we derive a similar result for the boundary control system introduced in Section 2.6.

Chapter 5 shows that funnel control is feasible for two new classes of infinite-dimensional systems: Section 5.1 covers systems with relative degree one and exponentially stable zero dynamics, Section 5.2 a large class of self-adjoint systems. In this section, we sharpen results for self-adjoint finite-dimensional systems before we treat the infinite-dimensional class. The results are applied to the boundary control system introduced in Section 2.6, which is an example of a self-adjoint system.

The second part of the thesis is concerned with the approach of balancing and truncation, and starts with the construction of various transformations in Chapter 6. The first section of this chapter recapitulates two canonical shift-realizations. Using these realizations we construct an output normalized realization on ℓ^2 in Section 6.2. In Section 6.3 we restrict the output normalized realization to a subspace and obtain an input normalized realization. The balanced realization in Section 6.4 results from interpolation between these two normalized realizations. Aside from the transformations we also address the truncation of the output normalized and the balanced realization.

Chapter 7 covers the $\mathcal{H}^\infty$-balancing and truncation process for Pritchard-Salamon systems. In Section 7.1 we introduce the type of Riccati-equations used in this thesis and recall corresponding results. In Sections 7.2 and 7.3 the methods of the preceding chapter are applied to a right factorization in order to obtain an $\mathcal{H}^\infty$-output normalized realization on ℓ^2 and its truncation. Finally, Section 7.4 contains the finite-dimensional robust controller design based on $\mathcal{H}^\infty$-balancing and truncation.

There is an appendix consisting of two sections: Section A.1 summarizes existing assertions on fractional powers of semigroup generators and interpolation, which are relevant for the boundary controlled heat equation. Section A.2 contains results on inhomogeneous Cauchy problems in Banach spaces connecting systems theory to differential equations.

A great part of the results presented in this thesis has been published in peer reviewed journals by the author together with Timo Reis: Section 4.2 is contained in [RS15b], Section 5.2 is the subject of [RS15a], and Chapter 6 is entirely contained in [RS14]. Furthermore, Sections 3.3, 4.1 and 5.1 are in the manuscript [IST15] which is currently undergoing a second revision.

2. Basic objects

In this chapter we recollect several definitions and fundamental results from the field of infinite-dimensional systems theory. Everything in this chapter is known up to Section 2.6, where we discuss a non-trivial example of a boundary control system. Most of the notations and conventions that we use are explained in this chapter. Further notation and a list of symbols and function spaces can be found in the appendix for quick reference.

2.1. Semigroups and rigged spaces

A *strongly continuous semigroup* in the Banach space $\mathcal{X}$ is a mapping $\mathfrak{A} : \mathbb{R}_{\geqslant 0} \to \mathcal{B}(\mathcal{X})$ that satisfies $\mathfrak{A}(0) = \mathrm{I}$, $\mathfrak{A}(s)\mathfrak{A}(t) = \mathfrak{A}(s+t)$ for all $s, t \geqslant 0$, and

$$\lim_{t \downarrow 0} \|\mathfrak{A}(t)x - x\|_{\mathcal{X}} = 0 \quad \forall\, x \in \mathcal{X}.$$

The growth bound of $\mathfrak{A}$ is

$$\omega_{\mathfrak{A}} := \inf \left\{ \omega \in \mathbb{R} \;\middle|\; \exists M > 0 \;\forall t \geqslant 0 : \;\|\mathfrak{A}(t)\|_{\mathcal{B}(\mathcal{X})} \leqslant M \mathrm{e}^{\omega t} \right\}.$$

The infimum in this definition is not always attained, but it is always less than infinity. The *generator* of a strongly continuous semigroup $\mathfrak{A}$ is the operator

$$Ax := \lim_{t \downarrow 0} \frac{1}{t}(\mathfrak{A}(t)x - x) \quad \forall\, x \in \operatorname{dom} A,$$

where $\operatorname{dom} A$ is defined as the set on which this limit exists. The generator is closed and densely defined, and $\mathbb{C}_{>\omega_{\mathfrak{A}}} \subset \rho(A)$.

An inevitable concept for semigroups are the rigged spaces. However, they are also relevant for operators that do not generate semigroups. We recap some basic facts that can be found in [Sta05, Section 3.6], [EN00, Section II.5].

Let $\mathcal{X}$ be a Banach space and $A : \operatorname{dom} A \subset \mathcal{X} \to \mathcal{X}$ be a densely defined operator with $\rho(A) \neq \emptyset$. Then the domain of A is a Banach space with respect to its natural graph norm $\|\cdot\|_{\operatorname{dom} A} := (\|\cdot\|_{\mathcal{X}}^2 + \|A\cdot\|_{\mathcal{X}}^2)^{\frac{1}{2}}$. This norm is equivalent to the expression $\|(\lambda - A)\cdot\|_{\mathcal{X}}$ for any $\lambda \in \rho(A)$. In situations where it is clear which operator A is used and the exact value of $\lambda \in \rho(A)$ is unimportant, it is convenient to write $\mathcal{X}_1 := \operatorname{dom} A$ and $\|\cdot\|_{\mathcal{X}_1} := \|(\lambda - A)\cdot\|_{\mathcal{X}}$ for some $\lambda \in \rho(A)$.

In the opposite direction, we take an arbitrary $\lambda \in \rho(A)$ and define the space $\mathcal{X}_{-1}$ to be the completion of $\mathcal{X}$ with respect to the norm $\|\cdot\|_{\mathcal{X}_{-1}} := \|(\lambda - A)^{-1}\cdot\|_{\mathcal{X}}$. Again, a different choice of $\lambda \in \rho(A)$ here leads to an equivalent norm. The operator $\lambda - A$ has a unique continuous extension $(\lambda - A)|_{\mathcal{X}}$ that maps $\mathcal{X}$ isometrically onto $\mathcal{X}_{-1}$. As a consequence, the operator

$$A|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}_{-1}, \qquad A|_{\mathcal{X}} := \lambda - (\lambda - A)|_{\mathcal{X}}, \tag{2.1}$$

is bounded and a continuous extension of A. If A generates a strongly continuous semigroup $\mathfrak{A}$ in $\mathcal{X}$ than $A|_{\mathcal{X}}$ generates a strongly continuous semigroup $t \mapsto \mathfrak{A}(t)|_{\mathcal{X}_{-1}}$ in $\mathcal{X}_{-1}$. Slightly abusing notation we denote this semigroup by $\mathfrak{A}|_{\mathcal{X}_{-1}}$. We point out that the rigged spaces remain the same (up to an equivalent norm) if we replace A by $A + B$ where $B \in \mathcal{B}(\mathcal{X})$ is a bounded perturbation.

If $\mathcal{X}$ is a Hilbert space, there is another representation of $\mathcal{X}_{-1}$ that is explained in [TW09, Section 2.10] and will be used throughout this thesis. Take note of the following conventions that we adopt from [TW09]: The scalar product $\langle \cdot , \cdot \rangle_{\mathcal{X}}$ of a Hilbert space $\mathcal{X}$ is defined to be linear in the first and anti-linear in the second component. When a Hilbert space is identified with its own dual via the Riesz isomorphism it is called a *pivot space*. Not all Hilbert spaces are considered as pivot spaces. The application of a functional $y \in \mathcal{X}'$ to $x \in \mathcal{X}$ is denoted by $\langle x , y \rangle_{\mathcal{X},\mathcal{X}'}$, and scalar multiplication on $\mathcal{X}'$ is defined by $\langle x , \lambda y \rangle_{\mathcal{X},\mathcal{X}'} := \bar{\lambda} \langle x , y \rangle_{\mathcal{X},\mathcal{X}'}$ for $y \in \mathcal{X}'$ and $x \in \mathcal{X}$. Furthermore, we define the reversed dual pairing $\langle y , x \rangle_{\mathcal{X}',\mathcal{X}} := \overline{\langle x , y \rangle_{\mathcal{X},\mathcal{X}'}}$ for $y \in \mathcal{X}'$ and $x \in \mathcal{X}$, which is linear in the first and anti-linear in the second component, just like the scalar product.

If $\mathcal{X}$ is a Hilbert space, the norm of the rigged space $\mathcal{X}_{-1}$ has the alternative representation

$$\|x\|_{\mathcal{X}_{-1}} := \sup_{\substack{z \in \operatorname{dom} A^*, \\ \|(\lambda - A)^* z\|_{\mathcal{X}} \leq 1}} |\langle x , z \rangle_{\mathcal{X}}| \quad \forall\, x \in \mathcal{X}.$$

From this expression it is not hard to see that $\|\cdot\|_{\mathcal{X}_{-1}}$ is equivalent to the norm $\|\cdot\|_{(\operatorname{dom} A^*)'}$ defined by

$$\|x\|_{(\operatorname{dom} A^*)'} := \sup_{\substack{z \in \operatorname{dom} A^*, \\ \|z\|_{\operatorname{dom} A^*} \leqslant 1}} |\langle x, z \rangle_{\mathcal{X}}| \quad \forall\, x \in \mathcal{X}.$$

When equipped with this norm, the space $\big(\mathcal{X}_{-1}, \|\cdot\|_{(\operatorname{dom} A^*)'}\big)$ is the so-called *dual space of* $(\operatorname{dom} A^*, \|\cdot\|_{\operatorname{dom} A^*})$ *with respect to the pivot space* $\mathcal{X}$. Indeed, it can be shown that the following mapping J is an isometric isomorphism from $\mathcal{X}_{-1}$ onto the dual space of $\operatorname{dom} A^*$: For $z \in \mathcal{X}_{-1}$ pick a sequence $(x_n) \in \mathcal{X}$ with $\|x_n - z\|_{\mathcal{X}_{-1}} \to 0$ as $n \to \infty$, then $Jz \in (\operatorname{dom} A^*)'$ is defined via

$$\langle Jz, \varphi \rangle_{(\operatorname{dom} A^*)', \operatorname{dom} A^*} := \lim_{n \to \infty} \langle x_n, \varphi \rangle_{\mathcal{X}} \quad \forall\, \varphi \in \operatorname{dom} A^*.$$

With this isomorphism we will always interpret $(\operatorname{dom} A^*)'$ as the dual space of $\operatorname{dom} A^*$ with respect to the pivot space $\mathcal{X}$. The bidual space $(\operatorname{dom} A^*)''$ is always identified with $\operatorname{dom} A^*$ itself. If a topological vector space $\mathcal{W}$ is continuously and densely embedded into the topological vector space $\mathcal{X}$, we write $\mathcal{W} \hookrightarrow \mathcal{X}$. The following Lemma taken from [TW09, Proposition 2.9.3] gives a sufficient condition for an operator to be extendable to $\mathcal{X}_{-1}$.

Lemma 2.1.1. *Let $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{X}_1$ and $\mathcal{X}_2$ be Hilbert spaces with $\mathcal{W}_1 \hookrightarrow \mathcal{X}_1$ and $\mathcal{W}_2 \hookrightarrow \mathcal{X}_2$. Assume $A \in \mathcal{B}(\mathcal{W}_1; \mathcal{X}_2)$ satisfies $A^*\mathcal{W}_2 \subset \mathcal{W}_1$. Then A has an extension $A|_{(\mathcal{W}_1)'} : (\mathcal{W}_1)' \to (\mathcal{W}_2)'$ given by $\langle A|_{(\mathcal{W}_1)'} w_1', w_2 \rangle_{\mathcal{W}_2', \mathcal{W}_2} := \langle w_1', A^* w_2 \rangle_{\mathcal{W}_1', \mathcal{W}_1}$ for $w_1' \in \mathcal{W}_1'$ and $w_2 \in \mathcal{W}_2$.*

2.2. System nodes

The direct product of two normed spaces $\mathcal{X}$ and $\mathcal{U}$ is denoted by $\mathcal{X} \times \mathcal{U}$ and equipped with the norm $\big\|[x\,,\, u]^\top\big\|_{\mathcal{X}\times\mathcal{U}} := \big(\|x\|_{\mathcal{X}}^2 + \|u\|_{\mathcal{U}}^2\big)^{1/2}$.

In this thesis we study dynamics that obey equations of the type

$$\dot{x}(t) = A\&B \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad y(t) = C\&D \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}.$$

Here, the expression $A\&B$ is simply the name of an operator that is defined on some subset of a product space $\mathcal{X} \times \mathcal{U}$. The symbols A and B hint towards special cases in which $A\&B$ can be split into two independent operators, $A : \operatorname{dom} A \subset \mathcal{X} \to \mathcal{X}$ and $B : \mathcal{U} \to \mathcal{X}$. The same is true for the operator $C\&D$, which can sometimes be split into two operators C and D. Let us render these objects more precisely by recalling several definitions from [Sta05].

Definition 2.2.1 (system node)**.** Let $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ be Hilbert spaces. A block operator

$$S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : \operatorname{dom} S \subset \mathcal{X} \times \mathcal{U} \;\to\; \mathcal{X} \times \mathcal{Y}$$

is an *operator node* if the following holds:

(i) S is a closed operator.

(ii) The operator $A : \operatorname{dom} A \subset \mathcal{X} \to \mathcal{X}$ defined by

$$Ax = A\&B \begin{bmatrix} x \\ 0 \end{bmatrix}, \qquad \operatorname{dom} A := \left\{ x \in \mathcal{X} \;\middle|\; \begin{bmatrix} x \\ 0 \end{bmatrix} \in \operatorname{dom} S \right\},$$

satisfies $\rho(A) \neq \varnothing$, and $\operatorname{dom} A$ is dense in $\mathcal{X}$.

(iii) The operator $A\&B$ can be extended to an operator $[A|_{\mathcal{X}}, B] \in \mathcal{B}(\mathcal{X} \times \mathcal{U}; \mathcal{X}_{-1})$, where $\mathcal{X}_{-1}$ is defined as in Section 2.1.

(iv) $\operatorname{dom} S = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{X} \times \mathcal{U} \;\middle|\; A|_{\mathcal{X}} + Bu \in \mathcal{X} \right\}.$

If, in addition, A is the generator of a strongly continuous semigroup $\mathfrak{A}$, then S is called a *system node*. We call A the *main operator*, and B the *control operator* of S. The operator $C \in \mathcal{B}(\operatorname{dom} A; \mathcal{U})$ defined by $Cx := C\&D \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ is called the *observation operators* of S. The *transfer function* of S at the point $\lambda \in \rho(A)$ is the operator

$$\mathbf{G}(\lambda) := C\&D \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1} B \\ \mathrm{I} \end{bmatrix} \quad \in \mathcal{B}(\mathcal{U}; \mathcal{Y}).$$

While the operator $A\&B$ is by definition always expendable into $A|_{\mathcal{X}}$ and B, the operator $C\&D$ can, in general, not be split. A class of operator nodes for which this is possible appears in the following definition adapted from [Sta05, Definition 5.1.1].

Definition 2.2.2 (compatible)**.** An operator node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ with main operator A is said to be *compatible with compatibility space* $\mathcal{W}$ if there exists a Hilbert space $\mathcal{W} \hookrightarrow \mathcal{X}$ such that $\operatorname{dom} A$ is continuously embedded into $\mathcal{W}$, the observation operator C has a bounded extension $C|_{\mathcal{W}} \in \mathcal{B}(\mathcal{W}; \mathcal{Y})$, and the control operator B satisfies $(\lambda - A|_{\mathcal{X}})^{-1} B\mathcal{U} \subset \mathcal{W}$ for some $\lambda \in \rho(A)$. In this case the expression

$$D := C\&D \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1} B \\ \mathrm{I} \end{bmatrix} - C|_{\mathcal{W}} (\lambda - A|_{\mathcal{X}})^{-1} B \tag{2.2}$$

is independent of $\lambda \in \rho(A)$ and an element of $\mathcal{B}(\mathcal{U}; \mathcal{Y})$ [Sta05, Lemma 5.1.4]. We call D the *feedthrough operator* induced by $C|_{\mathcal{W}}$.

Remark 2.2.3. (i) If $\operatorname{dom} A$ is not dense in $\mathcal{W}$ then the extension $C|_{\mathcal{W}}$ is not unique, and the feedthroughs induced by different extensions $C|_{\mathcal{W}}$ will in general not be the same; an example of this phenomenon is illustrated in Remark 2.6.5 (iii) and [Sta05, Remark 5.1.6].

(ii) An immediate consequence of (2.2) is that the transfer function of a compatible system node has the representation

$$\mathbf{G}(\lambda) = C|_{\mathcal{W}} (\lambda - A)^{-1} B + D \quad \forall\, \lambda \in \rho(A).$$

There is always an outstanding compatibility space that has nice characterization in terms of A and B:

Lemma 2.2.4. *Let S be a compatible operator node with compatibility space $\mathcal{W}$, main operator A and control operator B. Define*

$$(\mathcal{X} + B\mathcal{U})_1 := \left\{ \, x \in \mathcal{X} \;\middle|\; A\big|_{\mathcal{X}} x \in \mathcal{X} + B\mathcal{U} \, \right\}$$

with norm

$$\|x\|^2_{(\mathcal{X}+B\mathcal{U})_1} := \inf \left\{ \, \|x\|^2_{\mathcal{X}} + \|A\big|_{\mathcal{X}} x + Bu\|^2_{\mathcal{X}} + \|u\|^2_{\mathcal{U}} \;\middle|\; u \in \mathcal{U} \wedge A\big|_{\mathcal{X}} x + Bu \in \mathcal{X} \, \right\}.$$

Then S is also compatible with compatibility space $(\mathcal{X} + B\mathcal{U})_1$. Moreover, we have $(\mathcal{X} + B\mathcal{U})_1 = (\lambda - A|_{\mathcal{X}})^{-1}(\mathcal{X} + B\mathcal{U})$.

The main part of this lemma is [Sta05, Theorem 5.1.8]. The alternative characterization of the set $(\mathcal{X} + B\mathcal{U})_1$ is shown in [Sta05, Lemma 4.3.12].

By the following lemma proven in [Sta05, Lemma 4.7.8], the dynamics of a system node are well-defined for sufficiently regular input functions.

Lemma 2.2.5. *Let* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be a system node on* $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ *and let* $A, B, C\&D$ *and the semigroup* $\mathfrak{A}$ *be as in Definition 2.2.1. Then* $\mathfrak{A}$ *extends to a strongly continuous semigroup on* $(\operatorname{dom} A^*)'$*, and for each* $x_0 \in \mathcal{X}$ *and* $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}; \mathcal{U})$ *with* $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \operatorname{dom} S$*, the function*

$$x(t) := \mathfrak{A}(t)x_0 + \int_0^t \mathfrak{A}(t-\tau)\big|_{(\operatorname{dom} A^*)'} Bu(\tau)\,\mathrm{d}\tau \quad \forall\, t \geqslant 0,$$

satisfies $x \in \mathcal{C}^1(\mathbb{R}_{\geqslant 0}; \mathcal{X})$ *and* $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{C}(\mathbb{R}_{\geqslant 0}; \operatorname{dom} S)$*. Together with*

$$y(t) := C\&D \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \quad \forall\, t \geqslant 0,$$

the function $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] : [0, \infty) \to \mathcal{X} \times \mathcal{Y}$ *is the unique solution of the equation*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \quad \forall\, t \geqslant 0, \qquad x(0) = x_0. \tag{2.3}$$

Definition 2.2.6 (L^p-well-posedness)**.** An operator node S on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ is said to be *L^p-well-posed* with $p \in [1, \infty]$ if S is a system node and, for each $t > 0$, there exists a constant $M > 0$ such that all solutions $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ of the type described in the previous lemma satisfy

$$\|x(t)\|_{\mathcal{X}} + \|y\|_{L^p([0,t];\mathcal{Y})} \leqslant M \left(\|x(0)\|_{\mathcal{X}} + \|u\|_{L^p([0,t];\mathcal{U})} \right).$$

In Section 2.4 we will see that every well-posed operator node generates a so-called well-posed linear system. We conclude this section with a short motivation for this abstract concept: Let S be a well-posed system node and define for $t \geqslant 0$, $x_0 \in \operatorname{dom} A$ and $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}; \mathcal{U})$ the input-to-state map $\mathfrak{B}_t$, the state-to-output map $\mathfrak{C}_t$, and the input-output map $\mathfrak{D}_t$ by

$$\begin{aligned} \mathfrak{B}_t u|_{[0,t]} &:= \int_0^t \mathfrak{A}(t-\tau)\big|_{(\operatorname{dom} A^*)'} Bu(\tau)\,\mathrm{d}\tau, \qquad \mathfrak{C}_t x_0 := C\mathfrak{A}(\cdot)x_0, \\ \mathfrak{D}_t u|_{[0,t]} &:= C\&D \begin{bmatrix} \mathfrak{B}_t u|_{[0,t]} \\ u(t) \end{bmatrix}. \end{aligned} \tag{2.4}$$

Then the solution in Lemma 2.2.5 may be written as

$$\begin{bmatrix} x(t) \\ y|_{[0,t]} \end{bmatrix} := \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B}_t \\ \mathfrak{C}_t & \mathfrak{D}_t \end{bmatrix} \begin{bmatrix} x_0 \\ u|_{[0,t]} \end{bmatrix}. \tag{2.5}$$

The L^p-well-posedness then implies that we can extend the above operators so that the four operators

$$\begin{aligned} &\mathfrak{A}(t) : \mathcal{X} \to \mathcal{X}, && \mathfrak{B}_t : L^p([0,t];\mathcal{U}) \to \mathcal{X}, \\ &\mathfrak{C}_t : \mathcal{X} \to L^p([0,t];\mathcal{Y}), && \mathfrak{D}_t : L^p([0,t];\mathcal{U}) \to L^p([0,t];\mathcal{Y}), \end{aligned}$$

are continuous. Hence, we may use (2.5) to define the state $x(t)$ and output $y|_{[0,t]}$ for arbitrary $x_0 \in \mathcal{X}$ and $u \in L^p([0,t];\mathcal{U})$. The four operators above constitute a well-posed linear system in the sense of Section 2.4. For this thesis, which ultimately aims at output feedback without any knowledge of the internal states, the input-output map $\mathfrak{D}_t$ is of particular interest. Therefore it receives some extra attention in the following section.

2.3. Time-invariant causal operators and transfer functions

This section summarizes some facts about input-output maps regardless of any underlying system and its internal states.

For a function $u : \mathbb{R} \to \mathcal{U}$ and $J \subset \mathbb{R}$ we define the projections

$$(\pi_J u)(s) := \begin{cases} u(s), & s \in J, \\ 0, & s \in \mathbb{R}\backslash J, \end{cases}$$

and the special cases $\pi_+ := \pi_{[0,\infty)}$, $\pi_- := \pi_{(-\infty,0)}$. Furthermore, we use the shift operator

$$(\tau^t u)(s) := u(t+s), \quad \forall\, s \in \mathbb{R}.$$

The shift operator τ^t acting on $L^p_\omega(\mathbb{R}_{\leqslant 0};\mathcal{U})$ or $L^p_\omega(\mathbb{R}_{\geqslant 0};\mathcal{U})$ is denoted by τ^t_- and τ^t_+, respectively.

We will often implicitly extend a function u whose domain is a subset of $\mathbb{R}$ by zero when we apply the above operators. In the following definitions from [Sta05], the space $L^p_{\mathrm{c,loc}}(\mathbb{R};\mathcal{U})$ is used. It is defined as the set of all functions $u \in L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{U})$ whose support is bounded from below, (and not necessarily compact). $L^p_\omega(\mathbb{R}_{\geqslant 0};\mathcal{Y})$ is the exponentially weighted Lebesgue space. It consists of all functions u that satisfy $\mathrm{e}_\omega u \in L^p(\mathbb{R};\mathcal{U})$, where e_ω is the function $\mathrm{e}_\omega(t) := e^{\omega t}$.

Definition 2.3.1. Let $1 \leqslant p \leqslant \infty$, $\omega \in \mathbb{R}$, let $\mathcal{U}$ and $\mathcal{Y}$ be Banach spaces, and let $\mathfrak{D} : \mathrm{dom}(\mathfrak{D}) \subset L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{U}) \to L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{U})$ be a linear operator.

(i) $\mathfrak{D}$ is *time-invariant* if $\tau^t\mathfrak{D}u = \mathfrak{D}\tau^t u$ for all $u \in \mathrm{dom}\,\mathfrak{D}$ and all $t \in \mathbb{R}$.

(ii) A time-invariant operator $\mathfrak{D}$ is *causal* if $\pi_-\mathfrak{D}|_{L^p_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U})} = 0$.

(iii) $\mathrm{TIC}^p_\omega(\mathcal{U};\mathcal{Y})$ stands for the space of all bounded linear time-invariant causal operators $\mathfrak{D} : L^p_\omega(\mathbb{R};\mathcal{U}) \to L^p_\omega(\mathbb{R};\mathcal{Y})$.

(iv) $\mathrm{TIC}^p_{\mathrm{loc}}(\mathcal{U};\mathcal{Y})$ stands for the space of all linear time-invariant causal operators $\mathfrak{D} : L^p_{\mathrm{c,loc}}(\mathbb{R};\mathcal{U}) \to L^p_{\mathrm{c,loc}}(\mathbb{R};\mathcal{Y})$ with $\mathfrak{D}|_{L^p(I;\mathcal{U})} \in \mathcal{B}(L^p(I;\mathcal{U}); L^p(I;\mathcal{Y}))$ for every compact interval $I \subset \mathbb{R}$.

(v) The *Hankel operator* induced by $\mathfrak{D} \in \mathrm{TIC}^p_\omega(\mathcal{U};\mathcal{Y})$ is the operator

$$\mathfrak{H} : L^p_\omega(\mathbb{R}_{\leqslant 0};\mathcal{U}) \to L^p_\omega(\mathbb{R}_{\geqslant 0};\mathcal{Y}), \qquad \mathfrak{H} := \pi_+\mathfrak{D}|_{L^p_{\mathrm{loc}}(\mathbb{R}_{\leqslant 0};\mathcal{U})}.$$

Remark 2.3.2. If u is an $L^p_{\mathrm{c,loc}}(\mathbb{R};\mathcal{U})$-function, then the support of u is contained in $[\ell,\infty)$ for some $\ell \in \mathbb{R}$ and for every $t \in \mathbb{R}$ the norm $\left\|\pi_{(-\infty,t]}u\right\|_{L^p_\omega(\mathbb{R};\mathcal{U})}$ is finite. Hence, $\mathfrak{D}\pi_{(-\infty,t]}u$ is well-defined for $\mathfrak{D} \in \mathrm{TIC}^p_\omega(\mathcal{U};\mathcal{Y})$. The causality of $\mathfrak{D}$ guarantees that there is a unique function $y \in L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{Y})$ that satisfies

$$\pi_{(-\infty,t]}y := \pi_{(-\infty,t]}\mathfrak{D}\pi_{(-\infty,t]}u \quad \forall\, t \geqslant \ell.$$

With the assignment $\widetilde{\mathfrak{D}}u := y$ we define a mapping $\widetilde{\mathfrak{D}} : L^p_{\mathrm{c,loc}}(\mathbb{R};\mathcal{U}) \to L^p_{\mathrm{c,loc}}(\mathbb{R};\mathcal{Y})$, which coincides with $\mathfrak{D}$ on the intersection of $L^p_{\mathrm{c,loc}}(\mathbb{R};\mathcal{U})$ and $L^p_\omega(\mathbb{R};\mathcal{U})$. The mapping $\widetilde{\mathfrak{D}}$ can bee seen as a restriction of $\mathfrak{D}$ to $L^p_\omega(\mathbb{R};\mathcal{U})$- functions with compact support, followed by an extension. We will sometimes identify $\mathfrak{D}$ with $\widetilde{\mathfrak{D}}$ without explicit warning. Note however that we will only use the continuity of $\mathfrak{D}$ between the normed spaces $L^p_\omega(\mathbb{R};\mathcal{U})$ and $L^p_\omega(\mathbb{R};\mathcal{Y})$. We do not discuss continuity properties of $\widetilde{\mathfrak{D}}$ because the topology of $L^p_{\mathrm{c,loc}}(\mathbb{R};\mathcal{U})$ is rather involved and not locally convex.

Definition 2.3.3. An operator $\mathfrak{D} \in \mathrm{TIC}^p_{\mathrm{loc}}(\mathcal{U};\mathcal{Y})$ is said to be *stable* if it has a continuous extension to $\mathrm{TIC}^p_0(\mathcal{U};\mathcal{Y})$.

An operator $[\mathfrak{N}\,,\,\mathfrak{M}]^\top \in \mathrm{TIC}^p_0(\mathcal{U};\mathcal{Y}\times\mathcal{U})$ is called a *right factorization* of $\mathfrak{D} \in \mathrm{TIC}^p_{\mathrm{loc}}(\mathcal{U};\mathcal{Y})$ if $\mathfrak{M}$ has an inverse in $\mathrm{TIC}^p_{\mathrm{loc}}(\mathcal{U})$ such that $\mathfrak{D}u = \mathfrak{N}\mathfrak{M}^{-1}u$ for all $u \in L^p_{\mathrm{c,loc}}(\mathbb{R};\mathcal{U})$.

A stable operator $\mathfrak{D} \in \mathrm{TIC}^p_{\mathrm{loc}}(\mathcal{U};\mathcal{Y})$ will be identified with its extension. Also in the sense of extensions, we write $\mathfrak{D} = \mathfrak{N}\mathfrak{M}^{-1}$. Note that a right factorization is by definition stable.

Definition 2.3.4. The Laplace transform of a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U})$ is given by

$$\widehat{u}(s) = \int_0^\infty e^{-st}u(t)\,\mathrm{d}t$$

for all $s \in \mathbb{C}$ for which the integral converges absolutely. The domain of $\widehat{u}$ is a half plane $\mathbb{C}_{\geqslant\omega}$ for some $\omega \in \mathbb{R}$.

Recall the definition of Hardy spaces from [Dur70, Sta05]: For $p \in [1,\infty)$, the Hardy space $\mathcal{H}^p_\omega(\mathcal{U})$ is the set of all analytic functions $f : \mathbb{C}_{>\omega} \to \mathcal{U}$ with finite norm

$$\|f\|_{\mathcal{H}^p_\omega(\mathcal{U})} := \sup_{\alpha>\omega}\left(\int_{-\infty}^\infty \|f(\alpha+i\beta)\|^p_{\mathcal{U}}\,\mathrm{d}\beta\right)^{\frac{1}{p}},$$

and the space $\mathcal{H}^\infty_\omega(\mathcal{U};\mathcal{Y})$ is the set of all bounded analytic functions $f : \mathbb{C}_{>\omega} \to \mathcal{B}(\mathcal{U};\mathcal{Y})$ with norm

$$\|f\|_{\mathcal{H}^\infty_\omega(\mathcal{U};\mathcal{Y})} := \sup_{s\in\mathbb{C}_{>\omega}} \|f(s)\|_{\mathcal{B}(\mathcal{U};\mathcal{Y})}\,.$$

Theorem 2.3.5 (Paley-Wiener theorem)**.** *If $\mathcal{U}$ is a Hilbert space, then the Laplace transform,*

$$\widehat{\cdot} : L^2_\omega(\mathbb{R}_{\geqslant 0};\mathcal{U}) \to \mathcal{H}^2_\omega(\mathcal{U}),$$

is a homeomorphism with $\|\hat{u}\|_{\mathcal{H}^2_\omega(\mathcal{U})} = \sqrt{2\pi}\|u\|_{L^2_\omega(\mathbb{R}_{\geqslant 0};\mathcal{U})}$.

A proof of the Hilbert space-valued version of well-known Paley-Wiener theorem can be found in [Sta05, Theorem 10.3.4].

We define transfer functions for time-invariant causal operators in the sense of [Sta05, Definition 4.6.1]. The ambiguity between the transfer function of a system node and the transfer function of its input-output map is resolved in Lemma 2.4.6.

Definition 2.3.6. The *transfer function* $\widehat{\mathfrak{D}}$ of an operator $\mathfrak{D} \in \mathrm{TIC}^p_\omega(\mathcal{U};\mathcal{Y})$ is the operator valued function

$$\widehat{\mathfrak{D}}(s) = (u \mapsto (\mathfrak{D}(e_s u))(0)) \quad \forall\, s \in \mathbb{C}_{\geqslant\omega}.$$

In the next lemma, we summarize parts of [Sta05, Corollary 4.6.10] and [Sta05, Lemma 10.3.3].

Lemma 2.3.7. *Let* $\mathfrak{D} \in \mathrm{TIC}^p_\omega(\mathcal{U};\mathcal{Y})$. *Then the transfer function* $\widehat{\mathfrak{D}}$ *is an analytic* $\mathcal{B}(\mathcal{U};\mathcal{Y})$*-valued function in* $\mathbb{C}_{>\omega}$*, and, for each* $\alpha > \omega$*, it is in* $\mathcal{H}^\infty_\alpha(\mathcal{U};\mathcal{Y})$ *with* $\|\widehat{\mathfrak{D}}\|_{\mathcal{H}^\infty_\alpha(\mathcal{U};\mathcal{Y})} \leqslant \|\mathfrak{D}\|_{\mathrm{TIC}_\alpha(\mathcal{U};\mathcal{Y})}$. *Furthermore, for all* $u \in L^p_\omega(\mathbb{R}_{\geqslant 0};\mathcal{U})$ *the relation*

$$(\widehat{\mathfrak{D}u})(s) = \widehat{\mathfrak{D}}(s)\hat{u}(s), \quad \forall\, s \in \mathbb{C}_{\geqslant\omega},$$

holds.

The following result is a consequence of the Paley-Wiener theorem. A proof is given in [Sta05, Theorem 10.3.5].

Theorem 2.3.8. *If* $\mathcal{U}$ *and* $\mathcal{Y}$ *are Hilbert spaces, then the mapping* $\widehat{\cdot} : \mathrm{TIC}^2_\omega(\mathcal{U};\mathcal{Y}) \to \mathcal{H}^\infty_\omega(\mathcal{U};\mathcal{Y})$ *that associates each time-invariant causal operator to its transfer function is an isometric isomorphism.*

Definition 2.3.9 (regular)**.** A function $\widehat{\mathfrak{D}} \in \mathcal{H}^\infty_\omega(\mathcal{U};\mathcal{Y})$ is said to be *strongly regular* if the limit

$$Du := \lim_{s\to\infty, s\in\mathbb{R}} \widehat{\mathfrak{D}}(s)u$$

exists in $\mathcal{Y}$ for all $u \in \mathcal{U}$, and *uniformly regular* if the limit

$$D := \lim_{s\to\infty, s\in\mathbb{R}} \widehat{\mathfrak{D}}(s) \tag{2.6}$$

exists in $\mathcal{B}(\mathcal{U};\mathcal{Y})$.

This definition is taken from [Sta05, Definition 5.6.1]. By the uniform boundedness principle D defines a bounded linear operator from $\mathcal{U}$ to $\mathcal{Y}$, even in the strongly regular case.

2.4. Well-posed linear systems and their generators

In this section, $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$ are always Hilbert spaces.

Definition 2.4.1 (Well-posed linear system)**.** Let $p \in [1, \infty]$. An *ω-bounded L^p-well-posed linear system on* $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ is a quadruple $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ with the following properties:

(i) $t \mapsto \mathfrak{A}(t) \in \mathcal{B}(\mathcal{X})$ is a strongly continuous semigroup on $\mathcal{X}$ with growth bound $\omega_{\mathfrak{A}} < \omega$;

(ii) $\mathfrak{B} \in \mathcal{B}(L^p_\omega(\mathbb{R}_{\leqslant 0}; \mathcal{U}); \mathcal{X})$ satisfies $\mathfrak{A}(t)\mathfrak{B} = \mathfrak{B}\tau^t_-$ for all $t \geqslant 0$;

(iii) $\mathfrak{C} \in \mathcal{B}(\mathcal{X}; L^p_\omega(\mathbb{R}_{\geqslant 0}; \mathcal{Y}))$ satisfies $\mathfrak{C}\mathfrak{A}(t) = \tau^t_+\mathfrak{C}$ for all $t \geqslant 0$;

(iv) $\mathfrak{D} \in \mathcal{B}(L^p_\omega(\mathbb{R}; \mathcal{U}); L^p_\omega(\mathbb{R}; \mathcal{Y}))$ is continuous, causal, time-invariant and it satisfies $\pi_+\mathfrak{D}|_{L^p_\omega(\mathbb{R}_{\leqslant 0};\mathcal{U})} = \mathfrak{C}\mathfrak{B}$.

The growth bound of the system is defined as the growth bound of its semigroup. An *L^p-well-posed linear system* is a quadruple of operators that is an ω-bounded L^p-well-posed linear system for some $\omega \in \mathbb{R}$. Any L^p-well-posed linear system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is called a *realization of* $\mathfrak{D}$ and of $\widehat{\mathfrak{D}}$.

An L^p-well-posed linear system is called *observable* if $\ker \mathfrak{C} = \{0\}$, *controllable* if $\operatorname{ran} \mathfrak{B}$ is dense in $\mathcal{X}$, and *minimal* if it is both, controllable and observable.

Remark 2.4.2. An ω-bounded well-posed linear system is also α-bounded for every $\alpha > \omega_{\mathfrak{A}}$ [Sta05, Theorem 2.5.4 (iv)]. That is why there is usually no need to specify the bound ω when the growth bound of $\omega_{\mathfrak{A}}$ is known. An exception is the case where $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is 0-bounded, but $\mathfrak{A}$ is not exponentially stable.

By [Sta05, Lemma 4.3.5] and [Sta05, Lemma 4.4.1], well-posed linear systems have the following smoothing properties.

Lemma 2.4.3. *Let $p \in [1, \infty)$, let $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ be an ω-bounded L^p-well-posed linear system on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$, and denote the generator of $\mathfrak{A}$ by A. Then the following holds:*

(i) $\mathfrak{B}$ *maps* $W^{1,p}_{0,\omega}(\mathbb{R}_{\leqslant 0};\mathcal{U})$ *continuously into* $\operatorname{dom} A$*, and for all* $u \in W^{1,p}_{0,\omega}(\mathbb{R}_{\leqslant 0};\mathcal{U})$ *we have* $\mathfrak{B}\dot{u} = A\mathfrak{B}u$*.*

(ii) $\mathfrak{C}$ *maps* $\operatorname{dom} A$ *continuously into* $W^{1,p}_{\omega}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$*, and for all* $x \in \operatorname{dom} A$ *the derivative of* $\mathfrak{C}x$ *is the* L^p*-function* $\mathfrak{C}Ax$*.*

Definition 2.4.4 (Gramian)**.** Let $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ be a 0-bounded L^2-well-posed linear system on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$. The operator $\mathfrak{B}\mathfrak{B}^* \in \mathcal{B}(\mathcal{X})$ is the *controllability Gramian*, and $\mathfrak{C}^*\mathfrak{C} \in \mathcal{B}(\mathcal{X})$ the *observability Gramian* of $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$.

Definition 2.4.5. Let $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ be an L^p-well-posed linear system with growth bound $\omega_{\mathfrak{A}}$. The *main operator* of the system is the generator A of the semigroup $\mathfrak{A}$. For all $u \in \mathcal{U}$, the following expression is independent of $\lambda \in \mathbb{C}_{>\omega_{\mathfrak{A}}}$ [Sta05, Theorem 4.2.1]:

$$Bu := (\lambda - A|_{\mathcal{X}})\mathfrak{B}\mathrm{e}_\lambda u. \tag{2.7}$$

The operator $B \in \mathcal{B}(\mathcal{U}; (\operatorname{dom} A^*)')$ defined by (2.7) is the *control operator* of the system. The *observation operator* $C \in \mathcal{B}(\operatorname{dom} A; \mathcal{Y})$ of the system is defined by

$$Cx := (\mathfrak{C}x)(0) \quad \forall\, x \in \operatorname{dom} A.$$

To every well-posed linear system there is a corresponding well-posed system node and vice versa:

Lemma 2.4.6. *Let* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ *be an* L^p*-well-posed linear system on* $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ *with growth bound* $\omega_{\mathfrak{A}}$*, main operator* A*, control operator* B *and observation operator* C*. Define the set*

$$\operatorname{dom} S := \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{X} \times \mathcal{U} \;\middle|\; A|_{\mathcal{X}} + Bu \in \mathcal{X} \right\}$$

Then for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom} S$*, the expression*

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} := C\left(x - (\lambda - A|_{\mathcal{X}})^{-1}Bu\right) + \widehat{\mathfrak{D}}(\lambda)u$$

is independent of $\lambda \in \mathbb{C}_{>\omega_{\mathfrak{A}}}$ *and the block operator*

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : \operatorname{dom} S \subset \mathcal{X} \times \mathcal{U} \to \mathcal{X} \times \mathcal{Y},$$

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} := \begin{bmatrix} A|_{\mathcal{X}} x + Bu \\ C\left(x - (\lambda - A|_{\mathcal{X}})^{-1} Bu\right) + \widehat{\mathfrak{D}}(\lambda) u \end{bmatrix},$$

is an L^p-well-posed system node with main operator A, control operator B and observation operator C. Moreover, the transfer function $\mathbf{G}$ of this system node and the transfer function $\widehat{\mathfrak{D}}$ of $\mathfrak{D}$ satisfy

$$\widehat{\mathfrak{D}}(\lambda) = \mathbf{G}(\lambda) = C\&D \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1} B \\ \mathrm{I} \end{bmatrix} \qquad \forall\, \lambda \in \mathbb{C}_{>\omega_{\mathfrak{A}}}.$$

This follows from [Sta05, Theorem 4.7.13]; consult [Sta05, Definition 4.6.4] and [Sta05, Theorem 4.6.7] for the well-definition of $C\&D$ and the formula for the transfer function.

Summarizing [Sta05, Theorem 4.7.13] and [Sta05, Theorem 4.7.14], we may state the following lemma.

Lemma 2.4.7. *Let $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be an L^p-well-posed system node, and let ω be greater then the growth bound $\omega_{\mathfrak{A}}$ of the semigroup $\mathfrak{A}$. Then we can define the operators*

$$\mathfrak{B} : L^p_\omega(\mathbb{R}_{\leqslant 0}; \mathcal{U}) \to \mathcal{X}, \qquad \mathfrak{C} : \mathcal{X} \to L^p_\omega(\mathbb{R}_{\geqslant 0}; \mathcal{Y}), \qquad \mathfrak{D} : L^p_\omega(\mathbb{R}; \mathcal{U}) \to L^p_\omega(\mathbb{R}; \mathcal{Y})$$

by continuous extension of the mappings

$$\begin{aligned} \mathfrak{B}u &:= \int_{-\infty}^{0} \mathfrak{A}|_{(\operatorname{dom} A*)'}(-s) Bu(s)\,\mathrm{d}s && \forall\, u \in L^p_{\mathrm{c,loc}}(\mathbb{R}_{\leqslant 0}) \\ \mathfrak{C}x &:= C\mathfrak{A}(\cdot)x && \forall\, x \in \operatorname{dom} A, \\ \mathfrak{D}u &:= \left(t \mapsto C\&D \begin{bmatrix} \int_{-\infty}^{t} \mathfrak{A}|_{(\operatorname{dom} A*)'}(-s) Bu(s)\,\mathrm{d}s \\ u(t) \end{bmatrix} \right) && \forall\, t \in \mathbb{R}, u \in W^{2,p}_{\mathrm{c,loc}}(\mathbb{R}; \mathcal{U}), \end{aligned}$$

and $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is an ω-bounded, L^p-well-posed linear system. The system node associated to $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ via Lemma 2.4.6 is $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$.

Remark 2.4.8. (i) The last two Lemmas show that the well-posed linear system and the system node determine each other uniquely. Furthermore, the system node (and hence also the well-posed linear system) is uniquely determined by its main operator, control operator, observation operator and the value of the transfer function at some point $\lambda \in \mathbb{C}_{>\omega_{\mathfrak{A}}}$.

(ii) The fact that the main operator, control operator and observation operator of the node and the semigroup coincide implies that the operators $\mathfrak{B}_t$, $\mathfrak{C}_t$ and $\mathfrak{D}_t$ defined via (2.4) satisfy

$$\begin{aligned}
\mathfrak{B}_t u &= \mathfrak{B}\tau^t u && \forall\, u \in L^p([0,t];\mathcal{U}),\\
\mathfrak{C}_t x &= \pi_{[0,t]}\mathfrak{C}x && \forall\, x \in \mathcal{X},\\
\mathfrak{D}_t u &= \pi_{[0,t]}\mathfrak{D}u && \forall\, u \in L^p([0,t];\mathcal{U}).
\end{aligned}$$

Now we can formally define the state and output function as indicated at the end of Section 2.2.

Definition 2.4.9 (behavior)**.** Let $(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})$ be an L^p-well-posed linear system on $(\mathcal{U},\mathcal{X},\mathcal{Y})$. The *state* $x(t)$ at time t and the *output function* $y \in L^p_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$ *of* $(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})$ *with initial value* $x_0 \in \mathcal{X}$ *and input function* $u \in L^p_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U})$ are defined by

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} := \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B}\pi_-\tau^t \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix}. \tag{2.8}$$

The *behavior* of $(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})$ is defined as

$$\operatorname{bhv}(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D}) :=$$

$$\left\{ \begin{array}{l} (x,u,y) \in \mathcal{C}(\mathbb{R}_{\geqslant 0};\mathcal{X}) \times L^p_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U}) \times L^p_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{Y}) : \\ \begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B}\pi_-\tau^t \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} \begin{bmatrix} x(0) \\ u \end{bmatrix} \quad \forall\, t \geqslant 0. \end{array} \right\}$$

The state, the output, and the behavior of the associated system node are defined as the state, output, and behavior of $(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})$.

Remark 2.4.10. (i) For $u \in L^p_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U})$ we have $\pi_-\tau^t u \in L^p_\omega(\mathbb{R}_{\leqslant 0};\mathcal{U})$, so $\mathfrak{B}\pi_-\tau^t u$ is well-defined. By [Sta05, Theorem 2.12], the state $x(t)$ is a continuous function of t. The operator $\mathfrak{D}$ used here is an extension to $L^p_{\mathrm{c,loc}}(\mathbb{R};\mathcal{U})$.

(ii) For input functions of class $W^{2,p}_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U})$, Lemma 2.2.5 implies that the state $x(t)$ and the output function y of a system node are the unique solutions of the differential equation (2.3).

(iii) It is possible to define the state and output for input signals in $L^1_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U})$, even if the system is only L^p-well-posed for some $p \in (1,\infty)$. In this case the state defined by (2.8) is an element of the rigged space $\mathcal{X}_{-1}$, and the output y exists only in a distributional sense, see [Sta05, Definition 4.7.5]. However, for an L^p-well-posed system it is natural and logical to allow only for inputs of class $L^p_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U})$ in the behavior because these inputs lead to state functions in the state space $\mathcal{X}$ and outputs in $L^p_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U})$.

The state can be seen as the solution of a differential equation in the sense described in the Appendix by [Sta05, Theorem 4.3.1. (i)]:

Lemma 2.4.11. *Let $p \in [1,\infty)$, and let $(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})$ be an L^p-well-posed linear system on $(\mathcal{U},\mathcal{X},\mathcal{Y})$ with main operator A and control operator B. The state x corresponding to the input $u \in L^p_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U})$ and initial value $x_0 \in \mathcal{X}$ is the unique strong solution of*

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0,$$

in $\mathcal{X}$ in the sense of Definition A.2.1.

Definition 2.4.12. We say that the system node in Lemma 2.4.7 *generates* the well-posed linear system $(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})$. The *transfer function* of the well-posed linear system is defined as the transfer function of its generating system node (and is by Lemma 2.4.6 an extension of the transfer function of $\mathfrak{D}$). The well-posed linear system and its system node are said to be *strongly/uniformly regular* if their transfer function is *strongly/uniformly regular*.

If the system is strongly regular, the quadruple (A,B,C,D) consisting of the main operator A, the control operator B, the observation operator C, and the feedthrough D of the transfer function are said to be the *generators* of $(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})$.

Definition 2.4.13 (Cesàro extension)**.** Let $\mathfrak{C} \in \mathcal{B}(\mathcal{X};L^p_\omega(\mathbb{R}_{\geqslant 0};\mathcal{Y}))$ be the output map of an L^p-well-posed linear system with observation operator C for some $p \in [1,\infty)$. The *Cesàro extension* of C is defined as

$$C_{\mathrm{ex}}x := \lim_{t\to 0}\frac{1}{t}\int_0^t (\mathfrak{C}x)(\tau)\,\mathrm{d}\tau \quad \forall\, x \in \operatorname{dom} C_{\mathrm{ex}}, \tag{2.9}$$

and its natural domain $\operatorname{dom} C_{\mathrm{ex}}$ consists by definition of all $x \in \mathcal{X}$ for which the limit in (2.9) exists. The norm

$$\|x\|_{\operatorname{dom} C_{\mathrm{ex}}} := \|x\|_{\mathcal{X}} + \sup_{0<t<1} \left\| \frac{1}{t} \int_0^t (\mathfrak{C}x)(\tau)\, \mathrm{d}\tau \right\|_{\mathcal{Y}}, \qquad x \in \operatorname{dom} C_{\mathrm{ex}}$$

makes $\operatorname{dom} C_{\mathrm{ex}}$ a Banach space [Sta05, Theorem 5.4.3].

Lemma 2.4.14. *Let S be a uniformly regular L^p-well-posed system node with observation operator C and $p \in [1, \infty)$. Then S is compatible with the domain of the Cesàro extension C_{ex} as compatibility space. Moreover, the feedthrough of S induced by C_{ex} is the limit in* (2.6) *of the transfer function.*

This follows from the equivalence of (i) and (iv') in [Sta05, Theorem 5.6.5], with a view of [Sta05, Theorem 5.6.4]. All the systems treated in this thesis will be regular.

An important way to get from one realization to another is to use state space transformations. The easiest situation occurs when the transformation is a homeomorphism. In this case the proof of the following lemma is trivial, see e.g., [Sta05, Example 2.3.7]. We write $T\mathfrak{A}T^{-1}$ for the semigroup $t \mapsto T\mathfrak{A}(t)T^{-1}$, and $\mathfrak{A}|_{\mathcal{Z}}$ for the semigroup $t \mapsto \mathfrak{A}(t)|_{\mathcal{Z}}$.

Lemma 2.4.15. *Given a well-posed linear system $(\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1, \mathfrak{D}_1)$ on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$, a further Hilbert space $\mathcal{Z}$ and a boundedly invertible operator $T \in \mathcal{B}(\mathcal{X}; \mathcal{Z})$. Then $\mathfrak{A}_2 : \mathbb{R}_{\geqslant 0} \to \mathcal{B}(\mathcal{Z})$, $t \mapsto T\mathfrak{A}_1(t)T^{-1}$, $\mathfrak{B}_2 := T\mathfrak{B}_1$, $\mathfrak{C}_2 := \mathfrak{C}_1 T^{-1}$ and $\mathfrak{D}_2 := \mathfrak{D}_1$ constitute a well-posed linear system on $(\mathcal{U}, \mathcal{Z}, \mathcal{Y})$. The behavior of the two systems is related via*

$$(x, u, y) \in \operatorname{bhv}(\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1, \mathfrak{D}_1) \quad \Leftrightarrow \quad (Tx, u, y) \in \operatorname{bhv}(\mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{C}_2, \mathfrak{D}_2).$$

If $\mathfrak{D}$ is regular with feedthrough D, the generators of the new system are given by (A_2, B_2, C_2, D), where $\operatorname{dom} A_2 = T \operatorname{dom} A_1$, and

$$A_2 = TA_1T^{-1}, \quad B_2 = T|_{(\operatorname{dom} A_1^*)'} B_1, \quad C_2 = C_1 T^{-1}.$$

Here, $T|_{(\operatorname{dom} A_1^*)'}$ is the unique extension of T to an operator from $(\operatorname{dom} A_1^*)'$ to $(\operatorname{dom} A_2^*)'$.

Definition 2.4.16 (similarity)**.** Two well-posed linear systems $(\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1, \mathfrak{D})$ and $(\mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{C}_2, \mathfrak{D}_2)$ on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ and $(\mathcal{U}, \mathcal{Z}, \mathcal{Y})$, respectively, are *pseudo-similar* if $\mathfrak{D}_1 = \mathfrak{D}_2$ and there exists a closed, densely defined injective linear operator $T : \operatorname{dom} T \subset \mathcal{X} \to \operatorname{ran} T \subset \mathcal{Z}$ with the following properties: $\operatorname{ran} \mathfrak{B}_1 \subset \operatorname{dom} T$, $\operatorname{ran} \mathfrak{B}_2 \subset \operatorname{ran} T$, $\operatorname{dom} T$ is $\mathfrak{A}_1$-invariant, $\operatorname{ran} T$ is $\mathfrak{A}_2$-invariant and

$$\begin{aligned}
\mathfrak{A}_2(t) T x_1 &= T \mathfrak{A}_1(t) x_1, && \forall\, x_1 \in \operatorname{dom} T,\ t \in \mathbb{R}_{\geqslant 0},\\
\mathfrak{B}_2 u &= T \mathfrak{B}_1 u, && \forall\, u \in L^2(\mathbb{R}_{\leqslant 0}; \mathcal{U}),\\
\mathfrak{C}_2 T x_1 &= \mathfrak{C}_1 x_1, && \forall\, x_1 \in \operatorname{dom} T.
\end{aligned}$$

The systems $(\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1, \mathfrak{D}_1)$ and $(\mathfrak{A}_2,\ \mathfrak{B}_2,\ \mathfrak{C}_2,\ \mathfrak{D}_1)$ are said to be *similar* if T and T^{-1} are both bounded, and *unitary similar* if T is unitary.

We will now recall the concept of duality. Thereby, we will use the reflection operator around zero which is defined by

$$\text{Я} : L^2_{\mathrm{loc}}(\mathbb{R}; \mathcal{Y}) \to L^2_{\mathrm{loc}}(\mathbb{R}; \mathcal{Y}), \quad (\text{Я} y)(t) := y(-t) \quad \forall\, t \in \mathbb{R}, \tag{2.10}$$

for any Banach space $\mathcal{Y}$. The following lemma summarizes [Sta05, Theorem 6.2.3] and [Sta05, Theorem 6.2.13].

Lemma 2.4.17. *Let $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ be an L^2-well-posed linear system on the Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ with main operator A, control operator B and observation operator C. Define*

$$\left(\mathfrak{A}^d\,,\ \mathfrak{B}^d\,,\ \mathfrak{C}^d\,,\ \mathfrak{D}^d\right) := \left(\mathfrak{A}^*\,,\ \mathfrak{C}^* \text{Я}\,,\ \text{Я} \mathfrak{B}^*\,,\ \text{Я} \mathfrak{D}^* \text{Я}\right), \tag{2.11}$$

where the semigroup $\mathfrak{A}^*$ is defined by $\mathfrak{A}^*(t) := \mathfrak{A}(t)^*$ for all $t \geqslant 0$. Then (2.11) *is an L^2-well-posed linear system on $(\mathcal{Y}, \mathcal{X}, \mathcal{U})$. The main operator is A^*, the control operator is C^*, and the observation operator is B^*. The transfer function of* (2.11) *satisfies*

$$\widehat{\mathfrak{D}^d}(s) = \widehat{\mathfrak{D}}(\overline{s})^* \quad \forall\, s \in \rho(A^*).$$

Definition 2.4.18 (dual system)**.** Under the prerequisites of Lemma 2.4.17, the system in (2.11) is called the *dual* system of $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$.

2.5. Boundary control systems

A special type of control system often arises in the study of partial differential equations when the control acts on the boundary conditions; see (2.13) below for an example. Informally, the equations then look like

$$\dot{x}(t) = \Lambda x, \qquad u(t) \;= \Gamma x(t), \qquad y(t) = C\big|_{\mathcal{W}} x(t).$$

Since we have already seen that the operator C appears on various domains, we indicate the domain $\mathcal{W}$ here as well, even though it is superfluous in the current section.

Definition 2.5.1 (boundary control system)**.** Let $\mathcal{U}, \mathcal{X}, \mathcal{Y}$ be Hilbert spaces and let the Hilbert space $\mathcal{W}$ be continuously and densely injected into $\mathcal{X}$. The triple $(\Lambda, \Gamma, C|_{\mathcal{W}}) \in \mathcal{B}(\mathcal{W}, \mathcal{X}) \times \mathcal{B}(\mathcal{W}, \mathcal{U}) \times \mathcal{B}(\mathcal{W}, \mathcal{Y})$ is a *boundary control system* if the following conditions hold:

(i) $\ker \Gamma$ is dense in $\mathcal{X}$.

(ii) There is a $\lambda \in \mathbb{C}$ such that $(\lambda - \Lambda) : \ker \Gamma \to \mathcal{X}$ is bijective.

(iii) $\Gamma : \mathcal{W} \to \mathcal{U}$ is onto.

It is well-known that every boundary control system can be extended to a compatible operator node.

Lemma 2.5.2. *Let* $(\Lambda, \Gamma, C|_{\mathcal{W}}) \in \mathcal{B}(\mathcal{W}; \mathcal{X}) \times \mathcal{B}(\mathcal{W}; \mathcal{U}) \times \mathcal{B}(\mathcal{W}; \mathcal{Y})$ *be a boundary control system and* $D \in \mathcal{B}(\mathcal{U}; \mathcal{Y})$*. Then*

$$S := \begin{bmatrix} A|_{\mathcal{X}} & B \\ C|_{\mathcal{W}} & D \end{bmatrix}\Bigg|_{\operatorname{dom} S} : \operatorname{dom} S \subset \mathcal{X} \times \mathcal{U} \to \mathcal{X} \times \mathcal{Y}$$

is a compatible operator node, where

(i) $A := \Lambda|_{\ker \Gamma}$*, and* $A|_{\mathcal{X}}$ *is the extension of* A *in* (2.1)*;*

(ii) $B := (\Lambda - A|_{\mathcal{X}})\Gamma^{+}$*, where* $\Gamma^{+} \in \mathcal{B}(\mathcal{U}; \mathcal{W})$ *is an arbitrary right inverse of* Γ*;*

(iii) $\operatorname{dom} S := \left\{ \begin{bmatrix} w \\ u \end{bmatrix} \in \mathcal{W} \times \mathcal{U} \;\middle|\; u = \Gamma w \right\}.$

Furthermore, $\mathcal{W} = (\mathcal{X} + B\mathcal{U})_1$*, the norms of* $\mathcal{W}$ *and* $(\mathcal{X} + B\mathcal{U})_1$ *are equivalent, and*

$$\Lambda w = A\big|_{\mathcal{X}} w + B\Gamma w \quad \forall\, x \in \mathcal{W}. \tag{2.12}$$

Moreover, B *is strictly unbounded, which means* $\operatorname{ran} B \cap \mathcal{X} = \{0\}$.

This Lemma is a consequence of [Sta05, Theorem 5.2.13], see also [TW09, Proposition 10.1.2].

Remark 2.5.3. (i) An immediate consequence of this lemma is that for all $x \in \mathcal{W}$ there exists exactly one $u \in \mathcal{U}$ such that $A|_{\mathcal{X}} x + Bu \in \mathcal{X}$, and this u is given by $u = \Gamma x$.

(ii) The operator Λ is an extension of A to $\mathcal{W}$. However, Equation (2.12) shows that it does not equal the restriction to $\mathcal{W}$ of the extension $A|_{\mathcal{X}}$, cf. [Sta05, Remark 5.2.10].

Remark 2.5.4. A necessary condition for L^1-well-posedness of a system node with reflexive state space $\mathcal{X}$ is that the control operator satisfies $B \in \mathcal{B}(\mathcal{U}; \mathcal{X})$ [Sta05, Theorem 4.2.7]. In particular, a system node that emerges from a boundary control system on a Hilbert space can never be L^1-well-posed since its control operator is strictly unbounded by Lemma 2.5.2.

2.6. An example: The heat equation with boundary control

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ be a bounded domain with uniformly $\mathcal{C}^2$-boundary $\partial\Omega$ in the sense of [AF03, Chapter 4], and let $\nu : \partial\Omega \to \mathbb{R}^d$ be the outward normal derivative. Furthermore, we indicate the Riemann-Lebesgue volume measure on the manifold $\partial\Omega$ by σ_ξ. We will model a well-posed linear system that is informally written as the partial differential equation

$$\begin{aligned} \tfrac{\partial}{\partial t} x(\xi,t) &= \Delta x(\xi,t), && \forall\, (\xi,t) \in \Omega \times \mathbb{R}_{>0}, \\ u(t) &= \partial_\nu x(\xi,t), && \forall\, (\xi,t) \in \partial\Omega \times \mathbb{R}_{>0}, \\ y(t) &= \int_{\partial\Omega} x(\xi,t)\,\mathrm{d}\sigma_\xi, && \forall\, t \in \mathbb{R}_{>0}, \\ x(\xi,0) &= x_0(\xi), && \forall\, \xi \in \Omega, \end{aligned} \tag{2.13}$$

where $\Delta x(\xi,t) := \sum_{k=1}^{d} \frac{\mathrm{d}^2}{\mathrm{d}{\xi_k}^2} x(\xi,t)$ is the Laplacian. This is a well-known heat equation, but with a special type of boundary conditions. We have a scalar control $u(t) \in \mathbb{C}$ that acts similar to a Neumann boundary condition, and the scalar output $y(t) \in \mathbb{C}$ determines a Dirichlet-like boundary condition. The fact that $u(t)$ does not depend on ξ means that $\partial_\nu x(\xi,t)$ is forced to be constant in ξ. With the following operators the above system is a boundary control system.

Lemma 2.6.1. *Define $\mathcal{X} := L^2(\Omega)$ and*

$$\mathcal{W} := \left\{ x \in W^{2,2}(\Omega) \;\middle|\; \partial_\nu x|_{\partial\Omega} \equiv \frac{\int_\Omega \Delta x(\xi)\,\mathrm{d}\xi}{|\partial\Omega|}. \right\}, \tag{2.14}$$

and, for all $x \in \mathcal{W}$,

$$\Lambda x := \Delta x, \qquad \Gamma x := \frac{\int_\Omega \Delta x(\xi)\,\mathrm{d}\xi}{|\partial\Omega|}, \qquad C|_{\mathcal{W}} x := \int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi.$$

Then $(\Lambda, \Gamma, C|_{\mathcal{W}}) \in \mathcal{B}(\mathcal{W};\mathcal{X}) \times \mathcal{B}(\mathcal{W};\mathcal{U}) \times \mathcal{B}(\mathcal{W};\mathcal{Y})$ is a boundary control system.

Proof. Looking at (2.14) and the definition of Γ, we see that $A := \Lambda|_{\ker\Gamma}$ is the Laplacian with Neumann boundary conditions, more precisely

$$Ax = \Delta x \quad \forall\, x \in \operatorname{dom} A = \left\{ x \in W^{2,2}(\Omega) \;\middle|\; \partial_\nu x|_{\partial\Omega} \equiv 0 \right\}. \tag{2.15}$$

For this operator it is well-known that $\operatorname{dom} A$ is dense in $\mathcal{X}$ and that $\lambda - A$ is bijective for all $\lambda \in \mathbb{C}_{>0}$ [HT08, Theorem 7.13 (ii)]. Since Γ maps onto $\mathbb{R}$, all the properties required in Definition 2.5.1 are fulfilled. □

Remark 2.6.2. (i) The space $\mathcal{W}$ is equivalently characterized by

$$\mathcal{W} = \left\{ x \in W^{2,2}(\Omega) \;\middle|\; \exists\, u \in \mathbb{C} : \partial_\nu x|_{\partial\Omega} \equiv u \right\},$$

and Γ maps each $x \in \mathcal{W}$ to the constant value of $\partial_\nu x|_{\partial\Omega}$. This can be seen as follows: Assume that the normal derivative of $x \in W^{2,2}(\Omega)$ on the boundary is constant and equal to $u \in \mathbb{C}$. Then taking the scalar product of Δx with the constant function 1 and applying Gauß's theorem shows

$$\int_\Omega \Delta x(\xi) \cdot 1\,\mathrm{d}\xi = \int_{\partial\Omega} \partial_\nu x(\xi)\,\mathrm{d}\xi = u \cdot |\partial\Omega|,$$

whence

$$u = \frac{\int_\Omega \Delta x(\xi)\,\mathrm{d}\xi}{|\partial\Omega|} = \Gamma x.$$

(ii) If $d = 1$, the demanded $\mathcal{C}^2$-condition on $\partial\Omega$ is not well-defined. In this case it suffices that Ω is an interval (a, b) and the expressions have to be interpreted in the following way: $\partial\Omega := \{a, b\}$, $|\partial\Omega| := 2$, $\nu(a) := -1$, $\nu(b) := 1$, and $\int_{\partial\Omega} \varphi(\xi)\,\mathrm{d}\sigma_\xi := \varphi(a) + \varphi(b)$ for all $\varphi \in \mathcal{C}(a, b)$.

Lemma 2.6.3. *The following holds for the operator A in* (2.15)*, i.e. the Laplace operator with Neumann boundary condition:*

(i) The operator A is self-adjoint, nonpositive and has a compact resolvent. There is a real valued sequence $(\lambda_k)_{k\in\mathbb{N}_0}$ of eigenvalues of $-A$ such that (λ_k) is nondecreasing, $\lambda_0 = 0$, $\lambda_1 > 0$, and $\lambda_k \xrightarrow{k\to\infty} \infty$. In particular, $\sigma(A) = \{-\lambda_k \mid k \in \mathbb{N}_0\}$.

(ii) The eigenvectors of $-A$ form an orthonormal basis $(v_k)_{k\in\mathbb{N}_0}$ of $L^2(\Omega)$ with $v_k \in \operatorname{dom} A$ for all $k \in \mathbb{N}_0$, and there holds

$$\operatorname{dom} A = \left\{ \sum_{k=0}^{\infty} c_k v_k \;\middle|\; (c_k), (\lambda_k c_k) \in \ell^2(\mathbb{N}_0) \right\}, \tag{2.16}$$

$$\left\| \sum_{k=0}^{\infty} c_k v_k \right\|_{\operatorname{dom} A}^2 = \|(c_k)\|_{\ell^2(\mathbb{N}_0)}^2 + \|(\lambda_k c_k)\|_{\ell^2(\mathbb{N}_0)}^2, \tag{2.17}$$

and

$$Ax = -\sum_{k=0}^{\infty} \lambda_k \langle x\,,\, v_k\rangle_{L^2(\Omega)} \cdot v_k \quad \forall\, x \in \operatorname{dom} A. \tag{2.18}$$

(iii) The norm of $\operatorname{dom} A$ *is equivalent to the $W^{2,2}(\Omega)$-norm.*

Proof. Part (i) is [HT08, Theorem 7.13 (ii)]. The second part (ii) is a consequence of the spectral representation theorem for operators with pure point spectrum, see e.g. [Tri92, Section 4.5.1]. Regarding part (iii), it is easy to show that $\|\cdot\|_{\operatorname{dom} A} = \|(s - A)x\|_{\mathcal{X}}$ is an equivalent norm to $\|\cdot\|_{L^2(\Omega)} + \|\Delta\cdot\|_{L^2(\Omega)}$. Furthermore, Theorem 5.11 of [HT08] states that the latter norm is equivalent the $W^{2,2}(\Omega)$-norm. ☐

Due to the self-adjointness of A we have in particular $\operatorname{dom} A = \operatorname{dom} A^*$. In view of Section 2.1, $L^2(\Omega)$ is embedded into the rigged space $(\operatorname{dom} A)'$ in the following way: Each $x \in L^2(\Omega)$ is identified with the functional $\iota x \in (\operatorname{dom} A)'$, defined by

$$\langle \varphi\,,\,\iota x\rangle_{\operatorname{dom} A,(\operatorname{dom} A)'} := \langle \varphi\,,\,x\rangle_{L^2(\Omega)} = \int_\Omega \varphi(\xi)\overline{x(\xi)}\,\mathrm{d}\xi \quad \forall\,\varphi \in \operatorname{dom} A.$$

Note that the expression on the right is linear in φ and anti-linear in x. With our definition of multiplication in the dual space, this makes the injection ι linear. Furthermore, we may use the definition of the reversed pairing,

$$\langle \iota x\,,\,\varphi\rangle_{(\operatorname{dom} A)',\operatorname{dom} A} := \overline{\langle \varphi\,,\,\iota x\rangle_{\operatorname{dom} A,(\operatorname{dom} A)'}} = \int_\Omega x(\xi)\overline{\varphi(\xi)}\,\mathrm{d}\xi \quad \forall\,\varphi \in \operatorname{dom} A.$$

Lemma 2.6.4. *Let $\mathcal{X} = L^2(\Omega)$, $\mathcal{W}$ as in* (2.14)*, and the operator A as in* (2.15)*. In addition, define*

$$B : \mathbb{C} \to (\operatorname{dom} A)', \quad \langle Bu\,,\,\varphi\rangle_{(\operatorname{dom} A)',\operatorname{dom} A} := u\int_{\partial\Omega} \overline{\varphi(\xi)}\,\mathrm{d}\sigma_\xi \quad \forall\,\varphi \in \operatorname{dom} A, \tag{2.19}$$

and

$$C|_{\mathcal{W}} : \mathcal{W} \to \mathbb{C}, \quad C|_{\mathcal{W}}x := \int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \quad \forall\,x \in \mathcal{W}. \tag{2.20}$$

Then

$$S := \begin{bmatrix} A|_{\mathcal{X}} & B \\ C|_{\mathcal{W}} & 0 \end{bmatrix}\Bigg|_{\operatorname{dom} S}, \quad \operatorname{dom} S := \left\{ \begin{bmatrix} w \\ u \end{bmatrix} \in \mathcal{W} \times \mathcal{U} \;\middle|\; \partial_\nu w|_{\partial\Omega} \equiv u \right\}, \tag{2.21}$$

is the operator node corresponding to $(\Lambda, \Gamma, C|_{\mathcal{W}})$ via Lemma 2.5.2. In particular,

$$A|_{\mathcal{X}}w + B\Gamma w = \Lambda w = \Delta w \quad \forall\,x \in \mathcal{W}.$$

Proof. This follows from Lemma 2.5.2, we only need to calculate B: Let $u \in \mathbb{C}$ and let $\Gamma^+ \in \mathcal{B}(\mathbb{C}; W^{2,2}(\Omega))$ be some right inverse of Γ, i.e. $\Gamma^+ u = u\cdot w$ for some $w \in W^{2,2}(\Omega)$ with $\partial_\nu w|_{\partial\Omega} \equiv 1$ (for instance, take a solution of $\Delta w - w = 0$, $\partial_\nu w|_{\partial\Omega} \equiv 1$). Gauß's theorem now implies for all $\varphi \in \operatorname{dom} A$ that

$$\begin{aligned}
\langle Bu\,,\,\varphi\rangle_{(\operatorname{dom} A)',\operatorname{dom} A} &= \Big\langle (\Lambda - A|_X)\Gamma^+ u\,,\,\varphi\Big\rangle_{(\operatorname{dom} A)',\operatorname{dom} A} \\
&= \Big\langle \Lambda\Gamma^+ u\,,\,\varphi\Big\rangle_{(\operatorname{dom} A)',\operatorname{dom} A} - \Big\langle \Gamma^+ u\,,\,A\varphi\Big\rangle_{(\operatorname{dom} A)',\operatorname{dom} A} \\
&= \int_\Omega u\cdot(\Delta w(\xi))\overline{\varphi(\xi)}\,\mathrm{d}\xi - \int_\Omega u\cdot w(\xi)\overline{\Delta\varphi(\xi)}\,\mathrm{d}\xi
\end{aligned}$$

$$= u \int_{\partial\Omega} \partial_\nu w(\xi) \overline{\varphi(\xi)} \, \mathrm{d}\sigma_\xi = u \int_{\partial\Omega} \overline{\varphi(\xi)} \, \mathrm{d}\sigma_\xi.$$

This proves the assertion. □

Remark 2.6.5. (i) The adjoint of B satisfies $B^*\varphi = C\varphi$ for all φ in $\operatorname{dom} A$ because

$$\langle Bu \, , \, \varphi \rangle_{(\operatorname{dom} A)', \operatorname{dom} A} = \left\langle u \, , \, \int_{\partial\Omega} \varphi(\xi) \, \mathrm{d}\sigma_\xi \right\rangle_{\mathbb{C}} \qquad \forall \, \varphi \in \operatorname{dom} A, u \in \mathbb{C}.$$

System nodes with this property are sometimes called “collocated”.

(ii) The operator $C|_{\mathcal{W}}$ has an extension to $W^{\frac{1}{2}+\varepsilon,2}(\Omega)$ for any $\varepsilon > 0$ since for these spaces there exists a continuous trace operator that maps φ to $\varphi|_{\partial\Omega} \in L^2(\partial\Omega)$ [HT08, Theorem 4.24 (i)].

(iii) Instead of (2.20) we could define the output operator

$$\widetilde{C}|_{\mathcal{W}} x := \int_{\partial\Omega} x(\xi) \, \mathrm{d}\sigma_\xi + \frac{\int_\Omega \Delta x(\xi) \, \mathrm{d}\xi}{|\partial\Omega|} \qquad \forall \, x \in \mathcal{W},$$

and the feedthrough $\widetilde{D} := -1$. Then the lower line of S can be replaced by $[\widetilde{C}|_{\mathcal{W}} \, , \, \widetilde{D}]$ because, by the definition of $\mathcal{W}$ and $\operatorname{dom} S$,

$$\begin{bmatrix} \widetilde{C}|_{\mathcal{W}} & \widetilde{D} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} C|_{\mathcal{W}} & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \quad \forall \begin{bmatrix} x \\ u \end{bmatrix} \in \operatorname{dom} S.$$

However, we will shortly see that the feedthrough of the transfer function of S is zero. This justifies the choice of $C|_{\mathcal{W}}$ because we want the feedthrough of the transfer function to coincide with the feedthrough induced by $C|_{\mathcal{W}}$.

We are going to show that $W^{1,2}(\Omega)$ is another compatibility space for this operator node. To this end, we need the following lemma.

Lemma 2.6.6. *The following holds for the operator A defined in* (2.15)*:*

(i) For every $s > 0$ we have $\operatorname{dom}(s-A)^{\frac{1}{2}} = W^{1,2}(\Omega)$ and the norms of these spaces are equivalent. Furthermore, $(\frac{v_k}{s+\lambda_k})_{k\in\mathbb{N}_0}$ is an orthonormal basis of $W^{1,2}(\Omega)$, and $\operatorname{dom} A$ is densely and continuously embedded into $W^{1,2}(\Omega)$.

(ii) Moreover,

$$W^{1,2}(\Omega) = \left\{ \sum_{k=0}^{\infty} a_k v_k \;\middle|\; (\sqrt{\lambda_k} a_k) \in \ell^2(\mathbb{N}_0) \right\}, \tag{2.22}$$

$$\left\| \sum_{k=0}^{\infty} a_k v_k \right\|_{W^{1,2}(\Omega)}^2 = \|(a_k)\|_{\ell^2(\mathbb{N}_0)}^2 + \left\|(\sqrt{\lambda_k} a_k)\right\|_{\ell^2(\mathbb{N})}^2. \tag{2.23}$$

Proof. Since the operator $s - A$ is positive definite, part (i) follows from [HT08, Theorem 5.31 (ii)] in combination with [Tri92, Section 4.4.3]. The density claim is [HT08, Proposition 5.28 (i)]. Now for part (ii): An application of Gauß's divergence theorem shows that

$$\|x\|_{W^{1,2}(\Omega)} = \|x\|_{L^2(\Omega)} - \langle x \,,\, Ax\rangle_{L^2(\Omega)} \quad \forall\, x \in \operatorname{dom} A.$$

Using this and the spectral decomposition (2.18), we see that the following holds for all $x = \sum_{k=0}^{\infty} a_k v_k \in \operatorname{dom} A$

$$\begin{aligned}\|x\|_{W^{1,2}(\Omega)}^2 =& \|x\|_{L^2(\Omega)}^2 - \langle x \,,\, Ax\rangle_{L^2(\Omega)} = \|(a_k)\|_{\ell_2(\mathbb{N}_0)}^2 + \left\langle v \,,\, \sum_{k=0}^{\infty} \lambda_k \langle x \,,\, v_k\rangle\, v_k \right\rangle \\ &= \|(a_k)\|_{\ell_2(\mathbb{N}_0)} + \sum_{k=0}^{\infty} \lambda_k a_k \underbrace{|\langle x \,,\, v_k\rangle|^2}_{=|a_k|^2} = \|(a_k)\|_{\ell_2(\mathbb{N}_0)}^2 + \sum_{k=1}^{\infty} \lambda_k |a_k|^2.\end{aligned}$$

The representation (2.16) implies that linear combinations of $(v_k)_{k\in\mathbb{N}_0}$ are dense in $\operatorname{dom} A$. Since $\operatorname{dom} A$ is dense in $W^{1,2}(\Omega)$, we can infer from the above computations that $W^{1,2}(\Omega)$ is equal to the completion of $\operatorname{span}\{\, v_k \mid k \in \mathbb{N}_0 \,\}$ with respect to the norm

$$\left\| \sum_{k=0}^{\infty} a_k v_k \right\|^2 := \|(a_k)\|_{\ell_2(\mathbb{N}_0)}^2 + \left\| \left(\sqrt{\lambda_k} a_k\right) \right\|_{\ell_2(\mathbb{N})}^2.$$

Now, since $\lambda_k \to \infty$ as $k \to \infty$, the condition $\left(\sqrt{\lambda_k} a_k\right)_{k\in\mathbb{N}} \in \ell_2(\mathbb{N})$ implies $(a_k)_{k\in\mathbb{N}_0} \in \ell_2(\mathbb{N}_0)$, and we conclude (2.22). □

Lemma 2.6.7. *Let A and B be defined as in* (2.15) *and* (2.19)*, respectively. For* *all $s \in \rho(A)$, there holds*

$$(s-A)^{-1}B = \sum_{k=0}^{\infty} \frac{\int_{\partial\Omega} \overline{v_k(\xi)}\,\mathrm{d}\sigma_\xi}{s+\lambda_k} \cdot v_k \quad \in \quad W^{1,2}(\Omega). \tag{2.24}$$

This series converges in $W^{1,2}(\Omega)$ and

$$\sum_{k=0}^{\infty} \left| \int_{\partial\Omega} v_k(\xi)\,\mathrm{d}\sigma_\xi \right|^2 \frac{\lambda_k}{|s+\lambda_k|^2} < \infty. \tag{2.25}$$

Moreover, the space $W^{1,2}(\Omega)$ is a compatibility space for the system node defined in Lemma 2.6.4, and the feedthrough induced by

$$C|_{W^{1,2}(\Omega)} : W^{1,2}(\Omega) \to \mathbb{C}, \quad C|_{W^{1,2}(\Omega)}x := \int_{\partial\Omega} v_k(\xi)\,\mathrm{d}\sigma_\xi, \tag{2.26}$$

is zero.

Proof. Note that, just like C in Remark 2.6.5 (ii), the functional $Bu \in (\operatorname{dom} A)'$ extends to a continuous functional on $W^{1,2}(\Omega)$. Let $x \in L^2(\Omega)$. Then the series $\sum_{k=0}^{N} \langle x\,,\,v_k\rangle_{L^2(\Omega)}\, v_k$ converges in $L^2(\Omega)$ to x as $N \to \infty$. Due to the continuity of $(s-A)^{-\frac{1}{2}} \in \mathcal{B}(L^2(\Omega); \operatorname{dom}(s-A)^{\frac{1}{2}})$, the limit

$$\sum_{k=0}^{N} \frac{\langle x\,,\,v_k\rangle_{L^2(\Omega)}}{\sqrt{s+\lambda_k}} v_k = (s-A)^{-\frac{1}{2}} \sum_{k=0}^{N} \langle x\,,\,v_k\rangle_{L^2(\Omega)}\, v_k \to (s-A)^{-\frac{1}{2}}x, \quad N \to \infty,$$

holds in $\operatorname{dom}(s-A)^{\frac{1}{2}}$. This implies for every functional $\varphi \in (\operatorname{dom}(s-A)^{\frac{1}{2}})'$, (which is identified with the restricted functional on $\operatorname{dom} A$), that

$$\begin{aligned}
&\sum_{k=0}^{N} \frac{\langle x\,,\,v_k\rangle_{L^2(\Omega)}}{\sqrt{s+\lambda_k}} \langle v_k\,,\,\varphi\rangle_{\operatorname{dom} A,(\operatorname{dom} A)'} \\
&= \left\langle (s-A)^{-\frac{1}{2}} \sum_{k=0}^{N} \langle x\,,\,v_k\rangle_{L^2(\Omega)}\, v_k\,,\,\varphi \right\rangle_{\operatorname{dom} A,(\operatorname{dom} A)'} \\
&= \left\langle (s-A)^{-\frac{1}{2}} \sum_{k=0}^{N} \langle x\,,\,v_k\rangle_{L^2(\Omega)}\, v_k\,,\,\varphi \right\rangle_{\operatorname{dom}(s-A)^{\frac{1}{2}},(\operatorname{dom}(s-A)^{\frac{1}{2}})'} \\
&\to \left\langle (s-A)^{-\frac{1}{2}}x\,,\,\varphi \right\rangle_{\operatorname{dom}(s-A)^{\frac{1}{2}},(\operatorname{dom}(s-A)^{\frac{1}{2}})'} .
\end{aligned}$$

Thus, in particular for $Bu \in (\mathrm{dom}(s-A)^{\frac{1}{2}})'$ and all $x \in L^2(\Omega)$, the limit

$$\begin{aligned} \left\langle x\,,\, \sum_{k=0}^{N} \frac{\langle Bu\,,\, v_k\rangle}{\sqrt{s+\lambda_k}} v_k \right\rangle_{L^2(\Omega)} &= \sum_{k=0}^{N} \frac{\langle v_k\,,\, Bu\rangle_{\mathrm{dom}\,A,(\mathrm{dom}\,A)'}}{\sqrt{s+\lambda_k}} \langle x\,,\, v_k\rangle_{L^2(\Omega)} \\ &\to \left\langle (s-A)^{-\frac{1}{2}} x\,,\, Bu\right\rangle_{\mathrm{dom}(s-A)^{\frac{1}{2}},(\mathrm{dom}(s-A)^{\frac{1}{2}})'} \\ &= \left\langle x\,,\, (s-A)^{-\frac{1}{2}} Bu\right\rangle_{\mathcal{X}} \end{aligned}$$

holds. Since $L^2(\Omega)$ is a Hilbert space, we conclude that $\sum_{k=0}^{N} \langle Bu\,,\, v_k\rangle / \sqrt{s+\lambda_k} \cdot v_k$ converges in $L^2(\Omega)$, and the limit is $(s-A)^{-\frac{1}{2}} Bu$. Since $(s-A)^{-\frac{1}{2}}$ maps $L^2(\Omega)$ continuously onto $\mathrm{dom}(s-A)^{\frac{1}{2}} = W^{1,2}(\Omega)$, we conclude that

$$\begin{aligned} \sum_{k=0}^{N} \frac{\langle Bu\,,\, v_k\rangle}{s+\lambda_k} v_k &= (s-A)^{-\frac{1}{2}} \sum_{k=0}^{N} \frac{\langle Bu\,,\, v_k\rangle}{\sqrt{s+\lambda_k}} v_k \\ &\to (s-A)^{-\frac{1}{2}} (s-A)^{-\frac{1}{2}} Bu, \quad N \to \infty, \end{aligned}$$

in $W^{1,2}(\Omega)$. This proves, for all $s > 0$, the convergence of (2.24) in $W^{1,2}(\Omega)$, and (2.25) follows from the representation of $W^{1,2}(\Omega)$ in (2.22). For arbitrary $s \in \rho(A)$, the series in (2.25) is finite because it is finite for $s = 1$ and we have the estimate

$$\frac{1}{|s+\lambda_k|} \leqslant \frac{1}{1+\lambda_k} \cdot \underbrace{\sup_{n \in \mathbb{N}_0} \frac{1+\lambda_n}{|s+\lambda_n|}}_{<\infty} \qquad \forall\, k \in \mathbb{N}_0.$$

Now (2.24) follows from (2.25) together with (2.22) for arbitrary $s \in \rho(A)$.

The compatibility claim is a consequence of (2.24) and the fact that (2.26) is a continuous extension of $C|_{\mathcal{W}}$. The feedthrough is zero because of formula (2.2) and (2.21). □

Theorem 2.6.8. *Let A, B and $C|_{W^{1,2}(\Omega)}$ be defined as in* (2.15), (2.19) *and* (2.26), *and let (λ_k), (v_k) be as in Lemma 2.6.3. Define*

$$c_k := \left| \int_{\partial\Omega} v_k(\xi)\, \mathrm{d}\sigma_\xi \right|^2 \quad \forall\, k \in \mathbb{N}_0 \quad \text{and} \quad J_c := \{k \in \mathbb{N}_0 \mid c_k \neq 0\}.$$

Then the transfer function of the system node is

$$\mathbf{G}(s) = \sum_{k=0}^{\infty} \frac{c_k}{s+\lambda_k} = \sum_{k\in J_c} \frac{c_k}{s+\lambda_k} \quad \forall\, s \in \rho(A). \tag{2.27}$$

Furthermore, we have $0 \in J_c$*, and*

$$\left(\frac{c_k}{\lambda_k}\right) \in \ell^1(\mathbb{N}). \tag{2.28}$$

Proof. We express $(s-A)^{-1}B$ using the series in (2.24). Since this series converges in $W^{1,2}(\Omega)$, we may interchange the order of limit and application of $C|_{W^{1,2}(\Omega)}$ to obtain

$$\begin{aligned} C|_{W^{1,2}(\Omega)}(s-A)^{-1}B =& C\sum_{k=0}^{\infty} \frac{\int_{\partial\Omega} \overline{v_k(\xi)}\,\mathrm{d}\sigma_\xi}{s+\lambda_k} \cdot v_k = \sum_{k=0}^{\infty} \frac{\int_{\partial\Omega} \overline{v_k(\xi)}\,\mathrm{d}\sigma_\xi}{s+\lambda_k} \cdot C|_{W^{1,2}(\Omega)} v_k \\ =& \sum_{k=0}^{\infty} \frac{\left|\int_{\partial\Omega} v_k(\xi)\,\mathrm{d}\sigma_\xi\right|^2}{s+\lambda_k} = \sum_{k\in J_c} \frac{c_k}{s+\lambda_k}. \end{aligned}$$

Therefore, (2.27) holds on $\rho(A)$.

We have $0 \in J_c$ because the first eigenvector v_0 in Lemma 2.6.3 is a constant function; more precisely,

$$v_0(\cdot) \equiv \frac{1}{\sqrt{\int_\Omega 1\,\mathrm{d}\xi}}, \qquad \text{whence} \qquad c_0 = \frac{(\int_{\partial\Omega} 1\,\mathrm{d}\sigma_\xi)^2}{\int_\Omega 1\,\mathrm{d}\xi} > 0.$$

Finally, (2.28) is a consequence of (2.25):

$$\sum_{k=1}^{\infty} \frac{c_k}{\lambda_k} = \sum_{k=1}^{\infty} \frac{c_k\lambda_k}{\lambda_k^2} \leqslant \underbrace{\sup_{n\in\mathbb{N}} \frac{(1+\lambda_n)^2}{\lambda_n^2}}_{<\infty} \sum_{k=1}^{\infty} \frac{c_k\lambda_k}{(1+\lambda_k)^2} \overset{(2.25)}{<} \infty.$$

□

Remark 2.6.9. For every removable singularity $\lambda_k \in \sigma(A)$ of $\mathbf{G}(s)$, we have $k \in \mathbb{N}\backslash J_c$ which means that the eigenvector $v_k \in \operatorname{dom} A$ satisfies $Cv_k = 0$. The so-called *Hautus test* for well-posed linear systems, [Sta05, Corollary 9.6.2], then shows that our system is not observable. In other words, λ_k is an unobservable mode.

Whether such unobservable modes exist or not depends on the geometry of the underlying domain Ω. If, for example, $\Omega = [0, \pi] \subset \mathbb{R}$, then it is easy to see that $v(\xi) := \sin(\xi)$ is an eigenvector of A with eigenvalue -1. It lives in the kernel of C because $Cv = v(0) + v(\pi) = 0$. Therefore, $J_c \neq \mathbb{N}_0$ in this example.

Corollary 2.6.10. *For every $\omega > 0$, the series in* (2.27) *converges absolutely in $\mathcal{H}^\infty_{\geqslant\omega}(\mathbb{C};\mathbb{C})$; in particular, we have $\mathbf{G} \in \mathcal{H}^\infty_{\geqslant\omega}(\mathbb{C};\mathbb{C})$. Moreover, it is uniformly regular with feedthrough*

$$\lim_{\substack{s\to\infty,\\ s\in\mathbb{R}}} \mathbf{G}(s) = \lim_{\substack{s\to\infty,\\ s\in\mathbb{R}}} \sum_{k=0}^{\infty} \frac{c_k}{s+\lambda_k} = 0.$$

Proof. For $\omega > 0$, we have

$$\sum_{k=0}^{\infty} \sup_{s\in\mathbb{C}_{\geqslant\omega}} \left| \frac{c_k}{s+\lambda_k} \right| \leqslant \sup_{s\in\mathbb{C}_{\geqslant\omega}} \frac{c_0}{|s|} + \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k} \leqslant \frac{c_0}{\omega} + \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k}.$$

Thus, this series converges absolutely in $\mathcal{H}^\infty_{\geqslant\omega}(\mathbb{C};\mathbb{C})$ and $\mathbf{G} \in \mathcal{H}^\infty_{\geqslant\omega}(\mathbb{C};\mathbb{C})$. To prove the regularity, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} \frac{c_k}{\lambda_k} < \frac{\varepsilon}{2}$, and $t \in \mathbb{R}$ such that $\sum_{k=0}^{N} \frac{c_k}{t} < \frac{\varepsilon}{2}$. Then we have for all $s \geqslant t$,

$$\sum_{k=0}^{\infty} \frac{c_k}{s+\lambda_k} = \sum_{k=0}^{N} \frac{c_k}{s+\lambda_k} + \sum_{k=N+1}^{\infty} \frac{c_k}{s+\lambda_k} \leqslant \sum_{k=0}^{N} \frac{c_k}{t} + \sum_{k=N+1}^{\infty} \frac{c_k}{\lambda_k} < \varepsilon.$$

This proves that the transfer function converges to zero. □

Corollary 2.6.11. *Let $\mathbf{G}$ be the transfer function in Theorem 2.6.8, and let $\mathfrak{D} \in \mathrm{TIC}_{\mathrm{loc}}(\mathcal{U};\mathcal{Y})$ be the time-invariant causal operator associated to $\mathbf{G}$ in Theorem 2.3.8. Then*

$$(\mathfrak{D}u)(t) = \int_{-\infty}^{t} \sum_{k=0}^{\infty} c_k \mathrm{e}^{-\lambda_k(t-\tau)} u(\tau)\,\mathrm{d}\tau \quad \forall\, u \in L^2_\omega(\mathbb{R}_{\geqslant 0}),\ t \in \mathbb{R}. \tag{2.29}$$

Proof. A simple calculation gives

$$\int_0^\infty |\mathrm{e}^{-\omega\tau} c_k \mathrm{e}^{-\lambda_k\tau}|\,\mathrm{d}\tau = \int_0^\infty |c_k \mathrm{e}^{-(\omega+\lambda_k)\tau}|\,\mathrm{d}\tau = \frac{c_k}{\omega+\lambda_k} \quad \forall\, k \in \mathbb{N},$$

and

$$\int_0^\infty |\mathrm{e}^{-\omega\tau} c_0 \mathrm{e}^{-\lambda_0\tau}|\,\mathrm{d}\tau = \frac{c_0}{\omega}.$$

Hence, the summability of $(\frac{c_k}{\lambda_k})_{k\in\mathbb{N}}$ implies that the series

$$h := \sum_{k=0}^{\infty} (t \mapsto c_k e^{-\lambda_k t})$$

converges in $L^1_\omega(\mathbb{R}_{\geqslant 0})$, and

$$\|h\|_{L^1_\omega(\mathbb{R}_{\geqslant 0})} \leqslant \frac{c_0}{\omega} + \sum_{k=1}^{\infty} \frac{c_k}{\omega + \lambda_k}.$$

Now we apply the Laplace transform to h. Since the Laplace transform maps $L^1_\omega(\mathbb{R}_{\geqslant 0})$ continuously into $\mathcal{H}^\infty_\omega(\mathbb{C};\mathbb{C})$ it may be interchanged with the $L^1_\omega(\mathbb{R}_{\geqslant 0})$ limit in the definition of h and thus,

$$\widehat{h}(s) = \sum_{k=0}^{\infty} \int_0^\infty c_k e^{(-s-\lambda_k)\tau} \,\mathrm{d}\tau = \sum_{k=0}^{\infty} \frac{c_k}{s+\lambda_k} = \mathbf{G}(s) \quad \forall\, s \in \mathbb{C}_{\geqslant \omega}.$$

Let $u \in L^2_\omega(\mathbb{R}_{\geqslant 0})$ be given. By Lemma 2.3.7, we have

$$\widehat{\mathfrak{D}u}(s) = \mathbf{G}(s)\widehat{u}(s) \quad \forall\, s \in \mathbb{C}_{\geqslant \omega}.$$

Since this function is in $\mathcal{H}^2_\omega(\mathbb{C})$ we can apply the inverse Laplace transform to this equation and the convolution theorem for the Laplace transform [GLS90, Theorem 3.8.2] implies that

$$(\mathfrak{D}u)(t) = \int_0^t h(t-\tau)u(\tau)\,\mathrm{d}\tau \quad \forall\, t \geqslant 0.$$

This is (2.29). □

The well-posedness of the operator node in Lemma 2.6.4 was proven in [BGSW02]. We keep the following lemma for the record.

Lemma 2.6.12. *The operator node S in Lemma 2.6.4 is L^2-well-posed. The associated well-posed linear system, $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$, has the following additional properties:*

(i) The semigroup $\mathfrak{A} : \mathbb{R}_{\geqslant 0} \to \mathcal{B}(L^2(\Omega))$ is analytic and $\|\mathfrak{A}(t)\|_{\mathcal{B}(L^2(\Omega))} \leqslant 1$ for all

$t \geqslant 0$. *For all $t > 0$ the operator $\mathfrak{A}(t)$ maps $L^2(\Omega)$ into* $\operatorname{dom} A$ *and*

$$\exists\, M \geqslant 1: \ \|\mathfrak{A}(t)\|_{\mathcal{B}(L^2(\Omega);W^{2,2}(\Omega))} \leqslant M\left(1 + \frac{1}{t}\right).$$

(ii) For all $\delta \in \mathbb{R}_{>0}$ and $x \in L^2(\Omega)$, the infinite-time state-to-output map fulfills

$$\mathfrak{C}x|_{[\delta,\infty)} \in W^{1,\infty}([\delta,\infty)).$$

Proof. The well-posedness is stated in [BGSW02, Corollary 1]. A proof of analyticity and boundedness of $\mathfrak{A}$ is for example presented in [HT08, Chapter 5]. By [EN00, Chapter II, Theorem 4.6] the analyticity of $\mathfrak{A}$ implies that

$$\mathfrak{A}(\delta)x \in \operatorname{dom} A \quad \forall\, x \in L^2(\Omega),\ \delta \in \mathbb{R}_{>0},$$

and that there is a constant $m \geqslant 1$ such that

$$\|A\mathfrak{A}(t)\|_{L^2(\Omega)} = \frac{m}{t} \quad \forall\, t > 0.$$

Hence, we have

$$\|\mathfrak{A}(t)x\|_{L^2(\Omega)} + \|A\mathfrak{A}(t)x\|_{L^2(\Omega)} \leqslant \left(1 + \frac{m}{t}\right)\|x\|_{L^2(\Omega)} \quad \forall\, x \in L^2(\Omega),\ t > 0,$$

and the left hand side is equivalent to the $W^{2,2}(\Omega)$-norm of x by Lemma 2.6.3. Assertion (ii) can then be inferred from the relation

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{C}x(t) = C(\mathrm{I} - A)^{-1}\mathfrak{A}(t-\delta)(\mathrm{I} - A)A\mathfrak{A}(\delta)x \quad \forall\, t \in [\delta,\infty).$$

$\square$

2.7. Kalman compression

The principle of restricting a well-posed linear system to its controllable and observable subspace is described in [Sta05, Section 9.1]. In addition to this, we need to know what the generators of such a restriction look like.

Lemma 2.7.1. *Let $p \in [1,\infty]$ and $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ be an L^p-well-posed linear system on the Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$. Define $\mathcal{Z} := (\ker \mathfrak{C})^\perp$, equipped with the scalar product of $\mathcal{X}$. Then*

$$\left(\widetilde{\mathfrak{A}}, \quad \widetilde{\mathfrak{B}}, \quad \widetilde{\mathfrak{C}}, \quad \mathfrak{D}\right) := \left(\pi_{\mathcal{Z}} \mathfrak{A}|_{\mathcal{Z}}, \quad \pi_{\mathcal{Z}} \mathfrak{B}, \quad \mathfrak{C}|_{\mathcal{Z}}, \quad \mathfrak{D}\right)$$

is an observable L^p-well-posed linear system on $(\mathcal{U}, \mathcal{Z}, \mathcal{Y})$. Its main operator $\widetilde{A}$ satisfies

$$\operatorname{dom} \widetilde{A} = \pi_{\mathcal{Z}} \operatorname{dom} A, \qquad \operatorname{dom} \widetilde{A}^* = \mathcal{Z} \cap \operatorname{dom} A^*,$$

and

$$\begin{aligned} \widetilde{A} z &= \pi_{\mathcal{Z}} A x && \forall\, z \in \operatorname{dom} \widetilde{A}, \ \forall\, x \in \operatorname{dom} A \text{ with } \pi_{\mathcal{Z}} x = z, \\ \widetilde{A}^* z &= A^* z && \forall\, z \in \operatorname{dom} \widetilde{A}^*. \end{aligned}$$

The control operator $\widetilde{B}$ is given by

$$\langle \widetilde{B} u, z \rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} = \langle B u, z \rangle_{(\operatorname{dom} A^*)', \operatorname{dom} A^*} \quad \forall\, u \in \mathcal{U}, \ \forall\, z \in \operatorname{dom} \widetilde{A}^*.$$

The observation operator $\widetilde{C}$ satisfies

$$\widetilde{C} z = C x, \quad \forall\, z \in \operatorname{dom} \widetilde{A}, \qquad \forall\, x \in \operatorname{dom} A \text{ with } \pi_{\mathcal{Z}} x = z;$$

its Cesàro extension $\widetilde{C}_{\mathrm{ex}}$ fulfills

$$\widetilde{C_{\mathrm{ex}}} z = C_{\mathrm{ex}} z \quad \forall\, z \in \operatorname{dom} \widetilde{C_{\mathrm{ex}}} = \pi_{\mathcal{Z}} \operatorname{dom} C_{\mathrm{ex}} = \operatorname{dom} C_{\mathrm{ex}} \cap \mathcal{Z}.$$

Proof. The fact that $(\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \widetilde{\mathfrak{C}}, \mathfrak{D})$ is a well-posed linear system is easy and can be found in [Sta05, Corollary 9.1.10]. We need to determine the generators of this system. To this end, note that $\mathcal{Z}$ is an $\mathfrak{A}$-invariant, closed subspace, whence the generator $\widetilde{A}$ of the quotient semigroup $\widetilde{\mathfrak{A}}$ can be found in [EN00, Section 2.2.4]. Since we are in a Hilbert space setting, the adjoint semigroups $\mathfrak{A}^*$ and $\widetilde{\mathfrak{A}}^*$ are again strongly continuous [Sta05, Theorem 3.5.6]. The $\mathfrak{A}$-invariance of $\ker \mathfrak{C}$ implies the invariance of $\mathcal{Z}$ under $\mathfrak{A}^*$ and therefore, the relation

$$\langle \widetilde{\mathfrak{A}}(t) z, y \rangle_{\mathcal{Z}} = \langle \pi_{\mathcal{Z}} \mathfrak{A}(t) z, y \rangle_{\mathcal{X}} = \langle z, \mathfrak{A}(t)^* y \rangle_{\mathcal{Z}} \quad \forall\, z, y \in \mathcal{Z}$$

shows $\widetilde{\mathfrak{A}}^* = \mathfrak{A}^*|_{\mathcal{Z}}$. Thus, $\widetilde{A}^*$ is the generator of $\mathfrak{A}^*|_{\mathcal{Z}}$ which means by Lemma A.1.4 the part of A^* in $\mathcal{Z}$. The extension $\widetilde{A}|_{\mathcal{Z}} : \mathcal{Z} \subset (\operatorname{dom} \widetilde{A}^*)' \to (\operatorname{dom} \widetilde{A}^*)'$ reads

$$\langle \widetilde{A}|_{\mathcal{Z}} z, y\rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} = \langle z, A^* y\rangle_{\mathcal{X}} \quad \forall\, y \in \operatorname{dom} \widetilde{A}^*, z \in \mathcal{Z}.$$

We use this to calculate $\widetilde{B}$ via (2.7). Using that $\mathcal{Z}$ is A^*-invariant we obtain for all $z \in \operatorname{dom} \widetilde{A}^*$

$$\begin{aligned}
\langle \widetilde{B}u, z\rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} &= \langle \widetilde{\mathfrak{B}} e_\lambda u, (\bar{\lambda} - \widetilde{A}^*) z\rangle_{\mathcal{Z}} = \langle \pi_{\mathcal{Z}} \mathfrak{B} e_\lambda u, \bar{\lambda} z - A^* z\rangle_{\mathcal{X}} \\
&= \langle \mathfrak{B} e_\lambda u, \bar{\lambda} z - A^* z\rangle_{\mathcal{X}} = \langle (\lambda - A|_{\mathcal{X}}) \mathfrak{B} e_\lambda u, z\rangle_{\mathcal{X}} \\
&= \langle Bu, z\rangle_{(\operatorname{dom} A^*)', \operatorname{dom} A^*}.
\end{aligned}$$

We will show the claim about $\widetilde{C}$ for the Cesàro extension C_{ex} first. Let $z \in \operatorname{dom} \widetilde{C_{\mathrm{ex}}}$. Then, $z \in \mathcal{Z}$ by definition, and the limit

$$\frac{1}{t} \int_0^t (\mathfrak{C} z)(s)\, \mathrm{d}s = \frac{1}{t} \int_0^t (\widetilde{\mathfrak{C}} z)(s)\, \mathrm{d}s \xrightarrow{t \downarrow 0} \widetilde{C_{\mathrm{ex}}} z,$$

shows $z \in \operatorname{dom} C_{\mathrm{ex}} \cap \mathcal{Z}$. Trivially, this is a subset of $\pi_{\mathcal{Z}} \operatorname{dom} C_{\mathrm{ex}}$. Conversely, assume $z \in \pi_{\mathcal{Z}} \operatorname{dom} C_{\mathrm{ex}}$ and let $x \in \operatorname{dom} C_{\mathrm{ex}}$ be such that $z = \pi_{\mathcal{Z}} x$. Then

$$\frac{1}{t} \int_0^t (\widetilde{\mathfrak{C}} z)(s)\, \mathrm{d}s = \frac{1}{t} \int_0^t (\mathfrak{C} x)(s)\, \mathrm{d}s \xrightarrow{t \downarrow 0} C_{\mathrm{ex}} x, \tag{2.30}$$

which means $z \in \operatorname{dom} \widetilde{C_{\mathrm{ex}}}$. As we have already shown that this implies $z \in \operatorname{dom} C_{\mathrm{ex}} \cap \mathcal{Z}$, it follows that

$$\operatorname{dom} \widetilde{C_{\mathrm{ex}}} = \operatorname{dom} C_{\mathrm{ex}} \cap \mathcal{Z} = \pi_{\mathcal{Z}} \operatorname{dom} C_{\mathrm{ex}}.$$

The formula for the observation operator $C_{\mathrm{ex}}|_{\operatorname{dom} \widetilde{A}}$ is immediate from the special case $x \in \operatorname{dom} A$ in (2.30). □

Lemma 2.7.2. *Let $p \in [1, \infty]$ and $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ be an L^p-well-posed linear system on the Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$. Define the space $\mathcal{Z} := \overline{\operatorname{ran} \mathfrak{B}}$, equipped with the scalar product of $\mathcal{X}$. Then*

$$\left(\widetilde{\mathfrak{A}},\ \widetilde{\mathfrak{B}},\ \widetilde{\mathfrak{C}},\ \mathfrak{D}\right) := \left(\mathfrak{A}|_{\mathcal{Z}},\ \mathfrak{B},\ \mathfrak{C}|_{\mathcal{Z}},\ \mathfrak{D}\right)$$

is a controllable L^p-well-posed linear system on $(\mathcal{U}, \mathcal{Z}, \mathcal{Y})$. Its main operator $\widetilde{A}$

satisfies

$$\operatorname{dom} \widetilde{A} = \mathcal{Z} \cap \operatorname{dom} A, \qquad \operatorname{dom} \widetilde{A}^* = \pi_{\mathcal{Z}} \operatorname{dom} A^*$$

and

$$\begin{aligned} \widetilde{A}z &= Az && \forall\, z \in \operatorname{dom} \widetilde{A}, \\ \widetilde{A}^* z &= \pi_{\mathcal{Z}} A^* x && \forall\, z \in \operatorname{dom} \widetilde{A}^*, \ \forall\, x \in \operatorname{dom} A^* \text{ with } \pi_{\mathcal{Z}} x = z. \end{aligned}$$

For all $u \in \mathcal{U}$*, the control operator* $\widetilde{B}$ *satisfies*

$$\langle \widetilde{B}u, z \rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} = \langle Bu, x \rangle_{(\operatorname{dom} A^*)', \operatorname{dom} A^*}$$

for all $z \in \operatorname{dom} \widetilde{A}^*$ *and* $x \in \operatorname{dom} A^*$ *such that* $\pi_{\mathcal{Z}} x = z$*. The observation operator* $\widetilde{C}$ *equals the restriction of* C *to* $\operatorname{dom} \widetilde{A}$*, and its Cesàro extension* $\widetilde{C}_{\mathrm{ex}}$ *is given by*

$$\widetilde{C}_{\mathrm{ex}} z = Cz \quad \forall\, z \in \operatorname{dom} \widetilde{C} = \mathcal{Z} \cap \operatorname{dom} C.$$

Proof. The well-posedness of this is system is easy to show and contained in [Sta05, Corollary 9.1.10]. Since we are in a Hilbert space setting, the adjoint semigroups $\mathfrak{A}^*$ and $\widetilde{\mathfrak{A}}^*$ are again strongly continuous [Sta05, Theorem 3.5.6], and $\widetilde{A}^*$ generates the latter. A short calculation shows that $\widetilde{\mathfrak{A}}^* = \pi_{\mathcal{Z}} \mathfrak{A}^*|_{\mathcal{Z}}$. So $\widetilde{A}^*$ can alternatively be characterized as the quotient generator of the quotient semigroup, which by [EN00, Section 2.2.4] has the asserted representation. Consequently, the extension $\widetilde{A}|_{\mathcal{Z}} : \mathcal{Z} \subset (\operatorname{dom} \widetilde{A}^*)' \to (\operatorname{dom} \widetilde{A}^*)'$ satisfies

$$\forall\, z \in \mathcal{Z}, \ \forall\, y \in \operatorname{dom} \widetilde{A}^*, \ \forall\, x \in \operatorname{dom} A^* \text{ with } \pi_{\mathcal{Z}} x = y :$$

$$\langle \widetilde{A}|_{\mathcal{Z}} z, y \rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} = \langle z, \pi_{\mathcal{Z}} A^* x \rangle_{\mathcal{X}}.$$

We use this to resolve the expression $\widetilde{B}u = (\lambda - \widetilde{A})\widetilde{\mathfrak{B}} e_\lambda u$ for $u \in \mathcal{U}$: We take an arbitrary $z \in \operatorname{dom} \widetilde{A}^*$ and some $x \in \operatorname{dom} A^*$ with $\pi_{\mathcal{Z}} x = z$. Then

$$\begin{aligned} \langle \widetilde{B}u\,,\, z \rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} &= \langle \widetilde{\mathfrak{B}} e_\lambda u\,,\, (\bar{\lambda} - \widetilde{A}^*) z \rangle_{\mathcal{Z}} = \langle \mathfrak{B} e_\lambda u\,,\, \bar{\lambda} z - \pi_Z A^* x \rangle_{\mathcal{X}} \\ &= \langle \mathfrak{B} e_\lambda u\,,\, \bar{\lambda} x - A^* x \rangle_{\mathcal{X}} = \langle (\lambda - A|_{\mathcal{X}}) \mathfrak{B} e_\lambda u, x \rangle_X \\ &= \langle Bu, x \rangle_{(\operatorname{dom} A^*)'\,,\, \operatorname{dom} A^*}. \end{aligned}$$

The part about $\widetilde{C}$ and its extension is a direct consequence of the definition of the observation operator and the Cesàro extension (Definition 2.4.13), including the domain. □

Theorem 2.7.3 (Kalman compression). *Let $p \in [1, \infty]$, and let $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ be an ω-bounded L^p-well-posed linear system on the Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$. With the definitions*

$$\mathcal{M} := \pi_{(\ker \mathfrak{C})^\perp} \operatorname{ran} \mathfrak{B},$$
$$\widetilde{\mathfrak{A}} := \pi_{(\ker \mathfrak{C})^\perp} \mathfrak{A}|_{\overline{\mathcal{M}}}, \qquad \widetilde{\mathfrak{B}} := \pi_{(\ker \mathfrak{C})^\perp} \mathfrak{B}, \qquad \widetilde{\mathfrak{C}} := \mathfrak{C}|_{\overline{\mathcal{M}}},$$

the quadruple $(\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \widetilde{\mathfrak{C}}, \mathfrak{D})$ is a minimal L^p-well-posed linear system on $(\mathcal{U}, \overline{\mathcal{M}}, \mathcal{Y})$. The generator $\widetilde{A}$ of $\widetilde{\mathfrak{A}}$ has the domain $\operatorname{dom} \widetilde{A} = \overline{\mathcal{M}} \cap \pi_{(\ker \mathfrak{C})^\perp} \operatorname{dom} A$ and is given by

$$\widetilde{A}x = \pi_{(\ker \mathbb{C})^\perp} Az \quad \textit{for all } x \in \operatorname{dom} \widetilde{A} \textit{ and all } z \in \operatorname{dom} A \textit{ for which } x = \pi_{(\ker \mathfrak{C})^\perp} z.$$

The domain of the adjoint operator $\widetilde{A}^*$ is $\pi_{\overline{\mathcal{M}}}(\operatorname{dom} A^* \cap (\ker \mathfrak{C})^\perp)$. The control operator $\widetilde{B}$ is given by

$$\langle \widetilde{B}u, x \rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} = \langle Bu, z \rangle_{(\operatorname{dom} A^*)', \operatorname{dom} A^*}$$

for $z \in (\ker \mathfrak{C})^\perp \cap \operatorname{dom} A^*$ such that $\pi_{\overline{\mathcal{M}}} z = x$. The observation operator satisfies

$$\widetilde{C}x = Cz \quad \textit{for all } x \in \operatorname{dom} \widetilde{A} \textit{ and all } z \in \operatorname{dom} A \textit{ for which } x = \pi_{(\ker \mathfrak{C})^\perp} z,$$

Its Cesàro extension $\widetilde{C}_{\mathrm{ex}}$ equals C_{ex} with $\operatorname{dom} \widetilde{C}_{\mathrm{ex}} = \operatorname{dom} C_{\mathrm{ex}} \cap \overline{\mathcal{M}}$.

Proof. The theorem is proven by first applying Lemma 2.7.1 and then Lemma 2.7.2 with $\mathcal{Z} = \overline{\operatorname{ran} \pi_{(\ker \mathfrak{C})^\perp} \mathfrak{B}} = \overline{\mathcal{M}}$. The only thing that remains to be shown is that the projections $\pi_{\overline{\mathcal{M}}}$ and $\pi_{(\ker \mathfrak{C})^\perp}$ coincide on $\operatorname{ran} \mathfrak{B}$, or, in other words

$$\pi_{\overline{\mathcal{M}}} \mathfrak{B}u = \pi_{(\ker \mathfrak{C})^\perp} \mathfrak{B}u \quad \forall\, u \in L^p_\omega(\mathbb{R}_{\leqslant 0}; \mathcal{U})$$

Indeed, from $\overline{\mathcal{M}} = \overline{\operatorname{ran}(\pi_{(\ker \mathfrak{C})^\perp} \mathfrak{B})} \subset \overline{(\ker \mathfrak{C})^\perp} = (\ker \mathfrak{C})^\perp$ we deduce for arbitrary $u \in L^p_\omega(\mathbb{R}_{\leqslant 0}; \mathcal{U})$

$$\pi_{\overline{\mathcal{M}}} \mathfrak{B}u + \pi_{\overline{\mathcal{M}}^\perp} \mathfrak{B}u = \mathfrak{B}u = \pi_{(\ker \mathfrak{C})^\perp} \mathfrak{B}u + \pi_{\ker \mathfrak{C}} \mathfrak{B}u.$$

$$\Rightarrow \quad \pi_{\overline{\mathcal{M}}}\mathfrak{B}u - \pi_{(\ker\mathfrak{C})^\perp}\mathfrak{B}u = \pi_{\ker\mathfrak{C}}\mathfrak{B}u - \pi_{\overline{\mathcal{M}}^\perp}\mathfrak{B}u \in \overline{\mathcal{M}} \cap \overline{\mathcal{M}}^\perp = \{0\}.$$

□

2.8. Pritchard-Salamon systems

The so-called Pritchard-Salamon systems introduced in this section are a special type of compatible well-posed system nodes. They have the convenient property that the two operators $A\&B$ and $C\&D$ can be split uniquely into four operators A, B, C, D. In fact, the concept of a system node is not even needed for the control theory of Pritchard-Salamon systems. It is because of this and of the fact that Pritchard-Salamon system were historically developed before the theory of system nodes that one usually writes (A, B, C, D) instead of a block operator matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. We define Pritchard-Salamon systems in the sense of [PS87].

Definition 2.8.1 (Pritchard-Salamon system)**.** Let $\mathcal{U}, \mathcal{X}$, and $\mathcal{Y}$ be Hilbert spaces and let A generate a strongly continuous semigroup $\mathfrak{A}$ in $\mathcal{X}$. Furthermore, let $\mathcal{W}$ and $\mathcal{V}$ be Hilbert spaces with $\mathcal{X}_1 \subset \mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V} \subset \mathcal{X}_{-1}$, where the rigged spaces $\mathcal{X}_1$ and $\mathcal{X}_{-1}$ are defined as in Section 2.1. Then (A, B, C, D) is said to be a *Pritchard-Salamon system on* $(\mathcal{U}, (\mathcal{W}, \mathcal{X}, \mathcal{V}), \mathcal{Y})$ if the following conditions hold:

(i) $\mathfrak{A}$ extends to a strongly continuous semigroup $\mathfrak{A}|_{\mathcal{V}}$ on $\mathcal{V}$, and it restricts to a strongly continuous semigroup $\mathfrak{A}|_{\mathcal{W}}$ on $\mathcal{W}$;

(ii) $B \in \mathcal{B}(\mathcal{U}; \mathcal{V})$ is an *admissible control operator* for A, i.e. there exist $t, M > 0$ such that for all $u \in L^2([0,t]; \mathcal{U})$

$$\mathfrak{B}_t u := \int_0^t \mathfrak{A}|_{\mathcal{V}}(t-\tau)Bu(\tau)\,\mathrm{d}\tau \in \mathcal{W} \quad \text{and} \quad \|\mathfrak{B}_t u\|_{\mathcal{W}} \leqslant M\|u\|_{L^2([0,t];\mathcal{U})};$$

(iii) $C \in \mathcal{B}(\mathcal{W}; \mathcal{Y})$ is an *admissible observation operator for* A, i.e. there exist $t, M > 0$ such that for all $x \in \mathcal{W}$

$$\|C\mathfrak{A}(\cdot)x\|_{L^2([0,t];\mathcal{Y})} \leqslant M\|x\|_{\mathcal{V}}.$$

(iv) $D \in \mathcal{B}(\mathcal{U}; \mathcal{Y})$.

The generators of $\mathfrak{A}|_{\mathcal{W}}$ and $\mathfrak{A}|_{\mathcal{V}}$ are denoted by $A^{\mathcal{W}}$ and $A^{\mathcal{V}}$, respectively. The system is said to be *smooth* if $\operatorname{dom} A^{\mathcal{V}} \subset \mathcal{W}$.

Remark 2.8.2. (i) The space $\mathcal{X}$ in this definition is sometimes left out, see for example the definitions in [CLTZ94, vK93]. So strictly speaking our Definition gives only a subclass of the Pritchard-Salamon systems considered there. However, a smooth system in the sense of [CLTZ94, vK93] always fulfills our definition if we define $\mathcal{X}$ to be equal to $\mathcal{V}$. So for smooth systems there is no loss of generality. The reason why we include $\mathcal{X}$ is two-fold. Firstly $\mathcal{X}$ serves as a pivot space for the representations of $\mathcal{W}'$ and $\mathcal{V}'$, see also [vK93, p. 42]. Secondly, with our definition, systems of Pritchard-Salamon type become a proper subclass of L^2-well-posed linear systems, see Definition 2.8.6 below. This will allow us to apply the results of Chapter 6 to Pritchard-Salamon systems later on.

(ii) In [CZ94] the word “regular” was used instead of “smooth”. In this thesis we use regular in the sense of Definition 2.3.9, which was introduced by [Wei94b] and is something different.

(iii) “Admissible” control and observation operators for more general well-posed systems are defined with a different meaning, see [Sta05, Section 10.1].

(iv) The growth bounds of the semigroups $\mathfrak{A}|_{\mathcal{V}}$, $\mathfrak{A}|_{\mathcal{X}}$ and $\mathfrak{A}|_{\mathcal{W}}$ are in general not the same, an example is given in [CLTZ94].

In Chapter 3 we will encounter the special case where the control and observation operators are bounded with respect to $\mathcal{X}$. These systems are called state linear systems.

Definition 2.8.3 (state linear system)**.** A Pritchard-Salamon system on $(\mathcal{U}, (\mathcal{X}, \mathcal{X}, \mathcal{X}), \mathcal{Y})$ is called a *state linear system on* $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$.

Remark 2.8.4. Let A be the generator of a strongly continuous semigroup in $\mathcal{X}$, and $B \in \mathcal{B}(\mathcal{U}; \mathcal{X})$, $C \in \mathcal{B}(\mathcal{X}; \mathcal{Y})$, $D \in \mathcal{B}(\mathcal{U}; \mathcal{Y})$. Then it is easily seen that (A, B, C, D) is a state linear system on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$. In particular this shows that Definition 2.8.3 is equivalent to other definitions of state linear systems, e.g. in the monograph [CZ95].

Lemma 2.8.5. *Let* (A, B, C, D) *be a smooth Pritchard-Salamon system on* $(\mathcal{U}, (\mathcal{W}, \mathcal{X}, \mathcal{V}), \mathcal{Y})$ *and define*

$$\operatorname{dom} S := \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{X} \times \mathcal{U} \;\middle|\; A|_{\mathcal{X}} x + Bu \in \mathcal{X} \right\}.$$

Then

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} := \begin{bmatrix} A|_{\mathcal{X}} & B \\ C & D \end{bmatrix}\Bigg|_{\operatorname{dom} S}$$

is a compatible system node with compatibility space $\mathcal{W}$*.*

Moreover, $\operatorname{dom} A$ *is dense in* $\mathcal{W}$*,* $\operatorname{dom} S \subset \mathcal{W} \times \mathcal{U}$ *and the transfer function of this system node is uniformly regular with feedthrough* D*. In particular,* C *and* D *are uniquely determined by* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$*.*

Proof. By [CLTZ94, Lemma 2.12] there exist constants $M, \omega > 0$ such that for all $\lambda \in \mathbb{C}_{>\omega}$ the operator $(\lambda - A^{\mathcal{V}})^{-1}B$ maps into $\mathcal{W}$ and

$$\left\|(\lambda - A^{\mathcal{V}})^{-1}B\right\|_{\mathcal{B}(\mathcal{U};\mathcal{W})} \leqslant \frac{M}{\sqrt{\operatorname{Re}\lambda - \omega}}. \tag{2.31}$$

Now let $[x\,,\,u]^\top \in \operatorname{dom} S$. A short calculation shows that we can write x as

$$x = -(\lambda - A)^{-1}(A|_{\mathcal{X}} x + Bu) + (\lambda - A)^{-1}\lambda x + (\lambda - A^{\mathcal{V}})^{-1}Bu \tag{2.32}$$

for some $\lambda \in \mathbb{C}_{>\omega}$. Since all three summands on the right hand side are elements of $\mathcal{W}$, we conclude $x \in \mathcal{W}$ and $\operatorname{dom} S \subset \mathcal{W} \times \mathcal{U}$.

To show the closedness of the node let $[x_n\,,\,u_n]^\top$ be a sequence in $\operatorname{dom} S$ with

$$\begin{bmatrix} x_n \\ u_n \end{bmatrix} \xrightarrow{\mathcal{X}\times\mathcal{U}} \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{X} \times \mathcal{U} \quad \text{and} \quad \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\begin{bmatrix} x_n \\ u_n \end{bmatrix} \xrightarrow{\mathcal{X}\times\mathcal{Y}} \begin{bmatrix} z \\ y \end{bmatrix} \in \mathcal{X} \times \mathcal{Y}.$$

Since $A|_{\mathcal{X}}$ and B map continuously into the rigged space $\mathcal{X}_{-1}$ we conclude $A|_{\mathcal{X}} x + Bu = z$ in $\mathcal{X}_{-1}$, and since z is in $\mathcal{X}$, this implies $[x\,,\,u]^\top \in \operatorname{dom} S$. This shows in particular the closedness of $A\&B$. To show that $C\&D$ is closed we use the decomposition (2.32) on x_n: Since $A|_{\mathcal{X}} x_n + Bu_n$ converges in $\mathcal{X}$ and $(\lambda - A|_{\mathcal{V}})^{-1}B$

maps continuously into $\mathcal{W}$ we conclude that

$$x_n = -(\lambda - A)^{-1}(A|_{\mathcal{X}} x_n + Bu_n) + (\lambda - A)^{-1}\lambda x_n + (\lambda - A^{\mathcal{V}})^{-1}Bu_n$$

converges in $\mathcal{W}$ to

$$-(\lambda - A)^{-1}z + (\lambda - A)^{-1}\lambda x + (\lambda - A^{\mathcal{V}})^{-1}Bu = x.$$

Hence, $Cx_n \to Cx$ and, since the limit $Du_n \to Du$ is trivial, the closedness of $C\&D$ is shown.

All other conditions in Definition 2.2.1 are satisfied by assumption, so we do have a system node. The L^2-well-posedness follows from the estimates in (ii) and (iii) of Definition 2.8.1. Since $(\lambda - A^{\mathcal{V}})^{-1}B$ maps into $\mathcal{W}$, it is clear that $\mathcal{W}$ is a compatibility space and that D is the feedthrough associated to C via (2.2). Furthermore, the inequality (2.31) implies that the transfer function,

$$C\&D \begin{bmatrix} (\lambda - A^{\mathcal{V}})^{-1}B \\ \mathrm{I} \end{bmatrix} = C(\lambda - A^{\mathcal{V}})^{-1}B + D, \quad \forall\, \lambda \in \mathbb{C}_{>\omega},$$

converges to D with $\lambda \to \infty$. Hence, the node is uniformly regular and has feedthrough D.

Finally, the density of $\mathcal{X}_1$ in $\mathcal{W}$ is a consequences of the fact that $\mathfrak{A}$ restricts to a strongly continuous semigroup on $\mathcal{W}$, see [Sta05, Theorem 5.6.8 (ii) (c)]. □

Due to the well-posedness in this lemma, every Pritchard-Salamon system generates an L^2-well-posed linear system. This system will be called of Pritchard-Salamon type, more precisely, we make the following definition.

Definition 2.8.6 (Pritchard-Salamon type)**.** A uniformly regular ω-bounded L^2-well-posed linear system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ on the Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ is said to be *of Pritchard-Salamon type*, if there exists a compatibility space $\mathcal{W}$ and a Hilbert space $\mathcal{V}$ such that the main operator A, the control operator B, the extended observation operator $C|_{\mathcal{W}}$, and the feedthrough operator D induced by $C|_{\mathcal{W}}$ form a Pritchard Salamon system $(A, B, C|_{\mathcal{W}}, D)$ on $(\mathcal{U}, (\mathcal{W}, \mathcal{X}, \mathcal{V}), \mathcal{Y})$ in the sense of Definition 2.8.1. We call $(A, B, C|_{\mathcal{W}}, D)$ the *generators* of $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$.

The generators of such a system can easily be recovered from $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$: The operator A is of course the generator of $\mathfrak{A}$, and the feedthrough D is the limit of the transfer function. For $C|_{\mathcal{W}}$ we observe that $\mathfrak{C}w \in \mathcal{C}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$ for all $w \in \mathcal{W}$ and therefore $C|_{\mathcal{W}}w = (\mathfrak{C}w)(0)$. In order to determine B we can use a *Dirac sequence* $(d_n)_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}_{\leqslant 0};\mathbb{R}_{\geqslant 0})$. By this, we mean a sequence of functions $d_n \in L^2(\mathbb{R}_{\leqslant 0};\mathbb{R}_{\geqslant 0})$ satisfying $\operatorname{supp} d_n \subset [-\frac{1}{n}, 0]$ and $\int_{-\infty}^0 d_n(\tau)\,\mathrm{d}\tau = 1$. We see that $\mathfrak{B}ud_n$ converges to Bu in $\mathcal{V}$ for every $u \in \mathcal{U}$ because

$$\begin{aligned}
\|\mathfrak{B}ud_n - Bu\|_{\mathcal{V}} &= \left\| \int_{-\infty}^0 \mathfrak{A}(-\tau)|_{\mathcal{V}} Bu\, d_n(\tau)\,\mathrm{d}\tau - \int_{-\infty}^0 Bu\, d_n(\tau)\,\mathrm{d}\tau \right\|_{\mathcal{V}} \\
&\leqslant \int_{-\frac{1}{n}}^0 \|\mathfrak{A}(-\tau)|_{\mathcal{V}}Bu - Bu\|_{\mathcal{V}}\, d_n(\tau)\,\mathrm{d}\tau \\
&\leqslant \sup_{\tau\in[-\frac{1}{n},0]} \|\mathfrak{A}(-\tau)|_{\mathcal{V}}Bu - Bu\|_{\mathcal{V}} \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Duality concepts for Pritchard-Salamon systems require some explanation: We will always identify the dual space $\mathcal{X}'$ with $\mathcal{X}$ itself. We will interpret $\mathcal{W}'$ as the dual space of $\mathcal{W}$ with respect to the pivot space $\mathcal{X}$, see Section 2.1. The dual space of $\mathcal{V}$ will also be interpreted with respect to the pivot space $\mathcal{X}$ in the following sense: Every continuous functional v' on $\mathcal{V}$ is also a continuous functional on $\mathcal{X}$. Therefore, by the Riesz representation theorem we have a unique element $x_{v'} \in \mathcal{X}$ such that $\langle v', x\rangle_{\mathcal{V}',\mathcal{V}} = \langle x_{v'}, x\rangle_{\mathcal{X}}$ for all $x \in \mathcal{X}$. The element $v' \in \mathcal{V}'$ is identified with $x_{v'} \in \mathcal{X}$.

The adjoint of the generator $A^{\mathcal{W}} : \operatorname{dom} A^{\mathcal{W}} \subset \mathcal{W} \to \mathcal{W}$ of $\mathfrak{A}|_{\mathcal{W}}$ with respect to the duality pairing $\langle\cdot,\cdot\rangle_{\mathcal{W}',\mathcal{W}}$ is denoted by $(A^{\mathcal{W}})' : \operatorname{dom}(A^{\mathcal{W}})' \subset \mathcal{W}' \to \mathcal{W}'$. This operator generates the semigroup given by $(\mathfrak{A}(t)|_{\mathcal{W}})'$. The domain of this operator is

$$\operatorname{dom}(A^{\mathcal{W}})' = \left\{ x \in \mathcal{W}' \;\middle|\; \exists\, c > 0\ \forall\, y \in \operatorname{dom} A|_{\mathcal{W}} :\ |\langle x\,,\, Ay\rangle_{\mathcal{W}',\mathcal{W}}| \leqslant c\|y\|_{\mathcal{W}} \right\}$$

with norm $\|x\|_{\operatorname{dom}(A^{\mathcal{W}})'} = \|x\|_{\mathcal{W}'} + \|(A^{\mathcal{W}})'x\|_{\mathcal{W}'}$. For smooth Pritchard-Salamon systems, we have $\operatorname{dom}(A^{\mathcal{W}})' \hookrightarrow \mathcal{V}'$ [vK93, Theorem 2.17].

For a 0-bounded well-posed linear system of Pritchard-Salamon type the Gramians play an important role. We denote the adjoint of $\mathfrak{B} \in \mathcal{B}(L^2(\mathbb{R}_{\leqslant 0};\mathcal{U});\mathcal{W})$ by $\mathfrak{B}'$ and the adjoint of $\mathfrak{B} \in \mathcal{B}(L^2(\mathbb{R}_{\leqslant 0};\mathcal{U});\mathcal{X})$ by $\mathfrak{B}^*$. With the embedding $\mathcal{X} \subset \mathcal{W}'$ we than obtain $\mathfrak{B}'|_{\mathcal{X}} = \mathfrak{B}^*$. Similarly the adjoints $\mathfrak{C}' \in \mathcal{B}(L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y});\mathcal{V}')$ and $\mathfrak{C}^* \in \mathcal{B}(L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y});\mathcal{X})$ of $\mathfrak{C}$ with respect to $\mathcal{V}$ and $\mathcal{X}$, respectively, satisfy $\mathfrak{C}' = \mathfrak{C}^*|_{\mathcal{V}'}$.

For smooth Pritchard-Salamon systems Lemma 2.8 of [CZ94] states

$$\begin{aligned}
\mathfrak{B}\mathfrak{B}'|_{\operatorname{dom}(A^{\mathcal{W}})'} = \mathfrak{B}\mathfrak{B}^*|_{\operatorname{dom}(A^{\mathcal{W}})'} \quad &\in \mathcal{B}\left(\operatorname{dom}(A^{\mathcal{W}})'; \operatorname{dom} A^{\mathcal{V}}\right), \\
\mathfrak{C}^*\mathfrak{C}|_{\operatorname{dom} A^{\mathcal{V}}} = \mathfrak{C}'\mathfrak{C}|_{\operatorname{dom} A^{\mathcal{V}}} \quad &\in \mathcal{B}\left(\operatorname{dom} A^{\mathcal{V}}; \operatorname{dom}(A^{\mathcal{W}})'\right).
\end{aligned} \tag{2.33}$$

Definition 2.8.7 (impulse response)**.** The *impulse response* of an ω-bounded well-posed linear system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ of Pritchard-Salamon type with finite-dimensional input space $\mathcal{U}$ and control operator B is defined as the function

$$h := \mathfrak{C}B \in L^2_\omega(\mathbb{R}_{\geqslant 0}; \mathcal{B}(\mathcal{U}; \mathcal{Y})).$$

By [CLTZ94, Lemma 3.5 and Corollary 3.6] the impulse response satisfies $h = \mathfrak{C}B = C\mathfrak{B}$ and the Hankel operator has the representation

$$\mathfrak{C}\mathfrak{B}u = \int_{-\infty}^{0} h(-\tau)u(\tau)\,\mathrm{d}\tau \quad \forall\, u \in L^2_\omega(\mathbb{R}_{\leqslant 0}; \mathcal{U}).$$

Feedback

In order to control a system one typically applies a linear feedback. Informally speaking, this means the following: First, a new output $z(t) = Fx(t) + Gu(t) \in \mathcal{U}$ depending linearly on the state $x(t)$ and the input $u(t)$ of the system is created. In other words, the state and output equation in (2.3) receives a new output line

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \\
z(t) &:= \begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.
\end{aligned}$$

For this new system node to be a well-posed Pritchard-Salamon system we have to assume that F is an admissible observation operator in the sense of Definition 2.8.1. In the second step, the loop is closed, which means, the input $u(t)$ is made to equal the new output $z(t)$ plus some outer disturbance $\widetilde{u}(t)$, i.e.

$$\begin{aligned}
\dot{x}(t) &:= Ax(t) + B(z(t) + \widetilde{u}(t)), \\
z(t) &:= \begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) + \widetilde{u}(t) \end{bmatrix}.
\end{aligned}$$

In order to eliminate the auxiliary output z from these equation $\mathrm{I}-G$ should be invertible. This motivates the upcoming definition.

Definition 2.8.8 (admissible feedback pair)**.** The block operator $[F\,,\,G] \in \mathcal{B}(\mathcal{W}\times\mathcal{U};\mathcal{U})$ is an *admissible feedback pair* for the Pritchard-Salamon System (A,B,C,D) on $(\mathcal{U},(\mathcal{W},\mathcal{X},\mathcal{V}),\mathcal{Y})$ if $F\in\mathcal{B}(\mathcal{W};\mathcal{U})$ is an admissible observation operator for A, i.e. (A,B,F,G) is a Pritchard-Salamon system on $(\mathcal{U},(\mathcal{W},\mathcal{X},\mathcal{V}),\mathcal{U})$, and $\mathrm{I}-G$ is boundedly invertible in $\mathcal{B}(\mathcal{U})$.

Without loss of generality we could replace the admissible feedback pair $[F\,,\,G]$ by $[(\mathrm{I}-G)^{-1}F\,,\,0]$. That is why many authors only consider state feedback operators instead of feedback pairs. There are two reasons why we use the more general feedback pairs. The first is that the auxiliary output z should be allowed to depend on u via a feedthrough just like the original output $y = Cx + Du$ in (2.3) does. The second and more important one lies in the so-called closed-loop system introduced in the following lemma. This closed-loop system has to be defined with the feedthrough G and has several very important properties that will be described and exploited in Chapter 7.

Lemma 2.8.9. *If $[F\,,\,G] \in \mathcal{B}(\mathcal{W}\times\mathcal{U};\mathcal{U})$ is an admissible feedback pair for the smooth Pritchard-Salamon System (A,B,C,D) on $(\mathcal{U},(\mathcal{W},\mathcal{X},\mathcal{V}),\mathcal{Y})$, then*

$$A^{\mathcal{V}}_{\circlearrowleft} : \operatorname{dom} A^{\mathcal{V}} \subset \mathcal{V} \to \mathcal{V}, \qquad A^{\mathcal{V}}_{\circlearrowleft} := A^{\mathcal{V}} + B(\mathrm{I}-G)^{-1}F,$$

generates a strongly continuous semigroup $\mathfrak{A}^{\mathcal{V}}_{\circlearrowleft}$ in $\mathcal{V}$ which restricts to strongly continuous semigroups $\mathfrak{A}_{\circlearrowleft}$ and $\mathfrak{A}^{\mathcal{W}}_{\circlearrowleft}$ in $\mathcal{X}$ and $\mathcal{W}$, respectively. The generator $A_{\circlearrowleft}$ of $\mathfrak{A}_{\circlearrowleft}$ is the restriction of $A^{\mathcal{V}}_{\circlearrowleft}$ to

$$\operatorname{dom} A_{\circlearrowleft} := \left\{\ x \in \operatorname{dom} A^{\mathcal{V}} \ \middle|\ A^{\mathcal{V}}x + B(\mathrm{I}-G)^{-1}Fx \in \mathcal{X}\ \right\},$$

and the quadruple

$$\begin{aligned}&(A_{\circlearrowleft}, B_{\circlearrowleft}, C_{\circlearrowleft}, D_{\circlearrowleft}) :=\\ &\quad\left(A_{\circlearrowleft},\ B(\mathrm{I}-G)^{-1}, \begin{bmatrix} C + D(\mathrm{I}-G)^{-1}F \\ (\mathrm{I}-G)^{-1}F\end{bmatrix}, \begin{bmatrix} D(\mathrm{I}-G)^{-1} \\ (\mathrm{I}-G)^{-1}\end{bmatrix}\right)\end{aligned} \tag{2.34}$$

defines a smooth Pritchard-Salamon system on $(\mathcal{U},(\mathcal{W},\mathcal{X},\mathcal{V}),\mathcal{Y}\times\mathcal{U})$, the so-called closed-loop system.

If $[F\,,\,G] \in \mathcal{B}(\mathcal{W}\times\mathcal{U};\mathcal{U})$ is an admissible feedback pair, then $(\mathrm{I}-G)^{-1}F$ is an admissible observation operator for A. Hence the lemma follows from [vK93, Lemma 2.13].

Definition 2.8.10. An admissible feedback pair $[F\,,\,G] \in \mathcal{B}(\mathcal{W}\times\mathcal{U};\mathcal{U})$ for the smooth Pritchard-Salamon System (A, B, C, D) on $(\mathcal{U}, (\mathcal{W}, \mathcal{X}, \mathcal{V}), \mathcal{Y})$ is said to be *exponentially stabilizing* if the semigroups $\mathfrak{A}^{\mathcal{V}}_{\circlearrowleft}$ and $\mathfrak{A}^{\mathcal{W}}_{\circlearrowleft}$ that belong to the closed-loop system (2.34) are exponentially stable.

An example

Loosely speaking, the range of the control operator B and the domain of the observation operator C of a Pritchard-Salamon system must not be "too far apart". This is illustrated by the fact that substituting the observation operator of the example in Section 2.6 by a bounded operator creates a Pritchard-Salamon system:

Lemma 2.8.11. *Let the operators A and B be defined as in* (2.15) *and* (2.19)*, respectively. Furthermore, let $C \in \mathcal{B}(\mathcal{X};\mathbb{C})$. Then $(A, B, C, 0)$ is a Pritchard-Salamon system on $\big(\mathbb{C}, (L^2(\Omega), L^2(\Omega), W^{k,2}(\Omega)'), \mathbb{C}\big)$, where $k \in (\frac{1}{2}, 1)$. The transfer function is*

$$\mathbf{G}(s) = \sum_{k=0}^{\infty} \frac{c_k}{s+\lambda_k} \quad \forall\, s \in \rho(A),$$

where

$$c_k = \int_{\partial\Omega} \overline{v_k(\xi)}\,\mathrm{d}\sigma_\xi \cdot C v_k \quad \forall\, k \in \mathbb{N}_0.$$

Proof. By [Tri95, Theorem 4.3.3] there holds for the fractional powers and the interpolation functor defined in Section A.1,

$$\operatorname{dom}(\mathrm{I}-A)^{\frac{k}{2}} = \big[L^2(\Omega)\,,\,\operatorname{dom} A\big]_{\frac{k}{2}} = W^{k,2}(\Omega).$$

Thus, $\mathfrak{A}$ extends to a strongly continuous semigroup $\mathfrak{A}|_{\mathcal{X}_{-\frac{k}{2}}}$ on $\mathcal{X}_{-\frac{k}{2}} = W^{k,2}(\Omega)'$, and its generator has the domain $\operatorname{dom}(\mathrm{I}-A)^{1-\frac{k}{2}} = W^{2-k,2}(\Omega)$. As a consequence,

$$\int_0^t \big\|C\mathfrak{A}(\tau)\big|_{\mathcal{X}_{-\frac{k}{2}}}\big\|^2_{W^{k,2}(\Omega)'}\,\mathrm{d}\tau \leqslant \|C\|_{L^2(\Omega)'} \int_0^t \tau^{-k}\,\mathrm{d}\tau = \|C\|_{L^2(\Omega)'} \frac{t^{1-k}}{1-k}.$$

The formula for the transfer function follows by applying C to (2.24). $\square$

2.9. Kalman compression of Pritchard-Salamon systems

Since systems of Pritchard-Salamon type are well-posed linear systems, we can apply the Kalman compression from Section 2.7. However, those results do not yet show that the compressed system is again of Pritchard-Salamon type. In fact, the Kalman compression for Pritchard-Salamon systems should be interpreted in a different way. In this section let $\mathcal{U}$, $\mathcal{W}$, $\mathcal{X}$, $\mathcal{V}$, and $\mathcal{Y}$ be five Hilbert spaces.

Lemma 2.9.1. *Let* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ *be a well-posed linear system of Pritchard-Salamon type on* $(\mathcal{U}, (\mathcal{W}, \mathcal{X}, \mathcal{V}), \mathcal{Y})$. *Define*

$$\widetilde{\mathcal{W}} := \overline{\operatorname{ran}\mathfrak{B}}^{\mathcal{W}}, \quad \widetilde{\mathcal{X}} := \overline{\operatorname{ran}\mathfrak{B}}^{\mathcal{X}}, \quad \widetilde{\mathcal{V}} := \overline{\operatorname{ran}\mathfrak{B}}^{\mathcal{V}},$$

equipped with the norms of $\mathcal{W}$, $\mathcal{X}$ *and* $\mathcal{V}$, *respectively. Then*

$$\left(\widetilde{\mathfrak{A}}, \quad \widetilde{\mathfrak{B}}, \quad \widetilde{\mathfrak{C}}, \quad \mathfrak{D}\right) := \left(\mathfrak{A}|_{\widetilde{\mathcal{X}}}, \quad \mathfrak{B}, \quad \mathfrak{C}|_{\widetilde{\mathcal{X}}}, \quad \mathfrak{D}\right)$$

is a well-posed linear system of Pritchard-Salamon type on $(\mathcal{U}, (\widetilde{\mathcal{W}}, \widetilde{\mathcal{X}}, \widetilde{\mathcal{V}}), \mathcal{Y})$ *and controllable. Its generators are* $(\widetilde{A}, \widetilde{B}, \widetilde{C}, D)$, *where*

$$\begin{aligned} \widetilde{A}x &= Ax, \quad \forall\, x \in \operatorname{dom}\widetilde{A} = \widetilde{\mathcal{X}} \cap \operatorname{dom} A, \\ \widetilde{B} &= B, \\ \widetilde{C}w &= Cw, \quad \forall\, w \in \widetilde{W}. \end{aligned}$$

Proof. Since the embeddings

$$\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$$

are continuous and $\operatorname{ran}\mathfrak{B} \subset \mathcal{W}$, it is clear that we have the continuous and dense embeddings

$$\overline{\operatorname{ran}\mathfrak{B}}^{\mathcal{W}} \hookrightarrow \overline{\operatorname{ran}\mathfrak{B}}^{\mathcal{X}} \hookrightarrow \overline{\operatorname{ran}\mathfrak{B}}^{\mathcal{V}}.$$

Moreover, we have $\mathfrak{B} \in \mathcal{B}(L^2_\omega(\mathbb{R}_{\leqslant 0}; \mathcal{U}); \widetilde{\mathcal{W}})$ and $\mathfrak{C} \in \mathcal{B}(\widetilde{\mathcal{V}}; L^2_\omega(\mathbb{R}_{\geqslant 0}; \mathcal{Y}))$. Since $\operatorname{ran}\mathfrak{B}$ is $\mathfrak{A}|_{\mathcal{X}}$-invariant, it is also $\mathfrak{A}_{\mathcal{V}}$-invariant, and the same holds for the closures of $\operatorname{ran}\mathfrak{B}$. Therefore, $\mathfrak{A}$ induces strongly continuous semigroups $\mathfrak{A}|_{\widetilde{\mathcal{W}}}$, $\mathfrak{A}|_{\widetilde{\mathcal{X}}}$ and $\mathfrak{A}|_{\widetilde{\mathcal{V}}}$ on $\widetilde{\mathcal{W}}$, $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{V}}$, respectively. To see that B maps into $\widetilde{\mathcal{V}}$ we use a Dirac sequence

$(d_n)_{n\in\mathbb{N}} \subset L^2(\mathbb{R}_{\leqslant 0};\mathbb{R}_{\geqslant 0})$ with $\operatorname{supp} d_n \subset [-\frac{1}{n};0]$. Then, for all $u \in \mathcal{U}$, the limit

$$\begin{aligned}
\left\| \int_{-\infty}^{0} \mathfrak{A}(-\tau)|_{\mathcal{V}} B d_n(\tau) u \,\mathrm{d}\tau - Bu \right\|_{\mathcal{V}} &\leqslant \left\| \int_{-\infty}^{0} (\mathfrak{A}(-\tau)|_{\mathcal{V}} B - B) d_n(\tau) u \,\mathrm{d}\tau \right\|_{\mathcal{V}} \\
&\leqslant \int_{-\frac{1}{n}}^{0} \|(\mathfrak{A}(-\tau)|_{\mathcal{V}} B - B) d_n(\tau) u\|_{\mathcal{V}} \,\mathrm{d}\tau \\
&\leqslant \|u\|_{\mathcal{U}} \sup_{\tau\in[-\frac{1}{n},0]} \|(\mathfrak{A}(\tau)|_{\mathcal{V}} B - B)\|_{\mathcal{V}} \\
&\longrightarrow 0 \quad (n \to \infty),
\end{aligned}$$

shows that $Bu \in \overline{\operatorname{ran}\mathfrak{B}}^{\mathcal{V}}$. It is easily seen that $\widetilde{B}$, $\widetilde{C}$ and D are the remaining generators of the restricted system. □

If $\mathcal{Z}$ is a closed subspace of $\mathcal{X}$ the quotient space $\mathcal{X}/\mathcal{Z}$ is equipped with the norm

$$\|\tilde{z}\|_{\mathcal{X}/\mathcal{Z}} := \inf\left\{\|z - c\|_{\mathcal{X}} \mid c \in \mathcal{Z}, z \in \tilde{z}\right\},$$

Lemma 2.9.2. *Let $(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})$ be a well-posed linear system of Pritchard-Salamon type on $(\mathcal{U},(\mathcal{W},\mathcal{X},\mathcal{V}),\mathcal{Y})$. Define*

$$\widetilde{\mathcal{W}} := \mathcal{W}/\ker\mathfrak{C}|_{\mathcal{W}}, \qquad \widetilde{\mathcal{X}} := \mathcal{X}/\ker\mathfrak{C}|_{\mathcal{X}}, \qquad \widetilde{\mathcal{V}} := \mathcal{V}/\ker\mathfrak{C}|_{\mathcal{V}},$$

with the quotient norms $\|\cdot\|_{\widetilde{\mathcal{W}}}$, $\|\cdot\|_{\widetilde{\mathcal{X}}}$, $\|\cdot\|_{\widetilde{\mathcal{V}}}$, respectively. Then there holds

$$\widetilde{\mathcal{W}} \hookrightarrow \widetilde{\mathcal{X}} \hookrightarrow \widetilde{\mathcal{V}}.$$

We denote by $\tilde{\pi}_{\widetilde{\mathcal{W}}} : \mathcal{W} \to \widetilde{\mathcal{W}}$ the injection that maps each element of $\mathcal{W}$ to its equivalence class in $\widetilde{\mathcal{W}}$. Then

$$\left(\widetilde{\mathfrak{A}},\ \widetilde{\mathfrak{B}},\ \widetilde{\mathfrak{C}},\ \mathfrak{D}\right) := \left(\tilde{\pi}_{\widetilde{\mathcal{X}}}\mathfrak{A}|_{\widetilde{\mathcal{X}}},\ \tilde{\pi}_{\widetilde{\mathcal{X}}}\mathfrak{B},\ \mathfrak{C}|_{\widetilde{\mathcal{X}}},\ \mathfrak{D}\right)$$

is a well-posed linear system of Pritchard-Salamon type on $(\mathcal{U}$, $(\widetilde{\mathcal{W}},\widetilde{\mathcal{X}},\widetilde{\mathcal{V}})$, $\mathcal{Y})$ and observable. Its generators are $(\widetilde{A},\widetilde{B},\widetilde{C},D)$, where

$$\begin{aligned}
&\widetilde{A}\tilde{x} = \tilde{\pi}_{\widetilde{\mathcal{X}}} Az && \forall\, \tilde{x} \in \operatorname{dom}\widetilde{A} = \tilde{\pi}_{\widetilde{\mathcal{X}}} \operatorname{dom} A,\ \forall\, z \in \tilde{x} \cap \operatorname{dom} A, \\
&\widetilde{B} = \tilde{\pi}_{\widetilde{\mathcal{V}}} B,
\end{aligned}$$

$$\widetilde{C}\widetilde{w} = Cw \qquad\qquad \forall\, w \in \widetilde{w} \in \widetilde{W}.$$

Remark 2.9.3. The reason why we do not explicitly identify the quotient spaces, $\mathcal{X}/\ker \mathfrak{C}|_{\mathcal{X}}$ etc., with the orthogonal complements here is the following: The identity operator is not a continuous and dense injection from $(\ker \mathfrak{C}|_{\mathcal{X}})^{\perp}$ into $(\ker \mathfrak{C}|_{\mathcal{V}})^{\perp}$ because the first complement is taken with respect to the scalar product of $\mathcal{X}$ and the second with respect to the scalar product of $\mathcal{V}$. It is possible to construct an observable Pritchard-Salamon realization on these spaces as well. However the cost is that the embedding is no longer given by the identity but by a much more complicated mapping.

Proof of Lemma 2.9.2. First note that the identity given by

$$\mathrm{I} : \mathcal{W}/\ker \mathfrak{C}|_{\mathcal{W}} \to \mathcal{X}/\ker \mathfrak{C}|_{\mathcal{X}}, \qquad \widetilde{w} \mapsto \{x \in \mathcal{X} \mid \exists\, w \in \widetilde{w} : x - w \in \ker \mathfrak{C}|_{\mathcal{X}}\}$$

is indeed an injection. Furthermore, we have $\mathrm{I}\,\widetilde{\pi}_{\ker \mathfrak{C}|_{\mathcal{W}}} w = \widetilde{\pi}_{\ker \mathfrak{C}|_{\mathcal{X}}} w$ for all $w \in \mathcal{W}$. Therefore, $\|w_n - x\|_{\mathcal{X}} \to 0$ implies $\|\,\mathrm{I}\,\widetilde{\pi}_{\ker \mathfrak{C}|_{\mathcal{W}}} w_n - \widetilde{\pi}_{\ker \mathfrak{C}|_{\mathcal{X}}} x\|_{\widetilde{\mathcal{X}}} \to 0$ for any sequence (w_n) in $\mathcal{W}$ and $x \in \mathcal{X}$. Hence the density of $\mathcal{W}$ in $\mathcal{X}$ implies the density of $\widetilde{\mathcal{W}}$ in $\widetilde{\mathcal{X}}$. Furthermore, there holds for all $w \in \widetilde{\mathcal{W}}$

$$\begin{aligned}
\|\,\mathrm{I}\,\widetilde{w}\|_{\widetilde{\mathcal{X}}} &= \inf\{\|x - c\|_{\mathcal{X}} \mid c \in \ker \mathfrak{C}|_{\mathcal{X}}, x \in \mathrm{I}\,\widetilde{w}\}\\
&\leqslant \inf\{\|w - c_w\|_{\mathcal{X}} \mid c_w \in \ker \mathfrak{C}|_{\mathcal{W}}, w \in \widetilde{w}\}\\
&\leqslant \inf\{\|w - c_w\|_{\mathcal{W}} \mid c_w \in \ker \mathfrak{C}|_{\mathcal{W}}, w \in \widetilde{w}\}\\
&= \|\widetilde{w}\|_{\widetilde{\mathcal{W}}}.
\end{aligned}$$

In the second line we have used that $\widetilde{w} \subset \mathrm{I}\,\widetilde{w}$. This norm estimate shows that the embedding $\mathrm{I} : \widetilde{\mathcal{W}} \to \widetilde{\mathcal{X}}$ is continuous. Analogously, we see $\widetilde{\mathcal{X}} \hookrightarrow \widetilde{\mathcal{V}}$. These quotient spaces are Hilbert spaces because they are isometrically isomorphic to the corresponding orthogonal complement of $\ker \mathfrak{C}$ in the Hilbert spaces $\mathcal{W}$, $\mathcal{X}$ and $\mathcal{V}$, respectively.

The fact that $(\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \widetilde{\mathfrak{C}}, \mathfrak{D})$ is an observable well-posed linear system follows from Lemma 2.7.2 once we identify $\mathcal{X}/\ker \mathfrak{C}|_{\mathcal{X}}$ with $(\ker \mathfrak{C}|_{\mathcal{X}})^{\perp}$. It remains to determine its generators and to show that it is of Pritchard-Salamon type. We remark that, for $\widetilde{x} \in \widetilde{\mathcal{X}}$, the relation $\widetilde{x} \in \widetilde{\mathcal{W}}$ is true if and only if $\widetilde{x} \cap \mathcal{W} \neq \varnothing$, and analogously $\widetilde{v} \in \widetilde{\mathcal{X}}$ if and only if $\widetilde{v} \cap \mathcal{X} \neq \varnothing$. Therefore we have $\widetilde{\pi}_{\widetilde{\mathcal{X}}}\mathfrak{B} = \widetilde{\pi}_{\widetilde{\mathcal{W}}}\mathfrak{B}$, and the operator

$\mathfrak{C}|_{\widetilde{\mathcal{X}}}$ is continuously extendable to $\widetilde{\mathcal{V}}$. Let $x \in \widetilde{x} \in \widetilde{\mathcal{X}}$, then the computation

$$\widetilde{\mathfrak{A}}|_{\widetilde{\mathcal{V}}}(t)\widetilde{x} = \widetilde{\pi}_{\widetilde{\mathcal{V}}}\mathfrak{A}(t)\widetilde{x} = \widetilde{\pi}_{\widetilde{\mathcal{V}}}\mathfrak{A}(t)x = \widetilde{\pi}_{\widetilde{\mathcal{V}}}\underbrace{\mathfrak{A}(t)x}_{\in\mathcal{X}} = \widetilde{\pi}_{\widetilde{\mathcal{X}}}\mathfrak{A}(t)x = \widetilde{\mathfrak{A}}|_{\widetilde{\mathcal{X}}}(t)\widetilde{x}$$

shows that the semigroup $\mathfrak{A}|_{\widetilde{\mathcal{V}}}$ restricts to $\mathfrak{A}|_{\widetilde{\mathcal{X}}}$. An analogous computation shows that, in turn, $\mathfrak{A}|_{\widetilde{\mathcal{X}}}$ is an extension of $\mathfrak{A}|_{\widetilde{\mathcal{W}}}$. Furthermore, for all $w \in \widetilde{w} \in \widetilde{W}$ we have

$$\widetilde{C}\widetilde{w} = (\widetilde{\mathfrak{C}}\widetilde{w})(0) = (\mathfrak{C}w)(0) = Cw.$$

Let (d_n) be a Dirac sequence in $L^2(\mathbb{R}_{\leqslant 0}; \mathbb{R}_{\geqslant 0})$, and let $u \in \mathcal{U}$. Then

$$\left\|\widetilde{\mathfrak{B}}d_n u - \widetilde{\pi}_{\widetilde{\mathcal{V}}}Bu\right\|_{\widetilde{\mathcal{V}}} = \left\|\widetilde{\pi}_{\widetilde{\mathcal{W}}}\mathfrak{B}d_n u - \widetilde{\pi}_{\widetilde{\mathcal{V}}}Bu\right\|_{\widetilde{\mathcal{V}}} = \left\|\widetilde{\pi}_{\widetilde{\mathcal{V}}}\left(\mathfrak{B}d_n u - Bu\right)\right\|_{\widetilde{\mathcal{V}}},$$

which tends to zero for $n \to \infty$. This proves $\widetilde{B} = \widetilde{\pi}_{\widetilde{\mathcal{V}}}B$. □

Combining the last two lemmas we obtain a special version of Theorem 2.7.3

Lemma 2.9.4. *Let $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ be a well-posed linear system of Pritchard-Salamon type on $(\mathcal{U}, (\mathcal{W}, \mathcal{X}, \mathcal{V}), \mathcal{Y})$. With the definitions*

$$\mathcal{M} := \widetilde{\pi}_{\mathcal{W}/\ker\mathfrak{C}|_{\mathcal{W}}} \operatorname{ran}\mathfrak{B},$$

$$\widetilde{\mathcal{W}} := \overline{\mathcal{M}}^{\mathcal{W}}, \qquad \widetilde{\mathcal{X}} := \overline{\mathcal{M}}^{\mathcal{X}}, \qquad \widetilde{\mathcal{V}} := \overline{\mathcal{M}}^{\mathcal{V}},$$

$$\widetilde{\mathfrak{A}} := \widetilde{\pi}_{\mathcal{X}/\ker\mathfrak{C}}\mathfrak{A}|_{\widetilde{\mathcal{X}}}, \qquad \widetilde{\mathfrak{B}} := \widetilde{\pi}_{\mathcal{X}/\ker\mathfrak{C}}\mathfrak{B}, \qquad \widetilde{\mathfrak{C}} := \mathfrak{C}|_{\overline{\mathcal{M}}},$$

the quadruple $(\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \widetilde{\mathfrak{C}}, \mathfrak{D})$ is a well-posed linear system of Pritchard-Salamon type on $(\mathcal{U}, (\widetilde{\mathcal{W}}, \widetilde{\mathcal{X}}, \widetilde{\mathcal{V}}), \mathcal{Y})$ and minimal. The generator $\widetilde{A}$ of $\widetilde{\mathfrak{A}}$ is given by

$$\operatorname{dom}\widetilde{A} = \widetilde{\mathcal{X}} \cap \widetilde{\pi}_{\mathcal{X}/\ker\mathfrak{C}|_{\mathcal{X}}} \operatorname{dom} A,$$

$$\widetilde{A}\widetilde{x} = \widetilde{\pi}_{\mathcal{X}/\ker\mathfrak{C}|_{\mathcal{X}}} Az \quad \forall\, \widetilde{x} \in \operatorname{dom}\widetilde{A},\ \forall\, z \in \widetilde{x} \cap \operatorname{dom} A.$$

The domain of the adjoint operator $\widetilde{A}^*$ is $\widetilde{\pi}_{\mathcal{X}}(\operatorname{dom} A^* \cap (\mathcal{X}/\ker\mathfrak{C}))$. Analogous formulas hold for $\widetilde{A}|_{\widetilde{\mathcal{W}}}$ and $\widetilde{A}|_{\widetilde{\mathcal{V}}}$. The other generators are given by

$$\widetilde{B} = \widetilde{\pi}_{\mathcal{V}/\ker\mathfrak{C}|_{\mathcal{V}}} B \qquad \textit{and}$$

$$\widetilde{C}\widetilde{w} = Cw \quad \forall\, w \in \widetilde{\mathcal{W}},\ \forall\, w \in \widetilde{w}.$$

Proof. This follows by first applying Lemma 2.9.2, and then applying Lemma 2.9.1 to the resulting system. Note that the relation

$$\mathcal{M} = \widetilde{\pi}_{\mathcal{W}/\ker \mathfrak{C}|_{\mathcal{W}}} \operatorname{ran} \mathfrak{B} = \widetilde{\pi}_{\mathcal{X}/\ker \mathfrak{C}|_{\mathcal{X}}} \mathfrak{B} = \widetilde{\pi}_{\mathcal{V}/\ker \mathfrak{C}|_{\mathcal{V}}} \mathfrak{B}$$

holds. □

2.10. Notes and references

The theory of well-posed linear systems is standard nowadays. We have marked for each result in this chapter where the reader can find it. Most of it is taken from [Sta05], some Hilbert space specific results from [TW09].

Since Pritchard-Salamon theory was historically developed before the general theory of well-posed linear systems, the embedding of the Pritchard-Salamon systems into the well-posed linear systems described in Section 2.8 is rarely used, albeit well-known.

A detailed description of the generators of the Kalman compression as in Section 2.7 can not be found in the literature yet and neither can the more involved Kalman compression of Pritchard-Salamon systems in Section 2.9.

The example we have worked out in Section 2.6 is based on the examination of the same equation with different boundary condition in [BGSW02]. Additional properties of this configuration such as invariant zeros, transmission zeros and the root locus will be published in [RS15b].

3. State space transformations for systems with relative degree

In this chapter we consider systems whose relative degree is well-defined within the natural numbers. The zero dynamics form and the Byrnes-Isidori form developed in this chapter are two similar realizations. They both reveal the part of the behavior that can not be seen from the input-output map, the so-called zero dynamics. Moreover, the Byrnes-Isisdori form corresponds to the zero dynamics form of the dual system.

3.1. Relative degree

In the current chapter we are going to assume the existence of a relative degree in natural numbers. This means in particular that the control and the observation operator are bounded with respect to the state space $\mathcal{X}$. Throughout Chapter 3 we let the following presumption hold.

Presumption 3.1.1. *The Hilbert space $\mathcal{X}$ with scalar product $\langle \cdot\,, \cdot \rangle$ is real and $(A, B, C, 0)$ is a state linear system on $(\mathbb{R}, \mathcal{X}, \mathbb{R})$. Furthermore, $r \in \mathbb{N}$, and the control and observation operator are given by*

$$B : \mathbb{R} \to \mathcal{X}, \quad Bu := bu, \qquad C : \mathcal{X} \to \mathbb{R}, \quad Cx := \langle x\,, c \rangle\,,$$

with vectors $b \in \operatorname{dom} A^r$ and $c \in \operatorname{dom} {A^*}^r$ that satisfy

$$\left\langle A^{r-1}b\,, c \right\rangle \neq 0 \quad \text{and} \quad \left\langle A^{j}b\,, c \right\rangle = 0 \quad \forall\, j = 0, 1, \ldots, r-2. \tag{3.1}$$

Definition 3.1.2 (relative degree)**.** A state linear system on $(\mathbb{R}, \mathcal{X}, \mathbb{R})$ is said to be of *relative degree r* if it fulfills Presumption 3.1.1. In this case, we write $(A, B, C, 0) \in$

$\mathbf{\Sigma}_r$.

Remark 3.1.3. (i) The system node corresponding to $(A, B, C, 0)$ via Lemma 2.8.5 is

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} : \operatorname{dom} A \times \mathbb{R} \subset \mathcal{X} \times \mathbb{R} \to \mathcal{X} \times \mathbb{R}, \qquad \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} Ax + bu \\ \langle x\,,\, c \rangle \end{bmatrix}.$$

(ii) The adjoints of B and C satisfy $B^* = \langle \cdot\,,\, b \rangle$ and $C^* = c$. Therefore, a system is of relative degree r if and only if its dual system is; in other words

$$(A, B, C, 0) \in \mathbf{\Sigma}_r \quad \Leftrightarrow \quad (A^*, C^*, B^*, 0) \in \mathbf{\Sigma}_r.$$

(iii) The class $\mathbf{\Sigma}_r$ is invariant under similarity transformations; in other words, for every boundedly invertible operator T we have

$$(A, B, C, 0) \in \mathbf{\Sigma}_r \quad \Leftrightarrow \quad (TAT^{-1}, CT^{-1}, TB, 0) \in \mathbf{\Sigma}_r.$$

3.2. The zero dynamics form

Definition 3.2.1 (zero dynamics form)**.** Let $\mathcal{X}$ be a real Hilbert space. A state linear system $(A,\ B,\ C,\ D)$ on $(\mathbb{R}, \mathcal{X}, \mathbb{R})$ is said to be in *zero dynamics form* if and only if $\mathcal{X} = \mathbb{R}^r \times \mathcal{V}$ for some Hilbert space $\mathcal{V}$, and the operators A, B, C satisfy the following conditions:

(i) There exists an operator $Q : \operatorname{dom} Q \subset \mathcal{V} \to \mathcal{V}$ that generates a strongly continuous semigroup in $\mathcal{V}$;

(ii) The operator A has the domain $\operatorname{dom} A = \mathbb{R}^r \times \operatorname{dom} Q$, and there are bounded operators $p_0, \ldots, p_{r-1} : \mathbb{R} \to \mathbb{R}$, $\quad R : \mathbb{R} \to \mathcal{V}$, $\quad S : \mathcal{V} \to \mathbb{R}$ such that

$$A \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-2} \\ \alpha_{r-1} \\ \eta \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & p_0 & S \\ 1 & 0 & \cdots & 0 & p_1 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & 0 \\ \vdots & & \ddots & 0 & p_{r-2} & 0 \\ 0 & 0 & & 1 & p_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & R & Q \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-2} \\ \alpha_{r-1} \\ \eta \end{bmatrix} \tag{3.2}$$

for all $[\alpha_0\,,\,\alpha_1\,,\,\cdots\,,\,\alpha_{r-1}\,,\,\eta]^\top \in \operatorname{dom} A$;

(iii) There exists a $c_r \in \mathbb{R}\backslash\{0\}$ such that

$$Bu = \begin{bmatrix} u \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad C\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix} = \alpha_{r-1}c_r$$

for all $u \in \mathbb{R}$ and $[\alpha_0\,,\,\alpha_1\,,\,\cdots\,,\,\alpha_{r-1}\,,\,\eta]^\top \in \mathbb{R}^r \times \mathcal{V}$.

Remark 3.2.2. If $(A, B, C, 0) \in \mathbf{\Sigma}_r$ is in zero dynamics form, then A has a block operator structure of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & Q \end{bmatrix},$$

where $A_{11} : \mathbb{R}^r \to \mathbb{R}^r$, $A_{12} : \mathcal{V} \to \mathbb{R}^r$ and $A_{21} : \mathbb{R}^r \to \mathcal{V}$ are bounded operators, and only Q may be unbounded.

Proposition 3.2.3. *Every state linear system that is in zero dynamics form and has feedthrough zero belongs to the class* $\mathbf{\Sigma}_r$.

Proof. Let $(A, B, C, 0)$ be a system in zero dynamics form. Then

$$\left\{\ [\alpha_0\,,\,\ldots\,,\,\alpha_{r-\ell}\,,\,0\,,\,\ldots\,,\,0]^\top \in \mathbb{R}^r \times \mathcal{V} \ \middle|\ \alpha_0, \ldots, \alpha_{r-\ell} \in \mathbb{R}\ \right\} \subset \operatorname{dom} A^\ell$$

for $\ell \in \{1, \ldots, r\}$. Defining the vectors

$$b := [1\,,\,0\,,\,\ldots\,,\,0]^\top = B1, \qquad c := [0\,,\,\ldots\,,\,0\,,\,c_r\,,\,0]^\top,$$

we see $b \in \operatorname{dom} A^r$ and the special structure of A yields

$$\left\langle A^\ell b\,,\,c\right\rangle = 0 \quad \forall\,\ell \in \{0, \ldots, r-2\}, \quad \text{and} \quad \left\langle A^{r-1}b\,,\,c\right\rangle = c_r \neq 0.$$

It remains to prove $c \in \operatorname{dom} A^{*^r}$. An inductive argument shows that for all $\ell \in \{0, \ldots, r-1\}$ we have $c \in \operatorname{dom} A^{*^\ell}$ and

$$A^{*^\ell}c = [\gamma_0\,,\,\ldots\,,\,\gamma_{r-1}\,,\,0]^\top \quad \text{with} \quad \gamma_k = 0 \quad \forall\,k \in \{0, \ldots, r-2-\ell\}.$$

If we write $[\gamma_0\,,\,\dots\,,\,\gamma_{r-1}\,,\,0]^\top := {A^*}^{r-1}c$ and let $[\alpha_0\,,\,\dots\,,\,\alpha_{r-1}\,,\,\eta]^\top \in \operatorname{dom} A$, then the expression

$$\left\langle A\begin{bmatrix}\alpha_0\\ \vdots\\ \alpha_{r-1}\\ \eta\end{bmatrix}, \begin{bmatrix}\gamma_0\\ \vdots\\ \gamma_{r-1}\\ 0\end{bmatrix}\right\rangle_{\mathbb{R}^r\times\mathcal{V}} = \left\langle \begin{bmatrix}S\eta + p_0\alpha_{r-1}\\ \alpha_0 + p_1\alpha_{r-1}\\ \vdots\\ \alpha_{r-2} + p_{r-1}\alpha_{r-1}\end{bmatrix}, \begin{bmatrix}\gamma_0\\ \gamma_1\\ \vdots\\ \gamma_{r-1}\end{bmatrix}\right\rangle_{\mathbb{R}^r}$$

depends continuously on $[\alpha_0\,,\,\dots\,,\,\alpha_{r-1}\,,\,\eta]^\top$. This implies that $c \in \operatorname{dom} {A^*}^r$ and completes the proof of this proposition. $\square$

The goal of this section is to prove that every system of relative degree r can be put into zero dynamics form by a boundedly invertible transformation. To obtain this transformation, we choose a special representation of the state space $\mathcal{X}$. We define the subspace

$$\mathcal{S}_{A,b} := \operatorname{span}\{b\} \oplus \operatorname{span}\{Ab\} \oplus \cdots \oplus \operatorname{span}\{A^{r-1}b\},$$

where $\oplus$ indicates that the sum is direct. The Hilbert space $\mathcal{X}$ decomposes into the direct sum

$$\begin{aligned}\mathcal{X} &= \mathcal{S}_{A,b} \;\oplus\; \mathcal{S}^\perp_{A^*,c}\\ &= \operatorname{span}\{b\} \oplus \operatorname{span}\{Ab\} \oplus \cdots \oplus \operatorname{span}\{A^{r-1}b\}\\ &\qquad\qquad \oplus \{c\}^\perp \cap \{A^*c\}^\perp \cap \cdots \cap \{{A^*}^{r-1}c\}^\perp.\end{aligned} \tag{3.3}$$

This follows immediately from Presumption 3.1.1: Firstly, $b, Ab, \dots, A^{r-1}b$ are linearly independent, and secondly

$$\mathcal{S}^\perp_{A^*,c} \cap \mathcal{S}_{A,b} = \{0\}.$$

Hence, the sum $\mathcal{S}_{A,b} \oplus \mathcal{S}^\perp_{A^*,c}$ is direct and since $\mathcal{S}^\perp_{A^*,c}$ has by definition at most codimension r, equality in (3.3) follows. This means that every vector $x \in \mathcal{X}$ has a unique representation

$$x = \alpha_0 b + \cdots + \alpha_{r-1}A^{r-1}b + \eta, \quad \text{with} \quad \alpha_0, \dots, \alpha_{r-1} \in \mathbb{R},\; \eta \in \mathcal{S}^\perp_{A^*,c}.$$

In order to determine the coefficients $\alpha_k \in \mathbb{R}$ of this representation the next lemma exploits the relative degree property.

Lemma 3.2.4. *Define the functionals*

$$P^m : \mathcal{X} \to \mathbb{R}, \quad x \mapsto P^m x := P^m_{m+1} x - \sum_{j=m+2}^{r} P^m_j x\,, \qquad m = 0, \dots, r-1\,, \quad (3.4)$$

where

$$P^m_{m+1} : \mathcal{X} \to \mathbb{R}, \qquad x \mapsto P^m_{m+1} x := \frac{\left\langle x\,,\, {A^*}^{r-(m+1)} c\right\rangle}{\langle b\,,\, {A^*}^{r-1} c\rangle}\,, \qquad m = 0, 1, \dots, r-1,$$

and

$$P^m_j : \mathcal{X} \to \mathbb{R}, \qquad x \mapsto P^m_j x := \left(P^m_{m+1} A^{j-1} b - \sum_{k=m+2}^{j-1} P^m_k A^{j-1} b \right) \frac{\left\langle x\,,\, {A^*}^{r-j} c\right\rangle}{\langle b\,,\, {A^*}^{r-1} c\rangle}.$$

for $j = m+2, \dots, r$. Then the following holds:

(i) For all $\ell, m \in \{0, \dots, r-1\}$ we have

$$P^m A^\ell b \;=\; \begin{cases} 1, & \text{if } \ell = m, \\ 0, & \text{if } \ell \neq m; \end{cases} \quad (3.5)$$

$$P^m\, \mathcal{S}^{\perp}_{A^*,c} \;=\; \{0\}. \quad (3.6)$$

(ii) The operator

$$P_{\mathcal{S}^{\perp}_{A^*,c}} : \mathcal{X} \to \mathcal{X}, \qquad P_{\mathcal{S}^{\perp}_{A^*,c}} x := x - \sum_{j=0}^{r-1} (P^j x)\, A^j b, \quad (3.7)$$

is a projection onto $\mathcal{S}^{\perp}_{A^*,c}$, and every $x \in \mathcal{X}$ has a unique decomposition with respect to (3.3) of the form

$$x \;=\; (P^0 x) b + (P^1 x) A b + \dots + (P^{r-1} x) A^{r-1} b \;+\; P_{\mathcal{S}^{\perp}_{A^*,c}} x\,. \quad (3.8)$$

Proof. (i) Assertion (3.6) follows from the definitions of P^m and $\mathcal{S}^{\perp}_{A^*,c}$. From (3.1) we can easily deduce Assertion (3.5) for the cases $\ell = m$ and $\ell \in \{0, \dots, m-1\}$.

It remains to show (3.5) for $\ell \in \{m+1, \ldots, r-1\}$. By definition of P_j^m and (3.1) we have

$$P_j^m A^\ell b = 0 \quad \text{for all } j = \ell + 2, \ldots, r\,,$$

and therefore

$$\begin{aligned}
P^m A^\ell b &= P_{m+1}^m A^\ell b - \textstyle\sum_{j=m+2}^{r} P_j^m A^\ell b \\
&= P_{m+1}^m A^\ell b - P_{\ell+1}^m A^\ell b - \textstyle\sum_{j=m+2}^{\ell} P_j^m A^\ell b \\
&= P_{m+1}^m A^\ell b - \textstyle\sum_{j=m+2}^{\ell} P_j^m A^\ell b \\
&\qquad - \Big(P_{m+1}^m A^\ell b - \textstyle\sum_{k=m+2}^{\ell} P_k^m A^\ell b \Big) \underbrace{\frac{\big\langle A^\ell b\,,\, {A^*}^{r-1-\ell} c \big\rangle}{\langle b\,,\, {A^*}^{r-1} c\rangle}}_{=1} \\
&= 0\,.
\end{aligned}$$

(ii) By definition of $P_{\mathcal{S}^\perp_{A^*,c}}$ and (3.6) we have $P_{\mathcal{S}^\perp_{A^*,c}} x = x$ for all $x \in \mathcal{S}^\perp_{A^*,c}$, and by (3.5) we have

$$\operatorname{span}\{b\} \oplus \operatorname{span}\{Ab\} \oplus \cdots \oplus \operatorname{span}\{A^{r-1}b\} = \mathcal{S}_{A,b} \subset \ker P_{\mathcal{S}^\perp_{A^*,c}}\,.$$

Hence, in view of (3.3), $P_{\mathcal{S}^\perp_{A^*,c}}$ is a projection. Finally, (3.8) is a direct consequence of the definition of $P_{\mathcal{S}^\perp_{A^*,c}}$.

$\square$

Lemma 3.2.5. *With $P^0, \ldots, P^{r-1}$ and $P_{\mathcal{S}^\perp_{A^*,c}}$ defined as in Lemma 3.2.5 the operator*

$$T : \mathcal{X} \to \mathbb{R}^r \times \mathcal{S}^\perp_{A^*,c}\,, \qquad x \mapsto Tx := \begin{bmatrix} P^0 x \\ P^1 x \\ \vdots \\ P^{r-1} x \\ P_{\mathcal{S}^\perp_{A^*,c}} x \end{bmatrix},$$

is bounded and bijective with inverse

$$T^{-1}: \mathbb{R}^r \times \mathcal{S}^{\perp}_{A*,c} \to \mathcal{X}, \qquad \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix} \mapsto \sum_{j=0}^{r-1} \alpha_j A^j b \;+\; \eta.$$

Furthermore, with the orthogonal projector $\pi_{\mathcal{S}^{\perp}_{A*,c}} : \mathcal{X} \to \mathcal{X}$ *onto* $\mathcal{S}^{\perp}_{A*,c}$*, we have*

$$T^{-*}: \mathcal{X} \to \mathbb{R}^r \times \mathcal{S}^{\perp}_{A*,c}, \qquad x \mapsto \begin{bmatrix} \langle b\,,\, x\rangle \\ \langle Ab\,,\, x\rangle \\ \vdots \\ \langle A^{r-1}b\,,\, x\rangle \\ \pi_{\mathcal{S}^{\perp}_{A*,c}} x \end{bmatrix}. \tag{3.9}$$

and T^{-*} *maps* $\mathcal{S}^{\perp}_{A,b}$ *bijectively onto* $\{0\} \times \mathcal{S}^{\perp}_{A*,c}$*.*

Proof. The assertions about T and T^{-1} are a direct consequence of Lemma 3.2.4 (ii). The formula for T^{-*} holds because for all $[\alpha_0, \ldots, \alpha_{r-1}, \eta]^\top \in \mathbb{R}^r \times \mathcal{S}^{\perp}_{A*,c}$ and all $x \in \mathcal{X}$ we have

$$\left\langle T^{-1} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix}, x \right\rangle = \left\langle \sum_{j=0}^{r-1} \alpha_j A^j b \;+\; \eta\,,\, x \right\rangle = \left\langle \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix}, \begin{bmatrix} \langle b\,,\, x\rangle \\ \langle Ab\,,\, x\rangle \\ \vdots \\ \langle A^{r-1}b\,,\, x\rangle \\ \pi_{\mathcal{S}^{\perp}_{A*,c}} x \end{bmatrix} \right\rangle_{\mathbb{R}^r \times \mathcal{S}^{\perp}_{A*,c}}.$$

The last statement on T^{-*} follows from the fact that T is bijective together with formula (3.9). □

Lemma 3.2.6. *(i) For any* $m \in \{0, \ldots, r-1\}$ *and* P^m *as in* (3.4)*, the operator* $P^m A$ *is closable and densely defined. Its closure is the bounded linear functional*

$$\begin{aligned}
\overline{P^m A} &: \mathcal{X} \to \mathbb{R}, \\
x &\mapsto \frac{\left\langle x\,,\, {A^*}^{r-m} c\right\rangle}{\langle A^{r-1}b\,,\, c\rangle} \\
&\quad - \sum_{j=m+2}^{r} \left(P^m_{m+1} A^{j-1} b - \sum_{k=m+2}^{j-1} P^m_k A^{j-1} b \right) \frac{\left\langle x\,,\, {A^*}^{r+1-j} c\right\rangle}{\langle A^{r-1}b\,,\, c\rangle}.
\end{aligned} \tag{3.10}$$

(ii) With $P_{\mathcal{S}^\perp_{A^*,c}}$ as in (3.7), the operator

$$\begin{aligned}
A_{\circlearrowleft} :\ \operatorname{dom} A \cap \mathcal{S}^\perp_{A^*,c} \subset \mathcal{S}^\perp_{A^*,c} &\to \mathcal{S}^\perp_{A^*,c} \\
\eta &\mapsto A\eta - b\frac{\langle \eta\,,\, {A^*}^{r} c\rangle}{\langle b\,,\, {A^*}^{r-1} c\rangle}
\end{aligned} \tag{3.11}$$

is closed and densely defined in $\mathcal{S}^\perp_{A^*,c}$ and satisfies

$$A_{\circlearrowleft}\eta = A\eta - \left(P^0 A\eta\right) b = P_{\mathcal{S}^\perp_{A^*,c}} A\eta \quad \forall\, \eta \in \mathcal{S}^\perp_{A^*,c} \cap \operatorname{dom} A. \tag{3.12}$$

Proof. For $x \in \operatorname{dom} A$, a quick look at the definition of P^m in Lemma 3.2.4 reveals that the mapping defined in (3.10) coincides with $P^m Ax$. The right hand side of (3.10) is also defined for arbitrary $x \in \mathcal{X}$, hence $P^m A$ is closable and since its range is finite-dimensional, the closure $\overline{P^m A}$ is the continuous operator given by (3.10).

To prove (ii) we first show

$$A\eta = (P^0 A\eta) b + P_{\mathcal{S}^\perp_{A^*,c}} A\eta \qquad \forall\, \eta \in \mathcal{S}^\perp_{A^*,c} \cap \operatorname{dom} A\,. \tag{3.13}$$

If $r = 1$, then (3.13) follows immediately from the decomposition (3.3) and (3.7). Assume $r > 1$ and let $\eta \in \mathcal{S}^\perp_{A^*,c} \cap \operatorname{dom} A$. Then (3.3) and (3.7) yield

$$A\eta = \alpha_0 b + \alpha_1 Ab + \cdots + \alpha_{r-1} A^{r-1} b + P_{\mathcal{S}^\perp_{A^*,c}} A\eta \quad \text{with} \quad \alpha_i = P^i A\eta \in \mathbb{R}$$

and $P_{\mathcal{S}^\perp_{A^*,c}} A\eta \in \mathcal{S}^\perp_{A^*,c}$. Using this representation we obtain

$$0 = \langle \eta\,,\, A^* c\rangle = \langle A\eta\,,\, c\rangle \overset{(3.1)}{=} \alpha_{r-1} \left\langle A^{r-1} b\,,\, c\right\rangle,$$

and (3.1) moreover yields $\alpha_{r-1} = 0$. Next,

$$0 = \left\langle \eta\,,\, {A^*}^2 c \right\rangle = \langle A\eta\,,\, A^* c\rangle \overset{(3.1)}{=} \alpha_{r-2} \left\langle A^{r-1}b\,,\, c \right\rangle,$$

and (3.1) yields $\alpha_{r-2} = 0$. Proceeding in this way, we conclude

$$0 = \left\langle \eta\,,\, {A^*}^{r-1} c \right\rangle = \left\langle A\eta\,,\, {A^*}^{r-2} c \right\rangle \overset{(3.1)}{=} \alpha_1 \left\langle A^{r-1}b\,,\, c \right\rangle$$

and arrive at $0 = \alpha_{r-1} = \cdots \alpha_1$. This proves (3.13) and the second equality in (3.12). Note that for all $j = 2, \ldots, r$ we have

$$\left\langle A\eta\,,\, {A^*}^{r-j} c \right\rangle = \left\langle \eta\,,\, {A^*}^{r-j+1} c \right\rangle = 0,$$

whence, by definition, $P_j^0 A\eta = 0$. Now the definition of P^0 yields

$$P^0 A\eta = P_1^0 A\eta - \sum_{j=2}^{r} \underbrace{P_j^0 A\eta}_{=0} = \frac{\langle A\eta\,,\, {A^*}^{r-1} c\rangle}{\langle b\,,\, {A^*}^{r-1} c\rangle},$$

which proves the first equality in (3.12). This equation shows that $A_{\circlearrowleft}$ maps indeed into $\mathcal{S}^{\perp}_{A^*,c}$. Since A is closed and densely defined in $\mathcal{X}$ and the perturbation $\overline{P^0 A}$ is by (i) a bounded operator, it follows that $A_{\circlearrowleft}$ is a closed and densely defined operator in $\mathcal{S}^{\perp}_{A^*,c}$. This completes the proof of (ii). □

Remark 3.2.7. Equation (3.11) shows that the operator $A_{\circlearrowleft}$ may be interpreted as the main operator of a closed-loop system created by the feedback pair

$$\begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} \eta \\ u \end{bmatrix} := -\frac{\langle \eta\,,\, {A^*}^{r} c\rangle}{\langle b\,,\, {A^*}^{r-1} c\rangle} \qquad \forall \begin{bmatrix} \eta \\ u \end{bmatrix} \in P_{\mathcal{S}^{\perp}_{A^*,c}} \times \mathbb{R}.$$

That is why the space $\mathcal{S}^{\perp}_{A^*,c}$ is called feedback invariant, e.g. in [MR07].

Theorem 3.2.8. *Let T be the similarity transformation defined in Lemma 3.2.5. Then the system $(\widehat{A}, \widehat{B}, \widehat{C}, 0)$ defined by*

$$\widehat{A}x := TAT^{-1}x \quad \forall\, x \in T \operatorname{dom} A, \qquad \widehat{B} := TB, \qquad \widehat{C} := CT^{-1}.$$

is in zero dynamics form. More precisely, the operator $\widehat{A}$ has the domain $\mathbb{R}^r \times$

$\big(\mathcal{S}_{A^*,c}^{\perp} \cap \operatorname{dom} A\big)$ *and is given by*

$$\widehat{A}\begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_{r-2}\\ \alpha_{r-1}\\ \eta\end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & P^0A^rb & P^0A\\ 1 & 0 & \cdots & 0 & P^1A^rb & 0\\ 0 & 1 & \ddots & \vdots & \vdots & 0\\ \vdots & & \ddots & 0 & P^{r-2}A^rb & 0\\ 0 & 0 & & 1 & P^{r-1}A^rb & 0\\ 0 & 0 & \cdots & 0 & P_{\mathcal{S}_{A^*,c}^{\perp}}A^rb & P_{\mathcal{S}_{A^*,c}^{\perp}}A \end{bmatrix}\begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_{r-2}\\ \alpha_{r-1}\\ \eta\end{bmatrix} \tag{3.14}$$

for all $\alpha_0, \ldots, \alpha_{r-1} \in \mathbb{R}$ *and* $\eta \in \mathcal{S}_{A^*,c}^{\perp} \cap \operatorname{dom} A$*, and*

$$\widehat{B}u = \begin{bmatrix}u\\ 0\\ \vdots\\ 0\\ 0\end{bmatrix}, \qquad \widehat{C}\begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_{r-1}\\ \eta\end{bmatrix} = \left\langle \begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_{r-1}\\ \eta\end{bmatrix}, \begin{bmatrix}0\\ \vdots\\ 0\\ \langle A^{r-1}b\,,\, c\rangle\\ 0\end{bmatrix} \right\rangle_{\mathbb{R}^r\times\mathcal{S}_{A^*,c}^{\perp}}$$

for all $u \in \mathbb{R}$*,* $\alpha_0, \ldots, \alpha_{r-1} \in \mathbb{R}$ *and* $\eta \in \mathcal{S}_{A^*,c}^{\perp}$*.*

Proof. We first show that $T \operatorname{dom} A = \mathbb{R}^r \times \big(\mathcal{S}_{A^*,c}^{\perp} \cap \operatorname{dom} A\big)$. The standing assumption $b \in \operatorname{dom} A^r$ implies

$$T^{-1}\begin{bmatrix}\alpha_0 & \ldots & \alpha_{r-1} & \eta\end{bmatrix}^\top = \sum_{k=0}^{r-1} \alpha_k A^k b + \eta \qquad \in \quad \operatorname{dom} A$$

for all $\alpha_0, \ldots, \alpha_{r-1} \in \mathbb{R}$, $\eta \in \mathcal{S}_{A^*,c}^{\perp} \cap \operatorname{dom} A$. Conversely, we have for all $x \in \operatorname{dom} A$

$$P_{\mathcal{S}_{A^*,c}^{\perp}} x \;=\; x \;-\; \sum_{k=0}^{r-1} \underbrace{(P^k x) A^k b}_{\in \operatorname{dom} A} \qquad \in \quad \operatorname{dom} A.$$

$$\Rightarrow \qquad Tx = \begin{bmatrix}P^0x & \ldots & P^{r-1}x & P_{\mathcal{S}_{A^*,c}^{\perp}}x\end{bmatrix}^\top \;\in\; \mathbb{R}^r \times \big(\mathcal{S}_{A^*,c}^{\perp} \cap \operatorname{dom} A\big).$$

The decomposition (3.8) applied to the vector $A^r b$ reads

$$A^r b = \sum_{k=0}^{r-1} (P^k A^r b)\, A^k b + P_{\mathcal{S}_{A^*,c}^{\perp}} A^r b.$$

Using this we get for all $\begin{bmatrix}\alpha_0 & \alpha_1 & \dots & \alpha_{r-1} & \eta\end{bmatrix}^\top \in \mathbb{R}^r \times \mathcal{S}_{A^*,c}^\perp \cap \operatorname{dom} A$

$$
\begin{aligned}
T^{-1}\widehat{A}\begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_{r-1}\\ \eta\end{bmatrix} &= T^{-1}\begin{bmatrix} P^0A\eta + \alpha_{r-1}P^0A^rb\\ \alpha_0 + \alpha_{r-1}P^1A^rb\\ \vdots\\ \alpha_{r-2} + \alpha_{r-1}P^{r-1}A^rb\\ \alpha_{r-1}P_{\mathcal{S}_{A^*,c}^\perp}A^rb + P_{\mathcal{S}_{A^*,c}^\perp}A\eta\end{bmatrix}\\
&= (P^0A\eta)b + P_{\mathcal{S}_{A^*,c}^\perp}A\eta + \sum_{k=0}^{r-2}\alpha_kA^{k+1}b + \alpha_{r-1}\sum_{k=0}^{r-1}(P^kA^rb)A^kb\\
&\quad + \alpha_{r-1}P_{\mathcal{S}_{A^*,c}^\perp}A^rb\\
&\overset{(3.12)}{=} A\eta + A\sum_{k=0}^{r-2}\alpha_kA^kb + \alpha_{r-1}A^rb\\
&= A\left(\sum_{k=0}^{r-1}\alpha_kA^kb + \eta\right)\\
&= AT^{-1}\begin{bmatrix}\alpha_0 & \alpha_1 & \cdots & \alpha_{r-1} & \eta\end{bmatrix}^\top,
\end{aligned}
$$

whence (3.14) holds. Due to Lemma 3.2.6 (i) the operator P^0A is bounded. Obviously, all other operators in $\widehat{A}$ except for $P_{\mathcal{S}_{A^*,c}^\perp}A$ are bounded. It remains to show that $P_{\mathcal{S}_{A^*,c}^\perp}A$ generates a semigroup on $\mathcal{S}_{A^*,c}^\perp$, i.e. it fulfills Definition 3.2.1 (i). Because of the similarity to A it is clear that $\widehat{A}$ generates a semigroup on $\mathbb{R}^r \times \mathcal{S}_{A^*,c}^\perp$, see Lemma 2.4.15. With respect to the decomposition $\mathbb{R}^r \times \mathcal{S}_{A^*,c}^\perp$ the operator $\widehat{A}$ has the structure

$$
\widehat{A} = \begin{bmatrix}\widehat{A}_{11} & \widehat{A}_{12}\\ \widehat{A}_{21} & \widehat{A}_{22}\end{bmatrix},
$$

where the operators $\widehat{A}_{11} : \mathbb{R}^r \to \mathbb{R}^r$, $\widehat{A}_{12} : \mathcal{S}_{A^*,c}^\perp \to \mathbb{R}^r$, and $\widehat{A}_{21} : \mathbb{R}^r \to \mathcal{S}_{A^*,c}^\perp$ are bounded, and $\widehat{A}_{22} = P_{\mathcal{S}_{A^*,c}^\perp}A|_{\mathcal{S}_{A^*,c}^\perp}$. So the operator

$$
\operatorname{diag}(0, \widehat{A}_{22}) := \begin{bmatrix}0 & 0\\ 0 & \widehat{A}_{22}\end{bmatrix} = \widehat{A} - \begin{bmatrix}\widehat{A}_{11} & \widehat{A}_{12}\\ \widehat{A}_{21} & 0\end{bmatrix}
$$

differs from $\widehat{A}$ only by a bounded perturbation. In view of [EN00, Section III.1.3], it is therefore a semigroup generator whose domain equals $\operatorname{dom}\widehat{A}$. Obviously, $\{0\} \times$

$\mathcal{S}^{\perp}_{A*,c}$ is a closed, $\operatorname{diag}(0, \widehat{A}_{22})$-invariant subspace of $\mathbb{R}^r \times \mathcal{S}^{\perp}_{A*,c}$, and since the spectrum of $\widehat{A}_{22}$ is equal to the spectrum of $\operatorname{diag}(0, \widehat{A}_{22})$ up to the value 0, the condition (iv) of [Sta05, Theorem 3.14.4] is satisfied. This theorem implies that $\operatorname{diag}(0, \widehat{A}_{22})|_{\{0\}\times\mathcal{S}^{\perp}_{A*,c}}$ with domain

$$\operatorname{dom}\widehat{A} \cap (\{0\} \times \mathcal{S}^{\perp}_{A*,c}) = \{0\} \times (\mathcal{S}^{\perp}_{A*,c} \cap \operatorname{dom} A)$$

generates a strongly continuous semigroup on $\{0\} \times \mathcal{S}^{\perp}_{A*,c}$. Now the identification of $\mathcal{S}^{\perp}_{A*,c}$ with $\{0\} \times \mathcal{S}^{\perp}_{A*,c}$ and $P_{\mathcal{S}^{\perp}_{A*,c}}A$ with $\operatorname{diag}(0, \widehat{A}_{22})|_{\{0\}\times\mathcal{S}^{\perp}_{A*,c}}$ implies the claim.

Finally, the structures of $\widehat{B}$ and $\widehat{C}$ follow via

$$TB = Tb = \begin{bmatrix} P^0 b & P^1 b & \cdots & P^{r-1} b & P_{\mathcal{S}^{\perp}_{A*,c}} b \end{bmatrix}^\top \overset{(3.6),(3.5)}{=} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}^\top$$

and

$$CT^{-1}\begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{r-1} & \eta \end{bmatrix}^\top = \left\langle \sum_{k=0}^{r-1} \alpha_k A^k b + \eta \,,\, c \right\rangle \overset{(3.1)}{=} \alpha_{r-1}\left\langle A^{r-1} b \,,\, c \right\rangle.$$

for all $\begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{r-1} & \eta \end{bmatrix}^\top \in \mathbb{R}^r \times \mathcal{S}^{\perp}_{A*,c}$. $\square$

Proposition 3.2.9. *Let the system $(A, B, C, 0) \in \mathbf{\Sigma}_r$ be in zero dynamics form as in Definition 3.2.1. Let $\widetilde{\mathcal{V}}$ be another real Hilbert space, and let $(\widetilde{A}, \widetilde{B}, \widetilde{C}, 0) \in \mathbf{\Sigma}_r$ be in zero dynamics form as well with*

$$\widetilde{A} : \operatorname{dom}\widetilde{A} \subset \mathbb{R}^r \times \widetilde{\mathcal{V}} \to \mathbb{R}^r \times \widetilde{\mathcal{V}}, \quad \widetilde{C} : \mathbb{R}^r \times \widetilde{\mathcal{V}} \to \mathbb{R},$$

$$\widetilde{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \widetilde{p}_0 & \widetilde{S} \\ 1 & 0 & \cdots & 0 & \widetilde{p}_1 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & 0 \\ \vdots & & \ddots & 0 & \widetilde{p}_{r-2} & 0 \\ 0 & 0 & & 1 & \widetilde{p}_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & \widetilde{R} & \widetilde{Q} \end{bmatrix}, \quad C\begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix} = \alpha_{r-1}\widetilde{c}_r. \tag{3.15}$$

If the two systems are similar via a bounded and bijective similarity transformation $T : \mathbb{R}^r \times \mathcal{V} \to \mathbb{R}^r \times \widetilde{\mathcal{V}}$, then the entries of (3.2) and (3.15) are related as follows:

(i) $p_i = \widetilde{p}_i$ *for all* $i = 0, \ldots, r-1$*;*

(ii) There is a bijective mapping $\mathcal{T} \in \mathcal{B}(\mathcal{V}; \widetilde{\mathcal{V}})$ such that

$$(\widetilde{Q}, \widetilde{R}, \widetilde{S}) = (\mathcal{T} Q \mathcal{T}^{-1}, \mathcal{T} R, S\mathcal{T}^{-1}) \quad \text{with} \quad \operatorname{dom} \widetilde{Q} = \mathcal{T} \operatorname{dom} Q;$$

(iii) $\widetilde{c}_r = c_r$.

Proof. Simply applying $\widetilde{A}$ from (3.15) $r-1$ times to $\widetilde{B}$ yields

$$\widetilde{c}_r = \widetilde{C}\widetilde{A}^r\widetilde{B} = CT^{-1}TA^rT^{-1}TB = CA^rB = c_r,$$

which shows (iii).

The bounded bijective operator $T : \mathbb{R}^r \times \mathcal{V} \to \mathbb{R}^r \times \widetilde{V}$ admits a representation with respect to $\mathbb{R}^r \times \mathcal{V}$ and $\mathbb{R}^r \times \widetilde{V}$ of the form

$$T = \begin{bmatrix} T_{00} & T_{01} & \cdots & T_{0r} \\ T_{10} & T_{11} & \cdots & T_{1r} \\ \vdots & \vdots & & \vdots \\ T_{r0} & T_{r1} & \cdots & T_{rr} \end{bmatrix} \quad \text{with bounded} \quad \begin{array}{ll} T_{ij} : \mathbb{R} \to \mathbb{R}, & i, j \in \{0, \ldots, r-1\}, \\ T_{ir} : \mathcal{V} \to \mathbb{R}, & i \in \{0, \ldots, r-1\}, \\ T_{rj} : \mathbb{R} \to \widetilde{\mathcal{V}}, & j \in \{0, \ldots, r-1\}, \\ T_{rr} : \mathcal{V} \to \widetilde{\mathcal{V}}. & \end{array}$$

We calculate

$$\begin{bmatrix} T_{00} \\ T_{10} \\ \vdots \\ T_{r0} \end{bmatrix} = T \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = TB = \widetilde{B} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The relation

$$c_r\alpha_{r-1} = C \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix} = \widetilde{C}T \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix} = \underbrace{\widetilde{c}_r}_{=c_r} \sum_{k=0}^{r-1} T_{r-1,k}\alpha_k + T_{r-1,r}\eta$$

for all $[\alpha_0\,,\, \cdots\,,\, \alpha_{r-1}\,,\, \eta]^\top \in \mathbb{R}^r \times \mathcal{V}$ implies

$$\begin{bmatrix} T_{r-1,0} & \cdots & T_{r-1,r-1} & T_{r-1,r} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Hence,

$$T = \begin{bmatrix} 1 & T_{00} & T_{01} & \cdots & T_{0,r-1} & T_{0r} \\ 0 & T_{10} & T_{11} & \cdots & T_{1,r-1} & T_{1r} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & T_{r-2,0} & T_{r-2,1} & \cdots & T_{r-2,r-1} & T_{r-2,r} \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & T_{r1} & \cdots & T_{r,r-2} & 0 & T_{rr} \end{bmatrix}, \tag{3.16}$$

and we obtain the second column of T from

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \widetilde{A} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \overset{(3.16)}{=} \widetilde{A}T \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = TA \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = T \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} T_{01} \\ T_{11} \\ T_{21} \\ \vdots \\ T_{r1} \end{bmatrix}. \tag{3.17}$$

Analogously, we obtain the third column by

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \widetilde{A} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \overset{(3.17)}{=} \widetilde{A}T \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = TA \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = T \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} T_{02} \\ T_{12} \\ T_{22} \\ T_{32} \\ \vdots \\ T_{r2} \end{bmatrix}.$$

We proceed by calculating the first r columns of T in this way and arrive at

$$T = \begin{bmatrix} 1 & 0 & \dots & 0 & T_{0r} \\ 0 & 1 & & & T_{1r} \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & T_{rr} \end{bmatrix}. \tag{3.18}$$

Now the special structure of A, $\widetilde{A}$ and T in (3.2), (3.15) and (3.18), respectively, yields

$$\begin{bmatrix} S + T_{0r}Q \\ \\ \vdots \\ T_{r-2,r}Q \\ 0 \\ T_{rr}Q \end{bmatrix} = T \begin{bmatrix} S \\ 0 \\ \vdots \\ 0 \\ Q \end{bmatrix} = TA \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{I} \end{bmatrix} = \widetilde{A}T \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{I} \end{bmatrix} = \widetilde{A} \begin{bmatrix} T_{0r} \\ \vdots \\ T_{r-2,r} \\ 0 \\ T_{rr} \end{bmatrix} = \begin{bmatrix} \widetilde{S}T_{rr} \\ T_{0r} \\ \vdots \\ T_{r-1,r} \\ T_{r-2,r} \\ \widetilde{Q}T_{rr} \end{bmatrix}.$$

By successively comparing the blocks in order from $(r-2)$th to first and by finally considering the last entry, we see that

$$T_{r-2,r} = 0 = \cdots = T_{r0} = 0 \qquad \text{and} \qquad S = \widetilde{S}T_{rr}, \quad \widetilde{Q} = T_{rr}QT_{rr}^{-1}.$$

Finally, we summarize that the transformation has the form

$$T = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & T_{rr} \end{bmatrix}$$

This shows that the assertion of the Proposition holds with $\mathcal{T} := T_{rr}$. □

3.3. The Byrnes-Isidori form

Definition 3.3.1 (Byrnes-Isidori form)**.** Let $\mathcal{X}$ be a real Hilbert space. A state linear system $(A,\, B,\, C,\, D)$ on $(\mathbb{R}, \mathcal{X}, \mathbb{R})$ is said to be in *Byrnes-Isidori form* if and only if $\mathcal{X} = \mathbb{R}^r \times \mathcal{V}$ for some Hilbert space $\mathcal{V}$ and the operators A, B, C satisfy the following conditions.

(i) There exists an operator $Q : \operatorname{dom} Q \subset \mathcal{V} \to \mathcal{V}$ that generates a strongly continuous semigroup on $\mathcal{V}$;

(ii) The operator A has the domain $\operatorname{dom} A = \mathbb{R}^r \times \operatorname{dom} Q$, and there are bounded operators $p_0, \dots, p_{r-1} : \mathbb{R} \to \mathbb{R}$, $S : \mathcal{V} \to \mathbb{R}$, $R : \mathbb{R} \to \mathcal{V}$, such that

$$A \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-2} \\ \alpha_{r-1} \\ \eta \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & 0 & 0 \\ 0 & 0 & & 0 & 1 & 0 \\ p_0 & p_1 & \cdots & p_{r-2} & p_{r-1} & S \\ R & 0 & \cdots & 0 & 0 & Q \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-2} \\ \alpha_{r-1} \\ \eta \end{bmatrix}$$

for all $\begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{r-1} & \eta \end{bmatrix}^\top \in \operatorname{dom} A$;

(iii) There exists a $b_r \in \mathbb{R}\backslash\{0\}$ such that

$$Bu = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_r u \\ 0 \end{bmatrix}, \quad \text{and} \quad C \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix} = \alpha_0$$

for all $u \in \mathbb{R}$ and all $\begin{bmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_{r-1} & \eta \end{bmatrix}^\top \in \mathbb{R}^r \times \mathcal{V}$.

Lemma 3.3.2. *A system (A, B, C, D) is in Byrnes-Isidori form if and only if its dual system (A^*, C^*, B^*, D^*) is in zero dynamics form.*

Proof. This is obvious from the structure of the operators in Definition 3.3.1 and Definition 3.2.1. □

In order to transform a system (A, B, C, D) into Byrnes-Isidori form, we may therefore transform the dual system (A^*, C^*, B^*, D^*) into zero dynamics form via Theorem 3.2.8, and subsequently adjoin the result. We want to elucidate the transformation used in this process a little more and formulate this statement as a theorem.

We still assume Presumption 3.1.1 in this section. In (3.3) we split the space $\mathcal{X}$ into $\mathcal{S}_{A,b} \oplus \mathcal{S}_{A^*,c}^\perp$. Since the dual system of $(A, B, C, 0)$ is of relative degree r as well, we may analogously decompose $\mathcal{X}$ into

$$\begin{aligned}\mathcal{X} &= \mathcal{S}_{A*,c} \;\oplus\; \mathcal{S}_{A,b}^{\perp}\\ &= \operatorname{span}\{c\} \oplus \operatorname{span}\{A^*c\} \oplus \cdots \oplus \operatorname{span}\{{A^*}^{r-1}c\}\\ &\qquad\qquad\qquad\qquad \oplus \{b\}^{\perp} \cap \{Ab\}^{\perp} \cap \cdots \cap \{A^{r-1}b\}^{\perp}\,.\end{aligned}$$

Lemma 3.3.3. *Define the operators*

$$P^m : \mathcal{X} \to \mathbb{R}, \quad P^m x := P^m_{m+1}x - \sum_{j=m+2}^{r} P^m_j x\,, \quad m = 0, \ldots, r-1\,,$$

where

$$P^m_{m+1} : \mathcal{X} \to \mathbb{R}, \quad P^m_{m+1}x := \frac{\left\langle x\,,\, A^{r-(m+1)}b\right\rangle}{\langle c\,,\, A^{r-1}b\rangle}\,, \quad m = 0, 1, \ldots, r-1,$$

and

$$P^m_j : \mathcal{X} \to \mathbb{R}, \quad P^m_j x := \left(P^m_{m+1}{A^*}^{j-1}c - \sum_{k=m+2}^{j-1} P^m_k {A^*}^{j-1}c\right)\frac{\langle x\,,\, A^{r-j}b\rangle}{\langle c\,,\, A^{r-1}b\rangle}$$

for $j = m+2, \ldots, r$. Then the following holds:

(i) For any $\ell, m \in \{0, \ldots, r-1\}$ we have

$$\begin{aligned} P^m {A^*}^{\ell} c \;&=\; \begin{cases} 1, & \text{if } \ell = m,\\ 0, & \text{if } \ell \neq m\,; \end{cases}\\ P^m\, \mathcal{S}_{A,b}^{\perp} \;&=\; \{0\}. \end{aligned}$$

(ii) The operator

$$P_{\mathcal{S}_{A,b}^{\perp}} : \mathcal{X} \to \mathcal{X}, \quad P_{A,b}x := \Big(\mathrm{I} - \sum_{j=0}^{r-1} {A^*}^{j} c P^j\Big)x,$$

is a projection onto $\mathcal{S}_{A,b}^{\perp}$, and every $x \in \mathcal{X}$ has a unique decomposition of the form

$$x \;=\; (P^0x)c + (P^1x)A^*c + \cdots + (P^{r-1}x){A^*}^{r-1}c \;+\; P_{A,b}x\,.$$

Proof. This follows by replacing A, b, c by A^*, c, b, respectively, in Lemma 3.2.4. $\square$

Lemma 3.3.4. *With $P^0, \dots, P^{r-1}$ and $P_{\mathcal{S}_{A,b}^\perp}$ defined as in Lemma 3.3.3, the operator*

$$U : \mathcal{X} \to \mathbb{R}^r \times \mathcal{S}_{A,b}^\perp, \qquad x \mapsto Ux := \begin{bmatrix} P^0 x \\ P^1 x \\ \vdots \\ P^{r-1} x \\ P_{\mathcal{S}_{A,b}^\perp} x \end{bmatrix}, \tag{3.19}$$

is bounded and bijective with inverse

$$U^{-1} : \mathbb{R}^r \times \mathcal{S}_{A,b}^\perp \to \mathcal{X}, \qquad \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix} \mapsto \sum_{j=0}^{r-1} \alpha_j {A^*}^j c \ + \ \eta.$$

Furthermore, with the orthogonal projector $\pi_{\mathcal{S}_{A,b}^\perp} : \mathcal{X} \to \mathcal{X}$ onto $\mathcal{S}_{A,b}^\perp$, we have

$$U^{-*} : \mathcal{X} \to \mathbb{R}^r \times \mathcal{S}_{A,b}^\perp, \qquad x \mapsto \begin{bmatrix} \langle x \,,\, c \rangle \\ \langle x \,,\, A^* c \rangle \\ \vdots \\ \left\langle x \,,\, {A^*}^{r-1} c \right\rangle \\ \pi_{\mathcal{S}_{A,b}^\perp} x \end{bmatrix}. \tag{3.20}$$

and U^{-*} maps $\mathcal{S}_{A^*,c}^\perp$ bijectively onto $\{0\} \times \mathcal{S}_{A,b}^\perp$.

Proof. The assertions about U and its inverse follow directly from Lemma 3.3.3. The formula for U^{-*} follows since for all $[\alpha_0 \,, \cdots, \alpha_{r-1} \,, \eta]^\top \in \mathbb{R}^r \times \mathcal{S}_{A,b}^\perp$ and all $x \in \mathcal{X}$ we have

$$\left\langle x, U^{-1} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix} \right\rangle = \left\langle x \,,\, \sum_{j=0}^{r-1} \alpha_j {A^*}^j c \ + \ \eta \right\rangle = \left\langle \begin{bmatrix} \langle x \,,\, c \rangle \\ \langle x \,,\, A^* c \rangle \\ \vdots \\ \left\langle x \,,\, {A^*}^{r-1} c \right\rangle \\ \pi_{\mathcal{S}_{A,b}^\perp} x \end{bmatrix}, \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix} \right\rangle_{\mathbb{R}^r \times \mathcal{S}_{A,b}^\perp}.$$

The last statement on U^{-*} follows from the fact that U is bijective together with formula (3.20). $\square$

Theorem 3.3.5. *Define U and P^m as in Lemma 3.3.3 and* (3.19)*. The bounded and bijective operator $U^{-*}: \mathcal{X} \to \mathbb{R}^r \times \mathcal{S}_{A,b}^{\perp}$ converts the system $(A, B, C, 0) \in \mathbf{\Sigma}_r$ into the system*

$$(\widehat{A}, \widehat{B}, \widehat{C}, 0) := \left(U^{-*}AU^*, U^{-*}B, CU^*\right), \quad \textit{with} \quad \operatorname{dom} \widehat{A} := U^{-*} \operatorname{dom} A,$$

which is in Byrnes-Isidori form. More precisely, $\operatorname{dom} \widehat{A} = \mathbb{R}^r \times (\mathcal{S}_{A,b}^{\perp} \cap \operatorname{dom} A)$*,*

$$\widehat{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ p_0 & p_1 & \cdots & p_{r-2} & p_{r-1} & S \\ R & 0 & \cdots & 0 & 0 & Q \end{bmatrix},$$

with

$$\begin{aligned} p_i &= P^i A^{*^r} c & &\forall\, i \in \{0, \dots, r-1\} \\ R : \mathbb{R} \to \mathcal{S}_{A,b}^{\perp}, & & R\alpha &= \pi_{\mathcal{S}_{A,b}^{\perp}} A^r b \tfrac{\alpha}{\langle A^{r-1}b\,,\,c\rangle}, \\ S : \mathcal{S}_{A,b}^{\perp} \to \mathbb{R}, & & S\eta &= \left\langle \eta\,,\, P_{\mathcal{S}_{A,b}^{\perp}} A^{*^r} c \right\rangle \\ Q : \mathcal{S}_{A,b}^{\perp} \cap \operatorname{dom} A \to \mathcal{S}_{A,b}^{\perp}, & & Q\eta &= \pi_{\mathcal{S}_{A,b}^{\perp}} A\eta - R\,\langle \eta\,,\, c\rangle\,, \end{aligned} \tag{3.21}$$

and

$$\widehat{B}u = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \langle A^{r-1}b\,,\,c\rangle\, u \\ 0 \end{bmatrix}, \qquad \widehat{C} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix} = \alpha_0$$

for all $u \in \mathbb{R}$, $\alpha_0, \dots, \alpha_{r-1} \in \mathbb{R}$ and $\eta \in \mathcal{S}_{A,b}^{\perp}$. Moreover, Q generates a strongly continuous semigroup $\mathfrak{A}_Q$ in $\mathcal{S}_{A,b}^{\perp}$.

Proof. We have constructed U in a such away that it transforms $(A^*, C^*, B^*, 0)$ into zero dynamics form, i.e. by Theorem 3.2.8 we have on $\mathbb{R}^r \times \operatorname{dom} A^* \cap \mathcal{S}_{A,b}^\perp$

$$UA^*U^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & P^0 {A^*}^r c & P^0 A^* \\ 1 & 0 & \cdots & 0 & P^1 {A^*}^r c & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & 0 \\ \vdots & & \ddots & 0 & P^{r-2} {A^*}^r c & 0 \\ 0 & 0 & & 1 & P^{r-1} {A^*}^r c & 0 \\ 0 & 0 & \cdots & 0 & P_{\mathcal{S}_{A,b}^\perp} {A^*}^r c & P_{\mathcal{S}_{A,b}^\perp} A^* \end{bmatrix}.$$

For the operator $P_{\mathcal{S}_{A,b}^\perp} A^*$ restricted to $\mathcal{S}_{A,b}^\perp \cap \operatorname{dom} A^*$, we use the name

$$A_\circlearrowleft : \mathcal{S}_{A,b}^\perp \cap \operatorname{dom} A^* \subset \mathcal{S}_{A,b}^\perp \to \mathcal{S}_{A,b}^\perp, \quad A_\circlearrowleft := P_{\mathcal{S}_{A,b}^\perp} A^*.$$

By (3.12) (interpreted for the dual system) this operator fulfills

$$A_\circlearrowleft \eta = P_{\mathcal{S}_{A,b}^\perp} A^* \eta = A^* \eta - c \frac{\langle \eta\,,\, A^r b\rangle}{\langle c\,,\, A^{r-1} b\rangle} \quad \forall\, \eta \in \mathcal{S}_{A,b}^\perp \cap \operatorname{dom} A^*. \tag{3.22}$$

Recall that $(TL)^* = L^*T^*$ for any densely defined operator L and any bounded operator T, and if in addition T is boundedly invertible, we also have $(LT)^* = T^*L^*$ [Wei85, Section 4.4]. Hence, adjoining the equations above yields

$$U^{-*}AU^* = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ p_0 & p_1 & \cdots & p_{r-2} & p_{r-1} & \left(\alpha_{r-1} \mapsto \alpha_{r-1} P_{\mathcal{S}_{A,b}^\perp} {A^*}^r c\right)^* \\ (P^0 A^*)^* & 0 & \cdots & 0 & 0 & A_\circlearrowleft^* \end{bmatrix}.$$

It remains to show that the representations of Q, R and S in (3.21) are valid. The operator $\overline{P^0 A^*}$ is bounded, see Lemma 3.2.6 (i). Hence $(P^0 A^*)^* = (\overline{P^0 A^*})^*$. Furthermore, we have for all $\eta \in \mathcal{S}_{A,b}^\perp$ and all $\alpha \in \mathbb{R}$,

$$\left\langle \overline{P^0 A^*} \eta\,,\, \alpha \right\rangle_{\mathbb{R}} \overset{(3.10),(3.3)}{=} \left\langle \frac{\langle \eta\,,\, A^r b\rangle}{\langle c\,,\, A^{r-1} b\rangle}\,,\, \alpha \right\rangle_{\mathbb{R}} = \left\langle \eta\,,\, \alpha \frac{\pi_{\mathcal{S}_{A,b}^\perp} A^r b}{\langle A^{r-1} b\,,\, c\rangle} \right\rangle_{\mathcal{S}_{A,b}^\perp},$$

whence $(P^0 A^*)^* = R$. The formula for S is merely the definition of the adjoint of the mapping $\alpha_{r-1} \mapsto \alpha_{r-1} P_{\mathcal{S}_{A,b}^\perp} {A^*}^r c$. Lastly, we prove that $Q = A_\circlearrowleft^*$. Applying the orthogonal projection $\pi_{\mathcal{S}_{A,b}^\perp}$ to (3.22), we see

$$A_\circlearrowleft \eta = \pi_{\mathcal{S}_{A,b}^\perp} A^* \eta - \pi_{\mathcal{S}_{A,b}^\perp} c \frac{\langle \eta\,,\, A^r b\rangle}{\langle c\,,\, A^{r-1} b\rangle} \quad \forall\, \eta \in \mathcal{S}_{A,b}^\perp \cap \operatorname{dom} A^*.$$

The second summand in this representation is a bounded operator from $\mathcal{S}_{A,b}^\perp$ to itself. Therefore the adjoint of $A_\circlearrowleft$ in the Hilbert space $\mathcal{S}_{A,b}^\perp$ has the same domain as the adjoint of $\pi_{\mathcal{S}_{A,b}^\perp} A^*|_{\mathcal{S}_{A,b}^\perp}$, which is the set $\pi_{\mathcal{S}_{A,b}^\perp} \operatorname{dom} A$. Writing out the orthogonal projection shows for every $x \in \operatorname{dom} A$

$$\pi_{\mathcal{S}_{A,b}^\perp} x = x - \langle x\,,\, b\rangle\, b - \ldots - \left\langle x\,,\, A^{r-1} b\right\rangle A^{r-1} b \;\in\; \operatorname{dom} A,$$

whence $\operatorname{dom} A_\circlearrowleft^* = \pi_{\mathcal{S}_{A,b}^\perp} \operatorname{dom} A = \mathcal{S}_{A,b}^\perp \cap \operatorname{dom} A$. Now that the domain is determined, the calculation

$$\begin{aligned} \langle A_\circlearrowleft \eta\,,\, \nu\rangle_{\mathcal{S}_{A,b}^\perp} &= \left\langle \pi_{\mathcal{S}_{A,b}^\perp} A^* \eta - \pi_{\mathcal{S}_{A,b}^\perp} c \frac{\langle \eta\,,\, A^r b\rangle}{\langle c\,,\, A^{r-1} b\rangle}\,,\, \nu \right\rangle_{\mathcal{S}_{A,b}^\perp} \\ &= \langle A^* \eta\,,\, \nu\rangle - \frac{\langle \eta\,,\, A^r b\rangle}{\langle c\,,\, A^{r-1} b\rangle} \langle c\,,\, \nu\rangle \\ &= \left\langle \eta\,,\, \pi_{\mathcal{S}_{A,b}^\perp} A\nu \right\rangle_{\mathcal{S}_{A,b}^\perp} - \left\langle \eta\,,\, A^{r-1} b \frac{\langle \nu\,,\, c\rangle}{\langle A^r b\,,\, c\rangle} \right\rangle \\ &= \left\langle \eta\,,\, \pi_{\mathcal{S}_{A,b}^\perp} A\nu - \pi_{\mathcal{S}_{A,b}^\perp} A^r b \frac{\langle \nu\,,\, c\rangle}{\langle A^{r-1} b\,,\, c\rangle} \right\rangle_{\mathcal{S}_{A,b}^\perp} \end{aligned}$$

for all $\eta \in \mathcal{S}_{A,b}^\perp \cap \operatorname{dom} A^*$ and all $\nu \in \mathcal{S}_{A,b}^\perp \cap \operatorname{dom} A$ shows that

$$A_\circlearrowleft^* \nu = \pi_{\mathcal{S}_{A,b}^\perp} A\nu - \pi_{\mathcal{S}_{A,b}^\perp} A^r b \frac{\langle \nu\,,\, c\rangle}{\langle A^{r-1} b\,,\, c\rangle}.$$

This is the operator named Q in the theorem, and it generates a strongly continuous semigroup because its adjoint operator does. The proof is finished. □

The next corollary shows an interesting relation between the zero dynamics form and the Byrnes-Isidori form: The unbounded lower right operators in both forms are similar to each other and the spaces $\mathcal{S}_{A^*,c}^\perp$ and $\mathcal{S}_{A,b}^\perp$ are isomorphic.

Corollary 3.3.6. *The transformation U^* in Lemma 3.3.4 induces a bijective mapping,*

$$\widehat{U}^* : \mathcal{S}_{A,b}^{\perp} \to \mathcal{S}_{A^*,c}^{\perp}, \quad \eta \mapsto U^* \begin{bmatrix} 0 \\ \eta \end{bmatrix},$$

with inverse

$$\widehat{U}^{-*} : \mathcal{S}_{A^*,c}^{\perp} \to \mathcal{S}_{A,b}^{\perp}, \quad x \mapsto \pi_{\mathcal{S}_{A,b}^{\perp}} x.$$

The operator Q defined in (3.21) *on $\mathcal{S}_{A,b}^{\perp}$ is similar to the operator $A_{\circlearrowright}$ defined in* (3.11)*, and*

$$Q\eta = \widehat{U}^{-*} A_{\circlearrowright} \widehat{U}^* \eta \quad \forall\, \eta \in \operatorname{dom} Q.$$

Proof. The bijectivity of $\widehat{U}^*$ is immediate from Lemma 3.3.4. The definition of $A_{\circlearrowright}$ yields for all $\eta \in \operatorname{dom} Q \subset \mathcal{S}_{A,b}^{\perp}$,

$$\begin{aligned}
\widehat{U}^{-*} A_{\circlearrowright} \widehat{U}^* \eta &= \widehat{U}^{-*} \left(A\widehat{U}^* \eta - b \frac{\left\langle \widehat{U}^* \eta \,,\, {A^*}^r c \right\rangle}{\left\langle b \,,\, {A^*}^{r-1} c \right\rangle} \right) \\
&= \pi_{\mathcal{S}_{A,b}^{\perp}} A \widehat{U}^* \eta - \underbrace{\pi_{\mathcal{S}_{A,b}^{\perp}} b \frac{\left\langle \widehat{U}^* \eta \,,\, {A^*}^r c \right\rangle}{\left\langle b \,,\, {A^*}^{r-1} c \right\rangle}}_{=0} \\
&= \pi_{\mathcal{S}_{A,b}^{\perp}} A U^* \begin{bmatrix} 0 \\ \eta \end{bmatrix} = \begin{bmatrix} 0_{\mathbb{R}^r}^{\top} & \mathrm{I} \end{bmatrix} \widehat{A} \begin{bmatrix} 0 \\ \eta \end{bmatrix} = Q\eta,
\end{aligned}$$

which proves the assertion. □

Proposition 3.3.7. *Let $(A, B, C, 0) \in \mathbf{\Sigma}_r$ with Byrnes-Isidori form denoted as in Theorem 3.3.5. Then*

$$\rho(A) \cap \rho(Q) = \left\{ \lambda \in \rho(Q) \;\middle|\; \lambda^r - \sum_{k=0}^{r-1} p_k \lambda^k - S(\lambda - Q)^{-1} R \neq 0 \right\},$$

and the transfer function $\mathbf{G}$ of $(A, B, C, 0)$ is given by

$$\mathbf{G}(\lambda) = \frac{\left\langle A^{r-1} b \,,\, c \right\rangle}{\lambda^r - \sum_{k=0}^{r-1} p_k \lambda^k - S(\lambda - Q)^{-1} R} \quad \forall\, \lambda \in \rho(A) \cap \rho(Q). \tag{3.23}$$

Proof. Since the transformation into the Byrnes-Isidori form $(\widehat{A}, \widehat{B}, \widehat{C}, 0)$ in Theorem 3.3.5 is boundedly invertible, we have $\rho(A) = \rho(\widehat{A})$ and

$$\mathbf{G}(\lambda) = C(\lambda - A)^{-1}B = \widehat{C}(\lambda - \widehat{A})^{-1}\widehat{B}.$$

Let $\lambda \in \rho(Q)$. Then $\lambda - \widehat{A}$ is boundedly invertible if and only if the Schur complement

$$\begin{bmatrix} \lambda & -1 & \cdots & 0 & 0 \\ 0 & \lambda & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & -1 & 0 \\ 0 & 0 & & \lambda & -1 \\ -p_0 & -p_1 & \cdots & -p_{r-2} & \lambda - p_{r-1} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -S \end{bmatrix} (\lambda - Q)^{-1} \begin{bmatrix} -R & 0 & \cdots & 0 & 0 \end{bmatrix}$$

is boundedly invertible [Tre08, Theorem 2.3.3]. The latter is equivalent to

$$\lambda^r - \sum_{k=0}^{r-1} p_k \lambda^k - S(\lambda - Q)^{-1}R \neq 0. \tag{3.24}$$

Hence, the first claim follows from $\rho(A) = \rho(\widehat{A})$. Now let $\lambda \in \rho(Q) \cap \rho(A)$. Then (3.24) holds, and we may define the abbreviations

$$\alpha := \frac{\langle A^{r-1}b\,,\, c\rangle}{\lambda^r - \sum_{k=0}^{r-1} p_k \lambda^k - S(\lambda - Q)^{-1}R}, \qquad x := \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{r-1} \\ (\lambda - Q)^{-1}R \end{bmatrix} \alpha.$$

It is easily verified that

$$(\lambda - \widehat{A})x = \begin{bmatrix} \lambda & -1 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & -1 & 0 & 0 \\ 0 & 0 & & \lambda & -1 & 0 \\ -p_0 & -p_1 & \cdots & -p_{r-2} & \lambda - p_{r-1} & -S \\ -R & 0 & \cdots & 0 & 0 & \lambda - Q \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{r-2} \\ \lambda^{r-1} \\ (\lambda - Q)^{-1}R \end{bmatrix} \alpha = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \langle A^{r-1}b\,,\, c\rangle \\ 0 \end{bmatrix}.$$

Hence, $x = (\lambda - \widehat{A})^{-1}\widehat{B}$, and we obtain

$$\widehat{C}(\lambda - \widehat{A})^{-1}\widehat{B} = \widehat{C}x = \alpha.$$

This completes the proof of the proposition. $\square$

We now investigate the behavior of a system $(A, B, C, 0) \in \mathbf{\Sigma}_r$. We will show that the behavior of a system in Byrnes-Isidori form is closely related to the solution of a functional differential equation. The latter allows for a simpler representation of the input-output mapping. In this context we use the abbreviation "f.a.a." which means "for almost all", i.e. for all up to a null set.

Proposition 3.3.8. *Let $(A, B, C, 0) \in \mathbf{\Sigma}_r$, $x_0 \in \mathcal{X}$, $u \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}; \mathbb{R})$, and $y \in \mathcal{C}(\mathbb{R}_{\geqslant 0}; \mathbb{R})$ and use the notation of Theorem 3.3.5 for the Byrnes-Isidori form. Define, for fixed $\eta_0 \in \mathcal{S}^{\perp}_{A,b}$, the causal linear operator*

$$\begin{array}{rccl} \mathfrak{T}_{\eta_0} : & L^1_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}; \mathbb{R}) & \to & \mathcal{C}(\mathbb{R}_{\geqslant 0}; \mathbb{R}), \\ & y & \mapsto & \Big(t \mapsto S\mathfrak{A}_Q(t)\eta_0 + S\int_0^t \mathfrak{A}_Q(t-s)\, Ry(s)\,\mathrm{d}s\Big). \end{array}$$

Then the following are equivalent:

(i) $\exists\, x \in \mathcal{C}(\mathbb{R}_{\geqslant 0}; \mathcal{X})$ *with* $x(0) = x_0$ *and* $(x, u, y) \in \mathrm{bhv}\,(A, B, C, 0)$.

(ii) The function y is $r-1$-times continuously differentiable and satisfies

$$\begin{bmatrix} y(t) \\ \vdots \\ y^{(r-2)}(t) \\ y^{(r-1)}(t) \end{bmatrix} = \begin{bmatrix} \langle x_0\,,\, c\rangle \\ \vdots \\ \langle x_0\,,\, {A^*}^{r-2}c\rangle \\ \langle x_0\,,\, {A^*}^{r-1}c\rangle \end{bmatrix} + \begin{bmatrix} \int_0^t y^{(1)}(s)\,\mathrm{d}s \\ \vdots \\ \int_0^t y^{(r-1)}(s)\,\mathrm{d}s \\ \int_0^t \sum_{i=0}^{r-1} p_i y^{(i)}(s) + S\eta(s) + \langle A^{r-1}b\,,\, c\rangle u(s)\,\mathrm{d}s \end{bmatrix}, \tag{3.25}$$

$$\eta(t) = \mathfrak{A}_Q(t)\pi_{\mathcal{S}^{\perp}_{A,b}} x_0 + \int_0^t \mathfrak{A}_Q(t-s)\, Ry(s)\,\mathrm{d}s, \tag{3.26}$$

for all $t \geqslant 0$.

(iii) The function y is $r-1$-times continuously differentiable and its rth derivative satisfies

$$y^{(r)}(t) = \sum_{i=0}^{r-1} p_i\, y^{(i)}(t) + (\mathfrak{T}_{\eta_0} y)(t) + \left\langle A^{r-1} b\,,\, c \right\rangle u(t) \quad \text{f.a.a. } t \geqslant 0, \tag{3.27}$$

and,

$$\begin{bmatrix} y(0) \\ \vdots \\ y^{(r-1)}(0) \\ \eta_0 \end{bmatrix} = \begin{bmatrix} \langle x_0\,,\, c\rangle \\ \vdots \\ \langle x_0\,,\, {A^*}^{r-1} c\rangle \\ \pi_{\mathcal{S}_{A,b}^{\perp}} x_0 \end{bmatrix}. \tag{3.28}$$

The functions x and η in (i) and (ii) are related by the equation

$$x(t) = U^*[y(t)\,,\, \ldots\,,\, y^{(r-1)}(t)\,,\, \eta(t)]^\top \quad \forall\, t \geqslant 0,$$

with the transformation U defined in Lemma 3.3.4.

Proof. (i) $\Rightarrow$ (ii): Let $(x, u, y) \in \operatorname{bhv}(A, B, C, 0)$ with $x(0) = x_0$ and define the transformation U as in Lemma 3.3.4. By Lemma 2.4.15, the transformed function $\widehat{x} := U^{-*}x$ satisfies $(\widehat{x}, u, y) \in \operatorname{bhv}(\widehat{A}, \widehat{B}, \widehat{C}, 0)$. Owing to the boundedness of $\widehat{B}$ this implies by Lemma A.2.2 (ii) that

$$\widehat{x}(t) = \widehat{x}(0) + \int_0^t \widehat{A}\big|_{\mathbb{R}^r \times \mathcal{S}_{A,b}^{\perp}} x(s) + \widehat{B}u(s)\,\mathrm{d}s \quad \forall\, t \geqslant 0,$$

and that $\widehat{x} \in \mathcal{C}\left(\mathbb{R}_{\geqslant 0}; \mathbb{R}^r \times \mathcal{S}_{A,b}^{\perp}\right)$. Since the operators R, S, and $\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & 1 \\ p_0 & p_1 & \cdots & p_{r-1} \end{bmatrix}$ appearing in $\widehat{A}$ are bounded, Lemma A.2.3 implies that, if we write the function $\widehat{x}$ as

$$\widehat{x}(t) =: [\alpha_0(t)\,,\, \cdots\,,\, \alpha_{r-1}(t)\,,\, \eta(t)]^\top \quad \forall\, t \geqslant 0,$$

then

$$
\begin{bmatrix} \alpha_0(t) \\ \vdots \\ \alpha_{r-2}(t) \\ \alpha_{r-1}(t) \end{bmatrix} = \begin{bmatrix} \alpha_0(0) \\ \vdots \\ \alpha_{r-2}(0) \\ \alpha_{r-1}(0) \end{bmatrix} + \begin{bmatrix} \int_0^t \alpha_1(s)\,\mathrm{d}s \\ \vdots \\ \int_0^t \alpha_{r-1}(s)\,\mathrm{d}s \\ \int_0^t \sum_{i=0}^{r-1} p_i\alpha_i(s) + S\eta(s) + \langle A^{r-1}b\,,\, c\rangle\, u(s)\,\mathrm{d}s \end{bmatrix}, \tag{3.29}
$$

$$
\eta(t) = \eta(0) + \int_0^t Q\big|_{\mathcal{S}_{A,b}^{\perp}}\eta(s) + R\alpha_0(s)\,\mathrm{d}s \tag{3.30}
$$

for all $t \geqslant 0$, and

$$
\begin{bmatrix} \alpha_0(0) \\ \vdots \\ \alpha_{r-1}(0) \\ \eta(0) \end{bmatrix} = U^{-*}x(0) = \begin{bmatrix} \langle x_0\,,\, c\rangle \\ \vdots \\ \left\langle x_0\,,\, {A^*}^{r-1}c\right\rangle \\ \pi_{\mathcal{S}_{A,b}^{\perp}}x_0 \end{bmatrix}.
$$

Since

$$
y(t) = Cx(t) = \widehat{C}\widehat{x}(t) = \alpha_0(t) \quad \forall\, t \geqslant 0,
$$

we conclude from (3.29) that $y^{(i)} = \alpha_0^{(i)} = \alpha_i$ for all $i = 0, \ldots, r-1$. Lemma A.2.2 (i) shows that (3.30) implies (3.26) and therefore (ii) holds.

(ii) $\Rightarrow$ (iii): If (ii) holds, then the lower line of (3.25) shows that the function $y^{(r-1)} = \alpha_{r-1}$ is absolutely continuous. Therefore, it is almost everywhere differentiable and its derivative satisfies (3.27).

(iii) $\Rightarrow$ (i): Assume y satisfies (iii). Define $[\alpha_0\,, \ldots\,, \alpha_{r-1}] := [y\,, \ldots\,, y^{(r-1)}]$ and the function η by (3.26). Then (3.29) is fulfilled, and (3.30) as well because of Lemma A.2.2. Lemma 2.4.11 and Lemma A.2.3 therefore imply that

$$
\left(\begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{bmatrix}, u, y \right) \in \mathrm{bhv}(\widehat{A}, \widehat{B}, \widehat{C}, 0).
$$

Hence, the function

$$
x(\cdot) := U^*[\alpha_0(\cdot)\,, \ldots\,, \alpha_{(r-1)}(\cdot)\,, \eta(\cdot)]^\top
$$

satisfies $(x, u, y) \in \mathrm{bhv}(A, B, C, 0)$. Finally (3.28) yields the initial value $x(0) = U^* x_0$, and the proof of the proposition is complete. $\square$

Note that the right hand side of (3.27) may be interpreted as an ordinary differential term $\sum_{i=0}^{r-1} p_i\, y^{(i)}(t) + \langle A^{r-1} b\,,\, c\rangle\, u(t)$ which is perturbed by a functional term $(\mathfrak{T}_{\eta_0} y)(t)$. This structure will be exploited in Section 5.1 to control the system.

3.4. Notes and references

Our definition of relative degree is stronger than the frequency domain definition of [MR07]. More precisely, Presumption 3.1.1 implies by [MR07, Lemma 2.9] that the transfer function of the system fulfills

$$\lim_{s\to\infty, s\in\mathbb{R}} s^r C(s-A)^{-1} B \neq 0, \qquad \lim_{s\to\infty, s\in\mathbb{R}} s^{r-1} C(s-A)^{-1} B = 0. \tag{3.31}$$

This property is called relative degree r in [MR07, Definition 1.3]. A partial converse to the implication above is contained in [MR07, Lemma 2.9]: If the state linear systems $(A, B, C, 0)$ fulfills (3.31) and $\operatorname{ran} C^* \subset \operatorname{dom}(A^{*^{r-1}})$, then it has relative degree r in the sense of Definition 3.1.2.

The largest (infinite-dimensional) feedback invariant subspace of $\ker C$ has received much attention over the years. Many authors have studied existence and geometric and invariance properties, even under weaker assumptions than Presumption 3.1.1, e.g. [Cur84, Cur86, MR07, Zwa88]. It is well-known that under Presumption 3.1.1, this space is precisely $\mathcal{S}_{A,b}^{\perp}$, see [Zwa88, Section 4] and [MR07, Theorem 2.10]. The complete decomposition (3.3) has previously only been considered for systems with relative degree one, where it is simply $\mathcal{X} = \ker C \oplus \operatorname{ran} B$. This fact has been used in [Byr87, BLGS98, LZ91] for the purpose of high-gain control. Even for finite-dimensional systems, where it is easy to derive, the zero dynamics form in the sense of Definition 3.2.1 is not well documented.

Instead of that, the Byrnes-Isidori form for (nonlinear) finite-dimensional systems, which was first introduced in [BI91], is very popular and well understood, see [BIW14, IRT07, Isi95]. Despite this popularity, it has not been established for infinite-dimensional systems before. Furthermore, we believe the relation between the Byrnes-Isidori form and the zero dynamics form of the dual system that was established in Lemma 3.3.2 and Corollary 3.3.6 is a new insight and clarifies the relationship between two instruments that are often used to get a hold of the zero dynamics: the largest feedback invariant subspace of $\ker C$ and the Byrnes-Isidori form.

4. Zero dynamics

In this chapter we will analyze the behavior that leaves the output completely unaffected, the so-called zero dynamics.

Definition 4.0.1 (zero dynamics)**.** The *zero dynamics* of an L^p-well-posed linear system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is the subspace of the behavior defined by

$$\operatorname{zd}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) := \left\{ \, (x, u, y) \in \operatorname{bhv}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) \ \middle| \ y \equiv 0 \, \right\}.$$

If the system is of Pritchard-Salamon type with generators (A, B, C, D) we write

$$\operatorname{zd}(A, B, C, D) := \operatorname{zd}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}).$$

The system is said to have *exponentially stable zero dynamics* if and only if

$$\begin{gathered} \exists\, M, \mu > 0 \quad \forall\, (x, u, 0) \in \operatorname{zd}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) : \\ \|(x(t), u(t))\|_{\mathcal{X} \times \mathcal{U}} \leqslant M \mathrm{e}^{-\mu t} \|x(0)\| \quad \text{f.a.a. } t \geqslant 0, \end{gathered} \tag{4.1}$$

and *strongly stable zero dynamics* if and only if

$$\forall\, (x, u, 0) \in \operatorname{zd}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) : \quad \lim_{t \to \infty} \|(x(t), u(t))\|_{\mathcal{X} \times \mathcal{U}} = 0.$$

4.1. Zero dynamics for systems with relative degree

In this section we characterize the zero dynamics of systems with relative degree in the sense of Definition 3.1.2. We will prove that the zero dynamics are completely determined by the semigroup generator in the lower right corner of the Byrnes-Isidori form and the zero dynamics form, respectively. We start with the Byrnes-Isidori form.

Theorem 4.1.1. *Let Presumption 3.1.1 hold. Then, with the notation as in Theorem 3.3.5, the zero dynamics of the system* $(A, B, C, 0) \in \boldsymbol{\Sigma}_r$ *is given by*

$$\operatorname{zd}(A, B, C, 0) = \left\{ \left(U^* \begin{bmatrix} 0 \\ \mathfrak{A}_Q(\cdot)\eta_0 \end{bmatrix}, -\frac{S\mathfrak{A}_Q(\cdot)\eta_0}{\langle A^{r-1}b\,,\,c\rangle}, 0 \right) \;\middle|\; \eta_0 \in \mathcal{S}_{A,b}^{\perp} \right\}, \tag{4.2}$$

where $\mathfrak{A}_Q$ *denotes the semigroup generated by* Q *in* $\mathcal{S}_{A,b}^{\perp}$*.*

Proof. Let $(x, u, y) \in \operatorname{zd}(A, B, C, 0)$ and define $\eta_0 := \pi_{\mathcal{S}_{A,b}^{\perp}} x(0)$. By Proposition 3.3.8 $[y\,,\,\ldots\,,\,y^{(r-1)}]^\top$ and $\eta(\cdot) := \pi_{\mathcal{S}_{A,b}^{\perp}} x(\cdot)$ satisfy (3.25) and (3.26). Since $y \equiv 0$, we have $y^{(i)} \equiv 0$ for all $i = 0, \ldots, r-1$. Inserting this, we can easily solve (3.25) and (3.26) for u and η. We obtain $\eta(t) = \mathfrak{A}_Q(t)\eta_0$ and $u(t) = -\langle A^{r-1}b\,,\,c\rangle^{-1} S\eta(t)$. Since Proposition 3.3.8 also states that $x(t) = U^*[y(t)\,,\,\ldots\,,\,y^{(r-1)}(t)\,,\,\eta(t)]^\top$, the triple (x, u, y) belongs to the right hand side of (4.2).

Conversely, let $\eta_0 \in \mathcal{S}_{A,b}^{\perp}$ be given and define $x_0 := U^* \left[\begin{smallmatrix} 0 \\ \eta_0 \end{smallmatrix}\right]$. Then (3.20) shows

$$\begin{bmatrix} \langle x_0\,,\,c\rangle \\ \vdots \\ \left\langle x_0\,,\,{A^*}^{r-1}c\right\rangle \\ \pi_{\mathcal{S}_{A,b}^{\perp}} x_0 \end{bmatrix} = U^{-*}x_0 = \begin{bmatrix} 0 \\ \eta_0 \end{bmatrix}.$$

Using this equation, it can be seen that the functions

$$y(t) := 0, \qquad u(t) := -\frac{S\mathfrak{A}_Q(t)\eta_0}{\langle A^{r-1}b\,,\,c\rangle}, \qquad \eta(t) = \mathfrak{A}_Q(t)\eta_0 \qquad \forall\, t \geqslant 0,$$

satisfy (ii) of Proposition 3.3.8. Hence, this proposition implies that

$$\left(U^* \begin{bmatrix} 0 \\ \mathfrak{A}_Q(\cdot)\eta_0 \end{bmatrix}, -\frac{S\mathfrak{A}_Q(\cdot)\eta_0}{\langle A^{r-1}b\,,\,c\rangle}, 0 \right) = \left(U^* \begin{bmatrix} 0 \\ \eta(\cdot) \end{bmatrix}, u(\cdot), y(\cdot) \right) \in \operatorname{bhv}(A, B, C, 0).$$

Since $y \equiv 0$, the left hand side belongs to the zero dynamics and (4.2) is shown. □

Recall that the operator $A_{\circlearrowleft}$ in (3.11) is the lower right operator in the zero dynamics form. With the zero dynamics form it can be shown that this operator determines the zero dynamics as well. But since we already know that $A_{\circlearrowleft}$ is similar to Q, this is a simple corollary to Theorem 4.1.1.

Corollary 4.1.2. *Let Presumption 3.1.1 hold, define* $F\eta := -\frac{\langle \eta\,,\, {A^*}^r c\rangle}{\langle b\,,\, {A^*}^{r-1} c\rangle}$ *for* $\eta \in \mathcal{S}^{\perp}_{A^*,c}$*, and let* $\mathfrak{A}_{\circlearrowleft}$ *be the semigroup in* $\mathcal{S}^{\perp}_{A^*,c}$ *that is generated by the operator* $A_{\circlearrowleft}$ *defined by* (3.11)*. Then*

$$\operatorname{zd}(A,B,C,D) = \left\{ \left(\mathfrak{A}_{\circlearrowleft}(\cdot)\, x_0, F\mathfrak{A}_{\circlearrowleft}(\cdot)\, x_0\ , 0\right) \;\middle|\; x_0 \in \mathcal{S}^{\perp}_{A^*,c} \right\}.$$

Proof. The bijection $\widehat{U}^* : \mathcal{S}^{\perp}_{A,b} \to \mathcal{S}^{\perp}_{A^*,c}$ defined in Corollary 3.3.6 satisfies $\mathfrak{A}_Q(t) = \widehat{U}^{-*}\mathfrak{A}_{\circlearrowleft}(t)\widehat{U}^*$ for all $t \geqslant 0$. Observe that the right hand side of (4.2) is equal to

$$\left\{ \left(\widehat{U}^*\mathfrak{A}_Q(\cdot)\, \eta_0, -\frac{S\mathfrak{A}_Q(\cdot)\, \eta_0}{\langle A^{r-1}b\,,\, c\rangle}\ , 0\right) \;\middle|\; \eta_0 \in \mathcal{S}^{\perp}_{A,b} \right\}.$$

Therefore, the claim follows from Theorem 4.1.1 and the calculation

$$\begin{aligned}
\frac{S\widehat{U}^{-*}x}{\langle A^{r-1}b\,,\, c\rangle} &\overset{(3.21)}{=} \frac{\left\langle \widehat{U}^{-*}x\,,\, P_{\mathcal{S}^{\perp}_{A,b}} {A^*}^r c\right\rangle}{\langle A^{r-1}b\,,\, c\rangle} = \frac{\left\langle \widehat{U}^{-*}x\,,\, [0^{\top}_{\mathbb{R}^r}\ \ \mathrm{I}]\, U {A^*}^r c\right\rangle}{\langle A^{r-1}b\,,\, c\rangle} \\
&= \frac{\left\langle U^{-*}x\,,\, U {A^*}^r c\right\rangle}{\langle A^{r-1}b\,,\, c\rangle} = \frac{\left\langle x\,,\, {A^*}^r c\right\rangle}{\langle b\,,\, A^{r-1*} c\rangle} = -Fx
\end{aligned}$$

for all $x \in \mathcal{S}^{\perp}_{A^*,c}$. ∎

These findings allow for a characterization of the stability of the zero dynamics in terms of the operator Q or, equivalently, $A_{\circlearrowleft}$:

Lemma 4.1.3. *Let Presumption 3.1.1 hold. Then* $(A,B,C,0)$ *has exponentially stable zero dynamics if and only if the semigroup generated by the operator* Q *in Theorem 3.3.5 is exponentially stable, and the system has strongly stable zero dynamics if and only if this semigroup is strongly stable.*

Proof. Theorem 3.3.5 includes that Q generates a strongly continuous semigroup $\mathfrak{A}_Q$. If this semigroup is exponentially stable, then the assertion is an immediate consequence of Theorem 4.1.1. Assume on the other hand that $(A,B,C,0)$ has exponentially stable zero dynamics and let $\eta_0 \in \mathcal{S}^{\perp}_{A,b}$ be arbitrary. Then equation (4.2) shows that

$$\left(U^*\begin{bmatrix} 0 \\ \mathfrak{A}_Q(\cdot)\eta_0 \end{bmatrix}, -\frac{S\mathfrak{A}_Q(\cdot)\eta_0}{\langle A^{r-1}b\,,\, c\rangle}, 0\right) \in \operatorname{zd}(A,B,C,0).$$

Thus, the stability assumption (4.1) implies with $\widehat{U}^*$ as in Corollary 3.3.6 that

$$\forall\, t \geqslant 0 : \quad \left\| \widehat{U}^* \mathfrak{A}_Q(t)\eta_0 \right\|_{\mathcal{S}^{\perp}_{A^*,c}} = \left\| U^* \begin{bmatrix} 0 \\ \mathfrak{A}_Q(t)\eta_0 \end{bmatrix} \right\| \leqslant M \left\| U^* \begin{bmatrix} 0 \\ \eta_0 \end{bmatrix} \right\| \mathrm{e}^{-\mu t}.$$

Since $\widehat{U}^*$ is boundedly invertible, we conclude $\|\mathfrak{A}_Q(t)\eta_0\| \leqslant \|\widehat{U}^{-*}\| M \mathrm{e}^{-\mu t} \|\eta_0\|$. This shows the exponential stability of the semigroup because M and μ are by assumption independent of η_0. The part about strong stability follows in the same manner from Theorem 4.1.1. □

Corollary 4.1.4. *Let Presumption 3.1.1 hold and assume that $(A, B, C, 0)$ has exponentially stable zero dynamics. Then its transfer function $\mathbf{G}$ satisfies*

$$\mathbf{G}(\lambda) \neq 0 \quad \forall\, \lambda \in \rho(A) \cap \mathbb{C}_{\geqslant 0}$$

and

$$\rho(A) \cap \mathbb{C}_{\geqslant 0} = \left\{ \lambda \in \mathbb{C}_{\geqslant 0} \;\middle|\; \lambda^r - \sum_{k=0}^{r-1} P_k \lambda^k - S(\lambda - Q)^{-1} R \neq 0 \right\},$$

with $p_0, \dots, p_{r-1}, Q, R,$ and S as in Theorem 3.3.5.

Proof. By Lemma 4.1.3 the exponential stability of the zero dynamics is equivalent to the exponential stability of $\mathfrak{A}_Q$. By [CZ95, Theorem 5.15] this is equivalent to the conditions $\mathbb{C}_{\geqslant 0} \subset \rho(Q)$ and $\sup_{\lambda \in \mathbb{C}_{\geqslant 0}} \|(\lambda - Q)^{-1}\| < \infty$. Therefore, the denominator of the transfer function in (3.23) is finite at every point $\lambda \in \rho(A) \cap \mathbb{C}_{\geqslant 0}$ and the claim follows. □

In view of the internal loop form in Proposition 3.3.8 we may observe: If the zero dynamics are exponentially stable, then Lemma 4.1.3 implies that $\mathfrak{T}_{\eta_0}$ maps bounded functions to bounded functions.

4.2. Zero dynamics of the heat equation with boundary control

For general well-posed linear systems the zero dynamics are not necessarily characterized by a strongly continuous semigroup; a counterexample is in [MR07, Section 4]. However, we will prove in this section that the zero dynamics of the heat equation system introduced in Section 2.6 are entirely described by an exponentially stable, contractive and analytic semigroup. First we introduce the operator that will turn out to be the generator of this semigroup.

As in Section 2.6 Ω is a bounded domain with uniformly $\mathcal{C}^2$-boundary $\partial\Omega$. The following is a deep regularity result from [HT08, Proposition 5.26 (ii)].

Lemma 4.2.1. *Let $x \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$, satisfy*

$$\int_\Omega \nabla x(\xi) \cdot \overline{\nabla \varphi(\xi)} \, \mathrm{d}\xi = \int_\Omega f(\xi) \cdot \overline{\varphi(\xi)} \, \mathrm{d}\xi$$

for all $\varphi \in \mathcal{C}^\infty(\Omega)$ with $\partial_\nu \varphi|_{\partial\Omega} \equiv 0$. Then $x \in W^{2,2}(\Omega)$ and $\partial_\nu x|_{\partial\Omega} \equiv 0$.

Theorem 4.2.2. *Consider the operator*

$$\begin{aligned} A_0 &: \operatorname{dom} A_0 \subset L^2(\Omega) \to L^2(\Omega), \quad A_0 x := \Delta x, \\ \operatorname{dom} A_0 &:= \left\{ x \in W^{2,2}(\Omega) \;\middle|\; \begin{array}{l} \partial_\nu x|_{\partial\Omega} \equiv \frac{\int_\Omega \Delta x(\xi)\, \mathrm{d}\xi}{|\partial\Omega|} \\ \textit{and} \quad \int_{\partial\Omega} x(\xi)\, \mathrm{d}\sigma_\xi = 0. \end{array} \right\}. \end{aligned} \tag{4.3}$$

Then the following is true:

(i) A_0 is self-adjoint and has compact resolvent;

(ii) A_0 generates an analytic, contractive, and exponentially stable semigroup on $L^2(\Omega)$.

Proof. Step 1: We construct an associated sesquilinear form for A_0: Define the space

$$\mathcal{H} = \left\{ x \in W^{1,2}(\Omega) \;\middle|\; \int_{\partial\Omega} x(\xi)\, \mathrm{d}\sigma_\xi = 0 \right\}. \tag{4.4}$$

Then $\mathcal{H}$ is dense in $L^2(\Omega)$. We deduce from the Trace theorem, [HT08, Theorem 4.24], that $\mathcal{H}$ is the kernel of a continuous linear mapping, and therefore a closed subspace of $W^{1,2}(\Omega)$. More precisely, $\mathcal{H}$ is a Hilbert space inheriting the inner product of $W^{1,2}(\Omega)$. We define the sesquilinear form

$$a_0 : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad (x, z) \mapsto \int_\Omega \nabla x(\xi) \cdot \overline{\nabla z(\xi)} \, \mathrm{d}\xi, \tag{4.5}$$

which is continuous and symmetric. We prove that there is an $\alpha > 0$ with

$$\operatorname{Re} a_0(x,x) \geqslant \alpha \langle x,x \rangle_{\mathcal{H}} \quad \forall\, x \in \mathcal{H}. \tag{4.6}$$

Assume that this is false. Then there exists a bounded sequence (x_n) in $\mathcal{H}$ with

$$\|x_n\|_{W^{1,2}(\Omega)} = 1 \quad \forall\, n \in \mathbb{N}, \tag{4.7}$$

and

$$a_0(x_n,x_n) \overset{n\to\infty}{\longrightarrow} 0. \tag{4.8}$$

The Rellich-Kondrachov theorem, [HT08, Theorem 4.17 (i)], implies that there exists some $z \in L^2(\Omega)$ and a subsequence (x_{n_k}) with $\|z - x_{n_k}\|_{L^2(\Omega)} \overset{k\to\infty}{\longrightarrow} 0$. Together with (4.8) and (4.5) this implies that (x_{n_k}) is a Cauchy sequence in $W^{1,2}(\Omega)$. Thus we have $z \in W^{1,2}(\Omega)$ and $\|z - x_{n_k}\|_{W^{1,2}(\Omega)} \overset{k\to\infty}{\longrightarrow} 0$. Since differentiation as well as boundary evaluation are continuous with respect to the $W^{1,2}(\Omega)$ norm, it follows that $\nabla z = 0$ and $\int_{\partial\Omega} z(\xi)\,\mathrm{d}\xi = 0$. Hence, z is a constant function whose boundary integral vanishes. This implies $z = 0$, which is in contradiction to (4.7).

Step 2: With the definition of $\mathcal{H}$ and a_0 as in Step 1 and A_0 as in (4.3), we show that

$$\operatorname{dom} A_0 = \left\{ x \in \mathcal{H} \;\middle|\; \exists\, z \in L^2(\Omega) : \; a_0(x,\varphi) = \langle z, \varphi \rangle_{L^2(\Omega)} \quad \forall\, \varphi \in \mathcal{H} \right\}, \tag{4.9}$$

and

$$\langle A_0 x\,,\, \varphi \rangle_{L^2(\Omega)} = -a_0(x,\varphi) \quad \forall\, x,\varphi \in \operatorname{dom} A_0. \tag{4.10}$$

For $x \in \operatorname{dom} A_0$ the equation (4.10) follows with Green's formula since all $\varphi \in \mathcal{H}$ satisfy

$$\begin{aligned}
a_0(x,\varphi) &= \int_\Omega \nabla x(\xi) \cdot \overline{\nabla \varphi(\xi)}\,\mathrm{d}\xi \\
&= -\int_\Omega \Delta x(\xi) \cdot \overline{\varphi(\xi)}\,\mathrm{d}\xi + \int_{\partial\Omega} \partial_\nu x(\xi) \cdot \overline{\varphi(\xi)}\,\mathrm{d}\sigma_\xi \\
&= -\int_\Omega \Delta x(\xi) \cdot \overline{\varphi(\xi)}\,\mathrm{d}\xi + \tfrac{1}{|\partial\Omega|} \int_\Omega \Delta x(\xi)\,\mathrm{d}\xi \cdot \underbrace{\overline{\int_{\partial\Omega} \varphi(\xi)\,\mathrm{d}\sigma_\xi}}_{=0} \\
&= -\langle \Delta x\,,\, \varphi \rangle_{L^2(\Omega)}\,.
\end{aligned}$$

This computation also gives rise to the inclusion "$\subset$" in (4.9). To prove the converse inclusion, assume that $x \in \mathcal{H}$ and there exists some $z \in L^2(\Omega)$ with

$$\int_\Omega \nabla x(\xi) \cdot \overline{\nabla\varphi(\xi)}\,\mathrm{d}\xi = a_0(x,\varphi) = \langle z\,,\,\varphi\rangle_{L^2(\Omega)} \quad \forall\,\varphi \in \mathcal{H}. \tag{4.11}$$

Then (4.11) holds true for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$ in particular. Consequently, we have $z = -\Delta x$. We choose an $W^{2,2}(\Omega)$-function h with

$$\partial_\nu h\big|_{\partial\Omega} \equiv \frac{\int_\Omega \Delta x(\xi)\,\mathrm{d}\xi}{|\partial\Omega|},$$

and claim that $x - h$ fulfills

$$\int_\Omega \nabla(x-h)(\xi) \cdot \overline{\nabla\psi(\xi)}\,\mathrm{d}\xi = -\int_\Omega \Delta(x-h)(\xi) \cdot \overline{\psi(\xi)}\,\mathrm{d}\xi \quad \forall\,\psi \in W^{1,2}(\Omega).$$

Let $\psi \in W^{1,2}(\Omega)$. Then $\varphi := \psi - \frac{\int_{\partial\Omega}\psi(\xi)\,\mathrm{d}\xi}{|\partial\Omega|}$ is in $\mathcal{H}$ and $\nabla\psi = \nabla\varphi$. Thus we have

$$\begin{aligned}
&\int_\Omega \nabla(x-h)(\xi) \cdot \overline{\nabla\psi(\xi)}\,\mathrm{d}\xi \\
&\quad= \int_\Omega \nabla x(\xi) \cdot \overline{\nabla\varphi(\xi)}\,\mathrm{d}\xi - \int_\Omega \nabla h(\xi) \cdot \overline{\nabla\psi(\xi)}\,\mathrm{d}\xi \\
&\quad= \int_\Omega z(\xi) \cdot \overline{\varphi(\xi)}\,\mathrm{d}\xi - \int_\Omega \nabla h(\xi) \cdot \overline{\nabla\psi(\xi)}\,\mathrm{d}\xi \\
&\quad= \int_\Omega z(\xi) \cdot \overline{\varphi(\xi)}\,\mathrm{d}\xi + \int_\Omega \Delta h(\xi) \cdot \overline{\psi(\xi)}\,\mathrm{d}\xi - \int_{\partial\Omega} \partial_\nu h(\xi) \cdot \overline{\psi(\xi)}\,\mathrm{d}\xi \\
&\quad= -\int_\Omega \Delta x(\xi) \cdot \overline{\varphi(\xi)}\,\mathrm{d}\xi + \int_\Omega \Delta h(\xi) \cdot \overline{\psi(\xi)}\,\mathrm{d}\xi - \int_{\partial\Omega} \frac{\int_\Omega \Delta x(\zeta)\,\mathrm{d}\zeta}{|\partial\Omega|} \cdot \overline{\psi(\xi)}\,\mathrm{d}\xi \\
&\quad= -\int_\Omega \Delta x(\xi) \cdot \overline{\left(\varphi(\xi) + \frac{\int_{\partial\Omega}\psi(\xi)}{|\partial\Omega|}\right)}\,\mathrm{d}\xi + \int_\Omega \Delta h(\xi) \cdot \overline{\psi(\xi)}\,\mathrm{d}\xi \\
&\quad= \int_\Omega \Delta(h(\xi) - x(\xi)) \cdot \overline{\psi(\xi)}\,\mathrm{d}\xi.
\end{aligned}$$

Now Lemma 4.2.1 implies that $x - h \in W^{2,2}(\Omega)$ and $\partial_\nu(x-h)|_{\partial\Omega} = 0$. Hence, $x \in W^{2,2}(\Omega)$ and $\partial_\nu x|_{\partial\Omega} \equiv \frac{\int_\Omega \Delta x\,\mathrm{d}\xi}{|\partial\Omega|}$.

Step 3: We conclude statement (i) and (ii): Since we have the relations (4.6) and (4.10), and $a_0(\cdot,\cdot)$ is symmetric, Theorem VI.2.6 in [Kat80, p. 323] implies that A_0

is self-adjoint and negative definite. In particular, $0 \in \rho(A_0)$, and

$$A_0^{-1} L^2(\Omega) \subset \operatorname{dom} A_0.$$

Since $W^{2,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$ by the Rellich-Kondrachov theorem, we infer that A_0 has compact resolvent. Therefore, its spectrum consists of isolated eigenvalues [Kat80, Theorem 6.29, p.187], which are strictly negative. This shows that A_0 is a sectorial operator, and by [Sta05, Theorem 3.10.5], it generates an analytic semigroup $\mathfrak{A}_0(\cdot)$. The fact that its largest eigenvalue $-\omega_0$ is negative further implies that

$$\|\mathfrak{A}_0(t)\|_{\mathcal{B}(L^2(\Omega))} \leqslant \mathrm{e}^{-\omega_0 t} \quad \forall\, t \in \mathbb{R}_{\geqslant 0},$$

by [TW09, Proposition 2.6.5]. Hence, the semigroup is contractive and exponentially stable. ☐

As a consequence of this theorem the input-output-interchanged triple $(\Lambda, C|_{\mathcal{W}}, \Gamma)$ consisting of the operators defined in Lemma 2.6.1 is a system node as well (although not a well-posed one). In particular, $\Lambda|_{\ker C|_{\mathcal{W}}} = A_0$ is the generator of a strongly continuous semigroup $\mathfrak{A}_0$. The following result shows that this semigroup indeed gives a full characterization of the zero dynamics. Note that the analyticity of $\mathfrak{A}_0$ implies that $\mathfrak{A}_0(t)x_0 \in \operatorname{dom} A_0$ for all $t > 0$ and therefore the expression $\Gamma\mathfrak{A}_0(t)x_0$ is well-defined for all $t > 0$.

Theorem 4.2.3. *Consider the L^2-well-posed system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ in Lemma 2.6.12. Let A_0 be as in Theorem 4.2.2 and $\mathfrak{A}_0(\cdot)$ be the semigroup generated by A_0. The space*

$$\mathcal{Z} := \left\{ x_0 \in \mathcal{X} \;\middle|\; \Gamma\mathfrak{A}_0(\cdot)x_0 \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}; \mathcal{U}) \right\}$$

is an $\mathfrak{A}_0$-invariant, dense subspace of $\mathcal{X}$ and the zero dynamics are given by

$$\operatorname{zd}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) = \left\{ (\mathfrak{A}_0(\cdot)x_0, \Gamma\mathfrak{A}_0(\cdot)x_0, 0) \;\middle|\; x_0 \in \mathcal{Z} \right\}. \tag{4.12}$$

Proof. It is trivial that $\mathcal{Z}$ is an $\mathfrak{A}_0$-invariant linear vector space. Since Γ maps $\operatorname{dom} A_0$ continuously into $\mathcal{U}$, the function $\Gamma\mathfrak{A}_0(\cdot)x_0$ is bounded and continuous for all $x_0 \in \operatorname{dom} A_0$. Hence, $\operatorname{dom} A_0 \subset \mathcal{Z}$, which shows the density of $\mathcal{Z}$ in $\mathcal{X}$.

We first show the inclusion "$\subset$" in (4.12). Assume that $(x, u, 0) \in \mathcal{C}(\mathbb{R}_{\geqslant 0}; X) \times L^2_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}) \times L^2_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0})$ is in the zero dynamics of $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$. We know from

Lemma 2.6.6 that $\operatorname{dom}(\mathrm{I}-A)^{\frac{1}{2}}$ is the Sobolev space $W^{1,2}(\Omega)$. Therefore, with the complex interpolation functor $[\cdot\,,\,\cdot]_\theta$ in Definition A.1.2, we have

$$\operatorname{dom}(\mathrm{I}-A)^{\frac{3}{8}} \overset{(A.2)}{=} \Big[\mathcal{X}\,,\,\operatorname{dom}(\mathrm{I}-A)^{\frac{1}{2}}\Big]_{3/4} = \big[L^2(\Omega)\,,\,W^{1,2}(\Omega)\big]_{3/4} = W^{\frac{3}{4},2}(\Omega),$$

where the last equality follows from the interpolation of Sobolev spaces in [Tri95, Section 4.3.1, Theorem 1] and [Tri95, Equation 2.4.2/11]. Since Bu is for every $u \in \mathbb{C}$ a continuous functional on $W^{\frac{3}{4},2}(\Omega)$, we have

$$\operatorname{ran} B \subset (W^{\frac{3}{4},2}(\Omega))' = (\operatorname{dom}(\mathrm{I}-A)^{\frac{3}{8}})'.$$

Due to the smoothing property of $\mathfrak{A}$ described in (A.1) there exists an $M \geqslant 1$ such that

$$\Big\| \mathfrak{A}(t)\big|_{(\operatorname{dom}(\mathrm{I}-A)^{3/8})'} x \Big\|_{\operatorname{dom}(\mathrm{I}-A)^{1/2}} \leqslant M\left(1+t^{-7/8}\right) \quad \forall\, x \in (\operatorname{dom}(\mathrm{I}-A)^{3/8})',\ t>0,$$

and consequently $\|\mathfrak{A}(t)B\|_{\mathcal{B}(\mathcal{U};W^{1,2}(\Omega))} \leqslant M(1+t^{-7/8})$. Now let $T>0$. Then the last estimate and the fact that u is locally integrable yield

$$\begin{aligned}
&\int_0^T \int_0^t \|\mathfrak{A}(t-\tau)Bu(\tau)\|_{W^{1,2}(\Omega)}\,\mathrm{d}\tau\,\mathrm{d}t \\
&\quad\leqslant M\int_0^T\int_0^t \left(1+(t-\tau)^{-7/8}\right)\|u(\tau)\|_{\mathcal{U}}\,\mathrm{d}\tau\,\mathrm{d}t \\
&\quad= M\int_0^T\int_\tau^T \left(1+(t-\tau)^{-7/8}\right)\|u(\tau)\|_{\mathcal{U}}\,\mathrm{d}t\,\mathrm{d}\tau \\
&\quad\leqslant M\int_0^T \|u(\tau)\|_{\mathcal{U}}\,\mathrm{d}\tau \cdot \sup_{\tau\in(0,T)}\int_\tau^T 1+(t-\tau)^{-7/8}\,\mathrm{d}t \\
&\quad= M(T+8T^{1/8})\cdot\int_0^T \|u(\tau)\|_{\mathcal{U}}\,\mathrm{d}\tau.
\end{aligned}$$

Hence, Tonelli's theorem implies that the mapping $t \mapsto \mathfrak{B}_t u$ is in $L^1\left([0,T];W^{1,2}(\Omega)\right)$. The norm estimate on $\mathfrak{A}$ implies that $\mathfrak{A}(\cdot)x_0 \in L^1\left([0,T];W^{1,2}(\Omega)\right)$ as well. The state x is the sum of these two integrable functions and therefore we have $x \in L^1\left([0,T];W^{1,2}(\Omega)\right)$. Thus, the following holds for all $t>0$ and all $\varphi \in \operatorname{dom} A$ by

[TW09, Remark 4.1.2]:

$$\begin{aligned}\langle x(t) - x(0)\,,\, \varphi\rangle_{L^2(\Omega)} &= \int_0^t \langle x(t)\,,\, A^*\varphi\rangle_{L^2(\Omega)} + \langle u\,,\, B^*\varphi\rangle_{\mathcal{U}} \,\mathrm{d}\tau \\ &= -\int_0^t \langle \nabla x(t)\,,\, \nabla\varphi\rangle_{L^2(\Omega)} + \langle u\,,\, B^*\varphi\rangle_{\mathcal{U}} \,\mathrm{d}\tau.\end{aligned}$$

Since the right hand side depends continuously on φ with respect to the $W^{1,2}(\Omega)$ norm, this equation extends to all $\varphi \in W^{1,2}(\Omega)$. Since x is in the zero dynamics, we have $Cx(t) = 0$ for almost all $t > 0$, which means that $x(t)$ is in the domain of the sesquilinear form a_0 defined in (4.4). Hence, for $\varphi \in \operatorname{dom} A_0 \subset W^{1,2}(\Omega)$ the equation above becomes

$$\begin{aligned}\langle x(t) - x(0), \varphi\rangle_{L^2(\Omega)} &= -\int_0^t \langle \nabla x(t)\,,\, \nabla\varphi\rangle_{L^2(\Omega)} \,\mathrm{d}\tau = -\int_0^t a_0(x(t), \varphi)\,\mathrm{d}\tau \\ &= \int_0^t \langle x(t)\,,\, A_0^*\varphi\rangle_{L^2(\Omega)} \,\mathrm{d}\tau.\end{aligned}$$

This implies $x(t) = \mathfrak{A}_0(t)x(0)$ via Lemma A.2.2 (i). As a consequence, we have $x(t) \in \operatorname{dom} A_0$ and the derivative of x with respect to the $L^2(\Omega)$-norm satisfies

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x(t) = A_0 x(t) \in \mathcal{X} \quad \forall\, t > 0.$$

By definition of $\operatorname{dom} A_0$ and $\mathcal{W}$ in (2.14) we see that $\operatorname{dom} A_0 \subset \mathcal{W}$, and Lemma 2.5.2 therefore implies

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x(t) = A_0 x(t) = \Delta x(t) = A\big|_{\mathcal{X}} x(t) + Bu(t) \in \mathcal{X} \quad \forall\, t > 0.$$

Remark 2.5.2 now implies that $u(t)$ must equal $\Gamma x(t)$.

Now we proof the inclusion "$\supset$" in (4.12). Let $x_0 \in L^2(\Omega)$ and define $x(t) := \mathfrak{A}_0(t)x_0$. Since the semigroup $\mathfrak{A}_0$ is analytic we have $x(t) \in \operatorname{dom} A_0$ for all $t > 0$. Hence, $x(t)$ is an element of the space $\mathcal{W}$ defined in (2.14), and the derivative of x with respect to the $L^2(\Omega)$-norm satisfies

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x(t) = A_0 x(t) = \Delta x(t) \overset{(2.12)}{=} A|_{\mathcal{X}} x(t) + B\Gamma x(t) \quad \forall\, t > 0.$$

Since $x \in \mathcal{C}(\mathbb{R}_{\geqslant 0}; L^2(\Omega))$ and $\Gamma x(\cdot)$ is by assumption in $L^2(\mathbb{R}_{\geqslant 0};\mathcal{U})$, this implies that

$(x, u, Cx(\cdot))$ is in the behavior of $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ with Lemma A.2.2 (i). Finally, the representation of $\operatorname{dom} A_0$ in (4.3) shows that $Cx(t) = 0$ for all $t > 0$, hence $(x, u, 0)$ is in the zero dynamics. □

Remark 4.2.4. (i) Theorem 4.2.3 can be seen as an analog to Corollary 4.1.2. There it was shown that the zero dynamics of certain state linear systems are determined by a strongly continuous semigroup on some proper subspace of the state space, the codimension of which was determined by the relative degree. In contrast to this, the subspace $\mathcal{Z}$ that characterizes the zero dynamics in Theorem 4.2.3 is dense in $\mathcal{X}$.

(ii) For all $x_0 \in \mathcal{X}$, the function $u := \Gamma\mathfrak{A}_0(\cdot)x_0$ is well-defined and an element of $L^1_{\text{loc}}(\mathbb{R}_{\geqslant 0}; \mathcal{U})$. This can be shown as follows: The estimate (A.1) for the semigroup $\mathfrak{A}_0$ implies that

$$\exists\, M \geqslant 1\ \forall\, t > 0: \ \|\mathfrak{A}_0(t)\|_{\operatorname{dom}(-A_0)^\theta} \leqslant M(1 + t^{-\theta})e^{-\omega_{\mathfrak{A}_0} t}.$$

For $\theta \in (0, 1)$ the expression on the left is therefore integrable over finite intervals. Since $\operatorname{dom} A_0 \subset W^{2,2}(\Omega)$, the interpolation result in [Tri95, Theorem 4.3.1] implies

$$\operatorname{dom}(-A_0)^\theta \overset{\text{(A.2)}}{=} [\mathcal{X}\,,\, \operatorname{dom}(-A_0)]_\theta \subset \left[\mathcal{X}\,,\, W^{2,2}(\Omega)\right]_\theta = W^{2\theta,2}(\Omega).$$

Furthermore, the mapping

$$\Gamma : W^{2\theta,2}(\Omega) \to W^{2\theta-\frac{3}{2},2}(\partial\Omega), \quad x \mapsto (\xi \mapsto \partial_\nu x(\xi))\,, \quad 2\theta \geqslant 3/2,$$

is bounded [HT08, Theorem 4.24 (ii)]. Hence, for any choice of $\theta \in (\frac{3}{4}, 1)$ the function $u(\cdot) := \Gamma\mathfrak{A}_0(\cdot)x_0$ is in $L^1_{\text{loc}}(\mathbb{R}_{\geqslant 0}; \mathcal{U})$.

(iii) As mentioned in Remark 2.4.10 the behavior, and thereby the zero dynamics, can be defined in a weaker sense, such that it allows for inputs in $L^1_{\text{loc}}(\mathbb{R}_{\geqslant 0}; \mathcal{U})$. Under these conditions an analogous result to Theorem 4.2.3 will be published in [RS15b]. In that setup, the space $\mathcal{Z}$ is replaced by

$$\left\{\ x_0 \in \mathcal{X}\ \middle|\ \Gamma\mathfrak{A}_0(\cdot)x_0 \in L^1_{\text{loc}}(\mathbb{R}_{\geqslant 0}; \mathcal{U})\ \right\},$$

which is equal to the whole state space $L^2(\Omega)$ by part (ii) of this remark.

4.3. Notes and references

As mentioned in Section 3.4, the largest feedback invariant subspace in the kernel of C has been extensively studied in many publications [Cur84, Cur86, MR07, Zwa88]. Furthermore, [MR10] shows the relation to the invariant zeros of the system. But a meaningful definition of the zero dynamics in the sense of a subspace of the behavior was missing so far: In [BLGS98] the zero dynamics were simply defined to be the dynamics induced by the closed-loop operator on the largest feedback invariant space. This kind of definition is also referred to in [MR07, MR10]. By exploiting the zero dynamics form we have established this missing link for systems with natural relative degree.

Similarly, the word zero dynamics was used in some articles treating parabolic partial differential equations: The zero dynamics of a one-dimensional parabolic partial differential equation with boundary control and observation is mentioned in [BGH94]. For a multidimensional parabolic partial differential equation the authors of [BGIS06, BG09] also exploit the zero dynamics. In all three papers the authors simply define the zero dynamics to be the semigroup generated by the main operator restricted to the kernel of the observation operator, either knowing beforehand that this operator generates a strongly continuous semigroup or assuming so. In contrast, we have given a reasonable definition of zero dynamics for general well-posed linear systems and established the connection to the zero dynamics semigroup for our example of the heat equation with boundary control and observation. Interestingly enough, Theorem 4.2.3 shows that the zero dynamics are described by the zero dynamics semigroup restricted to a *dense* subspace.

The question, what trajectories the zero dynamics should cover, is of course a philosophical one: Allowing for distributional inputs or initial values and weaker concepts of solutions gives more and more trajectories. The setup presented here is consistent with the concept of L^p-well-posed linear systems. A description of our example that allows for inputs in $L^1_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U})$ will be given in [RS15b].

5. Funnel control

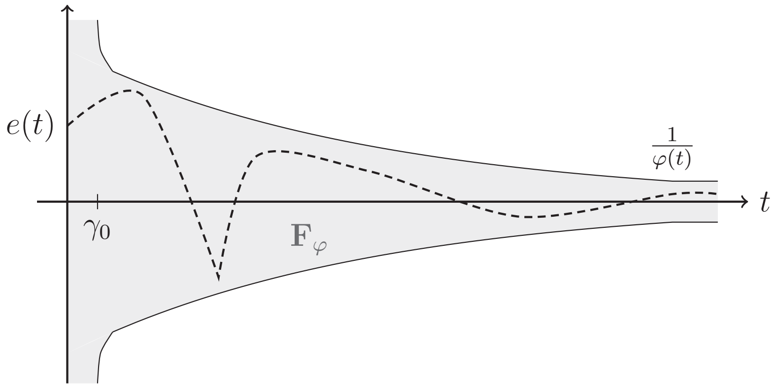

Figure 5.1.: Error evolution within the funnel $\mathbf{F}_\varphi$ with "width ∞" in $[0, \gamma_0]$. The boundary of the funnel is determined by the functions $\pm\frac{1}{\varphi}$.

In this section we are going to design a special time-varying, nonlinear output feedback in order to achieve two control objectives: The first one is approximate tracking, by the output y, of reference signals y_{ref} of class $W^{1,\infty}(\mathbb{R}_{\geqslant 0}; \mathbb{R})$. More precisely, for arbitrary $\lambda > 0$, the feedback strategy should ensure for every $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}_{\geqslant 0}; \mathbb{R})$ that the closed-loop system has a bounded solution and the tracking error $e(t) = y(t) - y_{\text{ref}}(t)$ satisfies $\|e(t)\| \leqslant \lambda$ for all t sufficiently large. The second control objective is prescribed transient behavior of the tracking error signal. We capture both objectives in the concept of a performance funnel

$$\mathbf{F}_\varphi := \left\{ \begin{bmatrix} t \\ e \end{bmatrix} \in \mathbb{R}_{\geqslant 0} \times \mathbb{R} \;\middle|\; |e|\varphi(t) < 1 \right\}, \tag{5.1}$$

which is determined by a function φ that belongs to the class

$$\Phi_{\gamma_0} := \left\{ \varphi \in W^{1,\infty}(\mathbb{R}_{\geqslant 0}) \;\middle|\; \begin{array}{l} \varphi|_{[0,\gamma_0]} \equiv 0\,, \text{ and} \\ \forall\, \delta > 0 :\ \inf\{\ \varphi(t)\ \mid t > \gamma_0 + \delta\ \} > 0 \end{array} \right\}$$

for some $\gamma_0 > 0$. In other words,

$$\varphi \in \Phi := \bigcup_{\gamma_0 > 0} \Phi_{\gamma_0}. \tag{5.2}$$

Note that the boundary of the funnel is determined by the reciprocal of φ as depicted in Figure 5. An output feedback strategy that forces the tracking error to evolve within the funnel $\mathbf{F}_\varphi$ will achieve both control objectives.

For example, if $\liminf_{t\to\infty} \varphi(t) > 1/\lambda$, then evolution within the funnel ensures that the first control objective is achieved. If φ is chosen as the function $t \mapsto \min\{t/T, 1\}/\lambda$, then evolution within the funnel ensures that the prescribed tracking accuracy $\lambda > 0$ is achieved within the prescribed time $T > 0$.

For $\varphi \in \Phi$, the *funnel controller* is

$$u(t) = \gamma\, k(t, y(t) - y_{\text{ref}}(t)) \cdot (y(t) - y_{\text{ref}}(t)), \qquad k(t, e) = \frac{\varphi(t)^2}{1 - \varphi(t)^2 e^2}, \tag{5.3}$$

where $\gamma \in \{-1, 1\}$ depends on the high gain amplification of the system.

Loosely speaking, funnel control exploits an inherent benign high-gain property of the system. The input can be interpreted as a proportional feedback $u(t) = -k(t)\, e(t)$ with the property that the gain $k(t)$ becomes large if $|e(t)|$ approaches the funnel boundary (equivalently, if $\varphi(t)|e(t)|$ approaches the value 1), thereby precluding contact with the funnel boundary. We emphasize that the gain is non-monotone and decreases as the error recedes from the funnel boundary.

The essence of the proof of the main result lies in showing that the closed-loop system possesses a solution and that the error does not hit the boundary of the performance funnel. We will prove this for two kinds of systems: First we consider systems with relative degree $r = 1$ and exponentially stable zero dynamics. Later we will consider systems with a special type of unbounded impulse response that is motivated by the example in Section 2.6. To this end we will use an approximation argument and have to sharpen results for the approximating finite-dimensional systems tightly before we can treat the infinite-dimensional class. Since this class is determined by its input-output map alone, we prove the existence of a global closed-loop solution, but cannot give a global bound on any norm of the state. However, such a global bound on the state space norm is established for the boundary control system introduced in Section 2.6 as an example.

5.1. Funnel control for systems with relative degree one

We consider systems that have relative degree $r = 1$ in the sense of Definition 3.1.2. In particular this implies that the *high gain amplification* CB is not zero. In this situation, the sign γ of the funnel controller in (5.3) is chosen as $\gamma = -\operatorname{sgn}(CB)$ to obtain the following theorem.

Theorem 5.1.1. *Consider a state linear system* $(A, B, C, 0)$ *on* $(\mathbb{R}, \mathcal{X}, \mathbb{R})$ *with relative degree* $r = 1$ *and exponentially stable zero dynamics. Let* $\varphi \in \Phi$ *specify the performance funnel* $\mathbf{F}_\varphi$ *and* $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geqslant 0}; \mathbb{R})$ *be the reference signal. There exists exactly one triple* $(x, u, y) \in \operatorname{bhv}(A, B, C, 0)$ *such that*

$$u(t) = -\frac{\operatorname{sgn}(CB)\varphi(t)^2}{1 - \varphi(t)^2 |y(t) - y_{\mathrm{ref}}(t)|^2}(y(t) - y_{\mathrm{ref}}(t)) \quad \forall\, t \geqslant 0.$$

Moreover,

$$\sup_{t \geqslant 0} \left(\|x(t)\|_{\mathcal{X}} + |u(t)| + |y(t)| \right) < \infty,$$

and

$$\exists\, \varepsilon \in (0,1) \quad \forall\, t > 0 : \quad |y(t) - y_{\mathrm{ref}}(t)|^2 \leqslant \varphi(t)^{-2} - \varepsilon. \tag{5.4}$$

Proof. We use the equivalence of (i) and (iii) in Proposition 3.3.8. In view of Proposition 3.3.8 (iii) we seek a function y that solves the equations

$$\begin{aligned} \dot{y}(t) &= p_0 y(t) + \int_0^t S\mathfrak{A}_Q(t-s) R y(s)\, \mathrm{d}s + S\mathfrak{A}_Q(t) \pi_{\mathcal{S}_{A,b}^{\perp}} x_0 + CBu(t), \\ u(t) &= \frac{-\operatorname{sgn} CB}{1 - \varphi(t)^2 |y(t) - y_{\mathrm{ref}}(t)|^2}(y(t) - y_{\mathrm{ref}}(t)), \\ y(0) &= Cx_0. \end{aligned} \tag{5.5}$$

Due to our assumption on the relative degree we have $CB \neq 0$. The exponential stability of the zero dynamics is by Lemma 4.1.3 equivalent to the exponential stability of the semigroup $\mathfrak{A}_Q$, which implies that the functions defined by $h(t) := S\mathfrak{A}_Q(t)R$ and $t \mapsto S\mathfrak{A}_Q(t)\pi_{\mathcal{S}_{A,b}^\perp} x_0$ are in $L^1 \cap L^\infty(\mathbb{R}_{\geqslant 0}; \mathbb{R})$. Introducing the auxiliary variable $e(t) := y(t) - y_{\mathrm{ref}}(t)$ as well as the abbreviation

$$f(t) := p_0 y_{\mathrm{ref}} + \int_0^t h(t-\tau) y_{\mathrm{ref}}(\tau) \,\mathrm{d}\tau - \dot{y}_{\mathrm{ref}}(t) + \mathfrak{A}_Q(t)\pi_{\mathcal{S}_{A,b}^\perp} x_0,$$

we can write (5.5) equivalently as

$$\begin{aligned} \dot{e}(t) &= p_0 e(t) + \int_0^t h(t-\tau) e(\tau) \,\mathrm{d}\tau - |CB| k(t, e(t)) e(t) + f(t), \quad \text{f.a.a. } t \geqslant 0, \\ e(0) &= Cx_0 - y_{\mathrm{ref}}(0). \end{aligned} \tag{5.6}$$

Note that this is a perturbed linear integrodifferential equation in the sense of [GLS90, Section 11.4]. The forcing function f is in $L^\infty(\mathbb{R}_{\geqslant 0})$ and the nonlinear perturbation, $k : \mathbf{F}_\varphi \to \mathbb{R}$, is continuous and locally Lipschitz in the second component. Therefore, a standard fixed point argument in the spirit of the Picard-Lindelöf theorem shows that there exists a unique local solution to this equation around every point $\left[\begin{smallmatrix} t_0 \\ e_0 \end{smallmatrix}\right] \in \mathbf{F}_\varphi$, cf. [GLS90, Theorem 11.4.1]. The fact that $\varphi(0) = 0$ guarantees that the point $\left[\begin{smallmatrix} 0 \\ Cx_0 - y_{\mathrm{ref}}(0) \end{smallmatrix}\right]$ is in $\mathbf{F}_\varphi$. Hence, there exists a solution to the initial value problem (5.6). We denote the maximal interval of existence for this solution by $[0, \omega)$ and the solution itself by $e : [0, \omega) \to \mathbb{R}$. Let γ_0 be such that $\varphi \in \Phi_{\gamma_0}$. The solution exists on $[0, \gamma_0]$ because the nonlinear term in (5.6) disappears on this interval. Therefore, we know $\omega > \gamma_0$, and we can choose an arbitrary $t_0 \in (\gamma_0, \omega)$. By definition of the class Φ_{γ_0} we have

$$0 < m := \inf_{t \in [t_0, \omega)} \frac{1}{\varphi(t)^2} \leqslant M := \sup_{t \in [t_0, \infty)} \frac{1}{\varphi(t)^2} < \infty.$$

Let L be the Lipschitz constant of the function $\varphi^{-2}|_{[t_0, \infty)}$, and define

$$\varepsilon := \min \left\{ \frac{m}{2}, \frac{\frac{1}{2}|CB| m}{Mp_0 + M \|h\|_{L^1(\mathbb{R}_{\geqslant 0})} + \sqrt{M} \|f\|_{L^\infty(\mathbb{R}_{\geqslant 0})} + L}, \frac{1}{\varphi(t_0)^2} - e(t_0)^2 \right\}.$$

We will prove that

$$|e(t)|^2 \leqslant \varphi(t)^{-2} - \varepsilon \quad \forall\, t \in [t_0, \omega). \tag{5.7}$$

To this end, we observe that

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t} e(t)^2 &= e(t)\dot{e}(t) \\
&= p_0 e(t)^2 + e(t) \int_0^t h(t-\tau) e(\tau) \,\mathrm{d}\tau + e(t) f(t) - |CB| k(t, e(t)) \cdot e(t)^2 \\
&\leqslant M p_0 + M \|h\|_{L^1(\mathbb{R}_{\geqslant 0})} + \sqrt{M} \|f\|_{L^\infty(\mathbb{R}_{\geqslant 0})} - |CB| k(t, e(t)) \cdot e(t)^2
\end{aligned}$$

for all $t \in [t_0, \omega)$. Assume that (5.7) is false and there exists a $t_1 \in [t_0, \omega)$ such that

$$e(t_1)^2 > \varphi(t_1)^{-2} - \varepsilon.$$

Then t_1 must be strictly greater than t_0 due to the definition of ε, and the continuity of φ and e implies that the maximum

$$t_\varepsilon := \max\left\{\, t \in [t_0, t_1) \;\middle|\; e(t)^2 = \varphi(t)^{-2} - \varepsilon \,\right\}$$

is attained. Furthermore, we have

$$e(t)^2 > \varphi(t)^{-2} - \varepsilon \quad \forall\, t \in (t_\varepsilon, t_1),$$

which implies

$$e(t)^2 > \frac{1}{\varphi(t)^2} - \varepsilon \geqslant m - \frac{m}{2} = \frac{m}{2} \quad \forall\, t \in (t_\varepsilon, t_1),$$

and

$$k(t, e(t)) = \frac{\varphi(t)^2}{1 - e(t)^2 \varphi(t)^2} \geqslant \frac{1}{\varepsilon} \quad \forall\, t \in (t_\varepsilon, t_1).$$

With our previous calculation and the definition of ε, we obtain

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t} e(t)^2 &\leqslant M p_0 + M \|h\|_{L^1(\mathbb{R}_{\geqslant 0})} + \sqrt{M} \|f\|_{L^\infty(\mathbb{R}_{\geqslant 0})} - |CB| k(t, e(t)) \cdot e(t)^2 \\
&\leqslant M p_0 + M \|h\|_{L^1(\mathbb{R}_{\geqslant 0})} + \sqrt{M} \|f\|_{L^\infty(\mathbb{R}_{\geqslant 0})} - |CB| \cdot \frac{m}{2\varepsilon} \\
&\leqslant -L.
\end{aligned}$$

Since L is the Lipschitz constant of $\varphi(\cdot)^{-2}$, this implies

$$e(t_1)^2 - e(t_\varepsilon)^2 \leqslant -2L(t_1 - t_\varepsilon) \leqslant -2|\varphi(t_1)^{-2} - \varphi(t_\varepsilon)^{-2}| \leqslant \varphi(t_1)^{-2} - \varphi(t_\varepsilon)^{-2},$$

and we have the contradiction

$$\varepsilon = \varphi(t_\varepsilon)^{-2} - e(t_\varepsilon)^2 \leqslant \varphi(t_1)^{-2} - e(t_1)^2 < \varepsilon.$$

Therefore, we have (5.7). Since furthermore $\varphi(t)^2 e(t)^2 < 1$ on the compact interval $[0, t_0]$, the inequality (5.4) holds after shrinking ε if necessary. This implies the boundedness of the functions e and $k(\cdot, e(\cdot))$.

We prove that the solution is global, i.e. that $\omega = \infty$. Seeking a contradiction, suppose that $\omega < \infty$. From (5.6) we see that $e \in W^{1,\infty}([0,\omega))$. Therefore, the limit $e(\omega) := \lim_{t \to \omega} e(t)$ exists, and because of the estimate (5.4) the point $\left[\begin{smallmatrix} \omega \\ e(\omega) \end{smallmatrix}\right]$ lies in the interior of $\mathbf{F}_\varphi$. But this means that the solution e can be extended further, in contradiction to the maximality of $[0, \omega)$. Hence, the maximal interval of existence of the solution e must be $[0, \infty)$.

Since e solves (5.6), the bounded functions $u(t) := k(t, e(t))e(t)$ and $y(t) = e(t) + y_{\mathrm{ref}}(t)$, $t \geqslant 0$ satisfy (5.5). Proposition 3.3.8 therefore implies that the function

$$x(t) := U^* \begin{bmatrix} y(t) \\ \mathfrak{A}_Q(t)\pi_{\mathcal{S}_{A,b}^\perp} x_0 + \int_0^t \mathfrak{A}_Q(t-s) R y(s)\,\mathrm{d}s \end{bmatrix} \quad \forall\, t \geqslant 0,$$

satisfies $(x, u, y) \in \mathrm{bhv}(A, B, C, 0)$. The exponential stability of $\mathfrak{A}_Q$ implies the boundedness of the function x and the proof is complete. □

5.2. Funnel control for self-adjoint systems

In this section we apply funnel control to systems that have an input-output map of the form

$$(\mathfrak{D}u)(t) = \int_{-\infty}^{t} \sum_{k=0}^{\infty} c_k \mathrm{e}^{-\lambda_k (t-\tau)} u(\tau)\,\mathrm{d}\tau, \quad u \in L^\infty_{\mathrm{c,loc}}(\mathbb{R}),$$

like the example in Section 2.6. However, the results are independent of that section and based solely on the following presumption, which is assumed to hold throughout

Section 5.2.

Presumption 5.2.1. *The real sequences $(c_k)_{k\in\mathbb{N}_0}$ and $(\lambda_k)_{k\in\mathbb{N}_0}$ have the following properties:*

(i) $c_0 > 0$ *and* $\lambda_0 = 0$*;*

(ii) $\lambda_k, c_k > 0$ *for all* $k \in \mathbb{N}$*;*

(iii) $(\lambda_k)_{k\in\mathbb{N}_0}$ *is nondecreasing;*

(iv) $\sum_{k=1}^{\infty} \frac{c_k}{\lambda_k} < \infty$.

First of all, we make sure that the operator $\mathfrak{D}$ above is a well-defined time-invariant causal operator, $\mathfrak{D} \in \mathrm{TIC}^{\infty}_{\mathrm{loc}}(\mathbb{C};\mathbb{C})$.

Lemma 5.2.2. *Let Presumption 5.2.1 hold. As $n \to \infty$, the functions*

$$h_n(t) := \sum_{k=0}^{n-1} c_k \mathrm{e}^{\lambda_k t}, \quad t \geqslant 0, \tag{5.8}$$

converge in $L^1_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0})$ to

$$h := \sum_{k=0}^{\infty} c_k \mathrm{e}^{-\lambda_k (\cdot)}, \tag{5.9}$$

and there holds

$$\|h\|_{L^1([0,t])} = c_0 t + \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k t}) \quad \forall\, t \geqslant 0.$$

The operators

$$\mathfrak{D}^{\{n\}} : L^{\infty}_{\mathrm{c,loc}}(\mathbb{R}) \to L^{\infty}_{\mathrm{c,loc}}(\mathbb{R}), \quad \mathfrak{D}^{\{n\}} u := \left(t \mapsto \int_{-\infty}^{t} h_n(t-\tau) u(\tau) \,\mathrm{d}\tau \right). \tag{5.10}$$

and

$$\mathfrak{D} : L^{\infty}_{\mathrm{c,loc}}(\mathbb{R}) \to L^{\infty}_{\mathrm{c,loc}}(\mathbb{R}), \quad \mathfrak{D} u := \left(t \mapsto \int_{-\infty}^{t} h(t-\tau) u(\tau) \,\mathrm{d}\tau \right), \tag{5.11}$$

are in $\mathrm{TIC}^{\infty}_{\mathrm{loc}}(\mathbb{C};\mathbb{C})$ and for all $t \geqslant 0$,

$$\left\| \mathfrak{D}|_{L^{\infty}([0,t])} - \mathfrak{D}^{\{n\}}|_{L^{\infty}([0,t])} \right\|_{\mathcal{B}(L^{\infty}([0,t]))} \to 0, \quad n \to \infty. \tag{5.12}$$

Proof. With the nonnegativity of c_k, a simple calculation gives

$$\int_0^t |c_k \mathrm{e}^{-\lambda_k \tau}| \,\mathrm{d}\tau = \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k t}) \quad \forall\, k \in \mathbb{N},$$

and

$$\int_0^t |c_0 \mathrm{e}^{-\lambda_0 \tau}| \,\mathrm{d}\tau = \int_0^t c_0 \,\mathrm{d}\tau = c_0 t.$$

Hence, condition (iv) in Presumption 5.2.1 implies that the series in (5.9) converges in $L^1([0,t])$, and we may interchange the order of integration and summation to obtain

$$\|h\|_{L^1([0,t])} = \int_0^t \sum_{k=0}^{\infty} c_k \mathrm{e}^{-\lambda_k \tau} = c_0 t + \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k t}).$$

Young's inequality [Bog07, Theorem 3.9.4] shows that $\mathfrak{D}^{\{n\}}$ and $\mathfrak{D}$ map $L^\infty_{\mathrm{c,loc}}(\mathbb{R}_{\geqslant 0})$ into itself. Owing to their convolution nature it is easily checked that $\mathfrak{D}^{\{n\}}$ and $\mathfrak{D}$ are time-invariant and causal, see [Sta05, Theorem A.3.7]. Young's inequality furthermore shows for all $u \in L^\infty([0,t])$ that

$$\begin{aligned}
\left\|\mathfrak{D}u - \mathfrak{D}^{\{n\}}u\right\|_{L^\infty([0,t])} &= \sup_{s\in[0,t]} \int_0^s (h - h_n)(s-\tau)u(\tau)\,\mathrm{d}\tau \\
&\leqslant \|h - h_n\|_{L^1([0,t])} \|u\|_{L^\infty([0,t])} \\
&\leqslant \sum_{k=n}^{\infty} \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k t}) \|u\|_{L^\infty([0,t])},
\end{aligned}$$

which implies (5.12) because $\frac{c_k}{\lambda_k}$ is summable.

□

It is well-known that the convolution with an integrable function results in a uniformly continuous function [Bog07, Corollary 3.9.6]. For the convolution kernel h_n, we sharpen this result by giving an estimate that is independent of $n \in \mathbb{N}$.

Lemma 5.2.3. *Let Presumption 5.2.1 hold and define* $\mathfrak{D}^{\{n\}}$ *and* $\mathfrak{D}$ *by* (5.10) *and* (5.11)*, respectively. Then, for all* $u \in L^\infty_{\mathrm{c,loc}}(\mathbb{R})$*, the function* $\mathfrak{D}^{\{n\}}u$ *is uniformly continuous. More precisely, for all* $t_1, t_2 \geqslant 0$ *and all* $n \in \mathbb{N}$

$$
\begin{aligned}
&|(\mathfrak{D}^{\{n\}}u)(t_1) - (\mathfrak{D}^{\{n\}}u)(t_2)| \\
&\qquad\qquad \leqslant \left(c_0|t_2 - t_1| + 2\sum_{k=1}^{\infty} \frac{c_k}{\lambda_k}\left(1 - \mathrm{e}^{-\lambda_k|t_2-t_1|}\right)\right) \cdot \sup_{s\leqslant\max\{t_1,t_2\}} |u(s)|.
\end{aligned}
$$

The same estimate holds with $\mathfrak{D}$ instead of $\mathfrak{D}^{\{n\}}$.

Proof. We assume without loss of generality that $t_1 \leqslant t_2$, and keeping in mind that the support of u is bounded from below, we calculate

$$
\begin{aligned}
&\left|(\mathfrak{D}^{\{n\}}u)(t_1) - (\mathfrak{D}^{\{n\}}u)(t_2)\right| \\
&\quad = \left| \int_{-\infty}^{t_1} h_n(t_1-\tau)u(\tau)\,\mathrm{d}\tau - \int_{-\infty}^{t_2} h_n(t_2-\tau)u(\tau)\,\mathrm{d}\tau \right| \\
&\quad \leqslant \left| \int_{-\infty}^{t_1} (h_n(t_1-\tau) - h_n(t_2-\tau))u(\tau)\,\mathrm{d}\tau \right| + \left| \int_{t_1}^{t_2} h_n(t_2-\tau)u(\tau)\,\mathrm{d}\tau \right| \\
&\quad \leqslant \left(\int_0^{\infty} |(h_n(\tau) - h_n(t_2-t_1+\tau))|\,\mathrm{d}\tau + \int_0^{t_2-t_1} |h_n(\tau)|\,\mathrm{d}\tau \right) \|u\|_{L^\infty((-\infty,t_2])} \\
&\quad = \left(\int_0^{\infty} \left| \sum_{k=0}^{n-1} c_k\mathrm{e}^{-\lambda_k\tau}(1-\mathrm{e}^{-\lambda_k(t_2-t_1)}) \right| \mathrm{d}\tau + \int_0^{t_2-t_1} \sum_{k=0}^{n-1} c_k\mathrm{e}^{-\lambda_k\tau}\,\mathrm{d}\tau \right) \|u\|_{L^\infty((-\infty,t_2])} \\
&\quad = \left(\int_0^{\infty} \sum_{k=1}^{n-1} c_k\mathrm{e}^{-\lambda_k\tau}(1-\mathrm{e}^{-\lambda_k(t_2-t_1)})\,\mathrm{d}\tau + \int_0^{t_2-t_1} \sum_{k=0}^{n-1} c_k\mathrm{e}^{-\lambda_k\tau}\,\mathrm{d}\tau \right) \|u\|_{L^\infty((-\infty,t_2])} \\
&\quad = \left(\sum_{k=1}^{n-1} \frac{c_k}{\lambda_k}(1-\mathrm{e}^{-\lambda_k(t_2-t_1)}) \right. \\
&\qquad\qquad \left. + c_0(t_2-t_1) + \sum_{k=1}^{n-1} \frac{c_k}{\lambda_k}(1-\mathrm{e}^{-\lambda_k(t_2-t_1)}) \right) \|u\|_{L^\infty((-\infty,t_2])} \\
&\quad = \left(2\sum_{k=1}^{\infty} \frac{c_k}{\lambda_k}(1-\mathrm{e}^{-\lambda_k(t_2-t_1)}) + c_0(t_2-t_1) \right) \|u\|_{L^\infty((-\infty,t_2])}.
\end{aligned}
$$

The estimate for $\mathfrak{D}$ can be shown by the same calculation with n replaced by ∞, or alternatively, with the convergence in (5.12). □

We are going to analyze a Volterra equation that is motivated by the following consideration: If $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is a well-posed realization of $\mathfrak{D}$ then, by (2.8), the output of this system with initial value x_0 and input u is $y = \mathfrak{D}u + \mathfrak{C}x_0$. In view of the funnel controller, it is therefore natural to consider a Volterra equation of the form

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,\mathrm{d}\tau + (\mathfrak{C}x_0)(t),$$
$$u(t) = -\frac{\varphi^2(t)}{1-\varphi^2(t)(y(t)-y_{\mathrm{ref}}(t))^2}(y(t)-y_{\mathrm{ref}}(t)).$$

It is convenient to formulate this equation in terms of the error $e := y - y_{\mathrm{ref}}$ and regard

$$f := \mathfrak{C}x_0 - y_{\mathrm{ref}}$$

as an inhomogeneity. This means, we seek a solution e to

$$e(t) = -\int_0^t h(t-\tau)\cdot k(\tau, e(\tau))\cdot e(\tau)\,\mathrm{d}\tau + f(t), \quad \forall\, t \geqslant 0$$

with

$$k(t,e) := \frac{\varphi(t)^2}{1-\varphi(t)^2\cdot e^2}. \tag{5.13}$$

In order to allow for problems where $\mathfrak{C}x_0$ is not bounded on $\mathbb{R}_{\geqslant 0}$, but $\mathfrak{C}x_0|_{[t_0,\infty)} \in W^{1,\infty}([t_0,\infty))$ for some t_0, the class of funnels in (5.2) has been chosen in such a way that φ, and thereby k, are zero on some small initial interval $[0,\gamma_0]$.

To solve the nonlinear Volterra equation globally, we will first treat the case where the kernel is given by the finite sum h_n, and then exploit the fact that $\mathfrak{D}^{\{n\}}$ approximates $\mathfrak{D}$ locally in the sup norm by Lemma 5.2.2.

5.2.1. Finite-dimensional systems

We treat the case where the convolution kernel is given by the finite sum $h_n(t) = \sum_{k=0}^{n-1} c_k \mathrm{e}^{\lambda_k t}$. The corresponding convolution operator,

$$\left(\mathfrak{D}^{\{n\}}u\right)(t) = \int_{-\infty}^t \sum_{k=0}^{n-1} c_k \mathrm{e}^{\lambda_k(t-\tau)}u(\tau)\,\mathrm{d}\tau,$$

has an n-dimensional realization that is of relative degree one and in Byrnes-Isidori form as the following lemma shows.

Lemma 5.2.4. *Under Presumption 5.2.1, define* h_n *and* $\mathfrak{D}^{\{n\}}$ *by* (5.8) *and* (5.10), *respectively. Then there exists some $R \in \mathbb{R}^{n-1}$ and a symmetric negative definite matrix $Q \in \mathbb{R}^{(n-1)\times(n-1)}$ such that, with the real numbers*

$$\Gamma^{\{n\}} := \sum_{k=0}^{n-1} c_k, \qquad p_0 := -\sum_{k=0}^{n-1} c_k \lambda_k,$$

the matrix $\mathcal{A} := \left[\begin{smallmatrix} p_0 & R^\top \\ R & Q \end{smallmatrix}\right]$ is negative semi-definite, and the n-dimensional system

$$(\mathcal{A}, \mathcal{B}, \mathcal{C}, 0) := \left(\begin{bmatrix} p_0 & R^\top \\ R & Q \end{bmatrix}, \begin{bmatrix} \Gamma^{\{n\}} \\ 0_{\mathbb{R}^{n-1}} \end{bmatrix}, \begin{bmatrix} 1 & 0_{\mathbb{R}^{n-1}}^\top \end{bmatrix}, 0 \right)$$

is a realization of $\mathfrak{D}^{\{n\}}$.

Proof. Define

$$A := \begin{bmatrix} -\lambda_0 & & \\ & \ddots & \\ & & -\lambda_{n-1} \end{bmatrix}, \qquad b := \begin{bmatrix} \sqrt{c_0} \\ \vdots \\ \sqrt{c_{n-1}} \end{bmatrix}. \tag{5.14}$$

Then the fact that $h_n(t) = b^\top e^{At} b$ for all $t \geqslant 0$, shows that $\big(A, b, b^\top, 0\big)$ is an n-dimensional realization of $\mathfrak{D}^{\{n\}}$. The relative degree is one, since $b^\top b \neq 0$. In order to transform this system into Byrnes-Isidori form, we use the following transformation. Choose $\widetilde{U} := [\widetilde{u}_1, \dots, \widetilde{u}_{n-1}]$ such that the matrix $[\frac{b}{\|b\|}, \widetilde{U}] \in \mathbb{R}^{n\times n}$ is unitary and define

$$T := \tfrac{1}{\|b\|} \begin{bmatrix} \frac{b}{\|b\|} & \widetilde{U} \end{bmatrix}.$$

The inverse of T is given by

$$T^{-1} = \|b\| \begin{bmatrix} \frac{b}{\|b\|} & \widetilde{U} \end{bmatrix}^\top.$$

A short calculation shows that the matrices

$$R := \widetilde{U}^\top A b \|b\|^{-1}, \qquad Q := \widetilde{U}^\top A \widetilde{U},$$

fulfill

$$\big(T^{-1}AT, T^{-1}b, b^\top T\big) = \left(\begin{bmatrix} p_0 & R^\top \\ R & Q \end{bmatrix}, \begin{bmatrix} \Gamma^{\{n\}} \\ 0_{\mathbb{R}^{n-1}} \end{bmatrix}, \begin{bmatrix} 1 & 0_{\mathbb{R}^{n-1}}^\top \end{bmatrix} \right).$$

It is clear from the definition of A that $Q = \widetilde{U}^\top A \widetilde{U} \leqslant 0$. Suppose that

$$v^\top \widetilde{U}^\top A \widetilde{U} v = 0 \quad \text{for some} \quad v \in \mathbb{R}^{n-1} \backslash \{0\}.$$

Then the fact that $\lambda_1, \ldots, \lambda_{n-1} > 0$ yields $\widetilde{U} v \in \operatorname{span}\{e_1\}$ and, by our choice of $\widetilde{U}$, we have $v^\top \widetilde{U}^\top b = 0$. Hence the first entry of b is zero, which contradicts the fact that $c_0 > 0$. Thus, Q must be negative definite. This completes the proof. $\square$

Remark 5.2.5. (i) The system node $\left[\begin{smallmatrix} A & b \\ b^\top & 0 \end{smallmatrix}\right]$ of the realization (5.14) is self-adjoint. This is the reason for the title of Section 5.2.

(ii) It can be shown that the realization considered in this lemma is *impedance passive* in the sense of [Sta02].

Since the realization in Lemma 5.2.4 is in Byrnes-Isidori form and Q is negative definite, the zero dynamics are exponentially stable by Lemma 4.1.3. The relative degree is obviously one, and therefore Theorem 5.1.1 can readily be applied. However, we crave more. In order to make use of the approximation in (5.12), we will show that the funnel control applied to this system results in a control function bounded by some constant that is *independent of* $n \in \mathbb{N}$. The crucial part that will lead us to this independence is the following lemma.

Lemma 5.2.6. *Let $A_{22} \in \mathbb{R}^{n-1 \times n-1}$ be symmetric and negative definite, and let $A_{12} \in \mathbb{R}^{1 \times n-1}$, $A_{11} \in \mathbb{R}$ be such that the matrix $A = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{smallmatrix}\right]$ is singular and negative semi-definite. Furthermore, define*

$$\begin{aligned} &\mathfrak{T} : L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}), \\ &\mathfrak{T}x := \left(t \mapsto A_{11} x(t) + \int_0^t A_{12} \mathrm{e}^{A_{22}(t-\tau)} A_{12}^\top x(\tau) \,\mathrm{d}\tau \right). \end{aligned}$$

Then the following claims hold:

(i) $A_{11} = A_{12} A_{22}^{-1} A_{12}^\top$.

(ii) For all $x \in L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0})$ and $t \geqslant 0$, there holds $\displaystyle\int_0^t x(\tau)(\mathfrak{T}x)(\tau) \,\mathrm{d}\tau \leqslant 0.$

(iii) For all $y \in W_0^{1,\infty}(\mathbb{R}_{\geqslant 0})$,

$$\|\dot{y} - \mathfrak{T}y\|_{L^\infty(\mathbb{R}_{\geqslant 0})} \leqslant \lim_{s\to 0} \frac{1}{s} \left(\begin{bmatrix} 1 & 0_{\mathbb{R}^{n-1}}^\top \end{bmatrix} \begin{bmatrix} s - A_{11} & -A_{12} \\ -A_{12}^\top & s - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0_{\mathbb{R}^{n-1}} \end{bmatrix} \right)^{-1} \|y\|_{W^{1,\infty}}.$$

Proof. (i) By using elementary row transformations and the singularity of A, we obtain

$$0 = \det \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{bmatrix} = \det(A_{22}) \cdot (A_{11} - A_{12}A_{22}^{-1}A_{12}^\top).$$

Then the result follows from $\det(A_{22}) \neq 0$, which holds true since A_{22} is negative definite.

(ii) By using the Cauchy-Schwarz and Young's inequality, we obtain for all $t \geqslant 0$

$$\begin{aligned} &\left| \int_0^t x(\sigma) \int_0^\sigma A_{12} \mathrm{e}^{A_{22}(\sigma-\tau)} A_{12}^\top x(\tau) \,\mathrm{d}\tau \,\mathrm{d}\sigma \right| \\ &\quad \leqslant \|x\|_{L^2([0,t])} \cdot \left\| \int_0^{(\cdot)} A_{12} \mathrm{e}^{A_{22}(\cdot-\tau)} A_{12}^\top x(\tau) \,\mathrm{d}\tau \right\|_{L^2([0,t])} \\ &\quad \leqslant \|x\|_{L^2([0,t])}^2 \cdot \|A_{12} \mathrm{e}^{A_{22}\cdot} A_{12}^\top\|_{L^1([0,t])} \\ &\quad \leqslant \|x\|_{L^2([0,t])}^2 \cdot (-A_{12}A_{22}^{-1}A_{12}^\top). \end{aligned}$$

This gives rise to the estimate

$$\begin{aligned} \int_0^t x(\tau)(\mathfrak{T}x)(\tau) \,\mathrm{d}\tau &= A_{11}\|x\|_{L^2([0,t])}^2 + \int_0^t x(\sigma) \int_0^\sigma A_{12} \mathrm{e}^{A_{22}(\sigma-\tau)} A_{12}^\top x(\tau) \,\mathrm{d}\tau \,\mathrm{d}\sigma \\ &\leqslant A_{11}\|x\|_{L^2([0,t])}^2 - A_{12}A_{22}^{-1}A_{12}^\top \|x\|_{L^2([0,t])}^2 \overset{\text{(i)}}{=} 0. \end{aligned}$$

(iii) Let $y \in W_0^{1,\infty}(\mathbb{R}_{\geqslant 0})$. Then integration by parts yields

$$\begin{aligned} &\int_0^t A_{12} \mathrm{e}^{A_{22}(t-\tau)} A_{12}^\top y(\tau) \,\mathrm{d}\tau \\ &\quad = A_{12} \mathrm{e}^{A_{22}t} \int_0^t \mathrm{e}^{-A_{22}\tau} A_{12}^\top y(\tau) \,\mathrm{d}\tau \\ &\quad = A_{12} \mathrm{e}^{A_{22}t} \left(-A_{22}^{-1} \mathrm{e}^{-A_{22}\tau} A_{12}^\top y(\tau) \Big|_{\tau=0}^{\tau=t} + A_{22}^{-1} \int_0^t \mathrm{e}^{-A_{22}\tau} A_{12}^\top \dot{y}(\tau) \,\mathrm{d}\tau \right) \end{aligned}$$

$$
\begin{aligned}
&= -A_{12}\mathrm{e}^{A_{22}t}A_{22}^{-1}\mathrm{e}^{-A_{22}t}A_{12}^\top y(t) + A_{12}\mathrm{e}^{A_{22}t}A_{22}^{-1}\int_0^t \mathrm{e}^{-A_{22}\tau}A_{12}^\top \dot{y}(\tau)\,\mathrm{d}\tau \\
&= -A_{12}A_{22}^{-1}A_{12}^\top y(t) + A_{12}\int_0^t A_{22}^{-1}\mathrm{e}^{A_{22}(t-\tau)}A_{12}^\top\,\dot{y}(\tau)\,\mathrm{d}\tau \\
&\overset{(i)}{=} -A_{11}y(t) + A_{12}\int_0^t A_{22}^{-1}\mathrm{e}^{A_{22}(t-\tau)}A_{12}^\top\,\dot{y}(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathfrak{T}(y)(t) &= A_{11}y(t) + \int_0^t A_{12}\mathrm{e}^{A_{22}(t-\tau)}A_{12}^\top y(\tau)\,\mathrm{d}\tau \\
&= \int_0^t A_{12}A_{22}^{-1}\mathrm{e}^{A_{22}(t-\tau)}A_{12}^\top\,\dot{y}(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Since A_{22} is symmetric and negative definite, the expression $A_{12}A_{22}^{-1}\mathrm{e}^{A_{22}t}A_{12}^\top$ is nonpositive for all $t \geqslant 0$ and

$$
\left\|A_{12}A_{22}^{-1}\mathrm{e}^{A_{22}(\cdot)}A_{12}^\top\right\|_{L^1(\mathbb{R}_{\geqslant 0})} = \int_0^\infty -A_{12}A_{22}^{-1}\mathrm{e}^{A_{22}\tau}A_{12}^\top\,\mathrm{d}\tau = A_{12}A_{22}^{-2}A_{12}^\top.
$$

Hence, we have

$$
\|\mathfrak{T}y\|_{L^\infty(\mathbb{R}_{\geqslant 0})} \leqslant A_{12}A_{22}^{-2}A_{12}^\top\,\|\dot{y}\|_{L^\infty(\mathbb{R}_{\geqslant 0})}
$$

and

$$
\|\dot{y}(\cdot) - \mathfrak{T}y\|_{L^\infty(\mathbb{R}_{\geqslant 0})} \leqslant (1 + A_{12}A_{22}^{-2}A_{12}^\top)\cdot\|y\|_{W^{1,\infty}(\mathbb{R}_{\geqslant 0})} \tag{5.15}
$$

for all $y \in W_0^{1,\infty}(\mathbb{R}_{\geqslant 0})$. The Schur complement [GvL83, p. 103] gives rise to the equation

$$
\begin{aligned}
&\left(\begin{bmatrix} 1 & 0_{\mathbb{R}^{n-1}}^\top \end{bmatrix}\begin{bmatrix} s - A_{11} & -A_{12} \\ -A_{12}^\top & s - A_{22}\end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0_{\mathbb{R}^{n-1}}\end{bmatrix}\right)^{-1} \\
&\quad = s - A_{11} - A_{12}(s - A_{22})^{-1}A_{12}^\top \\
&\quad \overset{(i)}{=} s - A_{12}(A_{22}^{-1} + (s - A_{22})^{-1})A_{12}^\top,
\end{aligned}
$$

and using de l'Hôpital's rule, we obtain

$$\begin{aligned}
&\lim_{s\to 0}\frac{1}{s}\left(\begin{bmatrix}1 & 0_{\mathbb{R}^{n-1}}^\top\end{bmatrix}\begin{bmatrix}s-A_{11} & -A_{12}\\ -A_{12}^\top & s-A_{22}\end{bmatrix}^{-1}\begin{bmatrix}1\\ 0_{\mathbb{R}^{n-1}}\end{bmatrix}\right)^{-1}\\
&\quad= 1 - A_{12}\lim_{s\to 0}\frac{1}{s}(A_{22}^{-1} + (s-A_{22})^{-1})A_{12}^\top\\
&\quad= 1 + A_{12}\lim_{s\to 0}(s-A_{22})^{-2}A_{12}^\top = 1 + A_{12}A_{22}^{-2}A_{12}^\top.
\end{aligned} \tag{5.16}$$

The combination of (5.15) with (5.16) gives the desired result.

□

Theorem 5.2.7. *Let Presumption 5.2.1 hold and define h_n and k by* (5.8) *and* (5.13)*, respectively. Let $t_0 > 0$ and $f \in W^{1,\infty}([t_0,\infty))$, and let $\varphi \in \Phi$ satisfy $\varphi(t_0) > 0$ and $|f(t_0)| < \frac{1}{\varphi(t_0)}$. Then, for all $n \in \mathbb{N}$, the Volterra equation*

$$e^{\{n\}}(t) = -\int_{t_0}^{t} h_n(t-\tau)\cdot k(\tau, e^{\{n\}}(\tau))\cdot e^{\{n\}}(\tau)\,\mathrm{d}\tau + f(t) \quad \forall\, t \geqslant t_0, \tag{5.17}$$

has a bounded, absolutely continuous solution $e^{\{n\}} : [t_0,\infty) \to \mathbb{R}$. There further exists a constant $\varepsilon > 0$ independent of n such that

$$|e^{\{n\}}(t)|^2 \leqslant \varphi(t)^{-2} - \varepsilon \quad \forall\, n \in \mathbb{N},\ t \geqslant t_0. \tag{5.18}$$

Proof. We define $\mathfrak{D}^{\{n\}}$ by (5.10) and the auxiliary functions

$$\begin{aligned}
f_0(t) &:= \begin{cases} \frac{t}{t_0} f(t_0), & t \in [0,t_0),\\ f(t), & t \geqslant t_0,\end{cases}\\
k_0(t,e) &:= \begin{cases} 0, & t \in [0,t_0),\\ \frac{\varphi(t)^2}{1-\varphi(t)^2 e^2}, & t \geqslant t_0.\end{cases}
\end{aligned}$$

Now we seek a solution to

$$e^{\{n\}}(t) = -\left(\mathfrak{D}^{\{n\}} k_0(\,\cdot\,, e^{\{n\}})\cdot e^{\{n\}}\right)(t) + f_0(t), \quad t \in [0,\infty). \tag{5.19}$$

For $t \in [0,t_0]$ the functions $e^{\{n\}}(t) = f_0(t)$ solves this equation because $k_0(t;\cdot) = 0$ on this interval. In view of the realization of $\mathfrak{D}^{\{n\}}$ given in Lemma 5.2.4, the solution of (5.19) can be extended beyond t_0 if and only if there is a solution z of the initial value problem

$$\begin{aligned}\dot{z}(t) &= \begin{bmatrix} p_0 & R^\top \\ R & Q \end{bmatrix} z(t) - \begin{bmatrix} \Gamma^{\{n\}} \\ 0_{\mathbb{R}^{n-1}} \end{bmatrix} k_0(t, e^{\{n\}}(t)) \cdot e^{\{n\}}(t), \\ e^{\{n\}}(t) &= \begin{bmatrix} 1 & 0_{\mathbb{R}^{n-1}}^\top \end{bmatrix} z(t) + f_0(t), \\ z(t_0) &= 0. \end{aligned} \tag{5.20}$$

The right hand side of this ordinary differential equation is defined on the open set

$$\mathbf{D} := \left\{ (t, z) \in [t_0, \infty) \times \mathbb{R}^n \;\middle|\; (t, z_1(t) + f(t)) \in \mathbf{F}_\varphi \right\} ,$$

with the performance funnel $\mathbf{F}_\varphi$ as in (5.1). It is readily verified that the right hand side of (5.20) satisfies a local Lipschitz condition with respect to $z(t)$ on the (relatively open) domain $\mathbf{D} \subset [t_0, \infty) \times \mathbb{R}^n$. Hence, by the standard theory of ordinary differential equations (see e.g. [Wal98, Theorem III.10.VI]), the initial-value problem (5.20) has a unique maximal solution

$$z^{\{n\}}(\cdot) \colon [t_0, \omega) \to \mathbb{R}^n, \quad t_0 < \omega \leqslant \infty,$$

and moreover,

$$\operatorname{graph}(z^{\{n\}}) := \{ (t, z^{\{n\}}(t)) |\; t \in [t_0, \omega) \} \subset \mathbf{D}$$

does not have compact closure in $\mathbf{D}$.

Now we show that the solution $e^{\{n\}}$ does not approach the boundary of $\mathbf{D}$. Define

$$y(t) := \begin{cases} 0, & t \in [0, t_0), \\ [1\,,\, 0_{\mathbb{R}^{n-1}}^\top]\, z^{\{n\}}(t), & t \in [t_0, \omega). \end{cases}$$

By Proposition 3.3.8 the function y satisfies, for almost all $t \in [t_0, \omega)$, the integrodifferential equation

$$\begin{aligned}\dot{y}(t) &= p_0 y(t) + R^\top \left(\int_0^t \mathrm{e}^{Q(t-\tau)} R y(\tau) \, \mathrm{d}\tau \right) - \Gamma^{\{n\}} k_0(t, e^{\{n\}}(t)) \cdot e^{\{n\}}(t), \\ &= \left(\mathfrak{T}^{\{n\}} y \right)(t) - \Gamma^{\{n\}} k_0(t, e^{\{n\}}(t)) \cdot e^{\{n\}}(t), \end{aligned} \tag{5.21}$$

where

$$\begin{aligned} \mathfrak{T}^{\{n\}} : L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}) &\to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}), \\ \left(\mathfrak{T}^{\{n\}} y\right)(t) &:= p_0 y(t) + R^\top \int_0^t \mathrm{e}^{Q(t-\tau)} R y(\tau)\,\mathrm{d}\tau. \end{aligned}$$

In order to prove that $\omega = \infty$, we will exploit two crucial properties of the operator $\mathfrak{T}^{\{n\}}$. Firstly, $\mathfrak{T}^{\{n\}}$ is negative semi-definite in the sense that

$$\forall\, t \geqslant 0,\ \forall\, e \in L^\infty([0,t]) : \quad \int_0^t e(\tau)(\mathfrak{T}^{\{n\}} e)(\tau)\,\mathrm{d}\tau \leqslant 0. \tag{5.22}$$

This follows from Lemma 5.2.6 (ii), because Q is a negative definite matrix. The second property is that

$$\left\| \dot{f}_0 - \mathfrak{T}^{\{n\}} f_0 \right\|_{L^\infty([0,\infty))} \leqslant \frac{\Gamma^{\{n\}}}{c_0} \cdot \|f_0\|_{W^{1,\infty}(\mathbb{R}_{\geqslant 0})}. \tag{5.23}$$

This holds because the transfer function of $\mathfrak{D}^{\{n\}}$ satisfies, for all $s \in \mathbb{C}_{>0}$,

$$\sum_{k=0}^n \frac{c_k}{s+\lambda_k} = \widehat{\mathfrak{D}^{\{n\}}}(s) = \begin{bmatrix} 1 \\ 0_{\mathbb{R}^{n-1}} \end{bmatrix}^\top \begin{bmatrix} s-p_0 & -R^\top \\ -R & s-Q \end{bmatrix}^{-1} \begin{bmatrix} \Gamma^{\{n\}} \\ 0_{\mathbb{R}^{n-1}} \end{bmatrix},$$

and by Lemma 5.2.6 (iii),

$$\begin{aligned} &\left\| \dot{f}_0 - \mathfrak{T}^{\{n\}} f_0 \right\|_{L^\infty(\mathbb{R}_{\geqslant 0})} \\ &\quad \leqslant \lim_{s\to 0} \frac{1}{s} \left(\begin{bmatrix} 1 \\ 0_{\mathbb{R}^{n-1}} \end{bmatrix}^\top \begin{bmatrix} s-p_0 & -R^\top \\ -R & s-Q \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0_{\mathbb{R}^{n-1}} \end{bmatrix} \right)^{-1} \|f_0\|_{W^{1,\infty}(\mathbb{R}_{\geqslant 0})} \\ &\quad = \lim_{s\to 0} \frac{1}{s} \cdot \left(\frac{\sum_{k=0}^n \frac{c_k}{s+\lambda_k}}{\Gamma^{\{n\}}} \right)^{-1} \cdot \|f_0\|_{W^{1,\infty}(\mathbb{R}_{\geqslant 0})} \\ &\quad = \frac{\Gamma^{\{n\}}}{c_0} \cdot \|f_0\|_{W^{1,\infty}(\mathbb{R}_{\geqslant 0})}. \end{aligned}$$

We use the representation (5.21) to show that the solution of (5.20) is global. Differentiating the second line of (5.20) shows for almost all $t \geqslant t_0$ that

$$\begin{aligned}\dot{e}^{\{n\}}(t) &= \dot{y}(t) + \dot{f}_0(t) \\ &= (\mathfrak{T}^{\{n\}}y)(t) - \Gamma^{\{n\}}k_0(t, e^{\{n\}}(t)) \cdot e^{\{n\}}(t) + \dot{f}_0(t) \\ &= (\mathfrak{T}^{\{n\}}e^{\{n\}})(t) + (\dot{f}_0(t) - (\mathfrak{T}^{\{n\}}f_0)(t)) - \Gamma^{\{n\}}k_0(t, e^{\{n\}}(t)) \cdot e^{\{n\}}(t).\end{aligned} \tag{5.24}$$

Now define

$$m := \inf_{t \in [t_0,\omega)} \varphi(t)^{-2}, \tag{5.25}$$

$$L := \text{Lipschitz constant of } \varphi|_{[t_0,\infty)}(\cdot)^{-2}, \tag{5.26}$$

$$M := \sup_{t \in [t_0,\omega)} \varphi(t)^{-1}, \tag{5.27}$$

$$\varepsilon := \min\left\{\frac{m}{2}, m\left(\frac{4M}{c_0}\|f_0\|_{W^{1,\infty}([0,\infty))} + \inf_{n\in\mathbb{N}} \frac{2L}{\Gamma^{\{n\}}}\right)^{-1}, \varphi(t_0)^{-2} - e^{\{n\}}(t_0)^2\right\}. \tag{5.28}$$

We show that (5.18) holds for all $t \in [t_0, \omega)$. Seeking a contradiction, we suppose that

$$\exists\, t_1 \in [t_0, \omega): \quad \varphi(t_1)^{-2} - (e^{\{n\}}(t_1))^2 < \varepsilon.$$

By continuity of φ and $e^{\{n\}}$, the maximum

$$t_\varepsilon := \max\left\{t \in [t_0, t_1) \,\middle|\, \varphi(t)^{-2} - (e^{\{n\}}(t))^2 = \varepsilon\right\}$$

is attained and

$$\forall\, t \in (t_\varepsilon, t_1): \quad \varphi(t)^{-2} - (e^{\{n\}}(t))^2 < \varepsilon.$$

Therefore, the definitions (5.25) and (5.28) imply

$$\forall\, t \in (t_\varepsilon, t_1): \quad (e^{\{n\}}(t))^2 > \varphi(t)^{-2} - \varepsilon \geqslant m - m/2 = m/2\,. \tag{5.29}$$

Moreover, for all $t \in (t_\varepsilon, t_1)$,

$$\frac{4M\|f_0\|_{W^{1,\infty}([0,\infty))}}{mc_0} + \frac{2L}{\Gamma^{\{n\}}m} \overset{(5.28)}{\leqslant} \frac{1}{\varepsilon} < \frac{1}{\varphi(t)^{-2} - (e^{\{n\}}(t))^2} \overset{(5.13)}{=} k(t, e^{\{n\}}(t)),$$

and thus

$$\forall\, t \in (t_\varepsilon, t_1): \quad \frac{2M}{c_0}\|f_0\|_{W^{1,\infty}([0,\infty))} - \frac{mk(t, e^{\{n\}}(t))}{2} \leqslant -\frac{L}{\Gamma^{\{n\}}}\,. \tag{5.30}$$

Finally, integrating $\frac{\mathrm{d}}{\mathrm{d}t}(e^{\{n\}}(t))^2$ and invoking (5.24), we get

$$\begin{aligned}
(e^{\{n\}}(t_1))^2 - (e^{\{n\}}(t_\varepsilon))^2 &= \int_{t_\varepsilon}^{t_1} \frac{\mathrm{d}}{\mathrm{d}\tau} e^{\{n\}}(\tau)^2 \,\mathrm{d}\tau = 2\int_{t_\varepsilon}^{t_1} e^{\{n\}}(\tau)\dot{e}^{\{n\}}(\tau)\,\mathrm{d}\tau \\
&= 2\int_{t_\varepsilon}^{t_1} e^{\{n\}}(\tau)(\mathfrak{T}^{\{n\}}e^{\{n\}})(\tau) + e^{\{n\}}(\tau)\left(\dot{f}_0(\tau) - (\mathfrak{T}^{\{n\}}e^{\{n\}})(\tau)\right) \\
&\qquad\qquad - \Gamma^{\{n\}}k(\tau, e^{\{n\}}(\tau))\,(e^{\{n\}}(\tau))^2\,\mathrm{d}\tau \\
&\overset{(5.22)}{\leqslant} 2\int_{t_\varepsilon}^{t_1} |e^{\{n\}}(\tau)|\left\|\dot{f}(\tau) - (\mathfrak{T}^{\{n\}}e^{\{n\}})(\tau)\right\|_\infty - \Gamma^{\{n\}}k(\tau, e^{\{n\}}(\tau))\,(e^{\{n\}}(\tau))^2\,\mathrm{d}\tau \\
&\overset{(5.27),(5.23)}{\leqslant} 2\int_{t_\varepsilon}^{t_1} M\frac{\Gamma^{\{n\}}}{c_0}\|f_0\|_{W^{1,\infty}[0,\infty)} - \Gamma^{\{n\}}k(\tau, e^{\{n\}}(\tau))\,(e^{\{n\}}(\tau))^2\,\mathrm{d}\tau \\
&\overset{(5.29)}{\leqslant} \int_{t_\varepsilon}^{t_1} \Gamma^{\{n\}}\left(\frac{2M}{c_0}\|f_0\|_{W^{1,\infty}[0,\infty)} - \frac{mk(\tau, e^{\{n\}}(\tau))}{2}\right)\mathrm{d}\tau \\
&\overset{(5.30)}{\leqslant} \int_{t_\varepsilon}^{t_1} -L\,\mathrm{d}\tau.
\end{aligned}$$

This implies

$$(e^{\{n\}}(t_1))^2 - (e^{\{n\}}(t_\varepsilon))^2 \leqslant -L(t_1 - t_\varepsilon) \overset{(5.26)}{\leqslant} -|\varphi(t_1)^{-2} - \varphi(t_\varepsilon)^{-2}|,$$

whence the contradiction

$$\varepsilon = \varphi(t_\varepsilon)^{-2} - (e^{\{n\}}(t_\varepsilon))^2 \leqslant \varphi(t_1)^{-2} - (e^{\{n\}}(t_1))^2 < \varepsilon.$$

This proves (5.18) since ε was chosen independently of n.

Finally, we show that $\omega = \infty$. Seeking a contradiction, suppose that $\omega < \infty$. Because of (5.18), the tuple $(t, e^{\{n\}}(t))$ is for all $t \in [t_0, \omega)$ in the set

$$\mathbf{K} := \left\{(t, e) \in \mathbf{F}_\varphi \,\middle|\, t \in [t_0, \omega],\ |e|^2 \leqslant \varphi(t)^{-2} - \varepsilon\right\} \subset \mathbf{F}_\varphi.$$

But the set $\mathbf{K}$ is compact, which contradicts the fact that the closure of the graph of $e^{\{n\}}|_{[t_0,\omega)}$ is not compact. Hence $\omega = \infty$. $\square$

We have shown that (5.17) possesses for each $n \in \mathbb{N}$ a solution $e^{\{n\}}$, and that these solutions are bounded away from the funnel boundary by a constant independent of n. We are now going use these findings to show that the set $\{e^{\{n\}} : n \in \mathbb{N}\}$ is equicontinuous.

Lemma 5.2.8. *The set of solutions* $\left\{ e^{\{n\}} \;\middle|\; n \in \mathbb{N} \right\}$ *to equation* (5.17) *that are given by Theorem 5.2.7, is uniformly equicontinuous. That is,*

$$\begin{gathered}\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, n \in \mathbb{N} \quad \forall\, t_1, t_2 \in [t_0, \infty) : \\ |t_1 - t_2| < \delta \Rightarrow |e^{\{n\}}(t_1) - e^{\{n\}}(t_2)| < \varepsilon.\end{gathered}$$

Proof. Define the input signal corresponding to $e^{\{n\}}$ by

$$u^{\{n\}}(t) := \begin{cases} -\frac{\varphi(t)^2}{1-(\varphi(t)e^{\{n\}}(t))^2} e^{\{n\}}(t), & t \in [t_0, \infty), \\ 0, & t \in [0, t_0), \end{cases} \tag{5.31}$$

so that (5.17) reads

$$e^{\{n\}}(t) = (\mathfrak{D}^{\{n\}} u^{\{n\}})(t) + f(t) \quad \forall\, t \in [t_0, \infty).$$

Then the uniform estimate (5.18) in Theorem 5.2.7 implies that there is a $C > 0$ with $\|u^{\{n\}}\|_{L^\infty([t_0,\infty))} < C$ for all $n \in \mathbb{N}$. By Presumption 5.2.1 (iv) there exists some $N \in \mathbb{N}$ with

$$\sum_{k=N+1}^{\infty} \frac{c_k}{\lambda_k} < \frac{\varepsilon}{8C}.$$

Since $f \in W^{1,\infty}(\mathbb{R}_{\geqslant 0})$ is uniformly continuous we may choose $\delta \in (0, \frac{\varepsilon}{4c_0 C})$ such that

$$|f(t_1) - f(t_2)| < \frac{\varepsilon}{4} \quad \text{for all } t_1, t_2 \geqslant 0 \text{ with } |t_1 - t_2| < \delta,$$

and

$$\sum_{k=1}^{N} \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k \delta}) < \frac{\varepsilon}{8C}.$$

For all $t_1, t_2 \in [t_0, \infty)$ with $|t_1 - t_2| < \delta$ we obtain with Lemma 5.2.3

$$
\begin{aligned}
&|e^{\{n\}}(t_1) - e^{\{n\}}(t_2)| \\
&\quad = |(\mathfrak{D}^{\{n\}} u^{\{n\}})(t_1) + f(t_1) - (\mathfrak{D}^{\{n\}} u^{\{n\}})(t_2) - f(t_2)| \\
&\quad \leqslant \underbrace{|f(t_1) - f(t_2)|}_{<\frac{\varepsilon}{4}} + \left|(\mathfrak{D}^{\{n\}} u^{\{n\}})(t_1) - (\mathfrak{D}^{\{n\}} u^{\{n\}})(t_2)\right| \\
&\quad \leqslant \frac{\varepsilon}{4} + \left(c_0 |t_1 - t_2| + 2 \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k} (1 - \mathrm{e}^{-\lambda_k \delta}) \right) \cdot \underbrace{\|u^{\{n\}}\|_{L^\infty(\mathbb{R}_{\geqslant 0})}}_{<C} \\
&\quad \leqslant \frac{\varepsilon}{4} + \Big(\underbrace{c_0 \delta}_{<\frac{\varepsilon}{4C}} + \underbrace{2 \sum_{k=1}^{N} \frac{c_k}{\lambda_k} (1 - \mathrm{e}^{-\lambda_k \delta})}_{\leqslant \frac{\varepsilon}{4C}} + \underbrace{2 \sum_{k=N+1}^{\infty} \frac{c_k}{\lambda_k}}_{\leqslant \frac{\varepsilon}{4C}} \Big) \cdot C < \varepsilon.
\end{aligned}
$$

□

5.2.2. Infinite-dimensional systems

Theorem 5.2.9. *Let Presumption 5.2.1 hold and define h by (5.9). Let $t_0 > 0$ and $f \in W^{1,\infty}([t_0, \infty))$, and let $\varphi \in \Phi$ satisfy $\varphi(t_0) > 0$ and $|f(t_0)| < \frac{1}{\varphi(t_0)}$. Then the equation*

$$
e(t) = -\int_{t_0}^{t} h(t - \tau) \cdot k(\tau, e(\tau)) \cdot e(\tau) \, \mathrm{d}\tau + f(t), \quad t \geqslant t_0, \tag{5.32}
$$

with

$$
k(t, e) = \frac{\varphi(t)^2}{1 - \varphi(t)^2 \cdot e^2}
$$

has a bounded, global solution $e \in \mathcal{BUC}([t_0, \infty))$, which is uniformly bounded away from the funnel boundary in the sense that

$$
\exists\, \varepsilon > 0 \quad \forall\, t \geqslant t_0: \quad |e(t)|^2 \leqslant \varphi(t)^{-2} - \varepsilon. \tag{5.33}
$$

Proof. Let $\left\{ e^{\{n\}} \;\middle|\; n \in \mathbb{N} \right\}$ be the set of solutions of (5.17) from Theorem 5.2.7. and let $t \geqslant t_0$ be arbitrary. Since the sequence $(e^{\{n\}}|_{[t_0,t]})_{n \in \mathbb{N}}$ is bounded by $1/\|\varphi\|_{L^\infty([t_0,\infty))}$ and, by Lemma 5.2.8, equicontinuous, we can conclude from the Arzelà-Ascoli theorem [Rud87, Theorem 11.28] that $(e^{\{n\}}|_{[t_0,t]})_{n \in \mathbb{N}}$ contains a subsequence $(e^{\{n_k\}}|_{[t_0,t]})_{k \in \mathbb{N}}$

that converges uniformly to some $e \in \mathcal{C}([t_0, t])$. The limit of (5.18) as $n \to \infty$ shows that (5.33) holds true. Hence, there is some $\delta > 0$ such that $\|\varphi^2 e^2\|_{L^\infty([t_0,t])} \leqslant 1 - \delta$, which is why the inputs u and $u^{\{n\}}$ defined by

$$u(\tau) := \begin{cases} -\frac{\varphi(\tau)^2}{1-(\varphi(\tau)e(\tau))^2} e(\tau), & \tau \in [t_0, \infty), \\ 0, & \tau \in [0, t_0), \end{cases}$$

and (5.31), respectively, are well-defined and satisfy

$$\begin{aligned} &\|u - u^{\{n_k\}}\|_{L^\infty([0,t])} \\ &= \left\| \frac{\varphi^2(e - e^{\{n_k\}}) + \varphi^4 e e^{\{n_k\}}(e - e^{\{n_k\}})}{(1 - \varphi^2 e^2)(1 - \varphi^2 (e^{\{n_k\}})^2)} \right\|_{L^\infty([t_0,t])} \\ &\leqslant \tfrac{1}{\delta^2} \left(\|\varphi\|^2_{L^\infty([t_0,t])} + \|\varphi\|^4_{L^\infty([t_0,t])} \|e\|_{L^\infty([t_0,t])} \|e^{\{n_k\}}\|_{L^\infty([t_0,t])} \right) \|e - e^{\{n_k\}}\|_{L^\infty([t_0,t])}. \end{aligned}$$

For $k \to \infty$ this implies $\lim_{k\to\infty} \|u - u^{\{n_k\}}|_{[0,t]}\|_{L^\infty([0,t])} = 0$. Furthermore, in the inequality

$$\begin{aligned} &\|e - (\mathfrak{D}u + f)\|_{L^\infty([t_0,t])} \\ &\quad = \|(e - e^{\{n_k\}}) - (\mathfrak{D}u + f) + (\mathfrak{D}^{\{n_k\}} u^{\{n_k\}} + f)\|_{L^\infty([t_0,t])} \\ &\quad \leqslant \|e - e^{\{n_k\}}\|_{L^\infty([t_0,t])} + \|\mathfrak{D}u - \mathfrak{D}^{\{n_k\}} u^{\{n_k\}}\|_{L^\infty([t_0,t])} \\ &\quad \leqslant \|e - e^{\{n_k\}}\|_{L^\infty([t_0,t])} + \|(\mathfrak{D} - \mathfrak{D}^{\{n_k\}})u + \mathfrak{D}^{\{n_k\}}(u - u^{\{n_k\}})\|_{L^\infty([t_0,t])} \\ &\quad \leqslant \|e - e^{\{n_k\}}\|_{L^\infty([t_0,t])} + \|\mathfrak{D} - \mathfrak{D}^{\{n_k\}}\|_{\mathcal{B}(L^\infty([0,t]))} \cdot \|u\|_{L^\infty([0,t])} \\ &\qquad + \|\mathfrak{D}^{\{n_k\}}\|_{\mathcal{B}(L^\infty([0,t]))} \|u - u^{\{n_k\}}\|_{L^\infty([0,t])}, \end{aligned}$$

the right hand side tends to zero because $\|\mathfrak{D} - \mathfrak{D}^{\{n_k\}}\|_{\mathcal{B}(L^\infty([0,t]))} \to 0$ as $k \to \infty$. This proves that the function e satisfies (5.32) on $[t_0, t]$. Since this construction was done with arbitrary $t \in [t_0, \infty)$, it enables us to construct a function $e : [t_0, \infty) \to \mathbb{R}$ that fulfills all the claims of the theorem. Finally, the uniform continuity of e is a consequence of the fact that e satisfies by (5.32) the convolution equation $e = \mathfrak{D}u + f$, and that $\mathfrak{D}u \in L^\infty([t_0, \infty))$ is bounded and uniformly continuous. □

Corollary 5.2.10. *Under Presumption 5.2.1, let $\gamma_0 > 0$, $\varphi \in \Phi_{\gamma_0}$ and a function*

$f \in W^{1,\infty}([\gamma_0,\infty))$ be given. Then the equation

$$e(t) = -\int_{\gamma_0}^{t} h(t-\tau)\cdot k(\tau, e(\tau))e(\tau)\,\mathrm{d}\tau + f(t), \qquad t \geqslant \gamma_0 \tag{5.34}$$

with k as in (5.13) *has a unique global solution $e \in \mathcal{BUC}([\gamma_0,\infty))$. This solution is uniformly bounded away from the funnel boundary in the sense that*

$$\exists\,\varepsilon > 0 \quad \forall\, t > \gamma_0 : \ |e(t)|^2 \leqslant \varphi(t)^{-2} - \varepsilon\,.$$

Proof. First of all it follows with standard fixed point arguments, see [GLS90, Chapter 12, Theorem 1.1], that for sufficiently small $t_0 > \gamma_0$ there exists a unique solution $e_0 \in \mathcal{BUC}([\gamma_0,t_0])$ of (5.34). Choosing t_0 small enough guarantees that the function $\widetilde{f} \in W^{1,\infty}([t_0,\infty))$ defined by

$$\widetilde{f}(t) = -\int_{\gamma_0}^{t_0} h(t-\tau)\cdot k(\tau, e_0(\tau))\cdot e_0(\tau)\,\mathrm{d}\tau + f(t) \quad \forall\, t \geqslant t_0,$$

satisfies the prerequisites of Theorem 5.2.9. This gives rise to the existence of a solution $\widetilde{e} \in \mathcal{BUC}([t_0,\infty))$ of the Volterra integral equation

$$\widetilde{e}(t) = -\int_{t_0}^{t} h(t-\tau)\cdot k(\tau, \widetilde{e}(\tau))\cdot \widetilde{e}(\tau)\,\mathrm{d}\tau + \widetilde{f}(t), \quad \forall\, t \geqslant t_0$$

Combined with e_0 on $[\gamma_0,t_0]$ this becomes a bounded and uniformly continuous solution of (5.34) on the entire interval $[\gamma_0,\infty)$.

In order to prove the uniqueness of the solution e we assume that, for some $t \in [\gamma_0,\infty)$, there are two functions $e_1, e_2 \in \mathcal{C}([\gamma_0,t])$ that solve (5.34). This means in particular that

$$\varphi(s)e_1(s) < 1 \quad \text{and} \quad \varphi(s)e_2(s) < 1 \quad \forall\, s \in [\gamma_0,t].$$

Define $t' := \inf\{\, \tau \in [\gamma_0,t] \mid e_1(\tau) \neq e_2(\tau) \,\}$. We show that $t' < t$ leads to a contradiction. Pick $\varepsilon > 0$ such that, for all τ in the compact interval $[\gamma_0,t]$, the following inequalities hold:

$$\varphi^2(\tau)e_1^2(\tau) \leqslant 1-\varepsilon^2, \quad \varphi^2(\tau)e_2^2(\tau) \leqslant 1-\varepsilon^2.$$

Further, choose δ such that

$$\int_0^\delta h(\tau)\ \mathrm{d}\tau < \frac{\varepsilon^4}{2\|\varphi\|^2_{L^\infty([0,t])}}.$$

Then defining for $i \in \{1,2\}$ the abbreviations

$$u_i := (t \mapsto k(t, e_i(t)) \cdot e_i(t)) = -\frac{\varphi^2}{1-\varphi^2 e_i^2} \cdot e_i,$$

we obtain for all $t \in [t', t'+\delta]$

$$\begin{aligned}
&|e_1(t) - e_2(t)| \\
&\quad \leqslant \int_{t'}^{t} |h(t-\tau)||u_1(\tau) - u_2(\tau)|\ \mathrm{d}\tau \\
&\quad \leqslant \int_0^{t-t'} |h(\tau)|\ \mathrm{d}\tau \cdot \sup_{\tau \in [t', t'+\delta]} |u_1(\tau) - u_2(\tau)| \\
&\quad \leqslant \frac{\varepsilon^4}{2\|\varphi\|^2_\infty} \cdot \left\| \frac{\varphi^2 + \varphi^4 e_1 e_2}{(1-\varphi^2 e_1^2)(1-\varphi^2 e_2^2)} (e_1 - e_2) \right\|_{L^\infty([t',t'+\delta])} \\
&\quad \leqslant \frac{\varepsilon^4}{2\|\varphi\|^2_\infty} \cdot \|\varphi\|^2_\infty \underbrace{\|1 - \varphi^2 e_1 e_2\|_{L^\infty([t',t'+\delta])}}_{<1+1} \cdot \underbrace{\left\| \frac{1}{1-\varphi^2 e_1^2} \right\|_{L^\infty([t',t'+\delta])}}_{<\frac{1}{\varepsilon^2}} \\
&\qquad \cdot \underbrace{\left\| \frac{1}{1-\varphi^2 e_2^2} \right\|_{L^\infty([t',t'+\delta])}}_{<\frac{1}{\varepsilon^2}} \cdot \|e_1 - e_2\|_{L^\infty([t',t'+\delta])} \\
&\quad < \|e_1 - e_2\|_{L^\infty([t',t'+\delta])}.
\end{aligned}$$

Now taking the supremum of all $t \in [t', t'+\delta]$ leads to the contradiction

$$\|e_1 - e_2\|_{L^\infty([t',t'+\delta])} < \|e_1 - e_2\|_{L^\infty([t',t'+\delta])}.$$

Thus, the corollary is true. $\square$

5.2.3. Funnel control for the heat equation with boundary control

Since the input-output map of the heat equation in Section 2.6 is of the type described in Lemma 5.2.2, we can apply the results from the previous section to this example. This readily provides a global solution to the closed-loop system. In addition, we are going to show that the corresponding state function is bounded.

Existence of a closed-loop solution

Theorem 5.2.11. *Let* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ *be the* L^2*-well-posed system corresponding to the heat equation as in Lemma 2.6.12. Let* $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geqslant 0})$, $x_0 \in L^2(\Omega; \mathbb{R})$ *be given. Pick any* $\varphi \in \Phi$ *and define the funnel feedback gain function* k *by* (5.13). *Then there exists a unique triple* $(x, u, y) \in \mathrm{bhv}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ *that satisfies* $x(0) = x_0$ *and*

$$u(t) = -k(t, y(t) - y_{\mathrm{ref}}(t))(y(t) - y_{\mathrm{ref}}(t)) \quad \forall\, t > 0. \tag{5.35}$$

Moreover,

(i) the input fulfills $u \in \mathcal{BUC}(\mathbb{R}_{>0})$*;*

(ii) the output function satisfies $y \in \mathcal{C}(\mathbb{R}_{>0})$ *and* $y|_{[\delta,\infty)} \in \mathcal{BUC}([\delta, \infty))$ *for all* $\delta > 0$*;*

(iii) the tracking error $e := y - y_{\mathrm{ref}}$ *evolves within the funnel* $\mathcal{F}_\varphi$ *with uniform distance to the funnel boundary in the sense that there is an* $\widetilde{\varepsilon} > 0$ *such that*

$$e(t)^2 \varphi(t)^2 \leqslant 1 - \widetilde{\varepsilon} \quad \forall\, t > 0.$$

Remark 5.2.12. For general $x_0 \in L^2(\Omega; \mathbb{R})$ the output signal y cannot be defined at the point zero. That is why the function y cannot be bounded on $\mathbb{R}_{\geqslant 0}$ in general. However, if x_0 is in $W^{1,2}(\Omega; \mathbb{R})$, then the upcoming Theorem 5.2.17 and the fact that $C \in \mathcal{B}(W^{1,2}(\Omega); \mathbb{C})$ imply that y is bounded on $\mathbb{R}_{\geqslant 0}$.

Proof. By Theorem 2.6.8 and Corollary 2.6.11, the input-output map $\mathfrak{D}$ has a representation

$$\mathfrak{D}u = \left(t \mapsto \int_0^t \sum_{k=0}^{\infty} c_k \mathrm{e}^{-\lambda_k (t-\tau)} u(\tau)\, \mathrm{d}\tau \right) \qquad \forall\, u \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}),$$

with c_k and λ_k fulfilling Presumption 5.2.1. Let $\gamma_0 > 0$ be such that $\varphi \in \Phi_{\gamma_0}$. By Lemma 2.6.12 (ii) we have $\mathfrak{C}x_0|_{[\gamma_0,\infty)} \in W^{1,\infty}([\gamma_0,\infty))$, which together with $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}_{\geqslant 0})$ implies that the function $f := \mathfrak{C}x_0 - y_{\text{ref}}$ fulfills $f|_{[\gamma_0,\infty)} \in W^{1,\infty}([\gamma_0,\infty))$. Thus, by Corollary 5.2.10, there exists a solution $e \in \mathcal{BUC}([\gamma_0,\infty))$ of the Volterra equation (5.34) with f as above. The corollary also states that

$$\exists\, \varepsilon > 0 \quad \forall\, t > \gamma_0 : \quad |e(t)|^2 \leqslant \varphi(t)^{-2} - \varepsilon.$$

Define the function

$$u(t) := \begin{cases} 0, & t \in [0,\gamma_0), \\ -k(t,e(t)) \cdot e(t), & t \geqslant \gamma_0. \end{cases}$$

The estimate above and the definition of k imply that the function $t \mapsto k(t,e(t))$ is bounded. Hence, u is bounded and a short calculation using the boundedness of k and the uniform continuity of e on $[\gamma_0,\infty)$ shows that u is uniformly continuous on $\mathbb{R}_{>0}$. So u satisfies (i).

With this u we define the functions x and y to be the state and output corresponding to the initial value x_0 by (2.8). Then we have

$$y(t) = (\mathfrak{C}x_0)(t) + (\mathfrak{D}u)(t) = (\mathfrak{C}x_0)(t), \quad \forall\, t \in (0,\gamma_0),$$

and

$$\begin{aligned} y(t) &= (\mathfrak{C}x_0)(t) + (\mathfrak{D}u)(t) = y_{\text{ref}}(t) + f(t) + (\mathfrak{D}u)(t) \\ &\overset{(5.34)}{=} y_{\text{ref}}(t) + e(t), \quad \text{f.a.a. } t \geqslant \gamma_0. \end{aligned} \tag{5.36}$$

This equation shows that y is continuous, and the restriction of y to $[\gamma_0,\infty)$ is in $\mathcal{BUC}([\gamma_0,\infty))$ since e and y_{ref} are. This implies (ii) because the uniform continuity on any compact interval $[\delta,\gamma_0]$ is trivial.

We claim that (5.35) holds for all $t > 0$. On $(0,\gamma_0)$, the function $\varphi \in \Phi_{\gamma_0}$ is zero by definition. Hence, $k(t, y(t) - y_{\text{ref}}(t)) = 0$ for $t \in [0,\gamma_0)$ and (5.35) is fulfilled by the definition of u. For $t \in [\gamma_0,\infty)$, inserting (5.36) into the definition of u shows (5.35).

It remains to prove (iii). Extending e to $\mathbb{R}_{>0}$ by $e := y - y_{\text{ref}}$, we get

$$\varphi(t)^2 e(t)^2 \leqslant 1 - \varphi(t)^2 \cdot \varepsilon \quad \forall\, t > 0$$

because $\varphi|_{(0,\gamma_0)} = 0$. Due to the continuity of e at γ_0 and the properties of φ, this implies that assertion (iii) holds for a suitable $\widetilde{\varepsilon} > 0$.

Finally, the uniqueness of the triple (x, u, y) follows from the uniqueness of the solution in Corollary 5.2.10 and the proof is complete. □

Boundedness and regularity of the closed-loop solution

Note that Theorem 5.2.11 does not yet say anything about the norm of the function x. In this section we will show that x is bounded in the norm of the state space $L^2(\Omega)$. To do this, we will exploit the fact that any constant output feedback stabilizes the system exponentially. Well-posedness of regular infinite-dimensional systems under output feedback is well understood, see [Wei94a]. The following lemma summarizes [Wei94a, Proposition 3.6, Theorem 6.1 & Theorem 7.2].

Lemma 5.2.13. *Let $\mathcal{X}$ be a Hilbert space and let $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ be a strongly regular L^2-well-posed linear system on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ with transfer function G. Let $K \in \mathcal{B}(\mathcal{Y};\mathcal{U})$ and $\omega \in \mathbb{R}$ be such that $\mathrm{I} + K\mathbf{G}(s)$ is invertible in $\mathcal{B}(\mathcal{U})$ for all $s \in \mathbb{C}_{\geqslant\omega}$ and*

$$\sup_{s\in\mathbb{C}_{\geqslant\omega}} \left\|(1 + K\mathbf{G}(s))^{-1}\right\|_{\mathcal{B}(\mathcal{U})} < \infty.$$

Let x and y be the state and output function corresponding to the initial value $x_0 \in \mathcal{X}$ and input $u \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U})$. With the function $v := u + Ky \in L^2_{\mathrm{loc}}(\mathcal{U})$, the state x satisfies

$$x(t) = \mathfrak{A}_K(t)x_0 + \mathfrak{B}_{K,t}v \quad \forall\, t \geqslant 0.$$

Here, $\mathfrak{A}_K$ is a strongly continuous semigroup on $\mathcal{X}$ generated by

$$A_K x = (A - KBC)x, \quad \operatorname{dom} A_K = \{\, x \in \operatorname{dom} C_{\mathrm{ex}} \mid (A - KBC_{\mathrm{ex}})x \in \mathcal{X} \,\}, \tag{5.37}$$

and

$$\mathfrak{B}_{K,t}v := \int_0^t \mathfrak{A}_K(t-\tau)|_{(\operatorname{dom} A_K^*)'} Bv(\tau)\,\mathrm{d}\tau,$$

where $\mathfrak{A}_K(t)|_{(\operatorname{dom} A_K^*)'}$ is the extension of $\mathfrak{A}_K(t)$ to $(\operatorname{dom} A_K^*)'$, and the integral is computed in $(\operatorname{dom} A_K^*)'$. In particular, the range of B is contained in this space.

Theorem 5.2.14. *Let $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ be the regular L^2-well-posed linear system on $(\mathbb{C}; L^2(\Omega); \mathbb{C})$ constructed from the heat equation in Lemma 2.6.12 and denote its* *transfer function by* $\mathbf{G}$*. For all* $K > 0$ *and all* $s \in \mathbb{C}_{>0}$*, the expression* $1 + K\mathbf{G}(s)$ *is nonzero and*

$$\sup_{s \in \mathbb{C}_{>0}} \left| \frac{1}{1 + K\mathbf{G}(s)} \right| < \infty.$$

The operator A_K *that is associated to* K *via Lemma 5.2.13 has the following properties:*

(i) $A_K x = \Delta x$*, and*

$$\operatorname{dom} A_K = \left\{ x \in W^{2,2}(\Omega) \;\middle|\; \partial_\nu x(\xi) = -K \int_{\partial\Omega} x(\zeta) \,\mathrm{d}\sigma_\zeta \quad \forall\, \xi \in \partial\Omega \right\}. \tag{5.38}$$

(ii) The operator A_K *is self-adjoint,* $\sigma(A_K) \subset (-\infty, 0)$*, and* A_K *has a compact resolvent. For all* $s \in \rho(A)$ *with* $K\mathbf{G}(s) \neq -1$ *we have* $s \in \rho(A_K)$ *and*

$$(s - A_K)^{-1} = (s - A)^{-1} - (s - A)^{-1} B \left(\frac{1}{K} + \mathbf{G}(s) \right)^{-1} C (s - A)^{-1}. \tag{5.39}$$

(iii) A_K *generates an exponentially stable analytic semigroup* $\mathfrak{A}_K$ *in* $L^2(\Omega)$*.*

Proof. By Corollary 2.6.10 the transfer function $\mathbf{G}$ is uniformly regular and has the feedthrough $D = 0$. Moreover, we have for all $k > 0$ and all $s \in \mathbb{C}_{>0}$,

$$\begin{aligned} |1 + K\mathbf{G}(s)| \geqslant \operatorname{Re}(1 + K\mathbf{G}(s)) &= 1 + K \sum_{k \in J_c} \operatorname{Re} \frac{c_k}{s + \lambda_k} \\ &\geqslant 1 + K \sum_{k \in J_c} c_k \frac{\operatorname{Re}(s + \lambda_k)}{|s + \lambda_k|^2} \geqslant 1. \end{aligned}$$

This shows that $1 + K\mathbf{G}(s)$ is boundedly invertible in the complex right half plane and therefore Lemma 5.2.13 applies. Now we prove the properties (i)–(iii):

(i) We show that the set defined in (5.38) is a subset of the domain given in (5.37). Let x be in the former set. Then x is in $\operatorname{dom} C_{\mathrm{ex}}$ because the trace operator is well-defined on $W^{2,2}(\Omega)$. Moreover, we have the following equation for all $\varphi \in \operatorname{dom} A^* = \operatorname{dom} A$:

$$\begin{aligned} &\langle Ax \,,\, \varphi \rangle_{L^2(\Omega)} - \langle BKCx \,,\, \varphi \rangle_{L^2(\Omega)} \\ &\quad = \int_\Omega x(\xi) \cdot \overline{\Delta\varphi(\xi)} \,\mathrm{d}\xi - K \int_{\partial\Omega} x(\zeta) \,\mathrm{d}\sigma_\zeta \int_{\partial\Omega} \overline{\varphi(\xi)} \,\mathrm{d}\sigma_\xi \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \Delta x(\xi) \cdot \overline{\varphi(\xi)} \, \mathrm{d}\xi - \int_{\partial\Omega} (\partial_\nu x(\xi)) \cdot \overline{\varphi(\xi)} \, \mathrm{d}\sigma_\xi \\
&\quad + \int_{\partial\Omega} x(\xi) \cdot \partial_\nu \overline{\varphi(\xi)} \, \mathrm{d}\sigma_\xi - \int_{\partial\Omega} K \int_{\partial\Omega} x(\zeta) \, \mathrm{d}\sigma_\zeta \, \overline{\varphi(\xi)} \, \mathrm{d}\sigma_\xi \\
&= \int_\Omega \Delta x(\xi) \cdot \overline{\varphi(\xi)} \, \mathrm{d}\xi \\
&\quad - \int_{\partial\Omega} \left(\partial_\nu x(\xi) \, \mathrm{d}\xi + K \int_{\partial\Omega} x(\zeta) \, \mathrm{d}\sigma_\zeta \right) \overline{\varphi(\xi)} \, \mathrm{d}\sigma_\xi \\
&= \int_\Omega \Delta x(\xi) \cdot \overline{\varphi(\xi)} \, \mathrm{d}\xi.
\end{aligned}
$$

This shows that $Ax - BKCx \in \mathcal{X}$ because it can be represented by the function $\Delta x \in L^2(\Omega)$. For the converse inclusion, take any $x \in \operatorname{dom} C_{\mathrm{ex}}$ with $Ax - BKCx \in \mathcal{X}$. Then x is by definition an element of the space $(\mathcal{X} + B\mathbb{C})_1$ defined in Lemma 2.2.4. By Lemma 2.5.2 this space equals the space $\mathcal{W}$ defined in (2.14). Therefore, $x \in W^{2,2}(\Omega)$, and Remark 2.5.3 (i) implies $\partial_\nu x \equiv -KCx = -K \int_{\partial\Omega} x(\xi) \, \mathrm{d}\xi$.

(ii) Let $s \in \rho(A)$ and $K\mathbf{G}(s) \neq -1$. Then, combining the Equations (6.14) and (7.3) of [Wei94a], we get that $s \in \rho(A_K)$ and that (5.39) holds. Since the resolvent of A is compact, this formula shows that the resolvent of A_K is compact as well. Therefore, the spectrum of A_K is a countable set of isolated eigenvalues [Kat80, Theorem 6.29, p.187]. With Gauß's theorem we get for all $x, z \in \operatorname{dom} A_K$

$$
\begin{aligned}
\langle A_K x \, , \, z \rangle_{L^2(\Omega)} &= \int_\Omega \Delta x(\xi) \cdot \overline{z(\xi)} \, \mathrm{d}\xi = - \int_\Omega \nabla x(\xi) \cdot \nabla \overline{z(\xi)} \, \mathrm{d}\xi + \int_{\partial\Omega} \partial_\nu x(\xi) \cdot \overline{z(\xi)} \, \mathrm{d}\sigma_\xi \\
&= - \int_\Omega \nabla x(\xi) \cdot \overline{\nabla z(\xi)} \, \mathrm{d}\xi - K \int_{\partial\Omega} x(\xi) \, \mathrm{d}\sigma_\xi \int_{\partial\Omega} \overline{z(\xi)} \, \mathrm{d}\sigma_\xi .
\end{aligned}
$$

By further reversing the roles of x and z in the above formula, we can conclude that

$$
\langle A_K x \, , \, z \rangle_{L^2(\Omega)} = \langle x \, , \, A_K z \rangle_{L^2(\Omega)} \quad \forall \, x, z \in \operatorname{dom} A_K .
$$

Since the spectrum of A_K consists of isolated eigenvalues, we have $\mathbb{R} \cap \rho(A_K) \neq \emptyset$. In other words, there exists some $\lambda \in \mathbb{R}$ such that $\lambda - A$ is onto. Thus, we conclude from [TW09, Proposition 3.2.4] that A_K is self-adjoint. Furthermore, A_K is nonpositive since for all $x \in \operatorname{dom} A_K$,

$$\langle A_K x \,,\, x\rangle_{L^2(\Omega)} = -\int_\Omega \nabla x(\xi) \cdot \overline{\nabla x(\xi)} \,\mathrm{d}\xi - K \left(\int_{\partial\Omega} x(\xi) \,\mathrm{d}\sigma_\xi \right)^2 \leqslant 0. \tag{5.40}$$

We show that zero is not an eigenvalue of A_K. Assume that $A_K x = 0$ for some function $x \in \operatorname{dom} A_K$, $x \not\equiv 0$. Then (5.40) implies $\nabla x = 0$ everywhere and $\int_{\partial\Omega} x(\xi) \,\mathrm{d}\sigma_\xi = 0$. Hence, x must be the constant zero function, which leads to a contradiction. Consequently zero is not an eigenvalue of A_K.

(iii) With the spectrum containing only isolated eigenvalues, statement (ii) implies $\sup_{\lambda \in \sigma(A_K)} \operatorname{Re}(\lambda) < 0$ and the claim follows with [TW09, Proposition 3.8.5]. □

In order to prove the boundedness and regularity results, we need to determine the domain of the operator root of the closed-loop generator A_K. To this end, we determine the symmetric sesquilinear form associated to A_K in the sense of [Kat80] because its domain is exactly the domain of $(-A_K)^{\frac{1}{2}}$, see Theorem A.1.5 or [Kat80, Section VI.2].

Lemma 5.2.15. *Let $K > 0$ and define A_K by* (5.38) *Then the bilinear form associated to A_K in the sense of Theorem A.1.5 has the domain* $\operatorname{dom} a_K = \operatorname{dom}(-A_K)^{\frac{1}{2}} = W^{1,2}(\Omega)$ *and is given by*

$$a_K(x, \psi) = \int_\Omega \nabla x(\xi) \overline{\nabla \psi(\xi)} \,\mathrm{d}\xi + K \int_{\partial\Omega} x(\xi) \,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega} \overline{\psi(\xi)} \,\mathrm{d}\sigma_\xi.$$

Proof. It is easy to see that a_K is a continuous, symmetric, nonnegative sesquilinear form on $W^{1,2}(\Omega)$. Hence, a_K fulfills the prerequisites of Theorem A.1.5. By this theorem it suffices to show that the domain $\operatorname{dom} A_K$ defined in (5.38) satisfies

$$\begin{aligned} &\operatorname{dom} A_K = \\ &\quad \left\{ x \in W^{1,2}(\Omega) \;\middle|\; \exists z \in L^2(\Omega) : a(x, \psi) = \langle z, \psi \rangle_{L^2(\Omega)} \ \forall\, \psi \in W^{1,2}(\Omega) \right\}. \end{aligned} \tag{5.41}$$

We show "$\subset$": Let $x \in \operatorname{dom} A_K$. Then $\Delta x \in L^2(\Omega)$, and the inclusion follows since for all $\psi \in W^{1,2}(\Omega)$ the following holds:

$$\begin{aligned} a_K(x, \psi) &= \int_\Omega \nabla x(\xi) \overline{\nabla \psi(\xi)} \,\mathrm{d}\xi + K \int_{\partial\Omega} x(\xi) \,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega} \overline{\psi(\xi)} \,\mathrm{d}\sigma_\xi \\ &= -\int_\Omega \Delta x(\xi) \overline{\psi(\xi)} \,\mathrm{d}\xi + \int_{\partial\Omega} \partial_\nu x(\xi) \overline{\psi(\xi)} \,\mathrm{d}\sigma_\xi + K \int_{\partial\Omega} x(\xi) \,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega} \overline{\psi(\xi)} \,\mathrm{d}\sigma_\xi \end{aligned}$$

$$\begin{aligned}
&= -\int_\Omega \Delta x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi + \left(-K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi + K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi\right)\int_{\partial\Omega}\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi \\
&= -\int_\Omega \Delta x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi.
\end{aligned}$$

Now for the inclusion "$\supset$": Let x be an element of the right hand set in (5.41). Then in particular for all compactly supported and smooth functions $\psi : \Omega \to \mathbb{C}$ the equation

$$\int_\Omega x(\xi)\overline{\Delta\psi(\xi)}\,\mathrm{d}\xi = -a_K(x,\psi) = -\int_\Omega z(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi$$

holds. This implies $\Delta x = -z \in L^2(\Omega)$. In order to show that x is in $W^{2,2}(\Omega)$, we pick some function $h \in W^{2,2}(\Omega)$ that satisfies $\partial_\nu h(\zeta) = -K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi$ for all $\zeta \in \partial\Omega$. Then for all $\psi \in W^{1,2}(\Omega)$ the following holds:

$$\begin{aligned}
&\int_\Omega \nabla(x-h)(\xi)\overline{\nabla\psi(\xi)}\,\mathrm{d}\xi \\
&\quad= \int_\Omega \nabla x(\xi)\overline{\nabla\psi(\xi)}\,\mathrm{d}\xi - \int_\Omega \nabla h(\xi)\overline{\nabla\psi(\xi)}\,\mathrm{d}\xi \\
&\quad= a_K(x,\psi) - K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega}\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi + \int_\Omega \Delta h(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi \\
&\qquad - \int_{\partial\Omega} \partial_\nu h(\xi)\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi \\
&\quad= a_K(x,\psi) + \int_\Omega \Delta h(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi \\
&\quad= \int_\Omega z(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi + \int_\Omega \Delta h(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi \\
&\quad= -\int_\Omega \Delta(x-h)(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi.
\end{aligned}$$

This implies by Lemma 4.2.1 that $x - h \in W^{2,2}(\Omega)$, and therefore we conclude $x \in W^{2,2}(\Omega)$. With this information we can finally apply Gauß's theorem, which yields

$$\begin{aligned}
a_K(x,\psi) &= \int_\Omega \nabla x(\xi)\overline{\nabla\psi(\xi)}\,\mathrm{d}\xi + K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega}\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi \\
&= -\int_\Omega \Delta x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi + \int_{\partial\Omega} \partial_\nu x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi + K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega}\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi.
\end{aligned}$$

The left hand side is by the previous considerations equal to $-\int_\Omega \Delta x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi$, so we have

$$\int_{\partial\Omega} \partial_\nu x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi + K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega} \overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi = 0 \quad \forall\, \psi \in W^{1,2}(\Omega).$$

This implies $\partial_\nu x \equiv -K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi$. □

Lemma 5.2.16. *Let $\theta \in [0,1]$ and denote by $(\cdot)'$ the duality with respect to the pivot space $L^2(\Omega)$. Then the extension $\mathfrak{A}_K(t)|_{(\operatorname{dom} A_K^*)'}$ maps $W^{\theta,2}(\Omega)'$ into $W^{1,2}(\Omega)$, and there are $c,\omega > 0$ such that*

$$\|\mathfrak{A}_K(t)|_{(\operatorname{dom} A_K^*)'} x\|_{W^{1,2}(\Omega)} \leqslant c\left(1 + t^{-\frac{1+\theta}{2}}\right) \mathrm{e}^{-\omega t}\|x\|_{W^{\theta,2}(\Omega)'} \quad \forall\, x \in W^{\theta,2}(\Omega)'.$$

Proof. This is an application of the complex interpolation functor $[\cdot\,,\,\cdot]_\theta$ in Definition A.1.2. With the self-adjointness of A_K it follows from (A.2), [Tri95, Section 4.3.1, Theorem 1] and [Tri95, Equation 2.4.2/11] that

$$\operatorname{dom}(-A_K)^{\theta/2} = [\mathcal{X}, \operatorname{dom}(-A_K)^{1/2}]_\theta = [L^2(\Omega), W^{1,2}(\Omega)]_\theta = W^{\theta,2}(\Omega) \quad \forall\, \theta \in [0,1].$$

Consequently, the dual spaces satisfy $(\operatorname{dom}(-A_K)^{\theta/2})' = W^{\theta,2}(\Omega)'\ \forall\, \theta \in [0,1]$. By Lemma A.1.1 and Lemma A.1.4 the semigroup $\mathfrak{A}_K|_{(\operatorname{dom} A_K^*)'}$ restricts to an analytic semigroup on $(\operatorname{dom}(-A_K)^{\theta/2})'$, whose generator has the domain $\operatorname{dom}(-A_K)^{1-\theta/2}$. Lemma A.1.1 implies further that this extended semigroup maps $(\operatorname{dom} A_K^{\theta/2})'$ into $\operatorname{dom} A_K^{1-\theta/2} \subset \operatorname{dom}(-A_K)^{1/2}$ and that

$$\begin{aligned}
&\exists\, c,\omega > 0 \quad \forall\, x \in (\operatorname{dom}(-A_K)^{\theta/2})' :\\
&\|\mathfrak{A}_K(t)|_{(\operatorname{dom} A_K^*)'} x\|_{\operatorname{dom}(-A_K)^{1/2}} \leqslant c\left(1 + t^{-\frac{1+\theta}{2}}\right) \mathrm{e}^{-\omega t}\|x\|_{(\operatorname{dom}(-A_K)^{\theta/2})'}.
\end{aligned}$$

A further use of $\operatorname{dom}(-A_K)^{1/2} = W^{1,2}(\Omega)$ gives rise to the desired result. □

Theorem 5.2.17. *The solution in Theorem 5.2.11 satisfies $x \in \mathcal{C}(\mathbb{R}_{>0}; W^{1,2}(\Omega))$,*

$$\sup_{t\geqslant 0} \|x(t)\|_{L^2(\Omega)} < \infty, \tag{5.42}$$

and there are $\omega, c > 0$ such that

$$\|x(t)\|_{W^{1,2}(\Omega)} < c\left(1 + t^{-\frac{1+\theta}{2}}\mathrm{e}^{-\omega t}\right) \quad \forall\, t > 0. \tag{5.43}$$

If $x_0 \in W^{1,2}(\Omega)$, *then* $x \in \mathcal{C}(\mathbb{R}_{\geqslant 0}; W^{1,2}(\Omega))$ *and*

$$\sup_{t\geqslant 0} \|x(t)\|_{W^{1,2}(\Omega)} < \infty.$$

Proof. Let γ_0 as in Theorem 5.2.11. By definition, u is equal to zero on $[0,\gamma_0]$, and the state x satisfies

$$x(t) = \mathfrak{A}(t)x_0, \quad \forall\, t \in [0,\gamma_0].$$

Consequently, the smoothing property of $\mathfrak{A}$ (Lemma 2.6.12) yields that

$$\|x(\gamma_0)\|_{W^{1,2}(\Omega)} \leqslant \|\mathfrak{A}(\gamma_0)\|_{\mathcal{B}(L^2(\Omega);W^{2,2}(\Omega))} \|x_0\|_{L^2(\Omega)} \leqslant c\left(1+\gamma_0^{-1}\right)\|x_0\|_{L^2(\Omega)},$$

and that $x \in \mathcal{C}\left((0,\gamma_0]; W^{1,2}(\Omega)\right)$. Obviously, if $x_0 \in W^{1,2}(\Omega)$ then x is actually continuous in the point 0 as well.

To analyze the behavior of x on $[\gamma_0,\infty)$ we exploit the exponential stability of the semigroup with constant output feedback. Choose any $K > 0$ and define $v(t) := u(t) + Ky(t)$. Then $v \in \mathcal{BUC}([\gamma_0,\infty))$ and, by Lemma 5.2.13, the function x satisfies

$$x(t) = \mathfrak{A}_K(t-\gamma_0)x(\gamma_0) + \mathfrak{B}_{K,t-\gamma_0} v(\cdot+\gamma_0). \tag{5.44}$$

We use Lemma 5.2.16 to show that $\mathfrak{B}_{K,t}$ has a smoothing effect. Let $w \in \mathcal{BUC}(\mathbb{R}_{\geqslant 0})$ and pick some $\theta \in (\frac{1}{2},1)$. Then B maps continuously into $W^{\theta,2}(\Omega)'$ because B^* is well-defined and continuous from $W^{\theta,2}(\Omega)$ into $\mathbb{C}$, see Remark 2.6.5. For the rest of this proof we use the notation $\|B\| := \|B\|_{\mathcal{B}(\mathbb{R};W^{\theta,2}(\Omega)')}$. Lemma 5.2.16 implies that $\mathfrak{A}_K(t-\tau)|_{W^{\theta,2}(\Omega)'} Bw(\tau)$ is in $W^{1,2}(\Omega)$ and

$$\left\|\mathfrak{A}_K(t-\tau)|_{W^{\theta,2}(\Omega)'} Bw(\tau)\right\|_{W^{1,2}(\Omega)} \leqslant c\left(1+(t-\tau)^{-\frac{1+\theta}{2}}\right) \mathrm{e}^{-\omega(t-\tau)}\|B\|\|w\|_\infty.$$

Since the real-valued function on the right hand side is integrable over $[0,t)$, the integral in $\mathfrak{B}_{K,t}w$ converges in $W^{1,2}(\Omega)$ and

$$\begin{aligned}
\|\mathfrak{B}_{K,t}w\|_{W^{1,2}(\Omega)} &\leqslant c\int_0^t \mathrm{e}^{-\omega(t-\tau)} + (t-\tau)^{-\frac{1+\theta}{2}} \mathrm{e}^{-\omega(t-\tau)}\,\mathrm{d}\tau \cdot \|B\|\|w\|_\infty \\
&= c\|B\|\|w\|_\infty \int_0^t \mathrm{e}^{-\omega\tau} + \tau^{-\frac{1+\theta}{2}}\mathrm{e}^{-\omega\tau}\,\mathrm{d}\tau \\
&\leqslant c\|B\|\|w\|_\infty \left(\frac{1-\mathrm{e}^{-\omega t}}{\omega} + \int_0^1 \tau^{-\frac{1+\theta}{2}}\mathrm{e}^{-\omega\tau}\,\mathrm{d}\tau + \int_1^\infty \tau^{-\frac{1+\theta}{2}}\mathrm{e}^{-\omega\tau}\,\mathrm{d}\tau\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant c\|B\|\|w\|_\infty \left(\frac{1-\mathrm{e}^{-\omega t}}{\omega} + \int_0^1 \tau^{-\frac{1+\theta}{2}} \,\mathrm{d}\tau + \int_1^\infty \mathrm{e}^{-\omega\tau} \,\mathrm{d}\tau \right) \\
&= c\|B\|\|w\|_\infty \left(\frac{1-\mathrm{e}^{-\omega t}}{\omega} + \frac{2}{1-\theta} + \frac{e^{-\omega}}{\omega} \right).
\end{aligned}
$$

This shows that $\mathfrak{B}_{K,t}w$ is a bounded function. We show continuity of this function with respect to the norm of $W^{1,2}(\Omega)$ at an arbitrary point $t \in \mathbb{R}_{\geqslant 0}$. We have

$$
\begin{aligned}
&\|\mathfrak{B}_{K,t+h}w - \mathfrak{B}_{K,t}w\|_{W^{1,2}(\Omega)} \\
&\quad= \left\| \int_0^{t+h} \mathfrak{A}(\tau)Bw(t+h-\tau)\,\mathrm{d}\tau + \int_0^t \mathfrak{A}(\tau)Bw(t-\tau)\,\mathrm{d}\tau \right\|_{W^{1,2}(\Omega)} \\
&\quad= \left\| \int_t^{t+h} \mathfrak{A}(\tau)Bw(t+h-\tau)\,\mathrm{d}\tau + \int_0^t \mathfrak{A}(\tau)B(w(t+h-\tau) - w(t-\tau))\,\mathrm{d}\tau \right\|_{W^{1,2}(\Omega)} \\
&\quad\leqslant \int_t^{t+h} \|\mathfrak{A}(\tau)Bw(t+h-\tau)\|_{W^{1,2}(\Omega)} \,\mathrm{d}\tau \\
&\qquad\qquad + \int_0^t \|\mathfrak{A}(\tau)B(w(t+h-\tau) - w(t-\tau))\|_{W^{1,2}(\Omega)} \,\mathrm{d}\tau \\
&\quad\leqslant c\|B\|\|w\|_\infty \int_t^{t+h} 1 + \tau^{-\frac{1+\theta}{2}} \,\mathrm{d}\tau \\
&\qquad\qquad + c\|B\| \int_0^t 1 + \tau^{-\frac{1+\theta}{2}} \,\mathrm{d}\tau \sup_{\tau\in[0,t+h]} |w(t+h-\tau) - w(t-\tau)| \\
&\quad\leqslant c\|B\|\|w\|_\infty \int_t^{t+h} 1 + \tau^{-\frac{1+\theta}{2}} \,\mathrm{d}\tau \\
&\qquad\qquad + c\|B\| \int_0^t 1 + \tau^{-\frac{1+\theta}{2}} \,\mathrm{d}\tau \sup_{\tau\in[0,t+h]} |w(t+h-\tau) - w(t-\tau)| \xrightarrow{h\to 0} 0
\end{aligned}
$$

because w is uniformly continuous and the function $1 + \tau^{-\frac{1+\theta}{2}}$ is integrable on the compact interval $[0, t+h]$. This proves that the mapping $t \mapsto \mathfrak{B}_{K,t}w$ is in $\mathcal{C}(\mathbb{R}_{\geqslant 0}; W^{1,2}(\Omega))$.

With $v(\cdot + \gamma_0)$ being in $\mathcal{BUC}(\mathbb{R}_{\geqslant 0})$, these results applied to (5.44) show $x \in \mathcal{C}(\mathbb{R}_{>0}; W^{1,2}(\Omega))$. Finally, the norm bounds (5.42) and (5.43) follow from the boundedness of $\mathfrak{B}_{K,t-\gamma_0}v(\cdot + \gamma_0)$ and the estimates

$$
\|\mathfrak{A}_K(t)x_0\|_{W^{1,2}(\Omega)} \leqslant c\left(1 + t^{-\frac{1}{2}}\right)\mathrm{e}^{-\omega t} \quad \text{and} \quad \|\mathfrak{A}_K(t)x_0\|_{L^2(\Omega)} \leqslant \mathrm{e}^{-\omega t}.
$$

□

5.3. Notes and references

The funnel control principle was first introduced in [IRS02] and has since than been extended to a variety of systems, including systems of relative degree greater than one [IRT06], nonlinear systems with hysteresis [IRS02, IRT07] and differential algebraic systems [BIR12a, BIR12b]. For all these systems the funnel control strategy is the same, namely to amplify the output error by the simple nonlinear gain function (5.13) and feed it back to the input. The challenge is to prove that the system properties guarantee the existence of a stable global solution to the closed-loop system. To the variety of systems for which this works, we have added the infinite-dimensional systems of relative degree one with stable zero dynamics and the (infinite-dimensional) systems whose input-output map is of the form (5.11). Thereby we have sharpened existing results on self-adjoint, finite-dimensional systems in Section 5.2.1 and provided a new proof for these systems.

The proof of Theorem 5.1.1 is not entirely new: Once the problem is written in the form (5.5), a slight modification of [IRS02, Theorem 7] can be applied to complete the proof. However, we chose to give a standalone proof that is much simpler than the one for nonlinear functional differential equations given in [IRS02].

The results on self-adjoint systems in Section 5.2 are already published in [RS15a].

6. State space transformations for systems with compact Hankel operator

In the subsequent sections we construct –by means of state space transformation– some very useful realizations of time-invariant causal operators with compact Hankel operators. The generators of these normalized and balanced realizations have matrix-like representations and they order the state coordinates according to their contribution to the input-output map. They are therefore suitable for approximating the input-output map by finite-dimensional realizations.

In contrast to the transformations in Chapter 3, the normalizing and balancing transformations in the present chapter far from being similarity transformations: Firstly, they are in general not continuous, and secondly they cut off any unobservable or uncontrollable part of the system.

6.1. Shift realizations

There are two canonical realizations of $\mathfrak{D} \in \mathrm{TIC}_0^2(\mathcal{U};\mathcal{Y})$ which are well-known as shift realizations. These are partial differential equations of transport type. We will use the results of Section 2.7 to construct minimal versions of the shift realizations and determine their generators. The Hankel operator

$$\mathfrak{H} : L^2(\mathbb{R}_{\leqslant 0};\mathcal{U}) \to L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y}), \qquad \mathfrak{H} = \pi_+\mathfrak{D}\big|_{L^2(\mathbb{R}_{\leqslant 0};\mathcal{U})},$$

plays an important role for these realizations and so do the shift operators

$$\tau_-^t : L^2(\mathbb{R}_{\leqslant 0};\mathcal{U}) \to L^2(\mathbb{R}_{\leqslant 0};\mathcal{U}), \qquad \tau_+^t : L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y}) \to L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})$$

defined in Section 2.3. Note that $(\tau_-^t)_{t\geqslant 0}$ and $(\tau_+^t)_{t\geqslant 0}$ are strongly continuous semigroups. We assume that $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ are Hilbert spaces, and we will make use of the reflection operator defined in (2.10).

Lemma 6.1.1. *Let $\mathfrak{D} \in \mathrm{TIC}_0^2(\mathcal{U};\mathcal{Y})$ with Hankel operator $\mathfrak{H}$ and define $\mathcal{Z} := (\ker \mathfrak{H})^\perp \subset L^2(\mathbb{R}_{\leqslant 0};\mathcal{U})$. The system*

$$\Big(\pi_{\mathcal{Z}}\tau_-|_{\mathcal{Z}},\quad \pi_{\mathcal{Z}},\quad \mathfrak{H}|_{\mathcal{Z}},\quad \mathfrak{D}\Big) \tag{6.1}$$

is a minimal 0*-bounded L^2-well-posed linear system on $(\mathcal{U},\mathcal{Z},\mathcal{Y})$. The main operator of this system is the following differential operator:*

$$A : \operatorname{dom} A \subset \mathcal{Z} \to \mathcal{Z}, \qquad \operatorname{dom} A = \pi_{\mathcal{Z}} W_0^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U}),$$

$$Az = \pi_{\mathcal{Z}}\dot{x} \quad \forall\, z \in \operatorname{dom} A,\ \forall\, x \in W_0^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U}) \text{ with } \pi_{\mathcal{Z}}x = z,$$

and its adjoint is

$$A^* : \operatorname{dom} A^* \subset \mathcal{Z} \to \mathcal{Z}, \qquad \operatorname{dom} A^* = \mathcal{Z} \cap W^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{U}),$$

$$A^* z = -\dot{z} \quad \forall\, z \in \operatorname{dom} A^*.$$

The control operator is the evaluation functional at zero, i.e.

$$B : \mathcal{U} \to (\operatorname{dom} A^*)', \qquad u \mapsto \left(\varphi \mapsto \langle \varphi(0)\,,\, u\rangle_{\mathcal{U}}\right).$$

The Hankel operator maps $\operatorname{dom} A$ *into* $W^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$ *and the observation operator is*

$$C : \operatorname{dom} A \to \mathcal{Y}, \qquad Cx = (\mathfrak{H}x)(0).$$

This system is called the exactly controllable shift realization of $\mathfrak{D}$ on $(\ker \mathfrak{H})^\perp$.

Proof. It is well-known that the so-called exactly controllable shift realization of $\mathfrak{D}$,

$$\Big(\tau_-,\quad \mathrm{I},\quad \mathfrak{H},\quad \mathfrak{D}\Big),$$

is an L^2-well-posed linear system on $(\mathcal{U}, L^2(\mathbb{R}_{\leqslant 0};\mathcal{U}),\mathcal{Y})$, see [Sta05, Example 2.6.5].

The generator of the left shift semigroup τ_- is the differential operator

$$\tfrac{\mathrm{d}}{\mathrm{d}\xi} : W_0^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U}) \subset L^2(\mathbb{R}_{\leqslant 0};\mathcal{U}) \to L^2(\mathbb{R}_{\leqslant 0};\mathcal{U}), \qquad x \mapsto \dot{x},$$

see [Sta05, Example 3.2.3 (iii)], whose adjoint is known to be

$$-\tfrac{\mathrm{d}}{\mathrm{d}\xi} : W^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U}) \subset L^2(\mathbb{R}_{\leqslant 0};\mathcal{U}) \to L^2(\mathbb{R}_{\leqslant 0};\mathcal{U}), \qquad x \mapsto -\dot{x}.$$

This together with Lemma 2.7.1 proves the well-posedness of (6.1) and the form of A and A^*. Since the observation operator is clear by Definition 2.4.5, it only remains to determine the control operator B. By (2.7) the control operator satisfies the following for all $\varphi \in (\ker \mathfrak{H})^\perp \cap W^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U})$, $u \in \mathcal{U}$:

$$\begin{aligned}
\langle \varphi\,,\, Bu \rangle_{\operatorname{dom} A^*,(\operatorname{dom} A^*)'} &= \left\langle (\bar{\lambda} - A^*)\varphi\,,\, \pi_{\mathcal{Z}}\, \mathrm{e}_\lambda u \right\rangle_{\mathcal{Z}} \\
&= \left\langle \bar{\lambda}\varphi + \dot{\varphi}\,,\, \mathrm{e}_\lambda u \right\rangle_{L^2(\mathbb{R}_{\leqslant 0};\mathcal{U})} \\
&= \int_{-\infty}^0 \left\langle \bar{\lambda}\varphi(t)\,,\, \mathrm{e}^{\lambda t} u \right\rangle_{\mathcal{U}} \,\mathrm{d}t + \int_{-\infty}^0 \left\langle \dot{\varphi}(t)\,,\, \mathrm{e}^{\lambda t} u \right\rangle_{\mathcal{U}} \,\mathrm{d}t \\
&= \langle \varphi(0)\,,\, u \rangle_{\mathcal{U}}\,.
\end{aligned}$$

Now the proof is complete. □

Remark 6.1.2. It can be shown that the exactly controllable shift realization belongs to a boundary control system in the sense of Lemma 2.5.2. Thus, every operator $\mathfrak{D} \in \mathrm{TIC}_0^2(\mathcal{U};\mathcal{Y})$ can be realized by a boundary control system.

Lemma 6.1.3. *Let $\mathfrak{D} \in \mathrm{TIC}_0^2(\mathcal{U};\mathcal{Y})$ with Hankel operator $\mathfrak{H}$ and define $\mathcal{Z} := \overline{\operatorname{ran} \mathfrak{H}}$. Then*

$$\Big(\tau_+|_{\mathcal{Z}}, \quad \mathfrak{H}, \quad \mathrm{I}_{\mathcal{Z}}, \quad \mathfrak{D}\Big) \tag{6.2}$$

is a minimal 0-bounded L^2-well-posed linear system on $(\mathcal{U}, \mathcal{Z}, \mathcal{Y})$. The main operator is

$$A : \operatorname{dom} A \subset \mathcal{Z} \to \mathcal{Z}, \qquad \operatorname{dom} A = W^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{Y}) \cap \mathcal{Z},$$

$$Az = \dot{z}.$$

The adjoint of this operator is the differential operator

$$A^* : \operatorname{dom} A^* \subset \mathcal{Z} \to \mathcal{Z}, \qquad \operatorname{dom} A^* = \pi_{\mathcal{Z}} W_0^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{Y}),$$

$$A^* z = -\pi_{\mathcal{Z}} \dot{x} \quad \forall\, z \in \operatorname{dom} A^*,\ \forall\, x \in W_0^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U}) \text{ with } \pi_{\mathcal{Z}} x = z.$$

The operator $\mathfrak{H}^*$ *maps* $W_0^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$ *into* $W^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U})$ *and the control operator of* (6.2) *satisfies*

$$B : \mathcal{U} \to (\operatorname{dom} A^*)', \quad \langle z\,,\, Bu\rangle_{\operatorname{dom} A^*,(\operatorname{dom} A^*)'} = \langle (\mathfrak{H}^* x)(0)\,,\, u\rangle_{\mathcal{U}}$$

for all $z \in \operatorname{dom} A^*$ *and all* $x \in W_0^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$ *with* $\pi_{\mathcal{Z}} x = z$*. The observation operator is given by*

$$C : \operatorname{dom} A \to \mathcal{Y}, \quad Cz = z(0).$$

We call the system (6.2) *the* exactly observable shift realization of $\mathfrak{D}$ on $\overline{\operatorname{ran}\mathfrak{H}}$*.*

Proof. Analogously to the previous proof we now apply Lemma 2.7.2 to the exactly observable shift realization on $L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})$,

$$\Big(\tau_+, \quad \mathfrak{H}, \quad \mathrm{I}, \quad \mathfrak{D}\Big),$$

which can be found in [Sta05, Example 2.6.5 (ii)]. It has the main operator

$$\tfrac{\mathrm{d}}{\mathrm{d}\xi} : W^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{Y}) \subset L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y}) \to L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y}), \qquad x \mapsto \dot{x},$$

see [Sta05, Example 3.2.3 (iii)], with adjoint

$$-\tfrac{\mathrm{d}}{\mathrm{d}\xi} : W_0^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{Y}) \subset L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y}) \to L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y}), \qquad x \mapsto -\dot{x}.$$

Therefore Lemma 2.7.2 yields the form of A and A^*. The verification of the operator C is straightforward from the definition of an observation operator. To calculate B we observe that the dual input-output map $\mathsf{R}\mathfrak{D}^*\mathsf{R}$ is in $\mathrm{TIC}_0^2(\mathcal{Y};\mathcal{U})$ by Lemma 2.4.17. The corresponding Hankel operator is

$$\mathsf{R}\mathfrak{H}^*\mathsf{R} : L^2(\mathbb{R}_{\leqslant 0};\mathcal{Y}) \to L^2(\mathbb{R}_{\geqslant 0};\mathcal{U}).$$

Since it is the input operator of the exactly observable shift realization of $\text{Я}\mathfrak{D}^*\text{Я}$,

$$\Big(\tau_+, \quad \text{Я}\mathfrak{H}^*\text{Я}, \quad \mathrm{I}, \quad \text{Я}\mathfrak{D}^*\text{Я}\Big),$$

we conclude from Lemma 2.4.3 (i) that $\mathfrak{H}^*$ maps $W_0^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$ to $W^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U})$ and that $\mathfrak{H}^*\dot{x}$ is the derivative of $\mathfrak{H}^*x$. Let $\lambda \in \mathbb{C}_{>0}$ and $u \in \mathcal{U}$. Then for all $z \in \operatorname{dom} A^*$ we take an arbitrary $x \in W_0^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$ with $\pi_{\mathcal{Z}}x = z$ and obtain

$$\begin{aligned}
\langle z\,,\, Bu\rangle_{\operatorname{dom} A^*,(\operatorname{dom} A^*)'} &= \Big\langle (\bar{\lambda} - A^*)z\,,\, \mathfrak{H}\mathrm{e}_\lambda u\Big\rangle_{\mathcal{Z}} \\
&= \Big\langle \bar{\lambda}z + \pi_{\mathcal{Z}}\dot{x}\,,\, \mathfrak{H}\mathrm{e}_\lambda u\Big\rangle_{\mathcal{Z}} \\
&= \Big\langle \bar{\lambda}x + \dot{x}\,,\, \mathfrak{H}\mathrm{e}_\lambda u\Big\rangle_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{U})} \\
&= \Big\langle \bar{\lambda}\mathfrak{H}^*x + \mathfrak{H}^*\dot{x}\,,\, \mathrm{e}_\lambda u\Big\rangle_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{U})} \\
&= \int_{-\infty}^{0} \Big\langle \bar{\lambda}(\mathfrak{H}^*x)(t)\,,\, \mathrm{e}^{\lambda t}u\Big\rangle_{\mathcal{U}}\,\mathrm{d}t + \int_{-\infty}^{0} \Big\langle \tfrac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{H}^*x)(t)\,,\, \mathrm{e}^{\lambda t}u\Big\rangle_{\mathcal{U}}\,\mathrm{d}t \\
&= \langle (\mathfrak{H}^*x)(0)\,,\, u\rangle_{\mathcal{U}}\,.
\end{aligned}$$

This is the desired expression for Bu. □

Remark 6.1.4. The adverb “exactly” indicates that the input operator of the exactly controllable shift realization is onto, which is stronger than controllability. Similarly it indicates in the exactly observable shift realization, that the adjoint of the output operator is onto, which is stronger than observability.

6.2. Output normalizing transformations

The exactly observable shift realization in the previous Section has the outstanding property that its observability Gramian is the identity operator. Similarly, the controllability Gramian of the exactly controllable shift realization is the identity operator. We call systems of this kind normalized, more precisely, we make the following definition.

Definition 6.2.1 (normalized system)**.** We say that a 0-bounded L^2-well-posed linear system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ on three Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ is *input normalized* if and only if $\mathfrak{B}\mathfrak{B}^* = \mathrm{I}_{\mathcal{X}}$, and *output normalized* if and only if $\mathfrak{C}^*\mathfrak{C} = \mathrm{I}_{\mathcal{X}}$.

Remark 6.2.2. Since the Gramians are solutions of so-called Lyapunov equations [Sta05, Section 10.4], input or output normalized systems are sometimes said to be “Lyapunov normalized”.

The shift realizations in the last section are the most popular normalized realizations. The goal of this section is to construct yet another realization that is output normalized and has the state space $\ell^2(\mathbb{N})$. This can for example be achieved by choosing an appropriate basis of $\overline{\operatorname{ran}\mathfrak{H}}$ if the Hankel operator is known and compact. In practice however, one is usually stuck with the generators of a system and an explicit representation of the Hankel operator is difficult to obtain. Therefore, we introduce amenable state space transformations that carry us from the generators of a given realization to a normalized realization.

For the rest of this chapter the following is a standing presumption.

Presumption 6.2.3. *We have the Hilbert spaces $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$, where $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional. $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is a 0-bounded L^2-well-posed linear system on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ with compact Hankel operator $\mathfrak{H} = \mathfrak{C}\mathfrak{B}$. Moreover, $\mathcal{X}_R$ and $\mathcal{X}_S$ are Hilbert spaces, and $R \in \mathcal{B}(\mathcal{X}_R, \mathcal{X})$, $S \in \mathcal{B}(\mathcal{X}_S, \mathcal{X})$ are operators such that the controllability and observability Gramians satisfy*

$$\mathfrak{B}\mathfrak{B}^* = RR^* \qquad \textit{and} \qquad \mathfrak{C}^*\mathfrak{C} = SS^*. \tag{6.3}$$

Remark 6.2.4. The factors may for instance be $R = \mathfrak{B}$, $S = \mathfrak{C}^*$, or $R = (\mathfrak{B}\mathfrak{B}^*)^{1/2}$, $S = (\mathfrak{C}^*\mathfrak{C})^{1/2}$. The motivation for the formulation of Presumption 6.2.3 is that the so-called “ADI method” [ORW13] directly provides factors R and S of the Gramians, which can be used.

A first consequence of Presumption 6.2.3 is that

$$\operatorname{ran} R = \operatorname{ran}\mathfrak{B}, \qquad \operatorname{ran}\mathfrak{C}^* = \operatorname{ran} S, \tag{6.4a}$$

$$\ker\mathfrak{B}^* = \ker R^*, \qquad \ker S^* = \ker\mathfrak{C}. \tag{6.4b}$$

The equations in (6.4a) are consequences of the fact that the operator square roots fulfill

$$\operatorname{ran} R = \operatorname{ran}\sqrt{RR^*} = \operatorname{ran}\sqrt{\mathfrak{B}\mathfrak{B}^*} = \operatorname{ran}\mathfrak{B} \qquad \text{and}$$
$$\operatorname{ran}\mathfrak{C}^* = \operatorname{ran}\sqrt{\mathfrak{C}^*\mathfrak{C}} = \operatorname{ran}\sqrt{SS^*} = \operatorname{ran} S,$$

see e.g. [Kat80, pp. 334–336]. The remaining assertions in (6.4b) follow by regarding the orthogonal complements in (6.4a). With this, the restricted operators

$$\begin{aligned} R &: (\ker R)^\perp \subset \mathcal{X}_R \to \overline{\operatorname{ran}\mathfrak{B}}, & S &: (\ker S)^\perp \subset \mathcal{X}_S \to (\ker \mathfrak{C})^\perp, \\ \mathfrak{B} &: (\ker \mathfrak{B})^\perp \subset L^2(\mathbb{R}_{\leqslant 0};\mathcal{U}) \to \overline{\operatorname{ran}\mathfrak{B}}, & \mathfrak{C} &: (\ker \mathfrak{C})^\perp \subset X \to \overline{\operatorname{ran}\mathfrak{C}}. \end{aligned}$$

are injective and have dense range. We denote their inverses (and adjoints of their inverses) by R^{-1}, $\mathfrak{B}^{-1}$, S^{-1} and $\mathfrak{C}^{-1}$ (R^{-*}, $\mathfrak{B}^{-*}$, S^{-*} and $\mathfrak{C}^{-*}$). Recall e.g. from [Sta05, Lemma 3.5.2] that any injective closed and densely defined operator T with dense range satisfies $(\operatorname{dom} T^{-1})^* = \operatorname{ran} T^*$, and $T^{-*} := (T^{-1})^* = (T^*)^{-1}$ is well defined.

Lemma 6.2.5. *The mappings*

$$\mathfrak{V} : \overline{\operatorname{ran} S^* R} \to \overline{\operatorname{ran}\mathfrak{H}}, \qquad \mathfrak{V} := \overline{\mathfrak{C}S^{-*}}|_{\overline{\operatorname{ran} S^* R}},$$

where $\overline{\mathfrak{C}S^{-*}}$ *is the continuous extension of* $\mathfrak{C}S^{-*}|_{\operatorname{ran} S^* R}$ *with respect to the norms of* $\mathcal{X}_S$ *and* $L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})$*, and*

$$\mathfrak{U} : (\ker S^* R)^\perp \to (\ker \mathfrak{H})^\perp, \qquad \mathfrak{U} := \mathfrak{B}^{-1} R|_{(\ker S^* R)^\perp}. \tag{6.5}$$

are unitary with inverses $\mathfrak{V}^* = \overline{S^*\mathfrak{C}^{-1}}$ *and* $\mathfrak{U}^* = R^{-1}\mathfrak{B}$*, respectively. Furthermore,*

$$\mathfrak{V}^*\mathfrak{H}\mathfrak{U}\, x = S^* R\, x \quad \forall\, x \in (\ker S^* R)^\perp. \tag{6.6}$$

Proof. From the fact that $\operatorname{ran} R = \operatorname{ran}\mathfrak{B}$ and

$$\|\mathfrak{C}S^{-*}x\|^2_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})} = \langle S^{-*}x, \mathfrak{C}^*\mathfrak{C}S^{-*}x\rangle_{\mathcal{X}} = \langle S^{-*}x, Sx\rangle_{\mathcal{X}} = \|x\|^2_{\mathcal{X}_S}$$

for all $x \in \operatorname{ran} S^*$, we deduce that $\mathfrak{C}S^{-*} : \operatorname{ran} S^* R \to \overline{\operatorname{ran}\mathfrak{H}}$ is an isometry with dense range and left inverse $S^*\mathfrak{C}^{-1}$. Therefore, it can be extended to a unitary operator $\mathfrak{V}$ between the closures of these two spaces. Analogously, we can deduce that the concatenation $R^*\mathfrak{B}^{-*} : \operatorname{ran}\mathfrak{B}^*\mathfrak{C}^* \to \operatorname{ran} R^* S$ satisfies

$$\|R^*\mathfrak{B}^{-*}x\|_{\mathcal{X}_R} = \|x\|_{L^2(\mathbb{R}_{\leqslant 0};\mathcal{U})} \quad \forall\, x \in \operatorname{ran}\mathfrak{B}^*,$$

and has a unitary extension that we denote by $\mathfrak{U}^* : (\ker \mathfrak{H})^\perp \to (\ker S^* R)^\perp$. Furthermore, because of (6.3), the identity $R^*\mathfrak{B}^{-*}x = R^{-1}\mathfrak{B}x$ holds for all $x \in \operatorname{ran}\mathfrak{B}^*\mathfrak{C}^*$, which is a dense subset of $(\ker\mathfrak{H})^\perp$. The operator $R^{-1}\mathfrak{B}|_{(\ker\mathfrak{H})^\perp}$ is defined on the complete space $(\ker\mathfrak{H})^\perp$ and it is closed because R^{-1} is closed. By the closed graph theorem it is continuous and hence, it must be equal to the unique unitary extension $\mathfrak{U}^*$ of $R^*\mathfrak{B}^{-*}$. This implies that its inverse is its adjoint, i.e. $\mathfrak{U} = \mathfrak{B}^{-1}R|_{(\ker S^*R)^\perp}$. The equation

$$\begin{aligned}\mathfrak{H}|_{(\ker\mathfrak{H})^\perp} &= \mathfrak{C}\pi_{(\ker S^*)^\perp}\mathfrak{B}|_{(\ker\mathfrak{H})^\perp} = \mathfrak{C}S^{-*}S^*RR^{-1}\mathfrak{B}|_{(\ker\mathfrak{H})^\perp}\\ &= \mathfrak{V}(S^*R)|_{(\ker S^*R)^\perp}\mathfrak{U}^*|_{(\ker\mathfrak{H})^\perp}\end{aligned}$$

shows (6.6) and completes the proof. □

As a consequence of this lemma and the compactness of $\mathfrak{H}$, the operator S^*R is compact as well. It therefore admits a *singular value decomposition* (sometimes called “canonical form for compact operators”) in the sense of [RS72, pp. 203]. That is, there are orthonormal systems $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{X}_R$ and $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{X}_S$ and a nonincreasing, positive null sequence $(\sigma_n)_{n\in\mathbb{N}}$ with

$$S^*R\,x = \sum_{n=1}^{\infty} \sigma_n \left\langle x\,,\, u_n\right\rangle_{\mathcal{X}_R} v_n \quad \forall x \in \mathcal{X}_R.$$

The numbers σ_n are called *singular values*, and (u_n, v_n) is the so-called *Schmidt pair* associated to σ_n. Note that we allow consecutive σ_n to be equal. A more convenient way of writing the singular value decomposition is

$$S^*R = V\Sigma U^*, \tag{6.7}$$

where the operators $\Sigma \in \mathcal{B}(\ell^2)$, $U \in \mathcal{B}(\ell^2;\mathcal{X}_R)$, $V \in \mathcal{B}(\ell^2;\mathcal{X}_S)$ are defined by

$$\Sigma(x_n)_{n\in\mathbb{N}} := (\operatorname{diag}(\sigma_n)_{n\in\mathbb{N}})\,(x_n)_{n\in\mathbb{N}} := (\sigma_n x_n)_{n\in\mathbb{N}} \tag{6.8}$$

and

$$U(x_n)_{n\in\mathbb{N}} := \sum_{n=0}^{\infty} x_n u_n, \qquad V(x_n)_{n\in\mathbb{N}} := \sum_{n=0}^{\infty} x_n v_n.$$

Here, we have assumed that there are infinitely many singular values, or, equivalently, that $\operatorname{ran} S^*R$ is infinite-dimensional. In case that this range is k-dimensional, the results in this chapter hold with ℓ^2 replaced by $\mathbb{C}^k$ and obvious modifications. In any case, there holds $\operatorname{ran} V = \overline{\operatorname{ran} S^*R}$, $\operatorname{ran} U = (\ker S^*R)^\perp$, $U^*U = V^*V = \mathrm{I}_{\ell^2}$, $VV^* = \pi_{\overline{\operatorname{ran} S^*R}}$ and $UU^* = \pi_{(\ker S^*R)^\perp}$. Note that the restrictions $U^*|_{\overline{\operatorname{ran} S^*R}}$ and $V^*|_{(\ker S^*R)^\perp}$ are both unitary. It can be seen that Σ is injective, self-adjoint and has dense range, and therefore we have

$$\Sigma = V^*S^*RU = U^*R^*SV.$$

Likewise, there is a singular value decomposition of the Hankel operator itself. The singular values of $\mathfrak{H}$ are called *Hankel singular values.* The following corollary to Lemma 6.2.5 shows that these coincide with the singular values of S^*R.

Corollary 6.2.6. *The singular values $(\sigma_n)_{n\in\mathbb{N}}$ of S^*R are the singular values of the Hankel operator $\mathfrak{H}$.*

Proof. The equalities

$$\begin{aligned}(S^*R)(S^*R)^*|_{\overline{\operatorname{ran} S^*R}} &= \mathfrak{V}^*\mathfrak{H}\mathfrak{H}^*\mathfrak{V}|_{\overline{\operatorname{ran} S^*R}} \quad \text{and}\\ (S^*R)^*(S^*R)|_{(\ker S^*R)^\perp} &= \mathfrak{U}^*\mathfrak{H}^*\mathfrak{H}\mathfrak{U}|_{(\ker S^*R)^\perp}\end{aligned}$$

show that v_i is an eigenvector of $S^*R(S^*R)^*$ to the eigenvalue $\sigma_i^2 > 0$ if and only if $\widetilde{v}_i := \mathfrak{V}v$ is an eigenvector of $\mathfrak{H}\mathfrak{H}^*$ corresponding to the same eigenvalue and, analogously, u_i is an eigenvector of $(S^*R)^*S^*R$ if and only if $\widetilde{u}_i := \mathfrak{U}u_i$ is an eigenvector of $\mathfrak{H}^*\mathfrak{H}$. Hence it follows that the singular values of S^*R and $\mathfrak{H}$ are equal. In particular,

$$\mathfrak{H}u = \sum_{i=1}^{\infty} \widetilde{v}_i\sigma_i\langle u, \widetilde{u}_i\rangle \quad \forall\, u \in L^2(\mathbb{R}_{\leqslant 0};\mathcal{U}) \tag{6.9}$$

is a singular value decomposition of $\mathfrak{H}$. $\square$

We will write the singular value decomposition in (6.9) as operator equation

$$\begin{aligned}\mathfrak{H} &= \widetilde{V}\Sigma\widetilde{U}^*, && \text{where}\\ \widetilde{U} &:= \mathfrak{U}U && \in \mathcal{B}(\ell^2; (\ker\mathfrak{H})^\perp),\\ \widetilde{V} &:= \mathfrak{V}V && \in \mathcal{B}(\ell^2; \overline{\operatorname{ran}\mathfrak{H}}),\end{aligned} \tag{6.10}$$

where $\mathfrak{U}$, $\mathfrak{V}$ are defined in Lemma 6.2.5, and $\Sigma \in \mathcal{B}(\ell^2)$ is precisely the operator defined in (6.8).

Lemma 6.2.7. *Let* $S^*R = V\Sigma U^*$ *be the singular value decomposition* (6.7) *and define*

$$\mathcal{M} := \pi_{(\ker S^*)^\perp} \operatorname{ran} R = \pi_{(\ker \mathfrak{C}^*)^\perp} \operatorname{ran} \mathfrak{B},$$

i.e. as in Theorem 2.7.3. The mapping

$$V^*S^*\big|_{\mathcal{M}} : \mathcal{M} \to \Sigma\ell^2$$

is an isomorphism with inverse given by

$$S^{-*}V(x_n) = \pi_{(\ker S^*)^\perp} RU\Sigma^{-1}(x_n) \quad \forall\,(x_n) \in \Sigma\ell^2. \tag{6.11}$$

Proof. With (6.7) and $VV^* = \pi_{\overline{\operatorname{ran} S^*R}}$ it can be seen that $V^*S^*|_{\mathcal{M}}$ is an isomorphism between the asserted spaces with inverse $S^{-*}V$. (The important part here is that the spaces were chosen correctly.) The singular value decomposition further shows immediately that V^*S^* is the left inverse of $\pi_{(\ker S^*)^\perp} RU\Sigma^{-1}$ on $\Sigma\ell^2$. To prove that it is a right inverse we calculate for given $y = \pi_{(\ker S^*)^\perp} Rx$ with $x \in \mathcal{X}_R$

$$\begin{aligned}\pi_{(\ker S^*)^\perp} RU\Sigma^{-1}V^*S^*y &= \pi_{(\ker S^*)^\perp} RU\Sigma^{-1}V^*S^*Rx = \pi_{(\ker S^*)^\perp} RUU^*x \\ &= \pi_{(\ker S^*)^\perp} R\pi_{(\ker S^*R)^\perp}x = \pi_{(\ker S^*)^\perp} Rx = y.\end{aligned}$$

□

Theorem 6.2.8. *Let* $S^*R = V\Sigma U^*$ *be the singular value decomposition of the operator* S^*R. *Then the operators*

$$\begin{aligned}\mathcal{T} : \mathcal{X} &\to \ell^2, & \mathcal{T}^+ : \Sigma\ell^2 \subset \ell^2 &\to \mathcal{X}, \\ x &\mapsto V^*S^*x, & x &\mapsto RU\Sigma^{-1}x\end{aligned}$$

are well-defined, and the following assertions hold true:

(i) There exists a constant $c > 0$ *such that, for all* $x \in \Sigma\ell^2$, $u \in L^2(\mathbb{R}_{\leqslant 0};\mathcal{U})$ *and* $t \geqslant 0$, *there holds*

$$\begin{aligned}&\|\mathcal{T}\mathfrak{A}(t)\mathcal{T}^+x\|_{\ell^2} \leqslant c\,\|x\|_{\ell^2}, \quad \|\mathcal{T}\mathfrak{B}u\|_{\ell^2} \leqslant c\,\|u\|_{L^2(\mathbb{R}_{\leqslant 0};\mathcal{U})}, \\ &\|\mathfrak{C}\mathcal{T}^+x\|_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})} \leqslant c\,\|x\|_{\ell^2}.\end{aligned}$$

(ii) With the unique continuous extensions

$$\overline{\mathcal{T}\mathfrak{A}\mathcal{T}^{+}} : \mathbb{R}_{\geqslant 0} \to \mathcal{B}(\ell^2),\ t \mapsto \overline{\mathcal{T}\mathfrak{A}(t)\mathcal{T}^{+}}, \quad and \quad \overline{\mathfrak{C}\mathcal{T}^{+}} \in \mathcal{B}(\ell^2; L^2(\mathbb{R}_{\geqslant 0}; \mathcal{Y})),$$

the quadruple

$$(\mathfrak{A}_o, \mathfrak{B}_o, \mathfrak{C}_o, \mathfrak{D}) := (\overline{\mathcal{T}\mathfrak{A}\mathcal{T}^{+}}, \mathcal{T}\mathfrak{B}, \overline{\mathfrak{C}\mathcal{T}^{+}}, \mathfrak{D}) \tag{6.12}$$

is a minimal 0*-bounded* L^2*-well-posed linear system on* $(\mathcal{U}, \ell^2, \mathcal{Y})$ *and output normalized. Furthermore, the system* (6.12) *is unitarily similar to the exactly observable shift realization of* $\mathfrak{D}$ *on* $\overline{\operatorname{ran}\mathfrak{H}}$ *via the unitary transformation* $\widetilde{V}$ *in* (6.10)*, and its controllability Gramian is* Σ^2.

In the following, we are going to refer to (6.12) as *the output normalized realization of* $\mathfrak{D}$ *on* ℓ^2. But there exist of course other output normalized realization on ℓ^2.

Proof. We are going to show that the mapping $\widetilde{V}$ in (6.10) transforms the shift realization (6.2) into the system (6.12). The boundedness of the operators in (i) then follows because $\widetilde{V}$ is unitary, and all the properties of the shift realization are preserved under this transformation by Lemma 2.4.15.

First note that, owing to the equality $\ker\mathfrak{C} = \ker S^*$ and Definition 2.4.1 (iii), we have the following expression for all $x \in \operatorname{ran} S^* R$

$$\begin{aligned} S^*\mathfrak{A}(t)S^{-*}x = S^*\pi_{(\ker\mathfrak{C})^\perp}\mathfrak{A}(t)\pi_{(\ker\mathfrak{C})^\perp}S^{-*}x = S^*\mathfrak{C}^{-1}\mathfrak{C}\mathfrak{A}(t)\mathfrak{C}^{-1}\mathfrak{C}S^{-*}x = \\ = \mathfrak{V}^*\mathfrak{C}\mathfrak{A}(t)\mathfrak{C}^{-1}\mathfrak{V}x = \mathfrak{V}^*\tau_+^t|_{\overline{\operatorname{ran}\mathfrak{H}}}\mathfrak{V}x. \end{aligned}$$

Furthermore, we can substitute (6.11) to obtain for all $x \in \operatorname{ran}\mathfrak{H}$

$$\begin{aligned} \widetilde{V}\mathcal{T}\mathfrak{A}(t)\mathcal{T}^{+}\widetilde{V}^*x &= \mathfrak{V}VV^*S^*\mathfrak{A}(t)\pi_{(\ker S^*)^\perp}RU\Sigma^{-1}V^*\mathfrak{V}^*x \\ &= \mathfrak{V}VV^*S^*\mathfrak{A}(t)S^{-*}VV^*\mathfrak{V}^*x = \\ &= \mathfrak{V}\pi_{\overline{\operatorname{ran} S^*R}}S^*\mathfrak{A}(t)S^{-*}\pi_{\overline{\operatorname{ran} S^*R}}\mathfrak{V}^*x \\ &= \mathfrak{V}\pi_{\overline{\operatorname{ran} S^*R}}\mathfrak{V}^*\tau_+^t\mathfrak{V}\pi_{\overline{\operatorname{ran} S^*R}}\mathfrak{V}^*x = \tau_+^t x, \end{aligned}$$

and by continuous extension it follows that this formula holds on the closure of

$\operatorname{ran}\mathfrak{H}$. Furthermore, one gets

$$\begin{aligned}\mathfrak{C}\mathcal{T}^{+}\widetilde{V}^{*}x &= \mathfrak{C}RU\Sigma^{-1}V^{*}\mathfrak{V}^{*}x = \mathfrak{C}\pi_{(\ker S^{*})^{\perp}}RU\Sigma^{-1}V^{*}\mathfrak{V}^{*}x \\ &= \mathfrak{C}S^{-*}VV^{*}\mathfrak{V}^{*}x = \mathfrak{V}\pi_{\overline{\operatorname{ran} S^{*}R}}\mathfrak{V}^{*}x = x.\end{aligned}$$

Again, continuous extension yields that $\mathfrak{C}RU\Sigma^{-1}$ is similar to $\mathrm{I}_{\overline{\operatorname{ran}\mathfrak{H}}}$ via the unitary transformation $V^{*}\mathfrak{V}^{*}$. The equation

$$\widetilde{V}\mathcal{T}\mathfrak{B} = \mathfrak{V}VV^{*}S^{*}\mathfrak{B} = \overline{\mathfrak{C}S^{-*}}S^{*}\mathfrak{B} = \overline{\mathfrak{C}S^{-*}}\pi_{\overline{\operatorname{ran} S^{*}R}}S^{*}\mathfrak{B} = \mathfrak{C}\mathfrak{B} = \mathfrak{H},$$

completes the proof of the asserted similarity.

Finally, the controllability Gramian computes to

$$\mathfrak{B}_{o}\mathfrak{B}_{o}^{*} = V^{*}S^{*}\mathfrak{B}\mathfrak{B}^{*}SV = V^{*}S^{*}RR^{*}SV = \Sigma^{2}.$$

□

Remark 6.2.9. It is relatively easy to check that the mapping $V^{*}S^{*} : \overline{\mathcal{M}} \to \ell^{2}$ is a pseudo-similarity transformation (Definition 2.4.16) between the output normalized system $(\mathfrak{A}_{o}, \mathfrak{B}_{o}, \mathfrak{C}_{o}, \mathfrak{D})$ and the realization $\left(\pi_{(\ker\mathfrak{C})^{\perp}}\mathfrak{A}|_{\overline{\mathcal{M}}}, \pi_{(\ker\mathfrak{C})^{\perp}}\mathfrak{B}, \mathfrak{C}|_{\overline{\mathcal{M}}}, \mathfrak{D}\right)$ in Theorem 2.7.3. It is shown in [RS14, Section 11] that the inverse of this pseudo-similarity transformation is the closure of the operator $\pi_{(\ker\mathfrak{C})^{\perp}}RU\Sigma^{-1} : \Sigma\ell^{2} \to \overline{\mathcal{M}}$.

The following corollary shows the relation between the generators of the last theorem and the generators of the exactly observable shift realization.

Corollary 6.2.10. *Define* $\widetilde{V}$ *as in the singular value decomposition* (6.10) *of* $\mathfrak{H}$. *The output normalized realization* (6.12) *on* ℓ^{2} *satisfies*

$$(\mathfrak{A}_{o}, \mathfrak{B}_{o}, \mathfrak{C}_{o}, \mathfrak{D}) = \left(\widetilde{V}^{*}\tau^{+}\widetilde{V}, \widetilde{V}^{*}\mathfrak{H}, \widetilde{V}, \mathfrak{D}\right), \tag{6.13}$$

and its generators are determined by the following relations:

$$\operatorname{dom} A_{o} = \left\{(x_{n}) \in \ell^{2} : \sum_{n=1}^{\infty} x_{n}\widetilde{v}_{n} \in W^{1,2}(\mathbb{R}_{\geqslant 0}; \mathcal{Y})\right\}, \tag{6.14a}$$

$$A_{o}(x_{n}) = \widetilde{V}^{*}\tfrac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{V}(x_{n}), \tag{6.14b}$$

$$\operatorname{dom} A_o^* = \left\{ (x_n) \in \ell^2 : \sum_{n=1}^{\infty} x_n \widetilde{v}_n \in \pi_{\overline{\operatorname{ran}\mathfrak{H}}} W_0^{1,2}(\mathbb{R}_{\geqslant 0}; \mathcal{Y}) \right\}, \tag{6.15a}$$

$$A_o^*(x_n) = -\widetilde{V}^* \tfrac{\mathrm{d}}{\mathrm{d}\xi} y \quad \forall\, y \in W_0^{1,2}(\mathbb{R}_{\geqslant 0}; \mathcal{Y}) \text{ with } \pi_{\overline{\operatorname{ran}\mathfrak{H}}} y = \widetilde{V}(x_n). \tag{6.15b}$$

For each $u \in \mathcal{U}$, the image $B_o u$ is an element of $(\operatorname{dom} A_o^*)'$ acting as

$$\langle B_o u\,,\, (x_n)\rangle_{(\operatorname{dom} A_o^*)', \operatorname{dom} A_o^*} = \left\langle u\,,\, (\mathfrak{H}^* \widetilde{V}(x_n))(0) \right\rangle_{\mathcal{U}} \tag{6.16}$$

Furthermore,

$$C_o(x_n) = \left(\sum_{n=1}^{\infty} x_n \widetilde{v}_n \right)(0) \quad \forall\, (x_n) \in \operatorname{dom} A_o. \tag{6.17}$$

All the series here are limits in the $L^2(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$ norm.

Proof. We have already shown in Theorem 6.2.8 that $\widetilde{V}$ is a unitary similarity transformation between (6.12) and (6.2), i.e. (6.13) holds. The generators of (6.12) are therefore obtained by applying the same similarity transformation to the generators of (6.2) given in Lemma 6.1.3, in the sense of Lemma 2.4.15. This proves the corollary. We only have to observe that

$$\operatorname{dom} A_o = \left\{ (x_n) \in \ell^2 \;\middle|\; \widetilde{V}(x_n) \in \overline{\operatorname{ran}\mathfrak{H}} \cap W^{1,2}(\mathbb{R}_{\geqslant 0}; \mathcal{Y}) \right\}$$

becomes (6.14a) because $\widetilde{V}(x_n)$ is always in $\overline{\operatorname{ran}\mathfrak{H}}$. □

Now we show that the generators of the output normalized realization on ℓ^2 can also be computed via the state space transformations in Theorem 6.2.8.

Theorem 6.2.11. *Let $\mathcal{T}$ and $\mathcal{T}^+$ be as in Theorem 6.2.8 Then the following is true for the generators A_o ,B_o and C_o of the output normalized realization (6.12) on ℓ^2:*

(i) The space $\mathcal{Z} := \mathcal{T}\mathfrak{B}W_0^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U})$ is a subset of $\Sigma\ell^2$ and a core *for A_o, i.e. it is dense in $(\operatorname{dom} A_o, \|\cdot\|_{\operatorname{dom} A_o})$. Moreover,*

$$A_o z = \mathcal{T}\widetilde{A}\pi_{(\ker S^*)^\perp}\mathcal{T}^+ z \quad \forall\, z \in \mathcal{Z}, \tag{6.18}$$

where the quotient operator $\widetilde{A}$ of A is defined as in Theorem 2.7.3.

(ii) The adjoint operator $(\mathcal{T}|_{\overline{\mathcal{M}}})^*$ *of* $\mathcal{T}|_{\overline{\mathcal{M}}}$ *is given by* $\pi_{\overline{\mathcal{M}}} SV$ *and maps* $\operatorname{dom} A_o^*$ *into* $\operatorname{dom} \widetilde{A}^*$. *The operator* $\mathcal{T}|_{\overline{\mathcal{M}}}$ *has a continuous extension* $\mathcal{T}_{-1} : (\operatorname{dom} \widetilde{A}^*)' \to (\operatorname{dom} A_o^*)'$ *given by*

$$\langle \mathcal{T}_{-1} x', (y_n) \rangle_{(\operatorname{dom} A_o^*)', \operatorname{dom} A_o^*} = \langle x' , \pi_{\overline{\mathcal{M}}} SV(y_n) \rangle_{\ell^2} \tag{6.19}$$

for all $x' \in (\operatorname{dom} \widetilde{A}^*)'$ *and* $(y_n) \in \operatorname{dom} A_o^*$.

(iii) The space $\mathcal{T}^+ \mathcal{Z}$ *is a subset of the domain of the Cesàro extension* C_{ex} *and the following formulas hold:*

$$A_o|_{\ell^2} x = \mathcal{T}_{-1} \widetilde{A}|_{\overline{\mathcal{M}}} \pi_{(\ker S^*)^\perp} \mathcal{T}^+ x, \qquad \forall\, x \in \Sigma \ell^2 \tag{6.20}$$

$$B_o = \mathcal{T}_{-1} \widetilde{B}, \tag{6.21}$$

$$C_o z = C_{\mathrm{ex}} \mathcal{T}^+ z \qquad \forall\, x \in \mathcal{Z}. \tag{6.22}$$

Moreover, A_o *and* $(A_o)_{-1}$ *are obtained by taking the closures of the respective operators above.*

Proof. We start with proving (i): Lemma 2.4.3 (i) states that $\mathfrak{B}$ maps the set $W_0^{1,2}(\mathbb{R}_{\leqslant 0}; \mathcal{U})$ into $\operatorname{dom} A$, so the relation

$$\mathcal{Z} = V^* S^* \mathfrak{B} W_0^{1,2}(\mathbb{R}_{\leqslant 0}; \mathcal{U}) \subset V^* S^* (\operatorname{dom} A \cap \operatorname{ran} R) \subset \Sigma \ell^2$$

holds. This means that for arbitrary $z \in \mathcal{Z}$, we may write $z = V^* S^* y$ with $y \in \operatorname{dom} A \cap \operatorname{ran} R$. Then $S^* y \in \operatorname{ran} S^* R$, and with VV^* being the identity on this set, one gets

$$\pi_{(\ker \mathfrak{C})^\perp} y = S^{-*} S^* y = S^{-*} V V^* S^* y = S^{-*} V z \overset{(6.11)}{=} \pi_{(\ker S^*)^\perp} R U \Sigma^{-1} z.$$

Recall that by Theorem 2.7.3, $\pi_{(\ker \mathfrak{C})^\perp} \mathfrak{A}|_{\overline{\mathcal{M}}}$ is a semigroup whose generator $\widetilde{A}$ has the domain $\overline{\mathcal{M}} \cap \pi_{(\ker \mathfrak{C})^\perp} \operatorname{dom} A$. Since $\pi_{(\ker \mathfrak{C})^\perp} y$ is in this domain, the calculation

$$\begin{aligned}
&\lim_{t \downarrow 0} \frac{1}{t} \left(\mathcal{T} \mathfrak{A}(t) \mathcal{T}^+ z - z \right) \\
&\quad = \lim_{t \downarrow 0} \frac{1}{t} \left(V^* S^* \mathfrak{A}(t) \pi_{(\ker S^*)^\perp} R U \Sigma^{-1} z - z \right) \\
&\quad = \lim_{t \downarrow 0} \frac{1}{t} \left(V^* S^* \mathfrak{A}(t) S^{-*} V (V^* S^*) y - (V^* S^*) y \right)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{t\downarrow 0} \frac{1}{t} V^* \left(S^* \pi_{(\ker S^*)^\perp} \mathfrak{A}(t) \pi_{(\ker S^*)^\perp} y - S^* \pi_{(\ker S^*)^\perp} y \right) \\
&= V^* S^* \lim_{t\downarrow 0} \frac{1}{t} \left(\pi_{(\ker S^*)^\perp} \mathfrak{A}(t) \pi_{(\ker S^*)^\perp} y - \pi_{(\ker S^*)^\perp} y \right) \\
&= V^* S^* \pi_{(\ker S^*)^\perp} \widetilde{A} \pi_{(\ker S^*)^\perp} y \\
&= V^* S^* \widetilde{A} \pi_{(\ker S^*)^\perp} R U \Sigma^{-1} z
\end{aligned}$$

shows $A_o z = V^* S^* \widetilde{A} S^{-*} V z$ and $V^* S^* (\operatorname{dom} A \cap \operatorname{ran} R) \subset \operatorname{dom} A_o$. By [EN00, Proposition II.1.7], $\mathcal{Z}$ is already a core for A_o if it is $\mathfrak{A}_o$-invariant and dense in $\operatorname{dom} A_o$. It is indeed invariant: We can write any $z \in \mathcal{Z}$ as $z = V^* S^* \mathfrak{B} u$ with $u \in W_0^{1,2}(\mathbb{R}_{\leqslant 0}; \mathcal{U})$ and the equality

$$\begin{aligned}
\mathfrak{A}_o z &= V^* S^* \mathfrak{A}(t) S^{-*} V z = V^* S^* \mathfrak{A}(t) S^{-*} V (V^* S^* \mathfrak{B} u) \\
&= V^* S^* \mathfrak{A}(t) \pi_{(\ker S^*)^\perp} \mathfrak{B} u = V^* S^* \mathfrak{A}(t) \mathfrak{B} u = V^* S^* \mathfrak{B} \tau_-^t u
\end{aligned}$$

holds. Now the left shift of u is obviously again in $W_0^{1,2}(\mathbb{R}_{\leqslant 0}; \mathcal{U})$ and the overall expression therefore in $\mathcal{Z}$. Regarding density, we have that the continuous mapping $V^* S^* \mathfrak{B}$ maps the dense subset $W_0^{1,2}(\mathbb{R}_{\leqslant 0}; \mathcal{U})$ of $L^2(\mathbb{R}_{\leqslant 0}; \mathcal{U})$ into a dense subset of its image, $\operatorname{ran} V^* S^* \mathfrak{B}$, which is $\Sigma \ell^2$. Since this is dense in ℓ^2, we conclude that $\mathcal{Z}$ is dense in ℓ^2 and in particular in $\operatorname{dom} A_o$. This proves (i).

Now we proof (ii). A simple calculation shows that the adjoint $(\mathcal{T}|_{\overline{\mathcal{M}}})^*$ of $\mathcal{T}|_{\overline{\mathcal{M}}} : \overline{\mathcal{M}} \to \ell^2$ equals $\pi_{\overline{\mathcal{M}}} S V$. In order to show that $(\mathcal{T}|_{\overline{\mathcal{M}}})^*$ maps $\operatorname{dom} A_o^*$ into $\operatorname{dom} \widetilde{A}^*$, we prove the following three auxiliary statements:

(I) For all $(x_n) \in \ell^2$ there holds $SV(x_n) = \mathfrak{C}^* \widetilde{V}(x_n)$ with $\widetilde{V}$ as in (6.10): Due to continuity, the equality

$$Sx = SS^* S^{-*} x = \mathfrak{C}^* \mathfrak{C} S^{-*} x = \mathfrak{C}^* \mathfrak{V} x,$$

which is true for all $x \in \operatorname{ran} S^* R$, must hold on $\overline{\operatorname{ran} S^* R} = \operatorname{ran} V$ as well, and the assertion follows because $\widetilde{V} = \mathfrak{V} V$.

(II) The operator $\mathfrak{C}^*$ maps $W_0^{1,2}(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$ into $\operatorname{dom} A^*$: This follows because $\mathfrak{C}^* Я$ is the input operator of the dual system, and therefore, $\mathfrak{C}^* Я$ maps $W_0^{1,2}(\mathbb{R}_{\leqslant 0}; \mathcal{Y})$ into $\operatorname{dom} A^*$ by Lemma 2.4.3.

(III) The last assertion is that $\pi_{\overline{\mathcal{M}}} \mathfrak{C}^* = \pi_{\overline{\mathcal{M}}} \mathfrak{C}^* \pi_{\overline{\operatorname{ran} \mathfrak{H}}}$: If we take an arbitrary $y \in$

$L^2(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$, then $(\overline{\operatorname{ran} \mathfrak{H}})^\perp = \ker \mathfrak{B}^* \mathfrak{C}^*$ shows that

$$\widetilde{y} := \mathfrak{C}^* \pi_{\overline{\operatorname{ran} \mathfrak{H}}^\perp} y \quad \in \quad \operatorname{ran} \mathfrak{C}^* \cap \ker \mathfrak{B}^* \quad \subset \quad (\ker \mathfrak{C})^\perp \cap (\operatorname{ran} \mathfrak{B})^\perp.$$

Hence, taking the scalar product with any $x \in \mathcal{M}$, which must be of the form $x = \pi_{(\ker \mathfrak{C})^\perp} b$ for some $b \in \operatorname{ran} \mathfrak{B}$ yields

$$\langle x\,,\, \widetilde{y} \rangle_{\mathcal{X}} = \langle b - \pi_{\ker \mathfrak{C}} b\,,\, \widetilde{y} \rangle_{\mathcal{X}} = \underbrace{\langle b\,,\, \widetilde{y} \rangle_{\mathcal{X}}}_{=0} - \underbrace{\langle \pi_{\ker \mathfrak{C}} b\,,\, \widetilde{y} \rangle_{\mathcal{X}}}_{=0} = 0.$$

It follows $\widetilde{y} \in \overline{\mathcal{M}}^\perp$, and therefore

$$\begin{aligned} \pi_{\overline{\mathcal{M}}} \mathfrak{C}^* y &= \pi_{\overline{\mathcal{M}}} \mathfrak{C}^* \pi_{\overline{\operatorname{ran} \mathfrak{H}}} y + \pi_{\overline{\mathcal{M}}} \mathfrak{C}^* \pi_{(\overline{\operatorname{ran} \mathfrak{H}})^\perp} y \\ &= \pi_{\overline{\mathcal{M}}} \mathfrak{C}^* \pi_{\overline{\operatorname{ran} \mathfrak{H}}} y + \pi_{\overline{\mathcal{M}}} \widetilde{y} \\ &= \pi_{\overline{\mathcal{M}}} \mathfrak{C}^* \pi_{\overline{\operatorname{ran} \mathfrak{H}}} y, \end{aligned}$$

which is what we wanted to show.

In order to prove our original claim, we pick $(x_n) \in \operatorname{dom} A_o^*$. Because $\widetilde{V}$ was the similarity transformation between (6.12) and the output normalized shift realization, we have $\widetilde{V} \operatorname{dom} A_o^* = \pi_{\overline{\operatorname{ran} \mathfrak{H}}} W_0^{1,2}(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$ by (6.15a). Hence, $\widetilde{V}(x_n) = \pi_{\overline{\operatorname{ran} \mathfrak{H}}} y$ for some $y \in W_0^{1,2}(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$, and with (I) and (III) we get

$$(\mathcal{T}|_{\overline{\mathcal{M}}})^*(x_n) = \pi_{\overline{\mathcal{M}}} S V(x_n) = \pi_{\overline{\mathcal{M}}} \mathfrak{C}^* \widetilde{V}(x_n) = \pi_{\overline{\mathcal{M}}} \mathfrak{C}^* \pi_{\overline{\operatorname{ran} \mathfrak{H}}} y = \pi_{\overline{\mathcal{M}}} \mathfrak{C}^* y.$$

Now, because of (II), the latter is an element of $\pi_{\overline{\mathcal{M}}}(\operatorname{dom} A^* \cap (\ker \mathfrak{C})^\perp)$, which was shown to be $\operatorname{dom} \widetilde{A}^*$ in Theorem 2.7.3. Finally, Lemma 2.1.1 implies that (6.19) is an extension of $\mathcal{T}|_{\overline{\mathcal{M}}}$ as claimed.

It remains to show (iii). Observe that, on the set $\mathcal{Z}$, the operators $A_o|_{\ell^2}$ and $\mathcal{T}_{-1} \widetilde{A}|_{\overline{\mathcal{M}}} \pi_{(\ker S^*)^\perp} \mathcal{T}^+$ reduce to their unextended versions and therefore coincide according to (i). Since $\mathcal{Z}$ is a core of the closed operator $A_o|_{\ell^2}$, whose domain contains $\Sigma \ell^2$, this shows that $\mathcal{T}_{-1} \widetilde{A}|_{\overline{\mathcal{M}}} \pi_{(\ker S^*)^\perp} \mathcal{T}^+$ is closable and its closure is $A_o|_{\ell^2}$. In particular, both operators coincide on the larger set $\Sigma \ell^2$. Hence, the assertion (6.20) is true. We make use of this fact to determine the control operator via (2.7). For any

$u \in \mathcal{U}$ and λ in $\rho(A_o) \cap \rho(\widetilde{A})$ it can be calculated by

$$\begin{aligned} B_o u &= (\lambda - A_o|_{\ell^2})\mathfrak{B}_o e_\lambda u = \left(\lambda - \mathcal{T}_{-1}\widetilde{A}|_{\overline{\mathcal{M}}}\pi_{(\ker S^*)^\perp}\mathcal{T}^+\right)\mathcal{T}\pi_{(\ker S^*)^\perp}\mathfrak{B}e_\lambda u \\ &= \mathcal{T}_{-1}(\lambda - \widetilde{A}|_{\overline{\mathcal{M}}})\pi_{(\ker S^*)^\perp}\mathfrak{B}e_\lambda u = \mathcal{T}_{-1}(\lambda - \widetilde{A}|_{\overline{\mathcal{M}}})\widetilde{\mathfrak{B}}e_\lambda u = \mathcal{T}_{-1}\widetilde{B}u. \end{aligned}$$

Here we have used that $\mathcal{T}$ maps $\mathcal{M}$ into $\Sigma\ell^2$ and that $\pi_{(\ker S^*)^\perp}\mathcal{T}^+\mathcal{T}$ is the identity on $\mathcal{M}$. Now for the output operator C_o: We take an element $z \in \mathcal{Z}$. Then there is an $x \in \operatorname{dom} A \cap \operatorname{ran} \mathfrak{B}$ such that $z = \mathcal{T}x$, but in general $\mathcal{T}^+z \neq x$. So the first thing we have to check is that $\mathcal{T}^+z$ is in the domain of C_{ex}. An immediate consequence of the definition of $\operatorname{dom} C_{\text{ex}}$ is that $\ker S^* = \ker \mathfrak{C} \subset (\operatorname{dom} C_{\text{ex}} \cap \ker C_{\text{ex}})$. Since $\operatorname{dom} C_{\text{ex}}$ is a linear space, we deduce

$$\pi_{(\ker \mathfrak{C})^\perp}\mathcal{T}^+z = \pi_{(\ker \mathfrak{C})^\perp}x = \underbrace{x}_{\in \operatorname{dom} C_{\text{ex}}} - \underbrace{\pi_{\ker \mathfrak{C}}x}_{\in \operatorname{dom} C_{\text{ex}}} \in \operatorname{dom} C_{\text{ex}}.$$

With this we get indeed

$$\mathcal{T}^+z = \pi_{(\ker \mathfrak{C})^\perp}\mathcal{T}^+z + \pi_{\ker \mathfrak{C}}\mathcal{T}^+z \in \operatorname{dom} C_{\text{ex}}.$$

Hence,

$$C_o z = \lim_{t\to 0} \frac{1}{t}\int_0^t (\mathfrak{C}\mathcal{T}^+z)(\tau)\,\mathrm{d}\tau = C_{\text{ex}}\mathcal{T}^+z.$$

□

Remark 6.2.12. If $\ker \mathfrak{C} = \{0\}$, i.e. the original system is observable, the projection $\pi_{(\ker S^*)^\perp}$ is just the identity and $\widetilde{A}$ may be replaced by A. In the non-observable case, one might be tempted to omit the projection in the expression $V^*S^*A\pi_{(\ker S^*)^\perp}RU\Sigma^{-1}$ as well, since A maps $\operatorname{dom} A \cap \ker S^*$ into $\ker S^*$ anyway. However, this is not allowed because for arbitrary $z \in \mathcal{Z}$, the vector $RU\Sigma^{-1}z$ will in general not be in the domain of A, even though the projected vector $\pi_{(\ker S^*)^\perp}RU\Sigma^{-1}z$ lies in $\pi_{(\ker S^*)^\perp} \operatorname{dom} A$.

Truncation

The output normalized realization on ℓ^2 proves beneficial for approximation of the input-output map, or more precisely the Hankel operator: An approximating sequence of finite dimensional systems arises by truncating the output normalized realization on ℓ^2. In order to define such truncations properly, an additional presumption has to be made.

Presumption 6.2.13. *Presumption 6.2.3 holds and, in addition, the compact Hankel operator has a special representation: There exists an* $h \in L^1(\mathbb{R}_{\geqslant 0}; \mathcal{B}(\mathcal{U};\mathcal{Y}))$ *such that, for all* $u \in L^2(\mathbb{R}_{\leqslant 0};\mathcal{U})$,

$$(\mathfrak{H}u)(t) = \int_{-\infty}^{0} h(t-\tau)u(\tau)\,\mathrm{d}\tau, \quad \textit{f.a.a.} \quad t \geqslant 0. \tag{6.23}$$

The function h is the so-called *impulse response* of the input output map $\mathfrak{D}$.

Remark 6.2.14. (i) A short calculation shows that the adjoint of $\mathfrak{H}$ is the mapping

$$\begin{aligned} \mathfrak{H}^* : &L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y}) \to L^2(\mathbb{R}_{\leqslant 0};\mathcal{U}), \\ &(\mathfrak{H}^* y)(t) = \int_0^{\infty} (h(\tau - t))^*\, y(\tau)\,\mathrm{d}\tau, \quad \text{f.a.a.} \quad t \geqslant 0, \end{aligned} \tag{6.24}$$

cf. [GO14, Lemma 4.9]. Note that our Hankel operator differs from the Hankel operator defined in [GO14] and [GCP88] by multiplication from the left with the reflection operator $\mathfrak{R}$ defined in (2.10).

(ii) The representation (6.23) implies compactness of the Hankel operator according to [GCP88, Appendix 1, p.895].

(iii) Recall that an operator is *nuclear* if only if it is compact and its singular values are summable. It has been proven in [Gui12, Corollary 5.1.18.] that nuclearity of the Hankel operator implies that a representation of the form (6.23) exists. Further characterizations of nuclearity of Hankel operators can be found in [CS01, Opm08, Opm10].

Lemma 6.2.15. *Under Presumption 6.2.3, the following implications hold:*

(i) If Presumption 6.2.13 holds, then $\mathfrak{D}$ *is strongly regular.*

(ii) If Presumption 6.2.13 holds, then the Schmidt pairs $(\widetilde{u}_i, \widetilde{v}_i)$ *of the Hankel operator satisfy* $\widetilde{u}_i \in W^{1,1}(\mathbb{R}_{\leqslant 0};\mathcal{U})$ *and* $\widetilde{v}_i \in W^{1,1}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$.

(iii) If Presumption 6.2.13 holds and moreover $h \in L^1 \cap L^2(\mathbb{R}_{\geqslant 0}; \mathcal{B}(\mathcal{U};\mathcal{Y}))$, *then the Schmidt pairs* $(\widetilde{u}_i, \widetilde{v}_i)$ *of the Hankel operator satisfy* $\widetilde{u}_i \in W^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U})$ *and* $\widetilde{v}_i \in W^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$.

(iv) If the system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ *is of Pritchard-Salamon type on* $(\mathcal{U}, (\mathcal{W}, \mathcal{X}, \mathcal{V}), \mathcal{Y})$ *with control operator* B*, then* (6.23) *holds with* $h = \mathfrak{C}B \in L^2(\mathbb{R}_{\geqslant 0}; \mathcal{B}(\mathcal{U}; \mathcal{Y}))$*. If, in addition,* $\mathfrak{C} \in \mathcal{B}(\mathcal{W}; L^1(\mathbb{R}_{\geqslant 0}; \mathcal{Y}))$ *then Presumption 6.2.13 is fulfilled.*

Proof. (i) We will prove the regularity with the help of the the output normalized shift realization of $\mathfrak{D}$. Since regularity is independent of the realization, we may use the non-minimal output normalized shift realization of $\mathfrak{D}$ on $L^2(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$. So let A, B and C be as in Lemma 6.1.3 with $\mathcal{Z}$ replaced by $L^2(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$, and let $u \in \mathcal{U}$ be arbitrary. We want to show that $(\mathrm{I} - A|_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})})^{-1}Bu$ is in $\operatorname{dom} C_{\mathrm{ex}}$. From (6.24) we get that

$$\langle Bu\,,\, \varphi\rangle = \int_0^\infty \langle u\,,\, h(t)^*\varphi(t)\rangle \,\mathrm{d}t.$$

We will use that $(\mathrm{I} - A|_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})})^{-1}Bu$ is the unique function x in $L^2(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$ that satisfies

$$\Big\langle (\mathrm{I} - A|_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})})x\,,\, \varphi\Big\rangle_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})} = \int_0^\infty \langle u\,,\, h(t)^*\varphi(t)\rangle_{\mathcal{U}} \,\mathrm{d}t \quad \forall\, \varphi \in W_0^{1,2}(\mathbb{R}_{\geqslant 0}; \mathcal{Y}),$$

or equivalently,

$$\int_0^\infty \langle h(t)u\,,\, \varphi(t)\rangle_{\mathcal{Y}} \,\mathrm{d}t = \langle x\,,\, (\mathrm{I} - A^*)\varphi\rangle_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})}\,.$$

Now we define the function

$$x(\xi) := \mathrm{e}^{\xi}\left(\int_\xi^\infty \mathrm{e}^{-\tau}h(\tau)u \,\mathrm{d}\tau\right) \quad \forall\, \xi \geqslant 0,$$

and claim that it solves the above equation. It can be shown by standard estimates that this function is in $L^1 \cap L^\infty(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$. Hence, Hölder's inequality implies that this defines indeed an $L^2(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$ function. Furthermore, the derivative,

$$\dot{x}(\xi) = x(\xi) - h(\xi)u,$$

is integrable as well. Therefore, we can use partial integration, and we obtain for all $\varphi \in W_0^{1,2}(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$

$$\langle x\,,\, (\mathrm{I} - A^*)\varphi\rangle_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})} = \int_0^\infty \langle x(\xi)\,,\, \varphi(\xi) + \dot{\varphi}(\xi)\rangle_{\mathcal{Y}} \,\mathrm{d}\xi$$

$$= \int_0^\infty \langle x(\xi)\,,\,\varphi(\xi)\rangle_{\mathcal{Y}} - \langle \dot{x}(\xi)\,,\,\varphi(\xi)\rangle_{\mathcal{Y}}\,\mathrm{d}\xi - \langle x(0)\,,\,\varphi(0)\rangle_{\mathcal{Y}}$$
$$= \int_0^\infty \langle h(\xi)u\,,\,\varphi(t)\rangle_{\mathcal{Y}}\,\mathrm{d}\xi.$$

This shows that $x = (\mathrm{I} - A|_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})})^{-1}Bu$. Since x is easily seen to be continuous, it follows that $x \in \operatorname{dom} C_{\mathrm{ex}}$ with $C_{\mathrm{ex}}x = x(0) = \int_0^\infty \mathrm{e}^{-\tau}h(\tau)u\,\mathrm{d}\tau$. By [Sta05, Theorem 5.6.5] the fact that $(\mathrm{I} - A|_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})})^{-1}Bu \in \operatorname{dom} C_{\mathrm{ex}}$ for all $u \in \mathcal{U}$ implies that $\mathfrak{D}$ is strongly regular.

The assertions (ii) and (iii) are proven in [GO14, Theorem 4.4] and [GO14, Lemma 4.11], respectively.

The first part of (iv) follows from [CLTZ94, Lemma 3.5 and Corollary 3.6], and the additional assumption guarantees that $h \in L^1 \cap L^2(\mathbb{R}_{\geqslant 0}; \mathcal{B}(\mathcal{U};\mathcal{Y}))$. □

For the rest of this Section, Presumption 6.2.13 is assumed to hold. The special case where $h \in L^2(\mathbb{R}_{\geqslant 0}; \mathcal{B}(\mathcal{U};\mathcal{Y}))$ is considerably easier and will arise in Chapter 7, where we treat Pritchard-Salamon systems.

The above lemma makes the following definition of the output normalized truncation possible.

Definition 6.2.16 (output normalized truncation)**.** Let Presumption 6.2.13 hold and denote by $(\sigma_n)_{n\in\mathbb{N}}$ the sequence of singular values of the Hankel operator $\mathfrak{H}$ with corresponding Schmidt pairs $(\widetilde{v}_j, \widetilde{u}_j)$. Choose r such that $\sigma_{r+1} \neq \sigma_r$ and denote by $\frac{\mathrm{d}}{\mathrm{d}\xi}$ the differential operator $\frac{\mathrm{d}}{\mathrm{d}\xi} : W^{1,1}(\mathbb{R}_{\geqslant 0};\mathcal{Y}) \to L^1(\mathbb{R}_{\geqslant 0};\mathcal{Y})$. The *$r$-th order output normalized truncation of* $\mathfrak{D}$ is the finite-dimensional system (A_r, B_r, C_r, D) defined by

$$\begin{aligned} A_r &= \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix} \in \mathbb{C}^{r,r}, \qquad & B_r &= \begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix} \in \mathcal{B}(\mathcal{U}, \mathbb{C}^r), \\ C_r &= \begin{bmatrix} c_1 & \cdots & c_r \end{bmatrix} \in \mathcal{B}(\mathbb{C}^r, \mathcal{Y}), \qquad & D &= \lim_{\lambda\to\infty} \widehat{\mathfrak{D}}(\lambda), \end{aligned} \tag{6.25}$$

and

$$\begin{aligned} a_{ij} &= \left\langle \widetilde{v}_i\,,\,\tfrac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{v}_j \right\rangle_{L^\infty(\mathbb{R}_{\geqslant 0};\mathcal{Y}),L^1(\mathbb{R}_{\geqslant 0};\mathcal{Y})} && \in \mathbb{C}, \\ b_i &= \sigma_i \left\langle \cdot\,,\,\widetilde{u}_i(0)\right\rangle_{\mathcal{U}} && \in \mathcal{B}(\mathcal{U}, \mathbb{C}), \\ c_j &= \widetilde{v}_j(0) && \in \mathcal{Y}. \end{aligned}$$

Remark 6.2.17. The output normalized truncation is called "balanced truncation" in [GO14]. We prefer the expression "output normalized" and reserve the name "balanced truncation" for the realization that actually is balanced in the sense of Definition 6.4.1.

The input-output map of the output normalized truncation approximates the input-output behavior of the original system. More precisely, Guiver and Opmeer have proven the following theorem in [GO14, Theorem 2.3 and Proposition 5.12]:

Theorem 6.2.18. *Under Presumption 6.2.13, the output normalized truncation of* (A, B, C, D) *is a minimal* 0*-bounded output-normalized state linear system on* $(\mathcal{U}$*,* $\mathbb{C}^r$*,* $\mathcal{Y})$ *with an exponentially stable semigroup. Its input-output map* $\mathfrak{D}_r$ *approximates* $\mathfrak{D}$ *in the sense that*

$$\|\mathfrak{D} - \mathfrak{D}_r\|_{\mathrm{TIC}_0^2(\mathcal{U};\mathcal{Y})} = \|\widehat{\mathfrak{D}} - \widehat{\mathfrak{D}}_r\|_{\mathcal{H}^\infty(\mathbb{C}_{\geqslant 0})} \leqslant 2 \sum_{\{n>r \mid \sigma_n \neq \sigma_k \forall k<n\}} \sigma_n. \tag{6.27}$$

Note that our presumptions do not guarantee summability of the Hankel singular values. If they are not summable, the right hand side in (6.27) is to be interpreted as infinity.

The remainder of the current section shows that the output normalized truncation is obtained by some kind of truncation of the output normalized realization on ℓ^2 and therefore deserves its name. If $h \in L^2(\mathbb{R}_{\geqslant 0}; \mathcal{B}(\mathcal{U};\mathcal{Y}))$, then Lemma 6.2.15 and Corollary 6.2.10 imply that $e_j \in \operatorname{dom} A_o$ for all $j \in \mathbb{N}$ and that the coefficients of the output normalized truncation satisfy

$$a_{ij} = \langle e_i \,,\, A_o e_j \rangle_{\ell^2} \,, \qquad c_j \;= C_o e_j.$$

Without the additional assumption on h the situation is much more difficult for the following reason. Since $\widetilde{v}_i$ is only in $W^{1,1}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$, the unit vector e_j is not in the domain of A_o described in (6.14b). It is therefore necessary to use the extension $A_o|_{\ell_2}$ on e_j instead. The functional $A_o|_{\ell_2} e_j \in (\operatorname{dom} A_o^*)'$ is by Lemma 2.1.1 defined through the adjoint (6.15b), and can, by partial integration, be shown to equal

$$\langle (x_n) \,,\, A_o|_{\ell_2} e_j \rangle_{\operatorname{dom} A_o^*, (\operatorname{dom} A_o^*)'} = -\int_0^\infty \left\langle \tfrac{\mathrm{d}}{\mathrm{d}\xi} \sum_{n=1}^\infty \widetilde{v}_n(\xi) x_n \,,\, \widetilde{v}_j(\xi) \right\rangle_{\mathcal{Y}} \mathrm{d}\xi,$$

$$= \int_0^\infty \sum_{n=1}^\infty \left\langle \widetilde{v}_n(\xi) x_n \,,\, \tfrac{\mathrm{d}}{\mathrm{d}\xi} \widetilde{v}_j(\xi) \right\rangle_{\mathcal{Y}} \mathrm{d}\xi,$$

with $\frac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{v}_j \in L^1(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$. This representation is valid for all (x_n) in the domain of A_o^*, which means $\sum_{n=1}^\infty \widetilde{v}_n x_n \in W_0^{1,2}(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$. Unfortunately, e_i is not in this domain. At a first glance, it may seem straight forward to extend this expression via

$$\langle e_i \,,\, A_o|_{\ell_2} e_j \rangle := \int_0^\infty \left\langle \widetilde{v}_i(\xi) \,,\, \tfrac{\mathrm{d}}{\mathrm{d}\xi} \widetilde{v}_j(\xi) \right\rangle_{\mathcal{Y}} \mathrm{d}\xi = \left\langle \widetilde{v}_i \,,\, \tfrac{\mathrm{d}}{\mathrm{d}\xi} \widetilde{v}_j \right\rangle_{L^\infty(\mathbb{R}_{\geqslant 0};\mathcal{Y}), L^1(\mathbb{R}_{\geqslant 0};\mathcal{Y})},$$

which is well-defined because $\widetilde{v}_i \in L^\infty(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$. To do this properly, however, we need to extend the functional $A_o|_{\ell_2} e_j$ to the set $\{(x_n) \in \ell_2 : \sum_{n=1}^\infty \widetilde{v}_n x_n \in W^{1,1}(\mathbb{R}_{\geqslant 0}; \mathcal{Y})\}$. Since $W_0^{1,2}(\mathbb{R}_{\geqslant 0}; \mathcal{Y}) \cap W_0^{1,1}(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$ is not dense in $W^{1,1}(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$, this extension can not simply be obtained from continuity. Instead we must explicitly declare the construction of this extension to be the following: First, the subspace of all functionals in $(W_0^{1,1}(\mathbb{R}_{\geqslant 0}; \mathcal{Y}))'$ that can be represented by an L^1-function is identified with the actual space $L^1(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$, and then it is embedded into the dual space of $W^{1,1}(\mathbb{R}_{\geqslant 0}; \mathcal{Y})$. For the operator B_o, which experiences the same difficulties only with the L^1-function $h(\cdot)u$ instead of $\frac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{v}_j$, one can either proceed in the same way or, more elegantly, by using the Cesàro extension of its adjoint. The next theorem formalizes the construction explained above.

Theorem 6.2.19. *Let Presumption 6.2.13 hold, and let A_o, B_o and C_o be the generators of the output normalized realization* (6.12)*. Then there exists a space $\mathcal{W}_o \hookrightarrow \ell_2$ such that the following holds true:*

(i) For all $i \in \mathbb{N}$ the canonical unit vector e_i is an element of $\mathcal{W}_o$;

(ii) $A_o|_{\ell_2} e_i \in \mathcal{W}_o'$ for all i, $\operatorname{ran} B_o \subset \mathcal{W}_o'$ and $\mathcal{W}_o \subset \operatorname{dom}(C_o)_{\mathrm{ex}}$;

(iii) The matrix entries a_{ij}, b_i, c_j of the output normalized truncation in Definition 6.2.16 satisfy

$$\begin{aligned} a_{ij} &= \langle e_i \,,\, A_o e_j \rangle_{\mathcal{W}_o, \mathcal{W}_o'} && \in \mathbb{C}, \\ b_i(\cdot) &= \langle e_i \,,\, B_o \cdot \rangle_{\mathcal{W}_o, \mathcal{W}_o'} = \langle (B_o^*)_{\mathrm{ex}} e_i \,,\, \cdot \rangle_{\mathcal{U}} && \in \mathcal{B}(\mathcal{U}; \mathbb{C}), \\ c_j &= (C_o)_{\mathrm{ex}} e_j && \in \mathcal{Y}. \end{aligned} \tag{6.28}$$

where $(B_o^*)_{\mathrm{ex}}$ *is the Cesàro extension of* B_o^*.

If, in addition, h *is in* $L^2(\mathbb{R}_{\geqslant 0};\mathcal{B}(\mathcal{U};\mathcal{Y}))$*, then it is possible to choose* $\mathcal{W}_o = \ell^2$.

Proof. Let us first assume that $h \in L^1 \cap L^2(\mathbb{R}_{\geqslant 0};\mathcal{B}(\mathcal{U};\mathcal{Y}))$ and set $\mathcal{W}_0 = \ell^2$. The representation of $\mathfrak{H}^*$ in (6.24) and B_o in (6.16) show that the following holds for all $(x_n) \in \operatorname{dom} A_o^*$ and $u \in \mathcal{U}$:

$$\langle B_o u\,,\,(x_n)\rangle_{(\operatorname{dom} A_o^*)',\operatorname{dom} A_o^*} = \int_0^\infty \Big\langle u\,,\,h^*(\tau)\widetilde{V}(x_n)\Big\rangle_{\mathcal{U}}\,\mathrm{d}\tau = \Big\langle \widetilde{V}^* h(\cdot)u, (x_n)\Big\rangle_{\ell^2}.$$

Recall that $(\operatorname{dom} A_o^*)'$ is by definition the dual space of $\operatorname{dom} A_o^*$ *with respect to the pivot space* ℓ^2. Whence ℓ^2 is by definition considered as a subset of $(\operatorname{dom} A_o^*)'$, and in this sense the functional $B_o u \in (\operatorname{dom} A_o^*)'$ is equivalent to the ℓ^2 sequence $\widetilde{V}^* h(\cdot)u$. In this way, we can make sense of the following scalar product:

$$\begin{aligned}\langle B_o u\,,\,e_i\rangle_{\ell^2} &= \Big\langle \widetilde{V}^* h(\cdot)u\,,\,e_i\Big\rangle_{\ell^2} = \langle h(\cdot)u\,,\,\widetilde{v}_i\rangle_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})}\\ &= \langle u\,,\,(\mathfrak{H}^*\widetilde{v}_i)(0)\rangle_{\mathcal{U}} = \Big\langle u\,,\,(\widetilde{U}\Sigma\widetilde{V}^*\widetilde{v}_i)(0)\Big\rangle_{\mathcal{U}}\\ &= \langle u\,,\,\sigma_i\widetilde{u}_i(0)\rangle_{\mathcal{U}} = b_i(u).\end{aligned}$$

As mentioned before this theorem, the other equations in (6.28) follow immediately from Corollary 6.2.10 since e_j is in $\operatorname{dom} A_o$. So the case where $h \in L^2(\mathbb{R}_{\geqslant 0};\mathcal{B}(\mathcal{U};\mathcal{Y}))$ is settled.

For $h \in L^1(\mathbb{R}_{\geqslant 0};\mathcal{B}(\mathcal{U};\mathcal{Y}))$, we show that the space

$$\mathcal{W}_o := \left\{\ (w_n) \in \ell_2\ \middle|\ \widetilde{V}(w_n) \in W^{1,1}(\mathbb{R}_{\geqslant 0};\mathcal{Y})\ \right\}$$

with the norm $\|(w_n)\|_{\mathcal{W}_o} := \|\widetilde{V}(w_n)\|_{W^{1,1}(\mathbb{R}_{\geqslant 0};\mathcal{Y})}$ has the asserted properties. Because the Schmidt vectors $\widetilde{v}_i = \widetilde{V}e_i$ are in $W^{1,1}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$, the vector e_i is by definition in $\mathcal{W}_o$ and (i) is true. We claim that the space

$$\mathring{\mathcal{W}}_o := \left\{ f \in (\operatorname{dom} A_o^*)'\ \middle|\ \begin{array}{l} \exists\, \mathring{f} \in L^1(\mathbb{R}_{\geqslant 0};\mathcal{Y}) \quad \forall\,(x_n) \in \operatorname{dom} A_o^*: \\ \langle f\,,\,(x_n)\rangle_{(\operatorname{dom} A_o^*)',\operatorname{dom} A_o^*} = \displaystyle\int_0^\infty \langle \mathring{f}(\xi), y(\xi)\rangle_{\mathcal{Y}}\,\mathrm{d}\xi \\ \text{for some } y \in W_0^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{Y}) \text{ with } \pi_{\overline{\operatorname{ran}\mathfrak{H}}}\,y = \widetilde{V}(x_n). \end{array} \right\}$$

with norm $\|f\|_{\mathring{\mathcal{W}}_o} := \|\mathring{f}\|_{L^1(\mathbb{R}_{\geqslant 0};\mathcal{Y})}$ is continuously embedded into $\mathcal{W}_o'$ via the injection

$$\iota : \mathring{\mathcal{W}}_0 \to \mathcal{W}_o', \quad \langle \iota f \,,\, (z_n)\rangle_{\mathcal{W}_o',\mathcal{W}_o} := \int_0^\infty \Big\langle \mathring{f}(\xi) \,,\, \sum_{n=1}^{\infty} z_n \widetilde{v}_n(\xi) \Big\rangle_{\mathcal{Y}} \,\mathrm{d}\xi \quad \forall\, (z_n) \in \mathcal{W}_o.$$

A simple estimate shows $|\langle \iota f \,,\, (z_n)\rangle_{\mathcal{W}_o',\mathcal{W}_o}| \leqslant \|\iota f\|_{L^1(\mathbb{R}_{\geqslant 0};\mathcal{Y})} \|(z_n)\|_{\mathcal{W}_o}$ and hence, ιf is a functional on $\mathcal{W}_o$. The estimate $\|\iota f\|_{\mathcal{W}_o'} \leqslant \|f\|_{\mathring{\mathcal{W}}_o}$ moreover shows the continuity of the embedding ι. To conclude the injectivity of ι, observe that the statement $\langle \iota f \,,\, (z_n)\rangle_{\mathcal{W}_o',\mathcal{W}_o} = 0$ for all $(z_n) \in \mathcal{W}_o$ is equivalent to $\int_0^\infty \langle \mathring{f}(\xi), w(\xi)\rangle_{\mathcal{Y}} \,\mathrm{d}\xi = 0$ for all $w \in W^{1,1}(\mathbb{R}_{\geqslant 0};\mathcal{Y}) \cap \operatorname{ran}\mathfrak{H}$, which implies $\mathring{f} \in (\operatorname{ran}\mathfrak{H})^\perp$. Hence, f is the zero functional on $\operatorname{dom} A_o^*$ if $\iota f = 0$. Note that the embedding of ℓ^2 into $(\operatorname{dom} A_o^*)'$ that we used in the previous case is inherent in the definition of $(\operatorname{dom} A_o^*)'$. The analogous embedding ι that we have now does not come automatically with any definition. We had to define it manually.

In order to prove that $A_o|_{\ell_2} e_j \in \mathcal{W}_o'$, it suffices now to show that $A_o|_{\ell_2} e_j \in \mathring{\mathcal{W}}_o$ for all $j \in \mathbb{N}$. Choose an arbitrary (z_n) in $\operatorname{dom} A_o^*$. Recalling the formula (6.15b) for A_o^*, we choose $y \in W_0^{1,2}(\mathbb{R}_{\geqslant 0};\mathcal{Y})$ with $\pi_{\overline{\operatorname{ran}\mathfrak{H}}} y = \widetilde{V}(z_n)$ and have

$$\begin{aligned}
\langle (z_n) \,,\, A_o|_{\ell_2} e_j \rangle_{\operatorname{dom} A_o^*,(\operatorname{dom} A_o^*)'} &= \langle A_o^*(z_n) \,,\, e_j \rangle_{\ell_2} \\
&= \Big\langle -\widetilde{V}^* \tfrac{\mathrm{d}}{\mathrm{d}\xi} y \,,\, e_j \Big\rangle_{\ell_2} \\
&= \Big\langle -\tfrac{\mathrm{d}}{\mathrm{d}\xi} y \,,\, \widetilde{V} e_j \Big\rangle_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})} \\
&= \int_0^\infty \Big\langle y(\xi) \,,\, \tfrac{\mathrm{d}}{\mathrm{d}\xi} \widetilde{v}_j(\xi) \Big\rangle_{\mathcal{Y}} \,\mathrm{d}\xi.
\end{aligned}$$

In the last line we have used partial integration between a $W^{1,1}$- and a $W_0^{1,2}$-function, which is justified by approximation with smooth functions. The equation above shows that $A_o|_{\ell_2} e_j$ is an element of $\mathring{\mathcal{W}}_o$ with $\mathring{A_o|_{\ell_2} e_j} = \frac{\mathrm{d}}{\mathrm{d}\xi} \widetilde{v}_j$. Now it is merely a matter of definition to see that $\langle e_i \,,\, \iota A_o|_{\ell_2} e_j \rangle_{\mathcal{W}_o,\mathcal{W}_o'}$ equals the desired formula for the matrix entries a_{ij}.

Similar to the previous case, the representations of B_o and $\mathfrak{H}^*$ in (6.16) and (6.24) imply for all $(x_n) \in (\operatorname{dom} A_o)^*$ and $u \in \mathcal{U}$

$$\langle B_o u \,,\, (x_n)\rangle_{(\operatorname{dom} A_o^*)',\operatorname{dom} A_o^*} = \Big\langle h(\cdot) u \,,\, \widetilde{V}(x_n) \Big\rangle_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})}.$$

Since h is in $L^1(\mathbb{R}_{\geqslant 0};\mathcal{B}(\mathcal{U};\mathcal{Y}))$, this shows $B_o u \in \mathring{\mathcal{W}}_o$ with $\mathring{B}_o u = hu$. Hence, the embedding ι gives

$$\begin{aligned}\langle B_o u, e_i\rangle_{\mathcal{W}_o',\mathcal{W}_o} &= \int_0^\infty \langle h(\xi)u, \widetilde{v}_i(\xi)\rangle_{\mathcal{Y}}\,\mathrm{d}\xi \overset{(6.24)}{=} \langle u, (\mathfrak{H}^*\widetilde{v}_i)(0)\rangle_{\mathcal{U}} \\ &= \langle u, \sigma_i\widetilde{u}_i(0)\rangle_{\mathcal{U}} = b_i(u).\end{aligned}$$

The alternative representation of b_i uses the Cesàro extension of B_o^*, which is defined as

$$(B_o^*)_{\mathrm{ex}}x := \lim_{t\to 0}\frac{1}{t}\int_0^t (\mathfrak{B}_o^* x)(-\tau)\,\mathrm{d}\tau \quad \forall\, x \in \operatorname{dom}(B_o^*)_{\mathrm{ex}}.$$

For e_i we have

$$\mathfrak{B}_o^* e_i \overset{(6.13)}{=} \mathfrak{H}^*\widetilde{V}e_i = \widetilde{U}\Sigma\widetilde{V}^*\widetilde{v}_i = \sigma_i\widetilde{u}_i \quad \in W^{1,1}(\mathbb{R}_{\leqslant 0};\mathcal{U}).$$

Since $\operatorname{dom}(B_o^*)_{\mathrm{ex}}$ is by definition the set where the limit above exists, we conclude that $e_i \in \operatorname{dom}(B_o^*)_{\mathrm{ex}}$ for all $i \in \mathbb{N}$ and that

$$\langle u, (B_o^*)_{\mathrm{ex}}e_i\rangle_{\mathcal{U}} = \langle u\,,\, \sigma_i\widetilde{u}_i(0)\rangle_{\mathcal{U}} = b_i(u)(0).$$

Analogously, $\mathcal{W}_o$ is contained in the domain of $(C_o)_{\mathrm{ex}}$ since $W^{1,1}$-functions are continuous and therefore posess a Cesàro limit at zero. Thus,

$$(C_o)_{\mathrm{ex}}e_j = \lim_{t\to 0}\frac{1}{t}\int_0^t (\mathfrak{C}_o e_j)(\tau)\,\mathrm{d}\tau = \lim_{t\to 0}\frac{1}{t}\int_0^t \widetilde{v}_j(\tau)\,\mathrm{d}\tau = \widetilde{v}_j(0) = c_i,$$

and the proof is complete. □

6.3. Input normalizing transformations

In analogy to Section 6.2, it is possible to construct an input normalized realization on ℓ^2, which is unitary similar to the input normalized shift realization on $(\ker\mathfrak{H})^\perp$ via the transformation $\widetilde{U}$ in (6.10). We will not carry this out here explicitly. Instead we only give one lemma highlighting another interesting aspect: The restriction of the output normalized realization to the subspace $\Sigma\ell^2$ is input normalized. This lemma is based on the same principle.

We equip the space $\Sigma\ell^2$, i.e. the image of Σ, with the scalar product

$$\langle \cdot\,,\,\cdot\rangle_{\Sigma\ell^2} := \langle \Sigma^{-1}\cdot\,,\,\Sigma^{-1}\cdot\rangle_{\ell^2}.$$

Lemma 6.3.1. *Let Presumption 6.2.3 hold. The output normalized system* (6.12) *restricted to $\Sigma\ell^2$, i.e.*

$$(\underline{\mathfrak{A}}, \underline{\mathfrak{B}}, \underline{\mathfrak{C}}, \mathfrak{D}) := (\mathfrak{A}_o|_{\Sigma\ell^2}, \mathfrak{B}_o, \mathfrak{C}_o|_{\Sigma\ell^2}, \mathfrak{D})\,, \tag{6.29}$$

is an input normalized 0*-bounded L^2-well-posed linear system on $(\Sigma\ell^2, \|\cdot\|_{\Sigma\ell^2})$. The generator $\underline{A}$ of $\underline{\mathfrak{A}}$ satisfies*

$$\begin{aligned} \operatorname{dom}\underline{A} = \mathcal{Z} = \{x \in \operatorname{dom}A_o \cap \Sigma\ell^2 : A_o x \in \Sigma\ell^2\}, \\ \underline{A} : \operatorname{dom}\underline{A} \subset \Sigma\ell^2 \to \Sigma\ell^2, \quad \underline{A}x = A_o x \quad \forall\, x \in \operatorname{dom}\underline{A}, \end{aligned} \tag{6.30}$$

where the space $\mathcal{Z} = V^*S^*\mathfrak{B}W_0^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U})$ is as in Theorem 6.2.11.

Proof. In analogy to the proof of Theorem 6.2.8, we claim that the operator

$$\widetilde{U}\Sigma^{-1} : \Sigma\ell^2 \to (\ker\mathfrak{H})^\perp,$$

with $\widetilde{U}$ as in the singular value decomposition (6.10), is a unitary similarity transformation between the system (6.29) and the input normalized shift realization (6.1) on $(\ker\mathfrak{H})^\perp$. We know that $\widetilde{U}$ is unitary. With respect to the scalar product of $\Sigma\ell^2$, the operator $\Sigma \in \mathcal{B}(\ell^2;\Sigma\ell^2)$ is unitary as well. So it suffices to show that $\widetilde{U}\Sigma^{-1}$ transforms (6.29) into (6.1). In Lemma 6.2.5 we proved that $R^{-1}\mathfrak{B}$ maps $(\ker\mathfrak{H})^\perp$ into $(\ker S^*R)^\perp$ and the restriction $R^{-1}\mathfrak{B}|_{(\ker\mathfrak{H})^\perp}$ was named $\mathfrak{U}^*$. Using the readily verified fact that $R^{-1}\mathfrak{B}$ also maps $\ker\mathfrak{H}$ into $\ker S^*R$, we therefore have

$$\pi_{(\ker S^*R)^\perp}(R^{-1}\mathfrak{B})u = \mathfrak{U}^*\pi_{(\ker\mathfrak{H})^\perp}u \quad \forall\, u \in L^2(\mathbb{R}_{\leqslant 0};\mathcal{U}),$$

and consequently for all $x \in \operatorname{ran}\mathfrak{B}$

$$\begin{aligned} V^*S^*x &= V^*S^*RR^{-1}\mathfrak{B}\mathfrak{B}^{-1}x = V^*S^*R\pi_{(\ker S^*R)^\perp}(R^{-1}\mathfrak{B})\mathfrak{B}^{-1}x \\ &= \Sigma U^*\mathfrak{U}^*\pi_{(\ker\mathfrak{H})^\perp}\mathfrak{B}^{-1}x = \Sigma\widetilde{U}^*\pi_{(\ker\mathfrak{H})^\perp}\mathfrak{B}^{-1}x. \end{aligned} \tag{6.31}$$

Using this and the $\mathfrak{A}$-invariance of $\operatorname{ran}\mathfrak{B}$, we get

$$\begin{aligned}\mathfrak{A}_o(t)|_{\Sigma\ell^2} &= V^*S^*\mathfrak{A}(t)RU\Sigma^{-1}\\ &= \Sigma\widetilde{U}^*\pi_{(\ker\mathfrak{H})^\perp}\mathfrak{B}^{-1}\mathfrak{A}(t)\mathfrak{B}\mathfrak{U}U\Sigma^{-1}\\ &= \Sigma\widetilde{U}^*\pi_{(\ker\mathfrak{H})^\perp}\tau_-^t|_{(\ker\mathfrak{H})^\perp}\widetilde{U}\Sigma^{-1}.\end{aligned}$$

This shows that the semigroup $\mathfrak{A}_o|_{\Sigma\ell^2}$ is unitarily similar to the strongly continuous semigroup of the shift realization on $(\ker\mathfrak{H})^\perp$. For the input operator, equation (6.31) immediately gives the asserted equality

$$V^*S^*\mathfrak{B} = \Sigma\widetilde{U}^*\pi_{(\ker\mathfrak{H})^\perp}\pi_{(\ker\mathfrak{B})^\perp} = \Sigma\widetilde{U}^*\pi_{(\ker\mathfrak{H})^\perp},$$

and finally, the output operator $\underline{\mathfrak{C}}$ equals

$$\mathfrak{C}RU\Sigma^{-1} = \mathfrak{C}\mathfrak{B}(\mathfrak{B}^{-1}R)U\Sigma^{-1} \overset{(6.5)}{=} \mathfrak{H}|_{(\ker\mathfrak{H})^\perp}\mathfrak{U}U\Sigma^{-1} = \mathfrak{H}|_{(\ker\mathfrak{H})^\perp}\widetilde{U}\Sigma^{-1}.$$

Altogether we have shown that (6.29) is unitary similar to the system (6.1). Therefore, its well-posedness follows from Lemma 2.4.15 and moreover, the unitary transformations keep the system input normalized.

Also by Lemma 2.4.15, the domain of $\underline{A}$ is given by the transformation $\Sigma\widetilde{U}^*$ applied to the domain of the exactly controllable shift realization. With (6.31) this becomes

$$\operatorname{dom}\underline{A} = \Sigma\widetilde{U}^*\pi_{(\ker\mathfrak{H})^\perp}W_0^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U}) = V^*S^*\mathfrak{B}W_0^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U}) = \mathcal{Z}.$$

On the other hand, we know that $\underline{\mathfrak{A}}$ is the restriction of $\mathfrak{A}_o$ and strongly continuous with respect to $\|\cdot\|_{\Sigma\ell^2}$. Hence, the generator $\underline{A}$ must be the part of A_o in $\Sigma\ell^2$ by Lemma A.1.4. The latter is by definition the last term in (6.30) and therefore the proof is complete. □

Remark 6.3.2. (i) We point out that the crucial property of the space $\mathcal{Z}$ is that $x \in \mathfrak{B}W_0^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U})$ does not only imply $x \in \operatorname{dom} A \cap \operatorname{ran}\mathfrak{B}$, but also $Ax \in \operatorname{ran}\mathfrak{B}$. This explains the fact that V^*S^*Ax is again an element of $\Sigma\ell^2$ and thus the relation (6.30).

(ii) A further similarity transformation with the unitary operator $\Sigma^{-1} : \Sigma\ell^2 \to \ell^2$

yields an input normalized system on the state space ℓ^2 as mentioned at the beginning of this section. This gives a completely analogous result to Theorem 6.2.8 with output normalization replaced by input normalization. For the upcoming interpolation step however, the present system with state space $\Sigma\ell^2$ is more convenient.

6.4. Balancing transformations

In the previously constructed realizations on ℓ^2 one of the Gramians was the identity operator and the other was the diagonal operator Σ^2. The most popular realization for finite-dimensional systems is the one in which both Gramians are equal to Σ. This realization is called balanced and is the eponym of the balanced truncation. For infinite-dimensional systems, equality of the Gramians can also be achieved, but with a lot of technical effort: The balanced realization has to be constructed by interpolating the input normalized and the output normalized realization.

Definition 6.4.1 (balanced system)**.** We say that a 0-bounded L^2-well-posed linear system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ on $(\mathcal{U}, \ell^2, \mathcal{Y})$ is *balanced* if and only if there exists a diagonal operator $\Sigma \in \mathcal{B}(\ell^2)$ such that the Gramians satisfy

$$\mathfrak{B}\mathfrak{B}^* = \mathfrak{C}^*\mathfrak{C} = \Sigma.$$

Theorem 6.4.2. *Let Presumption 6.2.3 hold and let $S^*R = V\Sigma U^*$ be the singular value decomposition of the operator S^*R. Then the operators*

$$\begin{aligned} T:\ \operatorname{ran} R \subset X &\to \ell_2, & T^+:\ \Sigma^{1/2}\ell_2 \subset \ell_2 &\to X, \\ x &\mapsto \Sigma^{-1/2}V^*S^*x, & x &\mapsto RU\Sigma^{-1/2}x \end{aligned} \tag{6.32}$$

are well-defined, and the following assertions are true:

(i) $\operatorname{ran} R = \operatorname{ran} \mathfrak{B}$*, and thus* $\mathfrak{A}(t) \operatorname{ran} R \subset \operatorname{ran} R$ *for all* $t \geqslant 0$*.*

(ii) There exists a constant $c > 0$ such that, for all $x \in \Sigma^{1/2}\ell_2$, $u \in L^2(\mathbb{R}_{\leqslant 0}; \mathcal{U})$ and $t \geqslant 0$,

$$\|T\mathfrak{A}(t)T^+x\|_{\ell_2} \leqslant c\ \|x\|_{\ell_2}, \qquad \|T\mathfrak{B}u\|_{\ell_2} \leqslant c\ \|u\|_{L^2(\mathbb{R}_{\leqslant 0};\mathcal{U})},$$

$$\|\mathfrak{C}T^+x\|_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})} \leqslant c\ \|x\|_{\ell_2}.$$

(iii) With the unique continuous extensions

$$\overline{T\mathfrak{A}T^+} : \mathbb{R}_{\geqslant 0} \to \mathcal{B}(\ell_2), \quad t \mapsto \overline{T\mathfrak{A}(t)T^+},$$

and

$$\overline{\mathfrak{C}T^+} \in \mathcal{B}(\ell_2; L^2(\mathbb{R}_{\geqslant 0}; \mathcal{Y})),$$

the quadruple

$$(\mathfrak{A}_b, \mathfrak{B}_b, \mathfrak{C}_b, \mathfrak{D}) := (\overline{T\mathfrak{A}T^+}, T\mathfrak{B}, \overline{\mathfrak{C}T^+}, \mathfrak{D}) \tag{6.33}$$

forms a minimal balanced 0*-bounded* L^2*-well-posed linear system on* $(\mathcal{U}, \ell_2, \mathcal{Y})$*.*

The idea behind the proof of Theorem 6.4.2 is to obtain the balanced system by interpolating between the output normalized realization (6.12) of $\mathfrak{D}$ on $(\ell^2, \langle\cdot,\cdot\rangle_{\ell^2})$ and its restriction to $(\Sigma\ell^2, \langle\cdot,\cdot\rangle_{\Sigma\ell^2})$ described in (6.29). So an important ingredient for the proof is the following auxiliary result about well-posedness of an interpolated system.

Lemma 6.4.3. *Let* $\underline{\mathcal{X}}$*,* $\mathcal{X}$ *and* $\overline{\mathcal{X}}$ *be Hilbert spaces with* $\underline{\mathcal{X}} \hookrightarrow \mathcal{X} \hookrightarrow \overline{\mathcal{X}}$*. Assume that there exists a positive operator* $\Sigma \in \mathcal{B}(\overline{\mathcal{X}})$ *such that* $\mathcal{X} = \operatorname{ran}\Sigma^{1/2}$*,* $\underline{\mathcal{X}} = \operatorname{ran}\Sigma$ *and*

$$\langle x, y\rangle_{\overline{\mathcal{X}}} = \langle \Sigma^{1/2}x, \Sigma^{1/2}y\rangle_{\mathcal{X}} = \langle \Sigma x, \Sigma y\rangle_{\underline{\mathcal{X}}} \quad \forall\, x, y \in \overline{\mathcal{X}}.$$

Let $(\overline{\mathfrak{A}}, \underline{\mathfrak{B}}, \overline{\mathfrak{C}}, \mathfrak{D})$ *and* $(\underline{\mathfrak{A}}, \underline{\mathfrak{B}}, \underline{\mathfrak{C}}, \mathfrak{D})$ *be two* 0*-bounded* L^2*-well-posed linear systems on the Hilbert spaces* $(\mathcal{U}, \overline{\mathcal{X}}, \mathcal{Y})$ *and* $(\mathcal{U}, \underline{\mathcal{X}}, \mathcal{Y})$ *respectively, with the same input map* $\underline{\mathfrak{B}}$*, the same input-output map* $\mathfrak{D}$ *and* $\underline{\mathfrak{A}} = \overline{\mathfrak{A}}|_{\underline{\mathcal{X}}}$*,* $\underline{\mathfrak{C}} = \overline{\mathfrak{C}}|_{\underline{\mathcal{X}}}$*. Then* $\mathcal{X}$ *is invariant under* $\overline{\mathfrak{A}}$ *and* $(\overline{\mathfrak{A}}|_{\mathcal{X}}, \underline{\mathfrak{B}}, \overline{\mathfrak{C}}|_{\mathcal{X}}, \mathfrak{D})$ *is a* 0*-bounded* L^2*-well-posed linear system on* $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$*. Moreover, the domain of the generator* A *of* $\mathfrak{A}$ *is the part of* $\overline{\mathfrak{A}}$ *in* $\mathcal{X}$ *and the domain of* $\underline{A}$ *is a core for* A*.*

Proof. The claim about well-posedness is a special case of Lemma 9.5.7 in [Sta05]. That the generator of a semigroup restricted to an invariant subspace is given by the part of the generator in the subspace, is Lemma A.1.4. To see that $\operatorname{dom}\underline{A}$ is a core, it suffices by [EN00, Proposition II.1.7] to see that it is invariant under $\mathfrak{A}$ and a dense subset of $\operatorname{dom} A$. The latter is true because $\underline{\mathcal{X}} \hookrightarrow \mathcal{X}$. □

Proof of Theorem 6.4.2. Part (i) has been shown before, see (6.4a) and Definition 2.4.1 (ii). It is restated in the theorem to underpin that the concatenations in (ii) are well-defined.

We apply the interpolation Lemma 6.4.3 to the output normalized system (6.12) on ℓ_2 and its restriction (6.29) to $\Sigma\ell_2$. This guarantees the well-posedness of the system

$$\big(\widehat{\mathfrak{A}},\ \widehat{\mathfrak{B}},\ \widehat{\mathfrak{C}},\ \mathfrak{D}\big) := \big(\mathfrak{A}_o|_{\Sigma^{1/2}\ell_2},\ \mathfrak{B}_o,\ \mathfrak{C}_o|_{\Sigma^{1/2}\ell_2},\ \mathfrak{D}\big) \tag{6.34}$$

on the interpolated state space $\Sigma^{1/2}\ell_2$. In particular, $\Sigma^{1/2}$ is invariant under $\mathfrak{A}_o$ and $\mathfrak{A}_o|_{\Sigma^{1/2}\ell_2}$ is strongly continuous with respect to $\|\cdot\|_{\Sigma^{1/2}\ell_2}$.

In order to determine the Gramians of this system, we calculate some adjoints with respect to $\langle\cdot,\cdot\rangle_{\Sigma^{1/2}\ell_2}$. For all $y \in L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})$ we have

$$\begin{aligned}
\langle \mathfrak{C}_o x\,,\, y\rangle_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})} &= \langle x\,,\, \mathfrak{C}_o^* y\rangle_{\ell_2}\\
&= \langle \Sigma^{-1/2}x\,,\, \Sigma^{-1/2}\Sigma\mathfrak{C}_o^* y\rangle_{\ell_2}\\
&= \langle x\,,\, \Sigma\mathfrak{C}_o^* y\rangle_{\Sigma^{1/2}\ell_2} \qquad\qquad \forall\, x \in \Sigma^{1/2}\ell_2,
\end{aligned}$$

and for all $u \in L^2(\mathbb{R}_{\leqslant 0};\mathcal{U})$

$$\begin{aligned}
\langle \mathfrak{B}_o u\,,\, x\rangle_{\Sigma^{1/2}\ell_2} &= \langle \Sigma^{-1/2}\mathfrak{B}_o u\,,\, \Sigma^{-1/2}x\rangle_{\ell_2}\\
&= \langle u\,,\, \mathfrak{B}_o^*\Sigma^{-1}x\rangle_{L^2(\mathbb{R}_{\leqslant 0};\mathcal{U})} \qquad \forall\, x \in \Sigma\ell_2.
\end{aligned}$$

Thus, the Gramians with respect to the scalar product $\langle\cdot,\cdot\rangle_{\Sigma^{1/2}\ell_2}$ are given by

$$\widehat{\mathfrak{C}}^*\widehat{\mathfrak{C}} = \Sigma\mathfrak{C}_o^*\mathfrak{C}_o = \Sigma\,\mathrm{id}_{\Sigma^{1/2}},$$

and

$$\begin{aligned}
\widehat{\mathfrak{B}}\widehat{\mathfrak{B}}^* x &= \mathfrak{B}_o\mathfrak{B}_o^*\Sigma^{-1}x = V^*S^*\mathfrak{B}\mathfrak{B}^*SV\Sigma^{-1}x = V^*S^*RR^*SV\Sigma^{-1}x\\
&= V^*S^*R\pi_{(\ker S^*R)^\perp}R^*SV\Sigma^{-1}x = V^*S^*RUU^*R^*SV\Sigma^{-1}x\\
&= \Sigma x \quad \forall\, x \in \Sigma\ell_2,
\end{aligned}$$

where the last equation can be extended to the whole space $\Sigma^{1/2}\ell_2$, because both of the operators $\widehat{\mathfrak{B}}\widehat{\mathfrak{B}}^*$ and Σ are in $\mathcal{B}(\Sigma^{1/2}\ell_2)$. The system (6.34) is therefore balanced.

The last step of the proof is to transfer this system to the favored state space ℓ_2

via another unitary transformation $\Sigma^{-1/2} : \Sigma^{1/2}\ell_2 \to \ell_2$. The result of this is the system

$$\begin{aligned}&\Big(\Sigma^{-1/2}\widehat{\mathfrak{A}}\Sigma^{1/2},\;\; \Sigma^{-1/2}\widehat{\mathfrak{B}},\;\; \widehat{\mathfrak{C}}\Sigma^{1/2},\;\; \mathfrak{D}\Big)\\ &= \Big(\Sigma^{-1/2}\overline{V^*S^*\mathfrak{A}RU\Sigma^{-1}}\Sigma^{1/2},\;\; \Sigma^{-1/2}V^*S^*\mathfrak{B},\;\; \overline{\mathfrak{C}RU\Sigma^{-1}}\Sigma^{1/2},\;\; \mathfrak{D}\Big)\end{aligned} \tag{6.35}$$

on ℓ_2. Since we are transforming unitarily with respect to the scalar products $\langle\cdot\,,\,\cdot\rangle_{\Sigma^{1/2}\ell_2}$ and $\langle\cdot\,,\,\cdot\rangle_{\ell_2}$, the Gramians do not change and the resulting system is still balanced. In order to complete the proof, it suffices to check that the operators defined in (6.33) and (6.35) are the same. For $\mathfrak{B}_o$ and $\mathfrak{D}$ there is nothing to prove. For $\mathfrak{A}_o(t)$ and $\mathfrak{C}_o$ it follows since all the operators are bounded with respect to the ℓ_2-norm and coincide on the dense subset $\Sigma\ell_2$ of ℓ_2. Thus, parts (ii) and (iii) are proven as well. □

Remark 6.4.4. (i) As an immediate consequence of Lemma 6.2.7 and the fact that $\Sigma^{1/2} : \ell^2 \to \Sigma^{1/2}\ell^2$ is an isomorphism the mapping

$$T|_{\mathcal{M}} : \mathcal{M} \subset \mathcal{X} \to \Sigma^{1/2}\ell^2, \quad T := \Sigma^{-1/2}V^*S^*,$$

is an isomorphism with inverse

$$T^+ : \Sigma^{1/2}\ell^2 \subset \ell^2 \to \mathcal{M}, \quad \pi_{(\ker S^*)^\perp}T^+ := \pi_{(\ker S^*)^\perp}RU\Sigma^{-1/2}.$$

(ii) It is shown in [RS14, Section 11] that the closure of T is a pseudo-similarity transformation between the system $(\mathfrak{A}_b, \mathfrak{B}_b, \mathfrak{C}_b, \mathfrak{D})$ and the Kalman-compressed realization $\big(\pi_{(\ker\mathfrak{C})^\perp}\mathfrak{A}|_{\overline{\mathcal{M}}}, \pi_{(\ker\mathfrak{C})^\perp}\mathfrak{B}, \mathfrak{C}|_{\overline{\mathcal{M}}}, \mathfrak{D}\big)$ in Theorem 2.7.3.

Theorem 6.4.5. *Let T and T^+ be as in Theorem 6.4.2. Then the following is true for the generators A_b ,B_b and C_b of the balanced realization in* (6.33)*:*

(i) The space $\mathcal{W} := T\mathfrak{B}W_0^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U})$ is a subset of $\Sigma^{1/2}\ell_2$ and a core for A_b. Moreover, $\pi_{(\ker S^*)^\perp}T^+\mathcal{W} \subset \pi_{(\ker\mathfrak{C})^\perp}\operatorname{dom} A$ and the quotient operator $\widetilde{A}$ of A defined in Theorem 2.7.3 satisfies

$$A_b x = T\widetilde{A}\pi_{(\ker S^*)^\perp}T^+x \quad \forall\, x \in \mathcal{W}. \tag{6.36}$$

(ii) There exists a space $\widetilde{\mathcal{W}} \subset \Sigma^{1/2}\ell_2 \cap \operatorname{dom} A_b^*$*, which is a core for* A_b^**, such that the adjoint*

$$(T|_{\overline{\mathcal{M}}})^* : \Sigma^{1/2}\ell_2 \subset \ell_2 \to \mathcal{X}, \qquad x \mapsto \pi_{\overline{\mathcal{M}}} S V \Sigma^{-1/2} x$$

fulfills $(T|_{\overline{\mathcal{M}}})^* \widetilde{\mathcal{W}} \subset \operatorname{dom} \widetilde{A}^*$*. For all* $x \in \widetilde{\mathcal{W}}$ *and* $u \in \mathcal{U}$ *the control operator fulfills*

$$\langle B_b u\,,\, x\rangle_{(\operatorname{dom} A_b^*)', \operatorname{dom} A_b^*} = \langle Bu, (T|_{\overline{\mathcal{M}}})^* x\rangle_{(\operatorname{dom} A^*)', \operatorname{dom} A^*}\,. \tag{6.37}$$

Consequently, $B_b u$ *is obtained by continuous extension of this functional to* $\operatorname{dom} A_b^*$*.*

(iii) The observation operator fulfills

$$C_b x = C_{\mathrm{ex}} T^+ x \quad \forall\, x \in \mathcal{W}.$$

Proof. By Remark 6.4.4 we have

$$\pi_{(\ker S^*)^\perp} T^+ \mathcal{W} = \pi_{(\ker S^*)^\perp} T^+ T \mathfrak{B} W_0^{1,2}(\mathbb{R}_{\leqslant 0}; \mathcal{U}) = \pi_{(\ker \mathfrak{C})^\perp} \mathfrak{B} W_0^{1,2}(\mathbb{R}_{\leqslant 0}; \mathcal{U}).$$

Theorem 2.7.3 and Lemma 2.4.3 (i) show that $\widetilde{A}$ maps this set into $\mathcal{M}$ and therefore the right hand side in (6.36) is well-defined. Recall that the system (6.34) in the proof of Theorem 6.4.2 was obtained by interpolation. Thus, by Lemma 6.4.3 the space $\operatorname{dom} \underline{A}$ is a core for $\widehat{A}$ and the domain of $\widehat{A}$ is the part of A_o in $\Sigma^{1/2}\ell_2$. This means in particular

$$\begin{aligned}
\operatorname{dom} \widehat{A} &= \left\{ x \in \Sigma^{1/2} \cap \operatorname{dom} A_o : A_o x \in \Sigma^{1/2}\ell_2 \right\}, \\
\widehat{A} z &= A_o z = V^* S^* \widetilde{A} \pi_{(\ker S^*)^\perp} R U \Sigma^{-1} x \quad \forall\, z \in \operatorname{dom} \underline{A}.
\end{aligned}$$

Formula (6.30) shows that $\mathcal{W} = \Sigma^{-1/2} \operatorname{dom} \underline{A}$. Since the semigroups $\mathfrak{A}_b$ and $\widehat{\mathfrak{A}}$ are unitarily similar via the transformation $\Sigma^{-1/2} \in \mathcal{B}(\Sigma^{1/2}\ell_2; \ell_2)$, their generators are related by

$$\begin{aligned}
\operatorname{dom} A_b &= \Sigma^{-1/2} \operatorname{dom} \widehat{A} = \Sigma^{-1/2} \left\{ x \in \Sigma^{1/2} \cap \operatorname{dom} A_o : A_o x \in \Sigma^{1/2}\ell_2 \right\} \\
A_b z &= \Sigma^{-1/2} \widehat{A} \Sigma^{1/2} z = \Sigma^{-1/2} V^* S^* \widetilde{A} \pi_{(\ker S^*)^\perp} R U \Sigma^{-1/2} z \quad \forall\, z \in \mathcal{W},
\end{aligned}$$

and $\mathcal{W}$ is a core for A_b. This shows all the assertions in (i).

We do not determine the domain of the adjoint A_b^* exactly, but we will prove that $\widetilde{\mathcal{W}} := \Sigma^{1/2} \operatorname{dom} A_o^*$ is a core for A_b^* and has the properties claimed in (ii). Take $y \in \widetilde{\mathcal{W}}$ and $x \in \operatorname{dom} A_b \subset \Sigma^{-1/2} \operatorname{dom} A_o$. Then the right hand side of the equation

$$\langle A_b x, y\rangle_{\ell_2} = \langle \Sigma^{-1/2} A_o \Sigma^{1/2} x, y\rangle_{\ell_2} = \langle x, \Sigma^{1/2} A_o^* \Sigma^{-1/2} y\rangle_{\ell_2}$$

is continuous in x, which implies $y \in \operatorname{dom} A_b^*$ and

$$A_b^* y = \Sigma^{1/2} A_o^* \Sigma^{-1/2} y \quad \forall\, y \in \widetilde{\mathcal{W}}. \tag{6.38}$$

So we have shown $\widetilde{\mathcal{W}} \subset \operatorname{dom} A_b^*$. We now prove that $\widetilde{\mathcal{W}}$ is dense in ℓ_2 and $\mathfrak{A}_b^*$-invariant. The continuity of $\Sigma^{1/2} \in \mathcal{B}(\ell_2)$ implies that $\Sigma^{1/2} \operatorname{dom} A_o^*$ is dense in $\Sigma^{1/2}\ell_2$ with respect to the topology of ℓ_2. Because $\Sigma^{1/2}\ell_2$ itself is dense in ℓ_2, it follows that $\widetilde{\mathcal{W}}$ is dense in ℓ_2. Furthermore, for $x \in \ell_2$ and $y \in \widetilde{\mathcal{W}} \subset \Sigma^{1/2}\ell_2$, the equation

$$\langle \mathfrak{A}_b(t) x, y\rangle_{\ell_2} = \langle \Sigma^{-1/2} \mathfrak{A}_o(t) \Sigma^{1/2} x, y\rangle_{\ell_2} = \langle x, \Sigma^{1/2} \mathfrak{A}_o^*(t) \Sigma^{-1/2} y\rangle_{\ell_2}$$

shows $\mathfrak{A}_b^*(t) y = \Sigma^{1/2} \mathfrak{A}_o^*(t) \Sigma^{-1/2} y$. This representation together with the definition of $\widetilde{\mathcal{W}}$ shows the $\mathfrak{A}_b$-invariance of $\widetilde{\mathcal{W}}$, since the $\mathfrak{A}_o(t)$ maps $\operatorname{dom} A_o^*$ into itself. So altogether $\widetilde{\mathcal{W}}$ must be a core of A_b^* by [EN00, Proposition II.1.7].

To complete the proof of (ii), we observe that $(T|_{\overline{\mathcal{M}}})^* = (\mathcal{T}|_{\overline{\mathcal{M}}})^* \Sigma^{-1/2}$ and therefore

$$(T|_{\overline{\mathcal{M}}})^* \widetilde{\mathcal{W}} = (\mathcal{T}|_{\overline{\mathcal{M}}})^* \Sigma^{-1/2} \Sigma^{1/2} \operatorname{dom} A_o^* = (\mathcal{T}|_{\overline{\mathcal{M}}})^* \operatorname{dom} A_o^*.$$

The latter set was shown to be a subset of $\operatorname{dom} \widetilde{A}^*$ in Theorem 6.2.11. Choose λ in the resolvent sets of A_b and A_o, and let $y \in \widetilde{\mathcal{W}}$ and $u \in \mathcal{U}$. Knowing from (i) that $y \in \operatorname{dom} A_b^*$ and using (6.38) we obtain with (6.21)

$$\begin{aligned}
\langle B_b u, y\rangle_{(\operatorname{dom} A_b^*)', \operatorname{dom} A_b^*} &= \langle (\lambda - A_b|_{\ell_2}) \mathfrak{B}_b e_\lambda u, y\rangle_{(\operatorname{dom} A_b^*)', \operatorname{dom} A_b^*} \\
&= \langle \mathfrak{B}_b e_\lambda u, (\bar\lambda - A_b^*) y\rangle_{\ell_2} \\
&= \langle \Sigma^{-1/2} \mathfrak{B}_o e_\lambda u, \Sigma^{1/2} (\bar\lambda - A_o^*) \Sigma^{-1/2} y\rangle_{\ell_2} \\
&= \langle \mathfrak{B}_o e_\lambda u, (\bar\lambda - A_o^*) \Sigma^{-1/2} y\rangle_{\ell_2} \\
&= \langle (\lambda - A_o|_{\ell_2}) \mathfrak{B}_o u, \Sigma^{-1/2} y\rangle_{(\operatorname{dom} A_o^*)', \operatorname{dom} A_o^*} \\
&= \langle B_o u, \Sigma^{-1/2} y\rangle_{(\operatorname{dom} A_o^*)', \operatorname{dom} A_o^*}
\end{aligned}$$

$$= \langle \widetilde{B}u, (T|_{\overline{\mathcal{M}}})^* y\rangle_{(\operatorname{dom}\widetilde{A}^*)',\operatorname{dom}\widetilde{A}^*}.$$

The functional $B_b u \in (\operatorname{dom} A_b^*)'$ is obtained by continuous extension of this expression to all $y \in \operatorname{dom} A_b^*$ because the core $\widetilde{\mathcal{W}}$ is dense in $\operatorname{dom} A_b^*$ with respect to the graph norm of A_b^*. Thus, the assertion in (ii) is shown, and (iii) follows from the fact that $\mathfrak{C}_b x = \mathfrak{C}_o \Sigma^{1/2} x$ for all $x \in \mathcal{W}$ and the definition of the observation operator. □

Remark 6.4.6. In view of Corollary 6.2.10 the generator A_b of $\mathfrak{A}_b$ is also equal to

$$\operatorname{dom} A_b = \Sigma^{-1/2}\left\{ (x_n) \in \Sigma^{1/2}\ell_2 \;\middle|\; \begin{array}{l} \widetilde{V}(x_n) \in W^{1,2}(\mathbb{R}_+;\mathcal{Y}) \quad \text{and} \\ \widetilde{V}^* \frac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{V}(x_n) \in \Sigma^{1/2}\ell_2 \end{array} \right\},$$
$$A_b x = \Sigma^{-1/2}\widetilde{V}^* \tfrac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{V}\Sigma^{1/2}x.$$

Truncation

Definition 6.4.7 (balanced truncation)**.** Let Presumption 6.2.13 hold and denote by $(\sigma_n)_{n\in\mathbb{N}}$ the sequence of singular values of the Hankel operator $\mathfrak{H}$ with corresponding Schmidt pairs $(\widetilde{v}_j, \widetilde{u}_j)$. Choose r such that $\sigma_{r+1} \neq \sigma_r$ and denote by $\frac{\mathrm{d}}{\mathrm{d}\xi}$ the differential operator $\frac{\mathrm{d}}{\mathrm{d}\xi} : W^{1,1}(\mathbb{R}_{\geqslant 0};\mathcal{Y}) \to L^1(\mathbb{R}_{\geqslant 0};\mathcal{Y})$. The *$r$-th order balanced truncation of* $\mathfrak{D}$ is the r-dimensional system (A_r, B_r, C_r, D) defined by the matrices in (6.25) with coefficients

$$a_{ij} = \frac{\sqrt{\sigma_j}}{\sqrt{\sigma_i}} \left\langle \widetilde{v}_i\,,\, \tfrac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{v}_j \right\rangle_{L^\infty(\mathbb{R}_{\geqslant 0};\mathcal{Y}),L^1(\mathbb{R}_{\geqslant 0};\mathcal{Y})} \quad \in \mathbb{C}, \tag{6.39a}$$
$$b_i = \sqrt{\sigma_i}\,\langle \cdot\,,\, \widetilde{u}_i(0)\rangle_{\mathcal{U}} \quad \in \mathcal{B}(\mathcal{U},\mathbb{C}), \tag{6.39b}$$
$$c_j = \frac{1}{\sqrt{\sigma_j}}\widetilde{v}_j(0) \quad \in \mathcal{Y}. \tag{6.39c}$$

Note that the output normalized truncation in Definition 6.2.16 and the balanced truncation are related by a simple state space transformation with the diagonal matrix $\Sigma_r := \operatorname{diag}(\sqrt{\sigma_1}\,, \ldots\,, \sqrt{\sigma_r}) \in \mathbb{C}^{r\times r}$. In particular they are realizations of the same input-output map.

It is shown in [RS14, Theorem 5.6] that the balanced truncation is obtained by truncating the balanced realization on ℓ^2 in analogy to Theorem 6.2.19. An easier way to obtain the balanced truncation is to determine the output normalized truncation and then transform it with the matrix Σ_r.

6.5. Notes and references

The balancing technique that makes use of the factorizations RR^* and SS^* of the Gramians was first introduced in [TP87] for finite-dimensional systems. The popularity of this balancing approach is mainly based on two facts: Firstly, there are various numerical methods for the determination of Gramians which directly provide the factors S and R instead of the Gramians themselves (see [Ant05, Chapter 6] for an overview). Secondly, this balancing approach can be easily modified to directly construct the truncated balanced realization without determining the parts of A_b, B_b, C_b which are truncated anyway. This is done by simply truncating the singular value decomposition of S^*R.

Balancing for infinite-dimensional systems has been considered in various articles [Cur03, CG86, GCP88, GLP90, Gui12, GO14, Obe86, Obe87, Obe91, OMS91].

The error bound in terms of neglected Hankel singular values has been first shown in [GCP88] for the class of systems with square integrable impulse response, and has recently been generalized to systems with nuclear Hankel operator [Gui12, GO14]. All mentioned approaches to balancing or output normalizing of infinite-dimensional systems have in common that they rely on a construction by means of the Schmidt pairs of the Hankel operator and not on transformations of the state space. The closest to our approach is [Sta05, Chapter 9], which defines pseudo-similarity transformations. In Sections 6.2–6.4 we have essentially worked out the concepts that are described on the level of input-to-state and state-to-output maps in [Sta05, Chapter 9], in more detail. The novelties in the present work are that we have used factors of the Gramians instead of the input-to-state and the state-to-output map, that we have applied the transformations to non-minimal systems and, most importantly, we have given detailed characterization of the generators and highlighted the relation to the balanced truncation defined in [Gui12, GO14].

The generators of a balanced realization on ℓ^2 have also been considered in [GCP88] for impulse responses in $L^1 \cap L^2$. However, the proof of [GCP88, Lemma 3.3] is flawed as it suggests without justification that the limit in $A_b \sum_{k=1}^{\infty} c_k \widetilde{v}_k$ is exchangeable with A_b.

By [Sta05, Theorem 9.2.5] two minimal realizations of the same input-output map are pseudo-similar. In particular, the output normalized and the balanced realization on ℓ^2 are pseudo-similar to $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ if this system is minimal. In

[RS14] we have shown that the corresponding pseudo-similarity transformations in this case are the mapping $\mathcal{T}$ in Theorem 6.2.8 and the closure of the mapping T in Theorem 6.4.2, respectively.

7. $\mathcal{H}^\infty$-balancing and truncation for Pritchard-Salamon systems

In this chapter we consider an ω-bounded Pritchard-Salamon system (A, B, C, D) on $(\mathcal{U},\ (\mathcal{W}, \mathcal{X}, \mathcal{V}),\ \mathcal{Y})$. We aim to construct a controller $\mathfrak{K} \in \mathrm{TIC}^2_{\mathrm{loc}}(\mathcal{Y}; \mathcal{U})$ that stabilizes the system in an input-output sense defined later. This controller should have a finite-dimensional realization in order to be implemented in practice. To this end, we use the approximation theory by balanced truncation that was already described in Section 6.2. The main obstacle is that this approximation is only valid for 0-bounded systems. The idea here is to stabilize the system first by an exponentially stabilizing feedback pair in the sense of Definition 2.8.10 and then perform balanced truncation on the closed-loop system described in (2.34), which is 0-bounded. We do not use an arbitrary exponentially stabilizing feedback pair, but the one that arises in the solution of the following linear quadratic minimization problem. Consider, for an initial value $x_0 \in \mathcal{W}$, the following set of admissible controls:

$$\mathcal{U}_{\mathrm{adm}}(x_0) := \left\{ u \in L^2(\mathbb{R}_{\geqslant 0}; \mathcal{U}) \;\middle|\; \begin{array}{l} \displaystyle\int_0^\infty \|x(t)\|_{\mathcal{V}}^2 \,\mathrm{d}t < \infty \quad \text{for all } x,\, y \\ \text{with } (x, u, y) \in \mathrm{bhv}(A, B, C, D) \\ \text{and } x(0) = x_0. \end{array} \right\}. \tag{7.1}$$

We try to find, for $\beta \in (0, 1]$, the minimizer of the set

$$\left\{ \|\mathfrak{C}x_0 + \mathfrak{D}u\|^2_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})} + \tfrac{1}{\beta^2}\|u\|^2_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{U})} \;\middle|\; u \in \mathcal{U}_{\mathrm{adm}}(x_0) \right\}.$$

In other words, if we assume for a moment that u is smooth, we try to minimize the functional

$$\int_0^\infty \|Cx(t) + Du(t)\|_{\mathcal{Y}}^2 + \tfrac{1}{\beta^2} \|u(t)\|_{\mathcal{U}}^2 \,\mathrm{d}t$$
$$\text{where } (x,u,y) \in \operatorname{bhv}(A,B,C,D), \quad x(0) = x_0,$$

over all $u \in \mathcal{U}_{\mathrm{adm}}(x_0)$. This problem has been studied extensively under various conditions, for example in [vK93, Mik02, PS87, Sta98c] to name but a few. The result is that there exists a self adjoint, so-called Riccati operator, $X_\infty \in \mathcal{B}(\mathcal{V};\mathcal{V}')$, such that

$$\langle x_0\,,\, \mathcal{X}_\infty x_0\rangle = \inf_{u\in\mathcal{U}_{\mathrm{adm}}(x_0)} \left(\|\mathfrak{C}x_0 + \mathfrak{D}u\|_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})}^2 + \tfrac{1}{\beta^2}\|u\|_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{U})}^2 \right), \tag{7.2}$$

and this Riccati operator solves an algebraic operator equation, known as control algebraic Riccati equation. However, it is usually not the only solution to this equation. Furthermore, an admissible feedback pair can be constructed from X_∞ such that the input-output map of the closed-loop system is in $\mathrm{TIC}_0^2(\mathcal{U};\mathcal{Y}\times\mathcal{U})$. This feedback pair is often assumed to be exponentially stabilizing which distinguishes a unique solution of the Riccati equation [vK93, Theorem 3.10]. The clue to this whole approach is that the closed-loop system $(\mathfrak{A}_{\circlearrowleft}, \mathfrak{B}_{\circlearrowleft}, \mathfrak{C}_{\circlearrowleft}, \mathfrak{D}_{\circlearrowleft})$ has some usable properties. Firstly, its Hankel operator fulfills Presumption 6.2.13. Secondly, if v is the minimizer of (7.2), then the (autonomous) output of the closed-loop system with initial value x_0 is

$$\mathfrak{C}_{\circlearrowleft} x_0 = \begin{bmatrix} \mathfrak{C}x_0 + \mathfrak{D}v \\ v \end{bmatrix}.$$

Therefore (7.2) implies that the observability Gramian $\mathfrak{C}_{\circlearrowleft}^*\mathfrak{C}_{\circlearrowleft}$ equals X_∞. Recall that knowledge of both Gramian is required for balanced truncation. The controllability Gramian is related to a dual, so-called filter algebraic Riccati equation. With these tools at hand, we can carry out balancing and truncation of the closed-loop system.

The procedure is called $\mathcal{H}^\infty$-balanced truncation because, following the idea of [MG91], the factor $\beta \in (0,1]$ makes it possible to consider the minimization problem above, together with a dual problem, as a special case of $\mathcal{H}^\infty$-control problem. This will be exploited in Section 7.4 to construct a robust controller.

7.1. Riccati equations

The notion of a control algebraic Riccati equation varies heavily in the literature, depending on the associated control problem. We use a definition in the spirit of [vK93]. In this chapter we use $(\cdot)^*$ for the Hilbert space adjoint of an operator and $(\cdot)'$ for the adjoint between the dual spaces (which are by Section 2.8 represented with respect to a pivot space).

Definition 7.1.1 (Riccati equations)**.** Let (A, B, C, D) be a smooth Pritchard-Salamon system on the Hilbert spaces $(\mathcal{U}, (\mathcal{W}, \mathcal{X}, \mathcal{V}), \mathcal{Y})$. An operator $X_\infty \in \mathcal{B}(\mathcal{V}; \mathcal{V}')$ is said to be a solution of the *HCARE ($\mathcal{H}^\infty$ Control Algebraic Riccati Equation)* if $X_\infty' = X_\infty$, $\langle X_\infty x, x\rangle_{\mathcal{V}',\mathcal{V}} \geqslant 0$ for all $x \in \mathcal{V}$, and the following equation holds for all $x, y \in \operatorname{dom} A^{\mathcal{V}} \subset \mathcal{W}$:

$$\begin{aligned}&\langle X_\infty x, A^{\mathcal{V}} y\rangle_{\mathcal{V}',\mathcal{V}} + \langle A^{\mathcal{V}} x, X_\infty y\rangle_{\mathcal{V},\mathcal{V}'} + \langle Cx, Cy\rangle_{\mathcal{Y}} \\ &\quad = \beta^2 \left\langle (\mathrm{I} + \beta^2 D^* D)^{-1}(D^* C + B' X_\infty)x \,,\, (D^* C + B' X_\infty)y\right\rangle_{\mathcal{U}}.\end{aligned} \tag{7.3}$$

For $\beta = 1$, the HCARE is simply called CARE (Control Algebraic Riccati Equation).

An operator $Y_\infty \in \mathcal{B}(\mathcal{W}'; \mathcal{W})$ is said to be a solution of the *HFARE ($\mathcal{H}^\infty$ Filter Algebraic Riccati Equation)* if $Y_\infty' = Y_\infty$, $\langle x, Y_\infty x\rangle_{\mathcal{W}',\mathcal{W}} \geqslant 0$ for all $x \in \mathcal{W}'$, and the following equation holds for all $x, y \in \operatorname{dom}(A^{\mathcal{W}})' \subset \mathcal{V}'$:

$$\begin{aligned}&\langle Y_\infty x, (A^{\mathcal{W}})' y\rangle_{\mathcal{W},\mathcal{W}'} + \langle (A^{\mathcal{W}})' x, Y_\infty y\rangle_{\mathcal{W}',\mathcal{W}} + \langle B' x \,,\, B' y\rangle_{\mathcal{U}} \\ &\quad = \beta^2 \left\langle (\mathrm{I} + D\beta^2 D^*)^{-1}(DB' + CY_\infty)x \,,\, (DB' + CY_\infty)y\right\rangle_{\mathcal{Y}}.\end{aligned} \tag{7.4}$$

For $\beta = 1$, the HFARE is simply called FARE (Filter Algebraic Riccati Equation).

Note that the HFARE is well-defined because the smoothness of the Pritchard-Salamon system implies by [vK93, Theorem 2.17 (iii)] that $\operatorname{dom}(A^{\mathcal{W}})' \subset \mathcal{V}'$. This means that the dual system is smooth as well.

The HCARE may be interpreted as the CARE with respect to a different scalar product. To this end, we introduce the space $\mathcal{U}_\beta$, which is defined as the input space $\mathcal{U}$ equipped with the new scalar product $\langle \cdot\,,\, \cdot\rangle_{\mathcal{U}_\beta} := \left\langle \frac{1}{\beta^2}\cdot\,,\, \cdot\right\rangle_{\mathcal{U}}$. Adjoint operators with respect to this scalar product are indicated by $(\cdot)^\odot$. The relation

$$\langle Du \,,\, y\rangle_{\mathcal{Y}} = \langle u \,,\, D^* y\rangle_{\mathcal{U}} = \left\langle u \,,\, \beta^2 D^* y\right\rangle_{\mathcal{U}_\beta} \qquad \forall\, u \in \mathcal{U}, y \in \mathcal{Y},$$

shows that the adjoint of D with respect to $\langle\cdot\,,\,\cdot\rangle_{\mathcal{U}_\beta}$ is $D^\odot = \beta^2 D^*$. An analogous calculation yields $B^\odot = \beta^2 B'$.

Lemma 7.1.2. *The operator X_∞ solves* (7.3) *if and only if it satisfies the following CARE for all $x, y \in \operatorname{dom} A^{\mathcal{V}}$:*

$$\begin{aligned}\langle X_\infty x, A^{\mathcal{V}} y\rangle_{\mathcal{V}',\mathcal{V}} + \langle A^{\mathcal{V}} x, X_\infty y\rangle_{\mathcal{V},\mathcal{V}'} + \langle Cx, Cy\rangle_{\mathcal{Y}} \\ = \Big\langle (\mathrm{I} + D^\odot D)^{-1}(D^\odot C + B^\odot X_\infty)x\,,\,(D^\odot C + B^\odot X_\infty)y\Big\rangle_{\mathcal{U}_\beta}.\end{aligned} \tag{7.5}$$

The operator Y_∞ solves (7.4) *if and only if $Y_2 := \beta^2 Y_\infty$ satisfies the following FARE for all $x, y \in \operatorname{dom}(A^{\mathcal{W}})'$:*

$$\begin{aligned}\langle \beta^2 Y_\infty x, (A^{\mathcal{W}})' y\rangle_{\mathcal{W},\mathcal{W}'} + \langle (A^{\mathcal{W}})' x, \beta^2 Y_\infty y\rangle_{\mathcal{W}',\mathcal{W}} + \langle B' x, B' y\rangle_{\mathcal{U}_\beta} \\ = \Big\langle (\mathrm{I} + DD^\odot)^{-1}(DB^\odot + C\beta^2 Y_\infty)x\,,\,(DB^\odot + C\beta^2 Y_\infty)y\Big\rangle_{\mathcal{Y}}.\end{aligned} \tag{7.6}$$

Proof. Note that (7.3) can equivalently be written as

$$\begin{aligned}&\langle X_\infty x, A^{\mathcal{V}} y\rangle_{\mathcal{V}',\mathcal{V}} + \langle A^{\mathcal{V}} x, X_\infty y\rangle_{\mathcal{V},\mathcal{V}'} + \langle Cx, Cy\rangle_{\mathcal{Y}} \\ &\quad = \frac{1}{\beta^2}\Big\langle (\mathrm{I} + \beta^2 D^* D)^{-1}(\beta^2 D^* C + \beta^2 B' X_\infty)x\,,\,(\beta^2 D^* C + \beta^2 B' X_\infty)y\Big\rangle_{\mathcal{U}},\end{aligned}$$

and multiplication of (7.4) by β^2 yields

$$\begin{aligned}&\langle \beta^2 Y_\infty x, (A^{\mathcal{W}})' y\rangle_{\mathcal{W},\mathcal{W}'} + \langle (A^{\mathcal{W}})' x, \beta^2 Y_\infty y\rangle_{\mathcal{W}',\mathcal{W}} + \beta^2\langle B' x, B' y\rangle_{\mathcal{U}} \\ &\quad = \Big\langle (\mathrm{I} + D\beta^2 D^*)^{-1}(D\beta^2 B' + C\beta^2 Y_\infty)x\,,\,(D\beta^2 B' + C\beta^2 Y_\infty)y\Big\rangle_{\mathcal{Y}}.\end{aligned}$$

As mentioned before, the adjoints with respect to $\langle\cdot\,,\,\cdot\rangle_{\mathcal{U}_\beta}$ satisfy $D^\odot = \beta^2 D^*$ and $B^\odot = \beta^2 B'$. This shows the claim. □

Throughout this chapter we will make the following hypothesis.

Presumption 7.1.3. *The quadruple (A, B, C, D) is a smooth Pritchard-Salamon system on the Hilbert spaces $(\mathcal{U}, (\mathcal{W}, \mathcal{X}, \mathcal{V}), \mathcal{Y})$. It generates the ω-bounded well-posed linear system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$. There exists a solution X_∞ to the HCARE such that, with the definitions*

$$L := (\mathrm{I} + \beta^2 D^* D), \qquad K := -L^{-1}(\beta^2 D^* C + \beta^2 B' X_\infty), \tag{7.7}$$

the block operator $[L^{\frac{1}{2}}K\,,\,\mathrm{I}-L^{\frac{1}{2}}] \in \mathcal{B}(\mathcal{W}\times\mathcal{U};\mathcal{U})$ *is an exponentially stabilizing admissible feedback pair for* (A,B,C,D). *Moreover, there exists a solution* Y_∞ *to the HFARE.*

Remark 7.1.4. We will not need any exponentially stabilizing condition on Y_∞.

Theorem 7.1.5. *Under Presumption 7.1.3 the following holds:*

(i) The closed-loop system

$$(A_\circlearrowleft, B_\circlearrowleft, C_\circlearrowleft, D_\circlearrowleft) := \\ \left((A^{\mathcal{V}} + BK)|_{\operatorname{dom} A_\circlearrowleft},\ BL^{-1/2},\ \begin{bmatrix} C+DK \\ K \end{bmatrix}, \begin{bmatrix} DL^{-1/2} \\ L^{-1/2} \end{bmatrix} \right), \tag{7.8}$$

with

$$\operatorname{dom} A_\circlearrowleft := \left\{\ x \in \operatorname{dom} A^{\mathcal{V}}\ \middle|\ A^{\mathcal{V}}x + BKx \in \mathcal{X}\ \right\},$$

is a 0*-bounded Pritchard-Salamon system on* $(\mathcal{U}, (\mathcal{W},\mathcal{X},\mathcal{V}), \mathcal{Y}\times\mathcal{U})$.

(ii) The input-output map of (7.8)*, which we denote by* $[\mathfrak{N}\,,\,\mathfrak{M}]^\top \in \mathrm{TIC}_0^2(\mathcal{U};\mathcal{Y}\times\mathcal{U})$*, satisfies*

$$\beta^2\mathfrak{N}^*\mathfrak{N} + \mathfrak{M}^*\mathfrak{M} = \mathrm{I}\,. \tag{7.9}$$

The operator $\mathfrak{M}$ *is invertible in* $\mathrm{TIC}^2_{\mathrm{loc}}(\mathcal{U};\mathcal{U})$ *and* $\mathfrak{D}u = \mathfrak{N}\mathfrak{M}^{-1}u$ *for all* $u \in L^2_{\mathrm{c,loc}}(\mathbb{R}_{\geqslant 0};\mathcal{U})$.

(iii) The operator X_∞ *equals the observability Gramian* Q *of* (7.8) *with respect to the output space* $\mathcal{Y}\times\mathcal{U}_\beta$*, i.e.* $X_\infty = Q := \mathfrak{C}_\circlearrowleft^\odot\mathfrak{C}_\circlearrowleft$.

(iv) The operator

$$\mathrm{I} + X_\infty\beta^2 Y_\infty\big|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X},$$

has an inverse, $(\mathrm{I}+X_\infty\beta^2Y_\infty)^{-1} \in \mathcal{B}(\mathcal{X})$*, and the controllability Gramian* P *of* (7.8) *satisfies*

$$(\mathrm{I}+X_\infty\beta^2Y_\infty)^{-1}\beta^2Y_\infty = P := \mathfrak{B}_\circlearrowleft\mathfrak{B}_\circlearrowleft^\odot = \beta^2\mathfrak{B}_\circlearrowleft\mathfrak{B}_\circlearrowleft^*.$$

Remark 7.1.6. (i) The operator $[\mathfrak{N}\,,\,\mathfrak{M}]^\top$ is a right factorization of $\mathfrak{D}$ in the sense of Definition 2.3.3. Such factorizations are usually considered to be "coprime" in some sense, see e.g. [CWW96, Mik06]. But coprimeness does not play a role here.

(ii) The reason for choosing the feedback pair $[L^{\frac{1}{2}}K\,,\,\mathrm{I}-L^{\frac{1}{2}}]$ instead of $[K\,,\,0]$ is that this choice "normalizes" the factorization in the sense that (7.9) holds, cf. [CZ95, Theorem 7.3.11].

(iii) In the theory of Riccati equations for state linear systems (i.e. when $\mathcal{W} = \mathcal{V} = \mathcal{X}$) it is often assumed that $D = 0$ without loss of generality, e.g. in [CZ95, CO06]. If D is zero, then the term D^*C in the definition of K vanishes, and thus $K \in \mathcal{B}(\mathcal{V};\mathcal{U})$. In general however, K is only in $\mathcal{B}(\mathcal{W};\mathcal{U})$. That is why we can not assume $D = 0$ without loss of full generality for all Pritchard-Salamon systems.

Proof of Theorem 7.1.5. (i) This follows from the assumption that the feedback pair is exponentially stabilizing, Lemma 2.8.9, and some standard estimates. It also contained in [Mik06, Lemma 4.4].

(ii) By [vK93, Theorem 3.10] our assumptions imply that, for every $x_0 \in \mathcal{W}$ and the admissible control set $\mathcal{U}_{\mathrm{adm}}(x_0)$ defined in (7.1), the function

$$v(\cdot) := K\mathfrak{A}_{\circlearrowright}(\cdot)x_0$$

solves the linear quadratic minimization problem (7.2). More precisely, there holds

$$\begin{aligned}\inf\left\{ \|\mathfrak{C}x_0 + \mathfrak{D}u\|^2_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})} + \|u\|^2_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{U}_\beta)} \;\middle|\; u \in \mathcal{U}_{\mathrm{adm}}(x_0) \right\} \\ = \|\mathfrak{C}x_0 + \mathfrak{D}v\|^2_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{Y})} + \|v\|^2_{L^2(\mathbb{R}_{\geqslant 0};\mathcal{U}_\beta)} = \langle x_0\,,\, X_\infty x_0\rangle_{\mathcal{V}}\,.\end{aligned}$$

Therefore, [Mik06, Lemma 4.4] can be invoked and implies the property (7.9). The fact that the closed-loop input-output map $[\mathfrak{N}\,,\,\mathfrak{M}]^\top$ is a right factorization of $\mathfrak{D}$ is well-known, see e.g. [Mik06, Corollary 5.2] or [CWW96].

(iii) As in Lemma 2.8.9 we denote the extension of $A_{\circlearrowright}$ to $\operatorname{dom} A^{\mathcal{V}}$ by $A^{\mathcal{V}}_{\circlearrowright}$. Since the closed loop system is exponentially stable, Lemma 2.8 of [CZ94] implies that Q is the unique solution of the following observability Lyapunov equation for all $x, y \in$

$\operatorname{dom} A^{\mathcal{V}}$:

$$
\begin{aligned}
0 &= \left\langle Qx\,,\, A^{\mathcal{V}}_{\circlearrowleft} y\right\rangle_{\mathcal{V}',\mathcal{V}} + \left\langle A^{\mathcal{V}}_{\circlearrowleft} x\,,\, Qy\right\rangle_{\mathcal{V},\mathcal{V}'} + \langle C_{\circlearrowleft} x\,,\, C_{\circlearrowleft} y\rangle_{\mathcal{Y}\times\mathcal{U}_\beta} \\
&\overset{(7.8)}{=} \left\langle Qx\,,\, A^{\mathcal{V}} y\right\rangle_{\mathcal{V}',\mathcal{V}} + \left\langle A^{\mathcal{V}} x\,,\, Qy\right\rangle_{\mathcal{V},\mathcal{V}'} + \langle Qx\,,\, BKy\rangle_{\mathcal{V}',\mathcal{V}} + \langle BKx\,,\, Qy\rangle_{\mathcal{V},\mathcal{V}'} \\
&\quad + \langle (C+DK)x\,,\, (C+DK)y\rangle_{\mathcal{Y}} + \langle Kx\,,\, Ky\rangle_{\mathcal{U}_\beta} \\
&= \left\langle Qx\,,\, A^{\mathcal{V}} y\right\rangle_{\mathcal{V}',\mathcal{V}} + \left\langle A^{\mathcal{V}} x\,,\, Qy\right\rangle_{\mathcal{V},\mathcal{V}'} + \langle Qx\,,\, BKy\rangle_{\mathcal{V}',\mathcal{V}} + \langle BKx\,,\, Qy\rangle_{\mathcal{V},\mathcal{V}'} \\
&\quad + \langle Cx\,,\, Cy\rangle_{\mathcal{Y}} + \langle D^*Cx\,,\, Ky\rangle_{\mathcal{U}} + \langle Kx\,,\, D^*Cy\rangle_{\mathcal{U}} \\
&\quad + \left\langle \left(\tfrac{1}{\beta^2} + D^*D\right) Kx\,,\, Ky\right\rangle_{\mathcal{U}} \\
&= \left\langle Qx\,,\, A^{\mathcal{V}} y\right\rangle_{\mathcal{V}',\mathcal{V}} + \left\langle A^{\mathcal{V}} x\,,\, Qy\right\rangle_{\mathcal{V},\mathcal{V}'} + \langle Cx\,,\, Cy\rangle_{\mathcal{Y}} \\
&\quad + \langle (B'Q + D^*C)x\,,\, Ky\rangle_{\mathcal{U}} + \langle Kx\,,\, (B'Q + D^*C)y\rangle_{\mathcal{U}} \\
&\quad + \tfrac{1}{\beta^2} \langle LKx\,,\, Ky\rangle_{\mathcal{U}} \\
&\overset{(7.7)}{=} \left\langle Qx\,,\, A^{\mathcal{V}} y\right\rangle_{\mathcal{V}',\mathcal{V}} + \left\langle A^{\mathcal{V}} x\,,\, Qy\right\rangle_{\mathcal{V},\mathcal{V}'} + \langle Cx\,,\, Cy\rangle_{\mathcal{Y}} \\
&\quad + \langle (B'Q + D^*C)x\,,\, Ky\rangle_{\mathcal{U}} - \beta^2 \left\langle L^{-1}(B'X_\infty + D^*C)x\,,\, (B'Q + D^*C)y\right\rangle_{\mathcal{U}} \\
&\quad - \langle (B'X_\infty + D^*C)x\,,\, Ky\rangle_{\mathcal{U}}\,.
\end{aligned}
$$

We see that with $Q = X_\infty$ this equation becomes the HCARE for X_∞. Thus, X_∞ solves the Lyapunov equation, and since there is only one solution, it must be equal to Q.

(iv) Before we turn to the controllability Gramian, we calculate a useful representation for the expression $C^{\odot}_{\circlearrowleft} C_{\circlearrowleft} : \mathcal{W} \to \mathcal{W}'$. We have

$$
\begin{aligned}
C^{\odot}_{\circlearrowleft} C_{\circlearrowleft} &= C'C + C'DK + K'D^*C + K'D^*DK + K'\frac{1}{\beta^2}K \\
&= C'C + C'DK + K'D^*C + \frac{1}{\beta^2}K'LK \\
&= C'C - C'DL^{-1}\beta^2(D^*C + B'X_\infty) - \beta^2(C'D + X_\infty B)L^{-1}D^*C \\
&\quad + \beta^2(C'D + X_\infty B)L^{-1}(D^*C + B'X_\infty) \\
&= C'C - C'D\beta^2 L^{-1}D^*C + X_\infty B\beta^2 L^{-1}B'X_\infty \\
&= C'\left(\mathrm{I} - \beta^2 DD^*\left(\mathrm{I} + \beta^2 DD^*\right)^{-1}\right) C + X_\infty B\beta^2 L^{-1}B'X_\infty \\
&= C'\left(\mathrm{I} + \beta^2 DD^*\right)^{-1} C + X_\infty B\beta^2 L^{-1}B'X_\infty
\end{aligned}
\tag{7.10}
$$

on $\mathcal{W}$. The second observation is that Y_∞ maps $\operatorname{dom}(A^{\mathcal{W}})'$ continuously into the space $\operatorname{dom} A^{\mathcal{V}}$. This follows because the HFARE together with the continuity of Y_∞ and B' on $\mathcal{V}'$ shows that the estimate

$$|\langle Y_\infty x\,,\,(A^{\mathcal{W}})'y\rangle_{\mathcal{W},\mathcal{W}'}| \leqslant c\,\|x\|_{\operatorname{dom}(A^{\mathcal{W}})'}\,\|y\|_{\mathcal{V}'} \quad \forall\, x,y \in \operatorname{dom}(A^{\mathcal{W}})'$$

holds for some constant $c > 0$, and this implies by definition that

$$Y_\infty x \in \operatorname{dom} A^{\mathcal{V}} \quad \forall\, x \in \operatorname{dom}(A^{\mathcal{W}})'.$$

Hence, the observability Lyapunov equation above still holds if we replace x and y by the expressions $\beta^2 Y_\infty x$ and $\beta^2 Y_\infty y$ for some $x, y \in \operatorname{dom}(A^{\mathcal{W}})'$. Together with (7.10) this means for all $x, y \in \operatorname{dom}(A^{\mathcal{W}})'$

$$\begin{aligned}
0 &= \left\langle X_\infty \beta^2 Y_\infty x\,,\, A^{\mathcal{V}}_{\circlearrowleft}\beta^2 Y_\infty y\right\rangle_{\mathcal{V}',\mathcal{V}} + \left\langle A^{\mathcal{V}}_{\circlearrowleft}\beta^2 Y_\infty x\,,\, X_\infty\beta^2 Y_\infty y\right\rangle_{\mathcal{V},\mathcal{V}'} \\
&\quad + \left\langle C_{\circlearrowleft}\beta^2 Y_\infty x\,,\, C_{\circlearrowleft}\beta^2 Y_\infty y\right\rangle_{\mathcal{Y}\times\mathcal{U}_\beta} \\
&= \left\langle X_\infty \beta^2 Y_\infty x\,,\, A^{\mathcal{V}}_{\circlearrowleft}\beta^2 Y_\infty y\right\rangle_{\mathcal{V}',\mathcal{V}} + \left\langle A^{\mathcal{V}}_{\circlearrowleft}\beta^2 Y_\infty x\,,\, X_\infty\beta^2 Y_\infty y\right\rangle_{\mathcal{V},\mathcal{V}'} \\
&\quad + \left\langle \left(\mathrm{I} + \beta^2 D D^*\right)^{-1} C\beta^2 Y_\infty x\,,\, C\beta^2 Y_\infty y\right\rangle_{\mathcal{Y}} \\
&\quad + \left\langle \beta^2 L^{-1} B' X_\infty \beta^2 Y_\infty x\,,\, B' X_\infty \beta^2 Y_\infty y\right\rangle_{\mathcal{U}}.
\end{aligned} \tag{7.11}$$

On the other hand we multiply the HFARE by β^2 and use the relation

$$\beta^2 L^{-1} = \beta^2 - \beta^4 D^* \left(\mathrm{I} + D\beta^2 D^*\right)^{-1} D$$

to obtain

$$\begin{aligned}
0 &= \langle \beta^2 Y_\infty x, (A^{\mathcal{W}})'y\rangle_{\mathcal{W},\mathcal{W}'} + \langle (A^{\mathcal{W}})'x, \beta^2 Y_\infty y\rangle_{\mathcal{W}',\mathcal{W}} \\
&\quad + \beta^2 \langle B'x\,,\, B'y\rangle_{\mathcal{U}} - \beta^4 \left\langle D^*(\mathrm{I} + D\beta^2 D^*)^{-1} D B'x\,,\, B'y\right\rangle_{\mathcal{Y}} \\
&\quad - \beta^4 \left\langle (\mathrm{I} + D\beta^2 D^*)^{-1} D B'x\,,\, C Y_\infty y\right\rangle_{\mathcal{Y}} \\
&\quad - \beta^4 \left\langle (\mathrm{I} + D\beta^2 D^*)^{-1} C Y_\infty x\,,\, D B'y\right\rangle_{\mathcal{Y}} \\
&\quad - \beta^4 \left\langle (\mathrm{I} + D\beta^2 D^*)^{-1} C Y_\infty x\,,\, C Y_\infty) y\right\rangle_{\mathcal{Y}} \\
&= \langle A^{\mathcal{V}} \beta^2 Y_\infty x, y\rangle_{\mathcal{V},\mathcal{V}'} + \langle x, A^{\mathcal{V}}\beta^2 Y_\infty y\rangle_{\mathcal{V}',\mathcal{V}}
\end{aligned}$$

$$
\begin{aligned}
&+ \beta^2 \left\langle L^{-1}B'x\,,\, B'y\right\rangle_{\mathcal{U}} \\
&- \beta^4 \left\langle L^{-1}B'x\,,\, D^*CY_\infty)y\right\rangle_{\mathcal{Y}} \\
&- \beta^4 \left\langle L^{-1}D^*CY_\infty x\,,\, B'y\right\rangle_{\mathcal{Y}} \\
&- \left\langle (\mathrm{I} + D\beta^2 D^*)^{-1}C\beta^2 Y_\infty x\,,\, C\beta^2 Y_\infty)y\right\rangle_{\mathcal{Y}}.
\end{aligned}
$$

Adding this equation to (7.11) gives

$$
\begin{aligned}
0 &= \left\langle X_\infty\beta^2 Y_\infty x\,,\, A_{\circlearrowleft}^{\mathcal{V}}\beta^2 Y_\infty y\right\rangle_{\mathcal{V}',\mathcal{V}} + \left\langle A_{\circlearrowleft}^{\mathcal{V}}\beta^2 Y_\infty x\,,\, X_\infty\beta^2 Y_\infty y\right\rangle_{\mathcal{V},\mathcal{V}'} \\
&\quad + \langle A^{\mathcal{V}}\beta^2 Y_\infty x, y\rangle_{\mathcal{V},\mathcal{V}'} + \langle x, A^{\mathcal{V}}\beta^2 Y_\infty y\rangle_{\mathcal{V}',\mathcal{V}} \\
&\quad + \beta^2 \left\langle L^{-1}B'x\,,\, B'y\right\rangle_{\mathcal{U}} + \beta^2 \left\langle L^{-1}B'X_\infty\beta^2 Y_\infty x\,,\, B'X_\infty\beta^2 Y_\infty y\right\rangle_{\mathcal{U}} \\
&\quad - \beta^4 \left\langle L^{-1}B'x\,,\, D^*CY_\infty)y\right\rangle_{\mathcal{Y}} \\
&\quad - \beta^4 \left\langle L^{-1}D^*CY_\infty x\,,\, B'y\right\rangle_{\mathcal{Y}} \\
&= \left\langle X_\infty\beta^2 Y_\infty x\,,\, A_{\circlearrowleft}^{\mathcal{V}}\beta^2 Y_\infty y\right\rangle_{\mathcal{V}',\mathcal{V}} + \left\langle A_{\circlearrowleft}^{\mathcal{V}}\beta^2 Y_\infty x\,,\, X_\infty\beta^2 Y_\infty y\right\rangle_{\mathcal{V},\mathcal{V}'} \\
&\quad + \langle A^{\mathcal{V}}\beta^2 Y_\infty x, y\rangle_{\mathcal{V},\mathcal{V}'} + \langle x, A^{\mathcal{V}}\beta^2 Y_\infty y\rangle_{\mathcal{V}',\mathcal{V}} \\
&\quad + \beta^2 \left\langle L^{-1}B'(\mathrm{I} + X_\infty\beta^2 Y_\infty)x\,,\, B'(\mathrm{I} + X_\infty\beta^2 Y_\infty)y\right\rangle_{\mathcal{U}} \\
&\quad - \beta^2 \left\langle L^{-1}B'x\,,\, (D^*C + B'X_\infty)\beta^2 Y_\infty)y\right\rangle_{\mathcal{Y}} \\
&\quad - \beta^2 \left\langle L^{-1}(D^*C + B'X_\infty)\beta^2 Y_\infty x\,,\, B'y\right\rangle_{\mathcal{Y}} \\
&= \left\langle X_\infty\beta^2 Y_\infty x\,,\, A_{\circlearrowleft}^{\mathcal{V}}\beta^2 Y_\infty y\right\rangle_{\mathcal{V}',\mathcal{V}} + \left\langle A_{\circlearrowleft}^{\mathcal{V}}\beta^2 Y_\infty x\,,\, X_\infty\beta^2 Y_\infty y\right\rangle_{\mathcal{V},\mathcal{V}'} \\
&\quad + \langle A_{\circlearrowleft}^{\mathcal{V}}\beta^2 Y_\infty x, y\rangle_{\mathcal{V},\mathcal{V}'} + \langle x, A_{\circlearrowleft}^{\mathcal{V}}\beta^2 Y_\infty y\rangle_{\mathcal{V}',\mathcal{V}} \\
&\quad + \beta^2 \left\langle L^{-1}B'(\mathrm{I} + X_\infty\beta^2 Y_\infty)x\,,\, B'(\mathrm{I} + X_\infty\beta^2 Y_\infty)y\right\rangle_{\mathcal{U}} \\
&= \left\langle (\mathrm{I} + X_\infty\beta^2 Y_\infty)x\,,\, A_{\circlearrowleft}^{\mathcal{V}}\beta^2 Y_\infty y\right\rangle_{\mathcal{V}',\mathcal{V}} + \left\langle A_{\circlearrowleft}^{\mathcal{V}}\beta^2 Y_\infty x\,,\, (\mathrm{I} + X_\infty\beta^2 Y_\infty)y\right\rangle_{\mathcal{V},\mathcal{V}'} \\
&\quad + \beta^2 \left\langle L^{-1}B'(\mathrm{I} + X_\infty\beta^2 Y_\infty)x\,,\, B'(\mathrm{I} + X_\infty\beta^2 Y_\infty)y\right\rangle_{\mathcal{U}}.
\end{aligned}
$$

Up to know we have not used the invertibility of $\mathrm{I} + X_\infty\beta^2 Y_\infty|_{\mathcal{X}}$ which we are going to show now. The operator $X_\infty|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ is nonnegative and possesses therefore an operator root $\sqrt{X_\infty} \in \mathcal{B}(\mathcal{X})$. The operator $\mathrm{I} + X_\infty\beta^2 Y_\infty|_{\mathcal{X}}$ is boundedly invertible if and only if $-1 \notin \sigma(X_\infty\beta^2 Y_\infty)$ which holds by Jacobson's lemma [Mü07, Corollary 30 in Section I.1] if and only if $-1 \notin \sigma(\sqrt{X_\infty}\beta^2 Y_\infty\sqrt{X_\infty})$. Since the operator $\sqrt{X_\infty}\beta^2 Y_\infty\sqrt{X_\infty}$ is nonnegative its spectrum is contained in $\mathbb{R}_{\geqslant 0}$. Thus,

$\mathrm{I} + X_\infty \beta^2 Y_\infty|_{\mathcal{X}}$ maps $\mathcal{X}$ onto itself and has a bounded inverse.

Analogous to the previous considerations for Y_∞ it can be shown that X_∞ maps $\operatorname{dom} A^{\mathcal{V}}$ continuously into $\operatorname{dom}(A^{\mathcal{W}})'$. With the bijectivity of $\mathrm{I} + X_\infty \beta^2 Y_\infty$ it follows that the image of $\operatorname{dom}(A^{\mathcal{W}})'$ under $\mathrm{I} + X_\infty \beta^2 Y_\infty$ is a dense subset of $\operatorname{dom}(A^{\mathcal{W}})'$. Hence, we deduce from the equation above that, for all x, y in this dense subset, the following controllability Lyapunov equation holds:

$$\begin{aligned}0 &= \Big\langle x\,,\, A^{\mathcal{V}}_{\circlearrowright}\beta^2 Y_\infty(\mathrm{I} + X_\infty\beta^2 Y_\infty)^{-1} y\Big\rangle_{\mathcal{V}',\mathcal{V}} + \Big\langle A^{\mathcal{V}}_{\circlearrowright}\beta^2 Y_\infty(\mathrm{I} + X_\infty\beta^2 Y_\infty)^{-1} x\,,\, y\Big\rangle_{\mathcal{V},\mathcal{V}'} \\ &\quad + \Big\langle L^{-1}B'x\,,\, B'y\Big\rangle_{\mathcal{U}^2_\beta}.\end{aligned}$$

Since the smoothness of the dual system implies $\operatorname{dom}(A^{\mathcal{W}}_{\circlearrowright})' = \operatorname{dom}(A^{\mathcal{W}})'$, this shows that $P := \beta^2 Y_\infty(\mathrm{I} + X_\infty\beta^2 Y_\infty)^{-1}$ solves for all $x, y \in \operatorname{dom}(A^{\mathcal{W}}_{\circlearrowright})'$ the controllability Lyapunov equation

$$0 = \Big\langle (A^{\mathcal{W}}_{\circlearrowright})'x\,,\, Py\Big\rangle_{\mathcal{W}',\mathcal{W}} + \Big\langle Px\,,\, (A^{\mathcal{W}}_{\circlearrowright})'y\Big\rangle_{\mathcal{W},\mathcal{W}'} + \Big\langle L^{-1}B'x\,,\, B'y\Big\rangle_{\mathcal{U}^2_\beta}.$$

Thus, P must be the controllability Gramian by [CZ94, Lemma 2.8]. □

7.2. $\mathcal{H}^\infty$-balancing

In this section we transform the system (A, B, C, D) in order to diagonalize the solutions X_∞ and Y_∞ of the two algebraic Riccati equations. To this end, we will diagonalize the Gramians of the closed-loop system $(A_{\circlearrowright}, B_{\circlearrowright}, C_{\circlearrowright}, D_{\circlearrowright})$ by the means developed in Section 6.2. In analogy to Presumption 6.2.3, we assume that the in- and output spaces are finite-dimensional and that we have some factors, S and R, of the Gramians at hand. The compactness of the Hankel operator in Presumption 6.2.3 is superfluous here, since it is automatically fulfilled for exponentially stable Pritchard-Salamon systems owing to Lemma 6.2.15.

Presumption 7.2.1. *Presumption 7.1.3 holds. In addition, $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional, $\mathcal{X}_R$ and $\mathcal{X}_S$ are two further Hilbert spaces, and the operators $R \in \mathcal{B}(\mathcal{X}_R; \mathcal{X})$ and $S \in \mathcal{B}(\mathcal{X}_S; \mathcal{X})$ satisfy*

$$RR^* = (\mathrm{I} + \beta^2 Y_\infty X_\infty)^{-1}\beta^2 Y_\infty, \qquad SS^* = X_\infty.$$

Remark 7.2.2. Solving a Lyapunov equation is numerically much easier than solving a Riccati equation. In practice, it therefore makes sense to compute the controllability Gramian $(\mathrm{I} + \beta^2 Y_\infty X_\infty)^{-1}\beta^2 Y_\infty$ of the closed-loop system instead of the solution Y_∞ of the HFARE. This can for example be done by solving the controllability Lyapunov equation with the ADI algorithm [ORW13]. This algorithm computes a factor R and motivates our presumption.

Lemma 7.2.3. *Let Presumption 7.2.1 hold. Then the Hankel operator of the closed-loop system* (7.8) *satisfies Presumption 6.2.13 with impulse response*

$$h = \mathfrak{C}_{\circlearrowright} B_{\circlearrowright} \in L^1 \cap L^2(\mathbb{R}_{\geqslant 0}; \mathcal{B}(\mathcal{U}; \mathcal{Y} \times \mathcal{U})).$$

The operators $Y_\infty X_\infty|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ and $S^* R : \mathcal{X} \to \mathcal{X}$ are compact. The nonincreasingly ordered sequence $(\nu_n)_{n\in\mathbb{N}}$ of the square-roots of the non-zero eigenvalues of $Y_\infty X_\infty$ is related to the singular values $(\sigma_n)_{n\in\mathbb{N}}$ of $S^* R$ by

$$\sigma_n = \frac{\beta \nu_n}{\sqrt{1 + \beta^2 \nu_n^2}} \quad \forall\, n \in \mathbb{N}. \tag{7.12}$$

Proof. Since the closed-loop semigroup is exponentially stable we conclude with standard estimates that

$$\mathfrak{C}_{\circlearrowright} \in L^1 \cap L^2(\mathbb{R}_{\geqslant 0}; \mathcal{B}(\mathcal{V}; \mathcal{Y} \times \mathcal{U})).$$

By [CLTZ94, Corollary 3.6], the function $\mathfrak{C}_{\circlearrowright} B$ is the impulse response, i.e. the Hankel operator has a representation of the form (6.23). With this and [GCP88, Appendix 1] it follows that the Hankel operator $\mathfrak{C}_{\circlearrowright}\mathfrak{B}_{\circlearrowright}$ is compact. This implies the compactness of the operator $\mathfrak{C}_{\circlearrowright}\mathfrak{B}_{\circlearrowright}\mathfrak{B}_{\circlearrowright}^{\odot}\mathfrak{C}_{\circlearrowright}^{\odot}$. Hence, the spectrum of this operator, with exception of the value zero, consists of countably many eigenvalues $(\sigma_n^2)_{n\in\mathbb{N}}$. Jacobson's Lemma [Mü07, Corollary 30 in Section I.1] implies that

$$\sigma(\mathfrak{C}_{\circlearrowright}\mathfrak{B}_{\circlearrowright}\mathfrak{B}_{\circlearrowright}^{\odot}\mathfrak{C}_{\circlearrowright}^{\odot})\backslash\{0\} = \sigma(\mathfrak{B}_{\circlearrowright}\mathfrak{B}_{\circlearrowright}^{\odot}\mathfrak{C}_{\circlearrowright}^{\odot}\mathfrak{C}_{\circlearrowright})\backslash\{0\},$$

and therefore, $\mathfrak{B}_{\circlearrowright}\mathfrak{B}_{\circlearrowright}^{\odot}\mathfrak{C}_{\circlearrowright}^{\odot}\mathfrak{C}_{\circlearrowright}$ is a compact operator as well. Since we have by Theorem 7.1.5 that

$$\mathfrak{B}_{\circlearrowright}\mathfrak{B}_{\circlearrowright}^{\odot}\mathfrak{C}_{\circlearrowright}^{\odot}\mathfrak{C}_{\circlearrowright} = (\mathrm{I} + \beta^2 Y_\infty X_\infty)^{-1}\beta^2 Y_\infty X_\infty = SS^* RR^*,$$

it follows that $Y_\infty X_\infty|_{\mathcal{X}} = (\mathrm{I} + \beta^2 Y_\infty X_\infty)|_{\mathcal{X}} \mathfrak{B}_{\circlearrowleft} \mathfrak{B}_{\circlearrowleft}^{\odot} \mathfrak{C}_{\circlearrowleft}^{\odot} \mathfrak{C}_{\circlearrowleft}$ is the concatenation of a compact and a continuous operator and therefore compact. Hence the, spectrum of this operator with exception of the value zero consists of eigenvalues and for all $n \in \mathbb{N}$,

$$\nu_n^2 \in \sigma(Y_\infty X_\infty) \qquad \Leftrightarrow \qquad \frac{\beta^2 \nu_n^2}{1 + \beta^2 \nu_n^2} \in \sigma(SS^* RR^*).$$

Another application of Jacobson's Lemma implies that $S^* RR^* S$ is compact and

$$\sigma(S^* RR^* S) \backslash \{0\} = \sigma(SS^* RR^*) \backslash \{0\} = \{\sigma_n^2 : n \in \mathbb{N}\}.$$

This implies (7.12) and the compactness of the operator root $(S^* RR^* S)^{\frac{1}{2}}$. Finally, the polar decomposition

$$S^* R = E (S^* RR^* S)^{\frac{1}{2}}$$

for some unitary operator E shows that $S^* R$ itself is compact and completes the proof. □

Definition 7.2.4 ($\mathcal{H}_\infty$-characteristic values)**.** Let Presumption 7.2.1 hold. Then the square roots of the countably many non-zero eigenvalues of $Y_\infty X_\infty$ are called *$\mathcal{H}_\infty$-characteristic values* of (A, B, C, D). We always order them non-increasingly in a sequence $(\nu_n)_{n \in \mathbb{N}}$.

Remark 7.2.5. With the diagonal operator $\Upsilon := \operatorname{diag}(\nu_n) \in \mathcal{B}(\ell^2)$ and Σ as in the singular value decomposition $S^* R = V \Sigma U^*$, the equation (7.12) implies that

$$\Upsilon = \tfrac{1}{\beta} \left(\mathrm{I} - \Sigma^2 \right)^{-\frac{1}{2}} \Sigma.$$

Now we apply the output normalizing transformations from Section 6.2 to the closed-loop system.

Lemma 7.2.6. *Let Presumption 7.2.1 hold and let $S^* R = V \Sigma U^*$ be the singular value decomposition of $S^* R$ with $\Sigma = \operatorname{diag}(\sigma_n) \in \mathcal{B}(\ell^2)$. Define the transformations*

$$\begin{aligned} T &:= V^* S^* \in \mathcal{B}\left(\mathcal{V}; \ell^2\right), \\ T^+ &:= R U \Sigma^{-1} \in \mathcal{B}\left(\Sigma \ell^2; \mathcal{W}\right), \end{aligned} \tag{7.13}$$

and let $(A_{\circlearrowleft}, B_{\circlearrowleft}, C_{\circlearrowleft}, D_{\circlearrowleft})$ be the closed-loop system in (7.8) *with input-output map*

$[\mathfrak{N}\,,\,\mathfrak{M}]^\top$. *We define the quotient operator* $\widetilde{A_\circlearrowleft}$ *as in Lemma 2.9.4, and as in Theorem 6.2.11 we can define* $A_{\circlearrowleft o}$ *to be the closure of the operator*

$$T\widetilde{A_\circlearrowleft}\pi_{\mathcal{W}/\ker S^*}T^+ z \quad \forall\, z \in \mathcal{Z} := T\mathfrak{B}_\circlearrowleft W_0^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U})$$

in ℓ^2. *Then the operators*

$$(A_{\circlearrowleft o}, B_{\circlearrowleft o}, C_{\circlearrowleft o}, D_{\circlearrowleft o}) := \left(\overline{T\widetilde{A_\circlearrowleft}\pi_{\mathcal{W}/\ker S^*}T^+}, TB_\circlearrowleft, C_\circlearrowleft T^+, D_\circlearrowleft\right) \tag{7.14}$$

generate a Pritchard-Salamon system on $(\mathcal{U}, (\Sigma\ell^2, \ell^2, \ell^2), \mathcal{Y}\times\mathcal{U})$, *which is a realization of* $[\mathfrak{N}\,,\,\mathfrak{M}]^\top$. *The output normalized truncation* $(A_{\circlearrowleft r}, B_{\circlearrowleft r}, C_{\circlearrowleft r}, D_\circlearrowleft)$ *of* $[\mathfrak{N}\,,\,\mathfrak{M}]^\top$ *in the sense of Definition 6.2.16 satisfies*

$$\begin{aligned}
\langle e_i\,,\, A_{\circlearrowleft r}e_j\rangle_{\mathbb{C}^r} &= \langle e_i\,,\, A_{\circlearrowleft o}e_j\rangle_{\ell^2} && \in \mathbb{C},\\
\langle e_i\,,\, B_{\circlearrowleft r}\cdot\rangle_{\mathbb{C}^r} &= \langle e_i\,,\, B_{\circlearrowleft o}\cdot\rangle_{\ell^2} = \langle (B_o^*)_{\mathrm{ex}}e_i\,,\,\cdot\rangle_{\mathcal{U}} && \in \mathcal{B}(\mathcal{U};\mathbb{C}),\\
C_{\circlearrowleft r}e_j &= (C_{\circlearrowleft o})_{\mathrm{ex}}e_j && \in \mathcal{Y}.
\end{aligned} \tag{7.15}$$

and its input-output map $[\mathfrak{N}_r\,,\,\mathfrak{M}_r]^\top$ *approximates* $[\mathfrak{N}\,,\,\mathfrak{M}]^\top$ *with the error bound*

$$\left\| \begin{bmatrix}\mathfrak{N}\\ \mathfrak{M}\end{bmatrix} - \begin{bmatrix}\mathfrak{N}_r\\ \mathfrak{M}_r\end{bmatrix} \right\|_{\mathrm{TIC}_0^2(\mathcal{U};\mathcal{Y}\times\mathcal{U})} \leqslant 2 \sum_{\{n>r\,|\,\sigma_n\neq\sigma_k\forall k<n\}} \sigma_n.$$

Proof. We apply Theorem 6.2.11 to the system (7.8). Note that the transformations in (7.13) are indeed bounded because the transformation $\mathfrak{V}$ and $\mathfrak{U}$ in Lemma 6.2.5 show

$$\begin{aligned}
T &= V^*S^* = V^*\mathfrak{V}^*\mathfrak{C}_\circlearrowleft \in \mathcal{B}\left(\mathcal{V};\ell^2\right),\\
T^+ &= RU\Sigma^{-1} = \mathfrak{B}_\circlearrowleft\mathfrak{U}U\Sigma^{-1} \in \mathcal{B}\left(\Sigma\ell^2;\mathcal{W}\right).
\end{aligned}$$

Theorem 6.2.11 states that $(A_{\circlearrowleft o}, B_{\circlearrowleft o}, C_{\circlearrowleft o}, D_{\circlearrowleft o})$ generates a 0-bounded well-posed linear system which is unitarily similar to the minimal output normalized shift realization of $[\mathfrak{N}\,,\,\mathfrak{M}]^\top$. We still need to prove that this system is of Pritchard-Salamon type. To this end, we take a closer look at the generators. According to Lemma 6.3.1, the part of $A_{\circlearrowleft o}$ in $\Sigma\ell^2$ is the operator

$$A_{\circlearrowleft o}^{\Sigma\ell^2} := T\widetilde{A_\circlearrowleft}\widetilde{\pi}_{\mathcal{W}/\ker S^*}T^+ z \quad \forall\, z \in T\mathfrak{B}_\circlearrowleft W_0^{1,2}(\mathbb{R}_{\leqslant 0};\mathcal{U})$$

and generates the restriction of the semigroup $\mathfrak{A}_{\circlearrowright o}$ to $\Sigma\ell^2$.

By Lemma 7.2.3 the impulse response of the closed-loop system is square integrable. Therefore, we may choose $\mathcal{W}_o = \ell^2$ in Theorem 6.2.19, which shows the boundedness of

$$B_{\circlearrowright o} = TBL^{-\frac{1}{2}} \in \mathcal{B}(\mathcal{U};\ell^2).$$

Furthermore, we have the output operator

$$C_{\circlearrowright o} = \begin{bmatrix} (C+DK)T^+ \\ KT^+ \end{bmatrix} \in \mathcal{B}(\Sigma\ell^2;\mathcal{Y}\times\mathcal{U}),$$

and the feedthrough

$$D_{\circlearrowright o} = \begin{bmatrix} DL^{-\frac{1}{2}} \\ L^{-\frac{1}{2}} \end{bmatrix}.$$

So these are indeed mappings between the correct spaces. The admissibility of B is equivalent to the boundedness of

$$T\mathfrak{B}_{\circlearrowright} = V^*S^*\mathfrak{B}_{\circlearrowright} \in \mathcal{B}\left(L^2(\mathbb{R}_{\leqslant 0};\mathcal{U});\Sigma\ell^2\right),$$

which is contained in Lemma 6.3.1, and the admissibility of C follows because the operator

$$\mathfrak{C}_{\circlearrowright}T^+ = \mathfrak{C}_{\circlearrowright}RU\Sigma^{-1} = \mathfrak{V}S^*RU\Sigma^{-1} = \mathfrak{V}V$$

admits a continuous extension to ℓ^2. Altogether, we see that $(A_{\circlearrowright o}, B_{\circlearrowright o}, C_{\circlearrowright o}, D_{\circlearrowright o})$ is a Pritchard-Salamon system.

Now the matrix coefficients in (7.15) follow directly from Theorem 6.2.19, and the error bound follows from Theorem 6.2.18. □

Remark 7.2.7. We point out that the two larger state spaces in the triple $(\Sigma\ell^2, \ell^2, \ell^2)$ coincide because $B_{\circlearrowright o}$ maps into ℓ^2. Therefore, there are only two different semigroup generators, $A_{\circlearrowright o}^{\Sigma\ell^2}$ and $A_{\circlearrowright o} = A_{\circlearrowright o}^{\ell^2}$.

Theorem 7.2.8. *Let Presumption 7.2.1 hold and define the quotient operator*

$$\widetilde{A} : \widetilde{\pi}_{\mathcal{X}/\ker S^*}\operatorname{dom} A \subset \mathcal{X} \to \mathcal{X},$$

$$\widetilde{A}\widetilde{x} := \widetilde{\pi}_{\mathcal{X}/\ker S^*}Az \quad \forall\, \widetilde{x} \in \operatorname{dom}\widetilde{A},\ \forall\, z \in \widetilde{x} \cap \operatorname{dom} A,$$

as in Lemma 2.9.4. Define the transformations T and T^+ and the space $\mathcal{Z}$ as in Lemma 7.2.6, and let A_∞ be the closure of the operator

$$T\widetilde{A}\,\widetilde{\pi}_{\mathcal{W}/\ker S^*}T^+z \quad \forall\, z \in \mathcal{Z}$$

in ℓ^2. Then

$$(A_\infty, B_\infty, C_\infty, D) := \left(\overline{T\widetilde{A}T^+}, TB, CT^+, D\right) \tag{7.16}$$

is a Pritchard-Salamon system on $(\mathcal{U}, (\Sigma\ell^2, \ell^2, \ell^2), \mathcal{Y})$ and a realization of $\mathfrak{D}$.

Proof. With the above assumptions we can define the closed-loop system $(A_\circlearrowleft, B_\circlearrowleft, C_\circlearrowleft, D_\circlearrowleft)$ with input-output map $[\mathfrak{N}\,,\,\mathfrak{M}]^\top$ as in Theorem 7.1.5, and the output normalized realization $(A_{\circlearrowleft o}, B_{\circlearrowleft o}, C_{\circlearrowleft o}, D_{\circlearrowleft o})$ of this mapping as in Lemma 7.2.6. The idea of the proof is to "undo" the feedback by application of the inverse feedback pair $[-KT^+\,,\,\mathrm{I} - L^{-\frac{1}{2}}] \in \mathcal{B}(\Sigma\ell^2 \times \mathcal{U}; \mathcal{U})$. We need to ensure that this process is compatible with the state space transformation, in other words that closing the loop and applying the transformations commute.

Note that the feedback pair above is admissible for $A_{\circlearrowleft o}$. Define the corresponding state feedback

$$K_o := \left(\mathrm{I} - (\mathrm{I} - L^{-\frac{1}{2}})\right)^{-1}\left(-KT^+\right) = -L^{\frac{1}{2}}KT^+ \in \mathcal{B}(\Sigma\ell^2; \mathcal{U}).$$

The admissibility implies by [vK93, Lemma 2.13 (ii)] that the operator defined via

$$A_{\circlearrowleft o}x + B_{\circlearrowleft o}K_o x \quad \forall\, x \in \left\{\, x \in \Sigma\ell^2 \cap \operatorname{dom} A_{\circlearrowleft o} \;\middle|\; A_{\circlearrowleft o}x + B_{\circlearrowleft o}K_o x \in \Sigma\ell^2 \,\right\}$$

generates a strongly continuous semigroup in $\Sigma\ell^2$ which extends to a strongly continuous semigroup on ℓ^2. The generator of this extended semigroup is denoted by $(A_{\circlearrowleft o} + B_{\circlearrowleft o}K_o)^{\ell^2}$. Moreover, by [vK93, Lemma 2.13] the domain of this operator equals $\operatorname{dom} A_{\circlearrowleft o}$ with equivalent norm, and it satisfies

$$(A_{\circlearrowleft o} + B_{\circlearrowleft o}K_o)^{\ell^2}x = A_{\circlearrowleft o}x + B_{\circlearrowleft o}K_o x \quad \forall\, x \in \operatorname{dom} A_{\circlearrowleft o} \cap \Sigma\ell^2.$$

The admissibility further implies that

$$\left((A_{\circlearrowleft o} + B_{\circlearrowleft o}K_o)^{\ell^2},\; B_{\circlearrowleft o}L^{\frac{1}{2}},\; C_{\circlearrowleft o,1} + D_{\circlearrowleft o,1}K_o,\; D_{\circlearrowleft o,1}L^{\frac{1}{2}}\right)$$

$$= \Big((A_{\circlearrowright o} + B_{\circlearrowright o}K_o)^{\ell^2}, TB, CT^+, D\Big)$$

is a Pritchard-Salamon system on $(\mathcal{U}, (\Sigma\ell^2, \ell^2, \ell^2), \mathcal{Y})$. Here $C_{\circlearrowright o,1}$ and $D_{\circlearrowright o,1}$ denote the first components of the block operators $C_{\circlearrowright o}$ and $D_{\circlearrowright o}$, respectively. What remains to be proven is that A_∞ coincides with $(A_{\circlearrowright o} + B_{\circlearrowright o}K_o)^{\ell^2}$. It suffices to verify this on the space $\mathcal{Z} \subset \operatorname{dom} A_{\circlearrowright o} \cap \Sigma\ell^2$, which is by Lemma 6.3.1 a core of $A_{\circlearrowright o}$, and therefore also a core of $(A_{\circlearrowright o} + B_{\circlearrowright o}K_o)^{\ell^2}$. With the notation of the Kalman compression as in Lemma 2.9.4, we have for all $z \in \mathcal{Z}$

$$\widetilde{\pi}_{\mathcal{W}/\ker S^*} T^+ z \in \operatorname{dom} \widetilde{A}_{\circlearrowright}^{\widetilde{\mathcal{W}}} \subset \operatorname{dom} \widetilde{A}^{\widetilde{\mathcal{V}}} \cap \widetilde{\mathcal{W}}.$$

Hence, $\ker S^* \cap \mathcal{W} = \ker \mathfrak{C}_{\circlearrowright}|_{\mathcal{W}} \subset \ker K$ yields

$$\begin{aligned}
(A_{\circlearrowright o} + B_{\circlearrowright o}K_o)^{\ell^2} z &= A_{\circlearrowright o} z + B_{\circlearrowright o}K_o z \\
&= T\widetilde{A_{\circlearrowright}}\widetilde{\pi}_{\mathcal{W}/\ker S^*} T^+ z - TBKT^+ z \\
&= T\widetilde{A}^{\widetilde{\mathcal{V}}}\widetilde{\pi}_{\mathcal{W}/\ker S^*} T^+ z + TBK\widetilde{\pi}_{\mathcal{W}/\ker S^*} T^+ z - TBKT^+ z \\
&= T\widetilde{A}\widetilde{\pi}_{\mathcal{W}/\ker S^*} T^+ z \\
&= A_\infty z,
\end{aligned}$$

and the proof is complete. □

Remark 7.2.9. (i) It can be shown that the compressed spaces $\widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{V}}$ defined in Lemma 2.9.4 satisfy

$$\overline{\widetilde{\pi}_{\mathcal{W}/\ker \mathfrak{C}|_{\mathcal{W}}} T^+ \Sigma\ell^2}^{\mathcal{W}} = \widetilde{\mathcal{W}}, \qquad \overline{\widetilde{\pi}_{\mathcal{V}/\ker \mathfrak{C}|_{\mathcal{V}}} T^+ \Sigma\ell^2}^{\mathcal{V}} = \widetilde{\mathcal{V}}.$$

(ii) For all $w \in \Sigma\ell^2$ we have

$$-\beta^2 B' X_\infty T^+ w = \beta^2 B' S S^* R U \Sigma^{-1} w = -\beta^2 B' S V e_i = -\beta^2 B_\infty^* w, \tag{7.17}$$

and since $B_\infty \in \mathcal{B}(\mathcal{U};\ell^2)$ this shows that the left hand side extends continuously to $-\beta^2 B_\infty^* \in \mathcal{B}(\ell^2;\mathcal{U})$. In regard of $K = -\beta^2(D^*C + B'X_\infty)$, this implies that if $D = 0$, then $KT^+ = -\beta^2 B' X_\infty T^+$ has a continuous extension to $\mathcal{B}(\ell^2;\mathcal{U})$.

Theorem 7.2.10. *Let Presumption 7.2.1 hold and let $(A_\infty, B_\infty, C_\infty, D)$ be the system in Theorem 7.2.8. Let $\Upsilon := \operatorname{diag}(\nu_n)$ be the diagonal operator built from* *the $\mathcal{H}^\infty$-characteristic values (ν_n) of (A, B, C, D). Then all unit vectors e_i satisfy $e_i \in \operatorname{dom} A_\infty$ and the following Riccati-like equations hold for all $i, j \in \mathbb{N}$:*

$$\begin{aligned} &\langle \mathrm{I}\, e_i, A_\infty e_j\rangle_{\ell^2} + \langle A_\infty e_i, \mathrm{I}\, e_j\rangle_{\ell^2} + \langle C_\infty e_i, C_\infty e_j\rangle_{\mathcal{Y}} \\ &\quad = \tfrac{1}{\beta^2} \left\langle (\mathrm{I} + \beta^2 D^* D)^{-1} (\beta^2 D^* C_\infty + \beta^2 B_\infty^* \mathrm{I}) e_i \,,\, (\beta^2 D^* C_\infty + \beta^2 B_\infty^* \mathrm{I}) e_j \right\rangle_{\mathcal{U}}, \end{aligned} \tag{7.18}$$

and

$$\begin{aligned} &\left\langle A_\infty \Upsilon^2 e_i \,,\, e_j \right\rangle_{\ell^2} + \left\langle e_i \,,\, A_\infty \Upsilon^2 e_j \right\rangle_{\ell^2} + \langle B_\infty^* e_i \,,\, B_\infty^* e_j\rangle_{\mathcal{U}} \\ &\quad = \left\langle (\mathrm{I} + D D^* \beta^2)^{-1} (D B_\infty^* \beta^2 + C_\infty \Upsilon^2) e_i \,,\, (D B_\infty^* \beta^2 + C_\infty \Upsilon^2) e_j \right\rangle_{\mathcal{Y}}. \end{aligned} \tag{7.19}$$

Remark 7.2.11. Since $(e_i)_{i\in\mathbb{N}}$ is in general not dense in $\operatorname{dom} A_\infty$, these equations do not imply that the operators I and Υ^2 solve the HCARE and HFARE in the sense of Definition 7.1.1.

Proof of Theorem 7.2.10. The idea of the proof is to use that the operators Σ^2 and I solve certain Lyapunov equations of the closed-loop system. This follows from the Lyapunov equations of the output normalized shift realization that are given in [Gui12], modulo a unitary transformation. Algebraically the upcoming calculations reverse the ones in the proof of Theorem 7.1.5. But because of the unboundedness of T^+ we have to be careful not to make any forbidden steps.

Recall from the definitions (7.8), (7.14), and the proof of Theorem 7.2.8 that we have

$$\begin{aligned} &(A_{\circlearrowright o}, B_{\circlearrowright o}, C_{\circlearrowright o}, D_{\circlearrowright o}) \\ &\quad = \left((A_\infty + TBKT^+)^{\ell^2}\big|_{\operatorname{dom} A_{\circlearrowright o}},\, TBL^{-1/2},\, \begin{bmatrix} CT^+ + DKT^+ \\ KT^+ \end{bmatrix},\, \begin{bmatrix} DL^{-1/2} \\ L^{-1/2} \end{bmatrix} \right). \end{aligned}$$

Let $\mathfrak{H}$ be the Hankel operator of this system and $\mathfrak{H} = \widetilde{V}\Sigma\widetilde{U}^*$ its singular value decomposition as in (6.10). By Corollary 6.2.10, the mapping $\widetilde{V}$ is unitary similarity transformation between this system and the exactly observable (and output normalized) shift realization of its input-output map on $\overline{\operatorname{ran}\mathfrak{H}}$. Since $\widetilde{V} e_i = \widetilde{v}_i \in W^{1,2}(\mathbb{R}_{\geqslant 0}; \mathcal{Y}) \cap \overline{\operatorname{ran}\mathfrak{H}}$, and the latter is the domain of the differential operator that generators the shift semigroup, we have

$$e_i = \widetilde{V}^* \widetilde{v}_i \in \operatorname{dom} A_{\circlearrowright o} \cap \Sigma \ell^2.$$

The shift realization satisfies the Lyapunov equation in [Gui12, Equation (5.105)], which becomes after application of $\widetilde{V}$

$$0 = \langle A_{\circlearrowright o} e_i\,,\, e_j\rangle_{\ell^2} + \langle e_i\,,\, A_{\circlearrowright o} e_j\rangle_{\ell^2} + \langle C_{\circlearrowright o} e_i\,,\, C_{\circlearrowright o} e_j\rangle_{\mathcal{U}_\beta}$$

for all $i, j \in \mathbb{N}$. Hence,

$$\begin{aligned}
0 &= \langle A_{\circlearrowright o} e_i\,,\, e_j\rangle_{\ell^2} + \langle e_i\,,\, A_{\circlearrowright o} e_j\rangle_{\ell^2} + \left\langle C_\infty e_i + DKT^+ e_i\,,\, C_\infty e_j + DKT^+ e_j\right\rangle_{\mathcal{Y}} \\
&\quad + \tfrac{1}{\beta^2}\left\langle KT^+ e_i\,,\, KT^+ e_j\right\rangle_{\mathcal{U}} \\
&= \langle A_{\circlearrowright o} e_i\,,\, e_j\rangle_{\ell^2} + \langle e_i\,,\, A_{\circlearrowright o} e_j\rangle_{\ell^2} + \langle C_\infty e_i\,,\, C_\infty e_j\rangle_{\mathcal{Y}} \\
&\quad + \left\langle \left(\tfrac{1}{\beta^2} + D^* D\right) KT^+ e_i\,,\, KT^+ e_j\right\rangle_{\mathcal{U}} \\
&\quad + \left\langle D^* C T^+ e_i\,,\, KT^+ e_j\right\rangle_{\mathcal{U}} + \left\langle KT^+ e_i\,,\, D^* C T^+ e_i\right\rangle_{\mathcal{U}} \\
&= \langle A_{\circlearrowright o} e_i\,,\, e_j\rangle_{\ell^2} + \langle e_i\,,\, A_{\circlearrowright o} e_j\rangle_{\ell^2} + \langle C_\infty e_i\,,\, C_\infty e_j\rangle_{\mathcal{Y}} + \tfrac{1}{\beta^2}\langle K_o e_i\,,\, K_o e_j\rangle_{\mathcal{U}} \\
&\quad + \left\langle D^* C_\infty e_i\,,\, KT^+ e_j\right\rangle_{\mathcal{U}} + \left\langle KT^+ e_i\,,\, D^* C_\infty e_j\right\rangle_{\mathcal{U}}.
\end{aligned}$$

With the two equations

$$\begin{aligned}
\left\langle D^* C_\infty e_i\,,\, KT^+ e_j\right\rangle_{\mathcal{U}} + \left\langle KT^+ e_i\,,\, D^* C_\infty e_j\right\rangle_{\mathcal{U}} &= -2\left\langle L^{-1}\beta^2 D^* C_\infty e_i\,,\, D^* C_\infty e_j\right\rangle_{\mathcal{U}} \\
- \left\langle D^* C_\infty e_i\,,\, L^{-1}\beta^2 B_\infty^* e_j\right\rangle_{\mathcal{U}} - \left\langle L^{-1}\beta^2 B_\infty^* e_i\,,\, D^* C_\infty e_j\right\rangle_{\mathcal{U}}
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{1}{\beta^2}\langle K_o e_i\,,\, K_o e_j\rangle_{\mathcal{U}} &= \beta^2\left\langle L^{-1} B_\infty^* e_i\,,\, B_\infty^* e_j\right\rangle_{\mathcal{U}} + \left\langle L^{-1}\beta^2 D^* C_\infty e_i\,,\, D^* C_\infty e_j\right\rangle_{\mathcal{U}} \\
&\quad + \left\langle D^* C_\infty e_i\,,\, L^{-1}\beta^2 B_\infty^* e_j\right\rangle + \left\langle L^{-1}\beta^2 B_\infty^* e_i\,,\, D^* C_\infty e_j\right\rangle_{\mathcal{U}},
\end{aligned}$$

(in which we have used (7.17)), the above becomes

$$\begin{aligned}
0 &= \langle A_{\circlearrowright o} e_i\,,\, e_j\rangle_{\ell^2} + \langle e_i\,,\, A_{\circlearrowright o} e_j\rangle_{\ell^2} + \langle C_\infty e_i\,,\, C_\infty e_j\rangle_{\mathcal{Y}} \\
&\quad - \left\langle L^{-1}\beta^2 D^* C_\infty e_i\,,\, D^* C_\infty e_j\right\rangle_{\mathcal{U}} + \left\langle L^{-1}\beta^2 B_\infty^* e_i\,,\, B_\infty^* e_j\right\rangle_{\mathcal{U}}
\end{aligned} \tag{7.20}$$

for all $i, j \in \mathbb{N}$. Using that

$$
\begin{aligned}
A_{\circlearrowright o} e_i &= A_\infty e_i - B_{\circlearrowright o} K_o e_i = A_\infty e_i + TBKT^+ e_i \\
&= A_\infty e_i - TBL^{-1}\beta^2 (B' X_\infty + D^* C) T^+ e_i \\
&\overset{(7.17)}{=} A_\infty e_i - B_\infty L^{-1} \beta^2 B_\infty^* e_i - B_\infty L^{-1} \beta^2 D^* C_\infty e_i
\end{aligned} \tag{7.21}
$$

for all e_i, we end up with

$$
\begin{aligned}
0 &= \left\langle (A_\infty - B_\infty L^{-1}\beta^2 D^* C_\infty) e_i \,,\, e_j \right\rangle_{\ell^2} + \left\langle e_i \,,\, (A_\infty - B_\infty L^{-1}\beta^2 D^* C_\infty) e_j \right\rangle_{\ell^2} \\
&\quad - \left\langle L^{-1}\beta^2 B_\infty^* e_i \,,\, B_\infty^* e_j \right\rangle_{\mathcal{U}} + \langle C_\infty e_i \,,\, C_\infty e_j \rangle_{\mathcal{Y}} - \left\langle L^{-1}\beta^2 D^* C_\infty e_i \,,\, D^* C_\infty e_j \right\rangle_{\mathcal{U}} \\
&= \langle A_\infty e_i \,,\, e_j \rangle_{\ell^2} + \langle e_i \,,\, A_\infty e_j \rangle_{\ell^2} + \langle C_\infty e_i \,,\, C_\infty e_j \rangle_{\mathcal{Y}} - \left\langle L^{-1}\beta^2 B_\infty^* e_i \,,\, B_\infty^* e_j \right\rangle_{\mathcal{U}} \\
&\quad - \left\langle L^{-1}\beta^2 D^* C_\infty e_i \,,\, B_\infty^* e_j \right\rangle_{\mathcal{U}} - \left\langle B_\infty^* e_j \,,\, L^{-1}\beta^2 D^* C_\infty e_i \right\rangle_{\mathcal{U}} \\
&\quad - \beta^2 \left\langle L^{-1} D^* C_\infty e_i \,,\, D^* C_\infty e_j \right\rangle_{\mathcal{U}} \\
&= \langle A_\infty e_i \,,\, e_j \rangle_{\ell^2} + \langle e_i \,,\, A_\infty e_j \rangle_{\ell^2} + \langle C_\infty e_i \,,\, C_\infty e_j \rangle_{\mathcal{Y}} \\
&\quad - \beta^2 \left\langle L^{-1} (D^* C_\infty + B_\infty^*) e_i \,,\, (D^* C_\infty + B_\infty^*) e_j \right\rangle_{\mathcal{U}} .
\end{aligned}
$$

This shows (7.18).

In order to prove the second equation we invoke the controllability Lyapunov equation [Gui12, Equation (5.106)] which implies with $\widetilde{V}^* \mathfrak{H}\mathfrak{H}^* \widetilde{V} = \Sigma^2$ that

$$
\begin{aligned}
0 &= \left\langle A_{\circlearrowright o} \Sigma^2 e_i \,,\, e_j \right\rangle_{\ell^2} + \left\langle e_i \,,\, A_{\circlearrowright o} \Sigma^2 e_j \right\rangle_{\ell^2} + \left\langle B_{\circlearrowright o}^{\odot} e_i \,,\, B_{\circlearrowright o}^{\odot} e_j \right\rangle_{\mathcal{U}_\beta} \\
&= \left\langle A_{\circlearrowright o} \Sigma^2 e_i \,,\, e_j \right\rangle_{\ell^2} + \left\langle e_i \,,\, A_{\circlearrowright o} \Sigma^2 e_j \right\rangle_{\ell^2} + \beta^2 \left\langle L^{-1} B_\infty^* e_i \,,\, B_\infty^* e_j \right\rangle_{\mathcal{U}}
\end{aligned}
$$

for all unit vectors e_i, e_j. Multiplying this equation by $(\frac{1}{\beta^2} + \nu_i)(1 + \beta^2 \nu_j)$ and using $\Sigma^2(\frac{1}{\beta^2} + \Upsilon^2) = \Upsilon^2$ we conclude that zero equals to

$$
\begin{aligned}
&\left\langle A_{\circlearrowright o} \Sigma^2 \left(\tfrac{1}{\beta^2} + \Upsilon^2\right) e_i \,,\, (1 + \beta^2 \Upsilon^2) e_j \right\rangle_{\ell^2} + \left\langle \left(\tfrac{1}{\beta^2} + \Upsilon^2\right) e_i \,,\, A_{\circlearrowright o} \Sigma^2 (1 + \beta^2 \Upsilon^2) e_j \right\rangle_{\ell^2} \\
&\qquad + \beta^2 \left\langle L^{-1} B_\infty^* \left(\tfrac{1}{\beta^2} + \Upsilon^2\right) e_i \,,\, B_\infty^* (1 + \beta^2 \Upsilon^2) e_j \right\rangle_{\mathcal{U}} \\
&\quad = \left\langle A_{\circlearrowright o} \Upsilon^2 e_i \,,\, (1 + \beta^2 \Upsilon^2) e_j \right\rangle_{\ell^2} + \left\langle (1 + \beta^2 \Upsilon^2) e_i \,,\, A_{\circlearrowright o} \Upsilon^2 e_j \right\rangle_{\ell^2} \\
&\qquad + \left\langle L^{-1} B_\infty^* \left(1 + \beta^2 \Upsilon^2\right) e_i \,,\, B_\infty^* (1 + \beta^2 \Upsilon^2) e_j \right\rangle_{\mathcal{U}} \\
&\quad = \left\langle A_{\circlearrowright o} \Upsilon^2 e_i \,,\, e_j \right\rangle_{\ell^2} + \left\langle e_i \,,\, A_{\circlearrowright o} \Upsilon^2 e_j \right\rangle_{\ell^2} + \left\langle A_{\circlearrowright o} \Upsilon^2 e_i \,,\, \beta^2 \Upsilon^2 e_j \right\rangle_{\ell^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left\langle \beta^2\Upsilon^2 e_i \,,\, A_{\circlearrowright o}\Upsilon^2 e_j \right\rangle_{\ell^2} + \left\langle L^{-1}B_\infty^* e_i \,,\, B_\infty^* e_j \right\rangle_{\mathcal{U}} + \left\langle L^{-1}B_\infty^*\beta^2\Upsilon^2 e_i \,,\, B_\infty^* e_j \right\rangle_{\mathcal{U}} \\
&+ \left\langle L^{-1}B_\infty^* e_i \,,\, B_\infty^*\beta^2\Upsilon^2 e_j \right\rangle_{\mathcal{U}} + \left\langle L^{-1}B_\infty^*\beta^2\Upsilon^2 e_i \,,\, B_\infty^*\beta^2\Upsilon^2 e_j \right\rangle_{\mathcal{U}} \\
= &\left\langle (A_{\circlearrowright o} + B_\infty L^{-1}\beta^2 B_\infty^*)\Upsilon^2 e_i \,,\, e_j \right\rangle_{\ell^2} + \left\langle e_i \,,\, (A_{\circlearrowright o} + B_\infty L^{-1}\beta^2 B_\infty^*)\Upsilon^2 e_j \right\rangle_{\ell^2} \\
&+ \left\langle A_{\circlearrowright o}\Upsilon^2 e_i \,,\, \beta^2\Upsilon^2 e_j \right\rangle_{\ell^2} + \left\langle \Upsilon^2 e_i \,,\, A_{\circlearrowright o}\beta^2\Upsilon^2 e_j \right\rangle_{\ell^2} + \left\langle L^{-1}B_\infty^* e_i \,,\, B_\infty^* e_j \right\rangle_{\mathcal{U}} \\
&+ \left\langle L^{-1}B_\infty^*\beta^2\Upsilon^2 e_i \,,\, B_\infty^*\beta^2\Upsilon^2 e_j \right\rangle_{\mathcal{U}} .
\end{aligned}
$$

Since $\Upsilon^2 e_i = \nu_i^2 e_i$ and $\beta^2\Upsilon^2 e_j = \nu_j^2 e_j$, we can use (7.20) multiplied by $\nu_i^2\beta^2\nu_j^2$ to substitute the second line of this equation and get

$$
\begin{aligned}
0 = &\left\langle (A_{\circlearrowright o} + B_\infty L^{-1}\beta^2 B_\infty^*)\Upsilon^2 e_i \,,\, e_j \right\rangle_{\ell^2} + \left\langle e_i \,,\, (A_{\circlearrowright o} + B_\infty L^{-1}\beta^2 B_\infty^*)\Upsilon^2 e_j \right\rangle_{\ell^2} \\
&+ \left\langle L^{-1}\beta^2 D^* C_\infty\Upsilon^2 e_i \,,\, D^* C_\infty\beta^2\Upsilon^2 e_j \right\rangle_{\mathcal{U}} - \left\langle C_\infty\Upsilon^2 e_i \,,\, C_\infty\beta^2\Upsilon^2 e_j \right\rangle_{\mathcal{Y}} \\
&+ \left\langle L^{-1}B_\infty^* e_i \,,\, B_\infty^* e_j \right\rangle_{\mathcal{U}} .
\end{aligned}
$$

Inserting (7.21) into this yields with the abbreviation $L^\diamond := \mathrm{I} + \beta^2 DD^*$, which satisfies $L^{-1}D^* = D^*(L^\diamond)^{-1}$,

$$
\begin{aligned}
0 = &\left\langle (A_\infty - B_\infty L^{-1}\beta^2 D^* C_\infty)\Upsilon^2 e_i \,,\, e_j \right\rangle_{\ell^2} + \left\langle e_i \,,\, (A_\infty - B_\infty L^{-1}\beta^2 D^* C_\infty)\Upsilon^2 e_j \right\rangle_{\ell^2} \\
&+ \left\langle (L^{-1} - \mathrm{I})B_\infty^* e_i \,,\, B_\infty^* e_j \right\rangle_{\mathcal{U}} + \langle B_\infty^* e_i \,,\, B_\infty^* e_j \rangle_{\mathcal{U}} \\
&+ \left\langle \left(DL^{-1}\beta^2 D^* - \mathrm{I}\right) C_\infty\Upsilon^2 e_i \,,\, C_\infty\beta^2\Upsilon^2 e_j \right\rangle_{\mathcal{Y}} \\
= &\left\langle A_\infty - B_\infty D^*(L^\diamond)^{-1}\beta^2 C_\infty)\Upsilon^2 e_i \,,\, e_j \right\rangle_{\ell^2} + \left\langle e_i \,,\, (A_\infty - B_\infty D^*(L^\diamond)^{-1}\beta^2 C_\infty)\Upsilon^2 e_j \right\rangle_{\ell^2} \\
&- \left\langle \beta^2(L^\diamond)^{-1}DB_\infty^* e_i \,,\, DB_\infty^* e_j \right\rangle_{\mathcal{Y}} + \langle B_\infty^* e_i \,,\, B_\infty^* e_j \rangle_{\mathcal{U}} \\
&- \left\langle (L^\diamond)^{-1}C_\infty\Upsilon^2 e_i \,,\, C_\infty\beta^2\Upsilon^2 e_j \right\rangle_{\mathcal{Y}} \\
= &\left\langle A_\infty\Upsilon^2 e_i \,,\, e_j \right\rangle_{\ell^2} + \left\langle e_i \,,\, A_\infty\Upsilon^2 e_j \right\rangle_{\ell^2} + \langle B_\infty^* e_i \,,\, B_\infty^* e_j \rangle_{\mathcal{U}} \\
&- \left\langle \beta^2(L^\diamond)^{-1}\left(DB_\infty^* + C_\infty\Upsilon^2\right) e_i \,,\, DB_\infty^* e_j \right\rangle_{\mathcal{Y}} \\
&- \left\langle (L^\diamond)^{-1}\left(DB_\infty^* + C_\infty\Upsilon^2\right) e_i \,,\, C_\infty\beta^2\Upsilon^2 e_j \right\rangle_{\mathcal{Y}} .
\end{aligned}
$$

This proves (7.19). □

7.3. $\mathcal{H}^\infty$-balanced truncation

Now we define the $\mathcal{H}^\infty$-balanced truncation in analogy to the balanced truncation in Definition 6.2.16. The main theorem in this section states that the $\mathcal{H}^\infty$-balanced truncation has essentially the same properties as the original system.

Definition 7.3.1. Let Presumption 7.2.1 hold, let $(\nu_n)_{n\in\mathbb{N}}$ be the $\mathcal{H}^\infty$-characteristic values of (A, B, C, D), and let $(A_\infty, B_\infty, C_\infty, D)$ be the system in (7.16). We define the *r-th $\mathcal{H}^\infty$-balanced truncation* of the Pritchard-Salamon system (A, B, C, D) to be the r-dimensional system (A_r, B_r, C_r, D) consisting of

$$A_r = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix} \in \mathbb{C}^{r\times r}, \qquad B_r = \begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix} \in \mathcal{B}(\mathcal{U}, \mathbb{C}^r),$$
$$C_r = \begin{bmatrix} c_1 & \cdots & c_r \end{bmatrix} \in \mathcal{B}(\mathbb{C}^r, \mathcal{Y}),$$

with coefficients

$$\begin{aligned} a_{ij} &:= \nu_i^{-\frac{1}{2}} \langle e_i \, , \, A_\infty e_j \rangle \, \nu_j^{\frac{1}{2}}, & & i, j \in \{1, \dots, r\}, \\ b_i u &:= \nu_i^{-\frac{1}{2}} \langle e_i \, , \, B_\infty u \rangle \, , & & i \in \{1, \dots, r\}, u \in \mathcal{U}, \\ c_j &:= C_\infty e_j \nu_j^{\frac{1}{2}}, & & j \in \{1, \dots, r\}. \end{aligned}$$

Recall that the $\mathcal{H}^\infty$-balanced system was constructed in the following way: First, the feedback loop was closed, then the closed-loop system was balanced, and then the loop was re-opened, before the system was finally truncated. The following theorem is a consequence of the fact that the truncation and the re-opening of the closed-loop in this process can be interchanged.

Theorem 7.3.2. *Let Presumption 7.2.1 hold and $(\nu_n)_{n\in\mathbb{N}}$ be the $\mathcal{H}^\infty$-characteristic values of (A, B, C, D). Choose $r \in \mathbb{N}$ such that $\nu_{r+1} \neq \nu_r$. Then the HCARE and the HFARE of the $\mathcal{H}^\infty$-balanced truncated system (A_r, B_r, C_r, D) are both solved by the matrix*

$$\Upsilon_r := \operatorname{diag}(\nu_1, \dots, \nu_r);$$

in other words,

$$\begin{aligned}\Upsilon_r A_r + A_r^*\Upsilon_r + C_r^* C_r &= \beta^2 \left(\Upsilon_r B_r + C^* D\right)\left(\mathrm{I} + \beta^2 D^* D\right)^{-1}\left(D^* C + B_r^* \Upsilon_r\right), \\ \Upsilon_r A_r^* + A_r\Upsilon_r + B_r B_r^* &= \beta^2 \left(\Upsilon_r C_r^* + B D^*\right)\left(\mathrm{I} + D\beta^2 D^*\right)^{-1}\left(D B^* + C_r \Upsilon_r\right).\end{aligned} \tag{7.22}$$

With L as in (7.7) and $K_r = -\beta^2 L^{-1}(B_r^*\Upsilon_r + D^* C_r)$, the matrix $A_r + B_r K_r$ generates an exponentially stable semigroup in $\mathbb{C}^r$, and the input-output map $[\mathfrak{N}_r\,,\,\mathfrak{M}_r]^\top$ of the closed-loop system,

$$\left(A_r + B_r K_r, B_r L^{-\frac{1}{2}}, \begin{bmatrix} C_r + D K_r \\ K_r \end{bmatrix}, \begin{bmatrix} D L^{-\frac{1}{2}} \\ L^{-\frac{1}{2}} \end{bmatrix}\right), \tag{7.23}$$

is a right factorization of the input-output map $\mathfrak{D}_r$ of (A_r, B_r, C_r, D). Moreover,

$$\beta^2 \mathfrak{N}_r^* \mathfrak{N}_r + \mathfrak{M}_r^* \mathfrak{M}_r = \mathrm{I}, \tag{7.24}$$

and

$$\left\| \begin{bmatrix} \mathfrak{N} \\ \mathfrak{M} \end{bmatrix} - \begin{bmatrix} \mathfrak{N}_r \\ \mathfrak{M}_r \end{bmatrix} \right\| \leqslant 2 \sum_{\{n>r \mid \nu_n \neq \nu_k \forall k<n\}} \frac{\beta \nu_n}{\sqrt{1+\beta^2 \nu_n^2}}. \tag{7.25}$$

Proof. The equations in (7.22) are a direct consequence of Theorem 7.2.10: Just multiply (7.18) by $\sqrt{\nu_i}\sqrt{\nu_j}$, and (7.19) by $\frac{1}{\sqrt{\nu_i}}\frac{1}{\sqrt{\nu_j}}$.

We prove that (7.23) is similar to the system $(A_{\circlearrowright r}, B_{\circlearrowright r}, C_{\circlearrowright r}, D_{\circlearrowright})$ in Lemma 7.2.6, i.e. the r-th output normalized truncation of the factor system in (7.8). More precisely, we show the following equations for all $i, j \in \{1, \dots, r\}$:

$$\left\langle e_i\,,\, \Upsilon_r^{\frac{1}{2}}(A_r + B_r K_r)\Upsilon_r^{-\frac{1}{2}} e_j \right\rangle_{\ell^2} = \langle e_i\,,\,(A_\infty + B_\infty K_o) e_j\rangle_{\ell^2},$$

$$\left\langle e_i\,,\, \Upsilon_r^{\frac{1}{2}} B_r L^{-\frac{1}{2}} \right\rangle_{\ell^2} = \left\langle e_i\,,\, B_\infty L^{-\frac{1}{2}} \right\rangle_{\ell^2},$$

$$\begin{bmatrix} (C_r + D K_r)\Upsilon_r^{-\frac{1}{2}} \\ K_r \Upsilon_r^{-\frac{1}{2}} \end{bmatrix} e_j = \begin{bmatrix} (C_\infty + D K_o) \\ K_o \end{bmatrix} e_j.$$

In view of Definition 7.3.1 these equations are immediate if the equation $K_r \Upsilon_r^{-\frac{1}{2}} e_j =$

$K_o e_j$ holds. This is equivalent to

$$(B_r^* \Upsilon_r + D^* C_r)\Upsilon_r^{-\frac{1}{2}} e_j = (B' X_\infty T^+ + D^* C T^+) e_j,$$

which holds because

$$\begin{aligned} B' X_\infty T^+ e_j &= B' S S^* R U \Sigma^{-1} e_j = B' S V e_j \overset{(7.17)}{=} B_\infty^* e_j = B_r^* \sqrt{\nu_j} e_j \\ &= B_r^* \Upsilon_r^{\frac{1}{2}} e_j. \end{aligned}$$

So $\Upsilon_r^{\frac{1}{2}}$ is indeed a similarity transformation between (7.23) and $(A_{\circlearrowright r}, B_{\circlearrowright r}, C_{\circlearrowright r}, D_{\circlearrowright})$. Since $A_{\circlearrowright r}$ generates by Theorem 6.2.18 an exponentially stable semigroup, so does $A_r + B_r K_r$. The error bound (7.25) also follows from Theorem 6.2.18.

Now that we know that K_r is an exponentially stabilizing feedback, we may apply Theorem 7.1.5 to the finite-dimensional system (A_r, B_r, C_r, D) to prove (7.24) and the lemma is shown. □

7.4. Robust control under right factor perturbation

In this section we show that a certain norm estimate guarantees robustness with respect to right factor perturbations. This result is independent of the rest of this thesis. We are going to apply it to the $\mathcal{H}^\infty$-balanced truncation in an ensuing corollary. For better reading, we write $\|\cdot\|$ for all norms in this section. This should not lead to confusion.

Definition 7.4.1. A controller $\mathfrak{K} \in \mathrm{TIC}^2_{\mathrm{loc}}(\mathcal{Y};\mathcal{U})$ is said to *stabilize* $\mathfrak{D} \in \mathrm{TIC}^2_{\mathrm{loc}}(\mathcal{U};\mathcal{Y})$ if $\mathrm{I} - \mathfrak{D}\mathfrak{K}$ has an inverse in $\mathrm{TIC}^2_0(\mathcal{Y};\mathcal{Y})$ and the operator

$$\begin{aligned} F(\mathfrak{D},\mathfrak{K}) &:= \begin{bmatrix} (\mathrm{I}-\mathfrak{D}\mathfrak{K})^{-1}\mathfrak{D} & (\mathrm{I}-\mathfrak{D}\mathfrak{K})^{-1}\mathfrak{D}\mathfrak{K} \\ \mathfrak{K}(\mathrm{I}-\mathfrak{D}\mathfrak{K})^{-1}\mathfrak{D} & \mathfrak{K}(\mathrm{I}-\mathfrak{D}\mathfrak{K})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathfrak{D}(\mathrm{I}-\mathfrak{K}\mathfrak{D})^{-1} & \mathfrak{D}(\mathrm{I}-\mathfrak{K}\mathfrak{D})^{-1}\mathfrak{K} \\ (\mathrm{I}-\mathfrak{K}\mathfrak{D})^{-1}\mathfrak{K}\mathfrak{D} & (\mathrm{I}-\mathfrak{K}\mathfrak{D})^{-1}\mathfrak{K} \end{bmatrix} \end{aligned} \tag{7.26}$$

is stable, i.e. $F(\mathfrak{D},\mathfrak{K}) \in \mathrm{TIC}^2_0(\mathcal{U}\times\mathcal{Y};\mathcal{Y}\times\mathcal{U})$.

Remark 7.4.2. $F(\mathfrak{D},\mathfrak{K})$ is the operator, that maps $[w_1\ w_2]^\top$ to $[z_1\ z_2]^\top$ in the closed-loop system depicted in Figure 7.4. Some literature requires boundedness of the

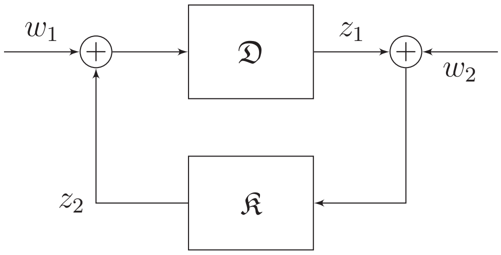

Figure 7.1.: Illustration of $[z_1\,,\ z_2]^\top = F(\mathfrak{D},\mathfrak{K})[w_1\,,\ w_2]^\top$.

operator

$$F(\mathfrak{D},\mathfrak{K}) + \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix}$$

instead of $F(\mathfrak{D},\mathfrak{K})$ for the definition of stability. This is obviously an equivalent condition and amounts to placing the signals z_1 and z_2 in Figure 7.4 behind the summations.

Theorem 7.4.3. *Let $\mathfrak{D}_r, \mathfrak{D} \in \mathrm{TIC}^2_{\mathrm{loc}}(\mathcal{U};\mathcal{Y})$, and let $[\mathfrak{N}\,,\ \mathfrak{M}]^\top \in \mathrm{TIC}^2_0(\mathcal{U};\mathcal{Y}\times\mathcal{U})$ be a right factorization of $\mathfrak{D}$ satisfying*

$$\beta^2\mathfrak{N}^*\mathfrak{N} + \mathfrak{M}^*\mathfrak{M} = \mathrm{I}\,. \tag{7.9}$$

for some $\beta \in (0,1]$. Let $[\mathfrak{N}_r\,,\ \mathfrak{M}_r]^\top \in \mathrm{TIC}^2_0(\mathcal{U};\mathcal{Y}\times\mathcal{U})$ be a right factorization of $\mathfrak{D}_r$ satisfying

$$\beta^2\mathfrak{N}_r^*\mathfrak{N}_r + \mathfrak{M}_r^*\mathfrak{M}_r = \mathrm{I} \tag{7.27}$$

and define

$$\varepsilon := \left\| \begin{bmatrix} \mathfrak{M} \\ \mathfrak{N} \end{bmatrix} - \begin{bmatrix} \mathfrak{M}_r \\ \mathfrak{N}_r \end{bmatrix} \right\|. \tag{7.28}$$

If $\mathfrak{K}_r \in \mathrm{TIC}^2_{\mathrm{loc}}(\mathcal{U};\mathcal{Y})$ stabilizes $\mathfrak{D}_r$ in such a way that, for some $\gamma > 0$,

$$\|F(\mathfrak{D}_r,\mathfrak{K}_r)\| \leqslant \gamma < \frac{1}{\varepsilon} - 1, \tag{7.29}$$

then $\mathfrak{K}_r$ stabilizes $\mathfrak{D}$ as well, and

$$\|F(\mathfrak{D},\mathfrak{K}_r)\| \leqslant \frac{\gamma + \gamma\varepsilon + \varepsilon}{1 - \gamma\varepsilon - \varepsilon}. \tag{7.30}$$

Before we proof this theorem, let us explain the ideas by means of state flow diagrams. We abbreviate the perturbations by $\Delta_{\mathfrak{M}} := \mathfrak{M} - \mathfrak{M}_{\mathrm{r}}$ and $\Delta_{\mathfrak{N}} := \mathfrak{N} - \mathfrak{N}_{\mathrm{r}}$. Then, $\mathfrak{D} = (\mathfrak{N}_{\mathrm{r}} + \Delta_{\mathfrak{N}})(\mathfrak{M}_{\mathrm{r}} + \Delta_{\mathfrak{M}})^{-1}$, so the plant $\mathfrak{D}$ may be replaced by the plant in Figure 7.2.

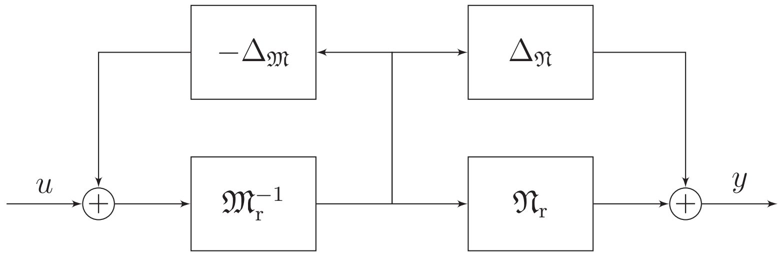

Figure 7.2.: Illustration of $y = \mathfrak{D}u = (\mathfrak{N}_r + \Delta_{\mathfrak{N}})(\mathfrak{M}_r + \Delta_{\mathfrak{M}})^{-1}u$.

Inserting this into the closed-loop system $F(\mathfrak{D}, \mathfrak{K}_{\mathrm{r}})$ gives the plant in Figure 7.3. The main assertion of Theorem 7.4.3 is that this plant is stable.

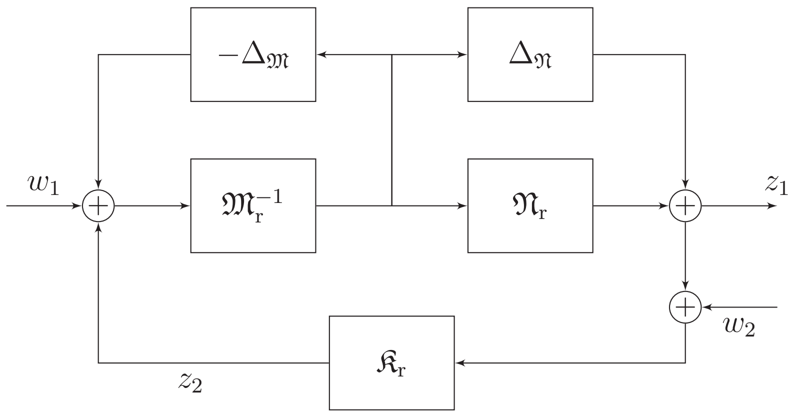

Figure 7.3.: Illustration of $[z_1\,,\, z_2]^\top = F(\mathfrak{D}, \mathfrak{K}_r)[w_1\,,\, w_2]^\top$ with $\mathfrak{D}$ as in Figure 7.2.

In order to prove this, we take a look at the reduced system and observe two things: Firstly, removing the perturbation $\Delta_{\mathfrak{M}}$ and $\Delta_{\mathfrak{N}}$ yields the closed-loop system $F(\mathfrak{D}_{\mathrm{r}}, \mathfrak{K}_{\mathrm{r}})$ which is stable by assumption; and secondly, the perturbations act like an output feedback of the auxiliary output z_3 marked in Figure 7.4.

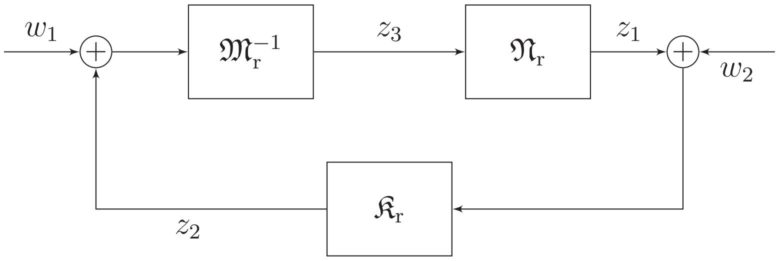

Figure 7.4.: Illustration of $[z_1\,,\, z_2]^\top = F(\mathfrak{N}_{\mathrm{r}}\mathfrak{M}_{\mathrm{r}}^{-1}, \mathfrak{K}_{\mathrm{r}})[w_1\,,\, w_2]^\top$.

Therefore, the following lemma about stability under feedback perturbation will be substantial in the proof. This lemma is readily verified and known as Small gain theorem.

Lemma 7.4.4 (Small gain theorem)**.** *Let $\mathfrak{K} \in \mathrm{TIC}_0^2(\mathcal{Y};\mathcal{U})$ and $\mathfrak{D} \in \mathrm{TIC}_0^2(\mathcal{U};\mathcal{Y})$ both be stable and assume*

$$\|\mathfrak{D}\|\,\|\mathfrak{K}\| < 1.$$

Then $\mathfrak{K}$ stabilizes $\mathfrak{D}$.

Proof of Theorem 7.4.3. We define the auxiliary operator

$$\mathfrak{F} := \mathfrak{M}_r^{-1}(\mathrm{I} - \mathfrak{K}_r\mathfrak{D}_r)^{-1}\begin{bmatrix}\mathrm{I} & \mathfrak{K}_r\end{bmatrix} \quad \in \quad \mathrm{TIC}_0^2(\mathcal{U}\times\mathcal{Y};\mathcal{U}),$$

which is the input-output map from $[w_1\,,\, w_2]^\top$ to z_3 in Figure 7.4. It is verified by looking at Figure 7.4 or short calculation that it fulfills the important equation

$$\begin{bmatrix}\mathfrak{N}_r\\ \mathfrak{M}_r\end{bmatrix}\mathfrak{F} := F(\mathfrak{D}_r, \mathfrak{K}_r) + \begin{bmatrix}0 & 0\\ \mathrm{I} & 0\end{bmatrix} \quad \in \quad \mathrm{TIC}_0^2(\mathcal{U}\times\mathcal{Y};\mathcal{Y}\times\mathcal{U}). \tag{7.31}$$

Now we proceed in three steps.

Step 1: We show that (7.29) implies

$$\|\mathfrak{F}\| < \frac{1}{\varepsilon}. \tag{7.32}$$

Equation (7.9) implies for all $u \in L^2(\mathbb{R};\mathcal{U})$

$$\left\|\begin{bmatrix}\mathfrak{M}u\\ \mathfrak{N}u\end{bmatrix}\right\|^2 = \langle\mathfrak{M}u, \mathfrak{M}u\rangle + \langle\mathfrak{N}u, \mathfrak{N}u\rangle \geqslant \langle\mathfrak{M}u, \mathfrak{M}u\rangle + \langle\beta^2\mathfrak{N}u, \mathfrak{N}u\rangle = \langle u, u\rangle = \|u\|^2,$$

and analogously, (7.27) implies

$$\left\| \begin{bmatrix} \mathfrak{M}_r u \\ \mathfrak{N}_r u \end{bmatrix} \right\| \geqslant \|u\|.$$

The second inequality gives

$$\begin{aligned}
\left\| \mathfrak{F} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| &\leqslant \left\| \begin{bmatrix} \mathfrak{N}_r \\ \mathfrak{M}_r \end{bmatrix} \mathfrak{M}_r^{-1} (\mathrm{I} - \mathfrak{K}_r \mathfrak{D}_r)^{-1} \begin{bmatrix} \mathrm{I} & \mathfrak{K}_r \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| \\
&\overset{(7.31)}{=} \left\| \left(F(\mathfrak{D}_r, \mathfrak{K}_r) + \begin{bmatrix} 0 & 0 \\ \mathrm{I} & 0 \end{bmatrix} \right) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| \\
&\leqslant (\gamma + 1) \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| \\
&\leqslant \frac{1}{\varepsilon} \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| \qquad \forall\, y_1, y_2 \in L^2(\mathbb{R}; \mathcal{Y}).
\end{aligned}$$

This proves (7.32).

Step 2: We show that (7.32) guarantees stability of $F(\mathfrak{D}, \mathfrak{K}_r)$. With the definition

$$\Delta := \begin{bmatrix} -\Delta_{\mathfrak{M}} \\ \Delta_{\mathfrak{N}} \end{bmatrix} := \begin{bmatrix} \mathfrak{M}_r - \mathfrak{M} \\ \mathfrak{N} - \mathfrak{N}_r \end{bmatrix},$$

and (7.28), the estimate (7.32) implies $\|\mathfrak{F}\| \|\Delta\| < 1$. Thus, by Lemma 7.4.4, the operator Δ stabilizes $\mathfrak{F}$. This means that $\mathrm{I} - \mathfrak{F}\Delta$ is invertible, that

$$\begin{aligned}
(\mathrm{I} - \mathfrak{F}\Delta)^{-1} &= \left(\mathrm{I} - \mathfrak{M}_r^{-1} (\mathrm{I} - \mathfrak{K}_r \mathfrak{D}_r)^{-1} \begin{bmatrix} \mathrm{I} & \mathfrak{K}_r \end{bmatrix} \begin{bmatrix} -\Delta_{\mathfrak{M}} \\ \Delta_{\mathfrak{N}} \end{bmatrix} \right)^{-1} \\
&= \left(\mathrm{I} + \mathfrak{M}_r^{-1} (\mathrm{I} - \mathfrak{K}_r \mathfrak{D}_r)^{-1} (\Delta_{\mathfrak{M}} - \mathfrak{K}_r \Delta_{\mathfrak{N}}) \right)^{-1} \\
&= ((\mathrm{I} - \mathfrak{K}_r \mathfrak{D}_r) \mathfrak{M}_r + \Delta_{\mathfrak{M}} - \mathfrak{K}_r \Delta_{\mathfrak{N}})^{-1} (\mathrm{I} - \mathfrak{K}_r \mathfrak{D}_r) \mathfrak{M}_r \\
&= (\mathfrak{M} - \mathfrak{K}_r \mathfrak{N})^{-1} (\mathrm{I} - \mathfrak{K}_r \mathfrak{D}_r) \mathfrak{M}_r
\end{aligned}$$

is in $\mathrm{TIC}_0^2(\mathcal{U}; \mathcal{U})$, and that

$$(\mathrm{I} - \mathfrak{F}\Delta)^{-1} \mathfrak{F} = \begin{bmatrix} (\mathfrak{M} - \mathfrak{K}_r \mathfrak{N})^{-1} & (\mathfrak{M} - \mathfrak{K}_r \mathfrak{N})^{-1} \mathfrak{K}_r \end{bmatrix}$$

$$= \mathfrak{M}^{-1}\left(\mathrm{I} - \mathfrak{K}_r\mathfrak{D}\right)^{-1}\begin{bmatrix}\mathrm{I} & \mathfrak{K}_r\end{bmatrix}$$

is in $\mathrm{TIC}_0^2(\mathcal{U}\times\mathcal{Y};\mathcal{U})$. In analogy to (7.31), multiplying this equation by $[\mathfrak{N}\,,\,\mathfrak{M}]^\top$ gives

$$\begin{bmatrix}\mathfrak{N}\\ \mathfrak{M}\end{bmatrix}(\mathrm{I} - \mathfrak{F}\Delta)^{-1}\mathfrak{F} = F(\mathfrak{D},\mathfrak{K}_r) + \begin{bmatrix}0 & 0\\ \mathrm{I} & 0\end{bmatrix}. \tag{7.33}$$

Since the left hand side is an element of $\mathrm{TIC}_0^2(\mathcal{U}\times\mathcal{Y};\mathcal{Y}\times\mathcal{U})$, this shows that $\mathfrak{K}_r$ stabilizes $\mathfrak{D}$.

Step 3: We prove (7.30). Combining equations (7.33) and (7.31), we obtain

$$\begin{aligned}
F(\mathfrak{D},\mathfrak{K}_r) - F(\mathfrak{D}_r,\mathfrak{K}_r) &= \begin{bmatrix}\mathfrak{N}\\ \mathfrak{M}\end{bmatrix}(\mathrm{I} - \mathfrak{F}\Delta)^{-1}\mathfrak{F} - \begin{bmatrix}\mathfrak{N}_r\\ \mathfrak{M}_r\end{bmatrix}\mathfrak{F}\\
&= \begin{bmatrix}\mathfrak{N}_r\\ \mathfrak{M}_r\end{bmatrix}(\mathrm{I} - \mathfrak{F}\Delta)^{-1}\mathfrak{F} + \begin{bmatrix}\Delta_{\mathfrak{N}}\\ \Delta_{\mathfrak{M}}\end{bmatrix}(\mathrm{I} - \mathfrak{F}\Delta)^{-1}\mathfrak{F} - \begin{bmatrix}\mathfrak{N}_r\\ \mathfrak{M}_r\end{bmatrix}\mathfrak{F}\\
&= \begin{bmatrix}\mathfrak{N}_r\\ \mathfrak{M}_r\end{bmatrix}\left((\mathrm{I} - \mathfrak{F}\Delta)^{-1} - \mathrm{I}\right)\mathfrak{F} + \begin{bmatrix}\Delta_{\mathfrak{N}}\\ \Delta_{\mathfrak{M}}\end{bmatrix}(\mathrm{I} - \mathfrak{F}\Delta)^{-1}\mathfrak{F}\\
&= \begin{bmatrix}\mathfrak{N}_r\\ \mathfrak{M}_r\end{bmatrix}\mathfrak{F}\Delta(\mathrm{I} - \mathfrak{F}\Delta)^{-1}\mathfrak{F} + \begin{bmatrix}\Delta_{\mathfrak{N}}\\ \Delta_{\mathfrak{M}}\end{bmatrix}(\mathrm{I} - \mathfrak{F}\Delta)^{-1}\mathfrak{F}\\
&= \left(F(\mathfrak{D}_r,\mathfrak{K}_r) + \begin{bmatrix}0 & 0\\ \mathrm{I} & 0\end{bmatrix}\right)\Delta(\mathrm{I} - \mathfrak{F}\Delta)^{-1}\mathfrak{F}\\
&\qquad\qquad + \begin{bmatrix}\Delta_{\mathfrak{N}}\\ \Delta_{\mathfrak{M}}\end{bmatrix}(\mathrm{I} - \mathfrak{F}\Delta)^{-1}\mathfrak{F}\\
&= \left(F(\mathfrak{D}_r,\mathfrak{K}_r)\Delta + \begin{bmatrix}\Delta_{\mathfrak{N}}\\ 0\end{bmatrix}\right)(\mathrm{I} - \mathfrak{F}\Delta)^{-1}\mathfrak{F}.
\end{aligned}$$

Since $[\mathfrak{N}\,,\,\mathfrak{M}]^\top$ is an isometry, we deduce for the norm

$$\begin{aligned}
\|F(\mathfrak{D},\mathfrak{K}_r)\| &\leqslant \|F(\mathfrak{D}_r,\mathfrak{K}_r)\| + \left\|F(\mathfrak{D}_r,\mathfrak{K}_r)\Delta + \begin{bmatrix}\Delta_{\mathfrak{N}}\\ 0\end{bmatrix}\right\|\|(\mathrm{I} - \mathfrak{F}\Delta)^{-1}\mathfrak{F}\|\\
&\leqslant \|F(\mathfrak{D}_r,\mathfrak{K}_r)\| + (\|F(\mathfrak{D}_r,\mathfrak{K}_r)\|\varepsilon + \varepsilon)\left\|\begin{bmatrix}\mathfrak{N}\\ \mathfrak{M}\end{bmatrix}(\mathrm{I} - \mathfrak{F}\Delta)^{-1}\mathfrak{F}\right\|
\end{aligned}$$

$$\leqslant \|F(\mathfrak{D}_r,\mathfrak{K}_r)\| + (\|F(\mathfrak{D}_r,\mathfrak{K}_r)\|\varepsilon + \varepsilon)\left\| F(\mathfrak{D},\mathfrak{K}_r) + \begin{bmatrix} 0 & 0 \\ \mathrm{I} & 0 \end{bmatrix} \right\|$$
$$\leqslant \|F(\mathfrak{D}_r,\mathfrak{K}_r)\| + (\|F(\mathfrak{D}_r,\mathfrak{K}_r)\|\varepsilon + \varepsilon)\left(\|F(\mathfrak{D},\mathfrak{K}_r)\| + 1\right).$$

Since assumption (7.29) guarantees

$$1 - \|F(\mathfrak{D}_r,\mathfrak{K}_r)\|\varepsilon - \varepsilon > \varepsilon,$$

we may conclude from the inequality above that

$$\|F(\mathfrak{D},\mathfrak{K}_r)\| \leqslant \frac{\|F(\mathfrak{D}_r,\mathfrak{K}_r)\|(1+\varepsilon) + \varepsilon}{1 - \|F(\mathfrak{D}_r,\mathfrak{K}_r)\|\varepsilon - \varepsilon}.$$

This proves the claim (7.30) □

If we have a robust controller for the reduced system in Theorem 7.3.2, then Theorem 7.4.3 shows that this controller will also stabilize the original, infinite-dimensional system. Fortunately, sufficient conditions for the existence of such a controller are well-known, see e.g. [TSH01, Chapter 14] or [MG90, Chapter 7]. In fact, the controller we are looking for is the solution to a special version of the so-called $\mathcal{H}^\infty$-four block problem, which is the following:

Problem ($\mathcal{H}^\infty$-four block problem)**.** *Given $\gamma > 0$ and*

$$\left[\begin{array}{c|c} \mathfrak{D}_{11} & \mathfrak{D}_{12} \\ \hline \mathfrak{D}_{21} & \mathfrak{D}_{22} \end{array}\right] \quad \in \quad \mathrm{TIC}^2_{\mathrm{loc}}(\mathcal{U}_1 \times \mathcal{U}_2; \mathcal{Y}_1 \times \mathcal{Y}_2),$$

find $\mathfrak{K} \in \mathrm{TIC}^2_{\mathrm{loc}}(\mathcal{Y}_2;\mathcal{U}_2)$ such that $(\mathrm{I} - \mathfrak{D}_{22}\mathfrak{K})$ has an inverse in $\mathrm{TIC}^2_0(\mathcal{Y}_2;\mathcal{Y}_2)$, and $\mathfrak{D}_{11} + \mathfrak{D}_{12}\mathfrak{K}(\mathrm{I} - \mathfrak{D}_{22}\mathfrak{K})^{-1}\mathfrak{D}_{21}$ is in $\mathrm{TIC}^2_0(\mathcal{U}_1;\mathcal{Y}_1)$ with

$$\|\mathfrak{D}_{11} + \mathfrak{D}_{12}\mathfrak{K}(\mathrm{I} - \mathfrak{D}_{22}\mathfrak{K})^{-1}\mathfrak{D}_{21}\| < \gamma. \tag{7.34}$$

If we define for our $\mathfrak{D}_r \in \mathrm{TIC}^2_{\mathrm{loc}}(\mathcal{U};\mathcal{Y})$ the auxiliary input-output map

$$\left[\begin{array}{c|c} \mathfrak{D}_{11} & \mathfrak{D}_{12} \\ \hline \mathfrak{D}_{21} & \mathfrak{D}_{22} \end{array}\right] := \left[\begin{array}{cc|c} \mathfrak{D}_r & 0 & \mathfrak{D}_r \\ 0 & 0 & \mathrm{I} \\ \hline \mathfrak{D}_r & \mathrm{I} & \mathfrak{D}_r \end{array}\right], \tag{7.35}$$

then (7.34) becomes the closed-loop map $F(\mathfrak{D}_r, \mathfrak{K})$ as defined in (7.26). This means, if $\mathfrak{K}_r$ solves the $\mathcal{H}^\infty$-four block problem for the auxiliary input-output map in (7.35), then it has the desired property $F(\mathfrak{D}_r, \mathfrak{K}_r) < \gamma$. If (A_r, B_r, C_r, D) is a realization of $\mathfrak{D}_r$, then

$$\left(A_r, \begin{bmatrix} B_r & 0 & \vline & B_r \end{bmatrix}, \begin{bmatrix} C_r \\ 0 \\ \hline C_r \end{bmatrix}, \begin{bmatrix} D & 0 & \vline & D \\ 0 & 0 & \vline & \mathrm{I} \\ \hline D & \mathrm{I} & \vline & D \end{bmatrix} \right) \tag{7.36}$$

is a realization of (7.35). Therefore, it suffices to solve the well-known finite-dimensional $\mathcal{H}^\infty$-four block problem for this auxiliary system. Solutions to this problem can for example be found in [Sto92, TSH01, ZDG96]. The idea of using the auxiliary input-output map is exploited in [MG90]. In order to solve this auxiliary four block problem, one has to assume that the diagonal of the feedthrough operator consists of zero matrices. For the system in (7.36), this means $D = 0$. Under this condition, we obtain our final corollary.

Corollary 7.4.5. *Let $\gamma > 1$, and let Presumption 7.2.1 hold with $D = 0$ and $\beta := (1 - \frac{1}{\gamma^2})^{\frac{1}{2}}$. Let $(\nu_n)_{n\in\mathbb{N}}$ be the $\mathcal{H}^\infty$-characteristic values of $(A, B, C, 0)$, and choose $r \in \mathbb{N}$ such that $\nu_{r+1} \neq \nu_r$. Assume that the r-th $\mathcal{H}^\infty$-balanced truncation $(A_r, B_r, C_r, 0)$ of $\mathfrak{D}$ is detectable (in the sense of [MG90]) and that $\nu_r < \gamma$. Then there is a controller $\mathfrak{K}_r$ that stabilizes the input-output map $\mathfrak{D}$ of $(A, B, C, 0)$ and has the r-dimensional state space realization*

$$\left(A_r - \beta^2 \Upsilon_r C_r^* C_r - B_r B_r^* \Upsilon_r \left(\mathrm{I} - \gamma^{-2}\Upsilon_r^2\right),\ \Upsilon_r C_r^*,\ -B_r^* \Upsilon_r \left(\mathrm{I} - \gamma^{-2}\Upsilon_r^2\right),\ 0 \right),$$

where $\Upsilon_r := \operatorname{diag}(\nu_1, \ldots, \nu_r)$. Moreover, the performance estimate in (7.30) holds.

Proof. The $\mathcal{H}^\infty$-balanced truncation $(A_r, B_r, C_r, 0)$ is by assumption detectable, and by Theorem 7.3.2 stabilizable. Moreover, by the same theorem, the operator Υ_r solves the HCARE and the HFARE in (7.22). A simple calculation shows that with $D = 0$ and $\beta := (1 - \frac{1}{\gamma^2})^{\frac{1}{2}}$, the Riccati equations in (7.22) reduce to the Riccati equations in [MG90, Proposition 7.3.3]. By our previous consideration, Υ_r therefore solves these equations and, with the additional condition $\nu_r < \gamma$, Proposition 7.3.3 of [MG90] states that the controller $\mathfrak{K}_r$ stabilizes $\mathfrak{D}$ with $F(\mathfrak{D}_r, \mathfrak{K}_r) < \gamma$. Finally, Theorem 7.4.3 shows that $\mathfrak{K}_r$ stabilizes $\mathfrak{D}$. □

Remark 7.4.6. If $D \neq 0$, there is still a way to solve the auxiliary $\mathcal{H}^\infty$-four block problem [Sto92, Section 5.6]. In contrast to the situation in Corollary 7.4.5, this method does not lead to the HCARE and HFARE in Definition 7.1.1 with $D \neq 0$ as one might hope. Thus, the $\mathcal{H}^\infty$-balancing and truncation method described in this chapter is not applicable in this case.

7.5. Notes and references

The overall idea of the $\mathcal{H}^\infty$-balancing and truncation procedures in Chapter 7 follows the finite-dimensional analog in [MG91], although there are some conceptional differences: We use a right factorization instead of a left factorization and we introduce a new scalar product instead of scaling the input-output map. This leads to a different measure of the error bound, which is neither better nor worse. The approach is based on the idea that the $\mathcal{H}^\infty$-four block problem for the auxiliary system (7.36) is closely related to the LQG problem for $(A, B, C, 0)$. According to [MG90, Chapter 7], it can also be interpreted as the problem of minimizing the so-called "entropy" of $(A, B, C, 0)$.

Coprime factorizations arising from exponentially stabilizing feedback are well understood for regular well-posed linear systems [CWW96]. There are extensions for non-exponentially stabilizing feedbacks as well [Sta98a, Mik02, Mik06]. The connection to Riccati equations is made in [Mik02, Mik06, CO06, OS14]. In particular, the construction of normalized factorizations is discussed in [CO06, Mik06].

The robustness results in Section 7.4 are standard algebraic calculations, similar to the ones for left factorizations in [Cur90], [CZ95, Chapter 9], or [ZDG96, Chapter 9].

A similar procedure for $\mathcal{H}^\infty$-balanced truncation can be carried out for discrete time systems, see [Sel15]. The works [CO04, Opm06, Opm07, OS14] make use of the discrete time theory and the Cayley transformation to transform the Riccati equations into simpler equations with bounded operators. In [Opm06, Opm07] this was exploited to construct an LQG-balanced truncation for continuous time infinite-dimensional systems. This approach can be exploited in various ways: either only the balancing and truncation is performed in discrete time, or even the controller is constructed in discrete time, before transforming back to continuous time. In any case, this method leads to a different controller since the Cayley transform does not commute with the process of truncating. Moreover, the state space realization of the resulting controller not as practical to compute as the one in Corollary 7.4.5.

A. Appendix

A.1. Analytic semigroups and interpolation

We recap some facts about analytic semigroups and fractional powers of their generators. These standard results are documented in e.g. [Sta05, Sections 3.9–3.10], [Paz83, Section 2.6].

An *analytic semigroup* in the Hilbert space $\mathcal{X}$ is a strongly continuous semigroup $\mathfrak{A}$ in $\mathcal{X}$ which can be extended to an analytic mapping $\mathfrak{A} : S_{0,\theta} \to \mathcal{B}(\mathcal{X})$ on a sector

$$S_{0,\theta} := \{ \lambda \in \mathbb{C} \mid \arg \lambda \in (-\theta, \theta) \} \quad \text{for some } \theta \in (0, \tfrac{\pi}{2}),$$

such that $\mathfrak{A}(s)\mathfrak{A}(t) = \mathfrak{A}(s+t)$ for all $s, t \in S_{0,\theta}$ and

$$\lim_{t \to 0,\ t \in S_{0,\theta}} \mathfrak{A}(t)x = x \quad \forall\, x \in \mathcal{X}.$$

A densely defined operator A is the generator of an analytic semigroup if and only if A is *sectorial* in the sense of [Sta05, Definition 3.10.2].

Let $A : \operatorname{dom} A \subset \mathcal{X} \to \mathcal{X}$ be the generator of an analytic semigroup $\mathfrak{A}$ in a Hilbert space $\mathcal{X}$ and let $\lambda \in \mathbb{C}_{\geqslant \omega_{\mathfrak{A}}}$. The negative powers of $\lambda - A$ are defined by the formula

$$(\lambda - A)^{-\alpha} x := \left(\int_0^\infty t^{\alpha-1} \mathrm{e}^{-t}\, \mathrm{d}t \right)^{-1} \int_0^\infty t^{\alpha-1} \mathrm{e}^{-\lambda t} \mathfrak{A}(t) x \, \mathrm{d}t, \quad \forall\, \alpha > 0,\ x \in \mathcal{X}.$$

This formula defines an injective, bounded linear operator. The positive positive powers of A are defined by

$$(\lambda - A)^{\alpha} : \operatorname{ran}(\lambda - A)^{-\alpha} \to \mathcal{X}, \quad (\lambda - A)^{\alpha} := \big((\lambda - A)^{-\alpha} \big)^{-1}, \quad \forall\, \alpha > 0.$$

These operators are closed and densely defined in $\mathcal{X}$. For $\alpha > 0$, the domain of

$(\lambda - A)^\alpha$ is equipped with the norm

$$\|x\|_\alpha := \|(\lambda - A)^\alpha x\|_{\mathcal{X}} ,$$

and named $\mathcal{X}_\alpha$. The space $\mathcal{X}_{-\alpha}$ is defined as the dual space of $\mathcal{X}_\alpha$ with respect to the pivot space $\mathcal{X}$, and therefore it is a subspace of the rigged space $\mathcal{X}_{-1} = (\operatorname{dom} A^*)'$ in Section 2.1. Different choices of $\lambda \in \mathbb{C}_{\geqslant \omega_{\mathfrak{A}}}$ yield equivalent norms. As a consequence of [Sta05, Lemma 3.10.9] we may state the following Lemma.

Lemma A.1.1. *Let $\mathfrak{A}$ be an analytic semigroup in $\mathcal{X}$, $\alpha \in [-1, 1]$ and define $\mathcal{X}_\alpha$ as above. The extension $\mathfrak{A}|_{\mathcal{X}_{-1}}$ restricts to an analytic semigroup on $\mathcal{X}_\alpha$. Furthermore,*

$$\exists\, M \geqslant 1: \ \left\|\mathfrak{A}_{\mathcal{X}_{-1}}(t)x\right\|_{\mathcal{X}_{\alpha+\theta}} \leqslant M \left(1 + t^{-\theta}\right) \mathrm{e}^{\omega_{\mathfrak{A}} t} \|x\|_{\mathcal{X}_\alpha} \quad \forall\, x \in \mathcal{X}_\alpha,\ t > 0. \qquad \text{(A.1)}$$

The estimate in (A.1) is an interpolation inequality. Let us briefly summarize the simplest form of the complex interpolation functor $[\cdot, \cdot]_\theta$ defined in [Tri95, Section 1.9.2].

Definition A.1.2 (complex interpolation functor)**.** Let $\mathcal{W}$, $\mathcal{X}$ be Banach spaces with $\mathcal{W} \hookrightarrow \mathcal{X}$. Define $S := \mathbb{C}_{>0} \backslash \mathbb{C}_{\geqslant 1}$ and $S_\alpha := \{\ \alpha + it \mid t \in \mathbb{R}\ \}$ for $\alpha = 0, 1$. Furthermore, define the function space

$$\mathcal{F}(S) := \left\{\ f \in \mathcal{C}(\overline{S}; \mathcal{X}) \cap \mathcal{H}^\infty(S; \mathcal{X}) \ \middle|\ f\big|_{S_0} \in \mathcal{C}(S_0; \mathcal{X}) \ \wedge\ f\big|_{S_1} \in \mathcal{C}(S_1; \mathcal{W})\ \right\}.$$

Then for $\theta \in (0, 1)$, the interpolation space of exponent θ is defined as

$$[\mathcal{X}\,,\, \mathcal{W}]_\theta := \{\ x \in \mathcal{X} \ \mid \exists\, f \in \mathcal{F}(S):\ f(\theta) = x\ \}$$

with norm

$$\|x\|_\theta := \inf \left\{\ \max \left\{\sup_{t \in S_0} \|f(t)\|_{\mathcal{X}}\, , \sup_{t \in S_1} \|f(t)\|_{\mathcal{W}}\right\}\ \middle|\ f \in \mathcal{F}(S) \wedge f(\theta) = x\ \right\}.$$

By [Tri95, Section 1.18.10] the following theorem holds.

Theorem A.1.3. *Let A be the self-adjoint generator of an analytic semigroup in*

$\mathcal{X}$, and let $0 \leqslant \beta < \alpha \leqslant 1$. For the spaces $\mathcal{X}_\alpha$, $\mathcal{X}_\beta$ defined as above, we have

$$\mathcal{X}_{(1-\theta)\alpha+\theta\beta} = [\mathcal{X}_\alpha\,,\,\mathcal{X}_\beta]_\theta \quad \forall\,\theta \in (0,1). \tag{A.2}$$

The next lemma, which is found for example in [EN00, p. 60], helps determine the generator of $\mathfrak{A}|_{\mathcal{X}_\alpha}$.

Lemma A.1.4. *Let the Banach space $\mathcal{W}$ be continuously embedded into the Banach space $\mathcal{X}$, and let A be the generator of a semigroup $\mathfrak{A}$ in $\mathcal{X}$. Assume that $\mathcal{W}$ is $\mathfrak{A}$-invariant and $t \mapsto \mathfrak{A}(t)|_{\mathcal{W}}$ is strongly continuous with respect to the norm of $\mathcal{W}$. Then the generator of $\mathfrak{A}|_{\mathcal{W}}$ is* the part of A in $\mathcal{W}$, *i.e. the operator*

$$A^{\mathcal{W}}x := Ax \quad \forall x \in \operatorname{dom} A^{\mathcal{W}} := \{\ x \in \mathcal{W} \cap \operatorname{dom} A \ \mid\ Ax \in \mathcal{W}\ \}.$$

Thus, for $\alpha \in [-1,1]$, the generator of the semigroup $\mathfrak{A}|_{\mathcal{X}_\alpha}$ is the part of $A|_{\mathcal{X}_{-1}}$ in $\mathcal{X}_\alpha$.

Another powerful means to determine the space $\mathcal{X}_{1/2}$ is Kato's Second representation theorem. We summarize Kato's First and Second representation theorem [Kat80, Section VI.2] in the following theorem.

Theorem A.1.5. *Let $\mathcal{W}$ and $\mathcal{X}$ be Hilbert spaces with $\mathcal{W} \hookrightarrow \mathcal{X}$, and let $a : \mathcal{W}\times\mathcal{W} \to \mathbb{C}$ be a continuous, hermitian symmetric sesquilinear form that fulfills*

$$\operatorname{Re} a(x,x) = a(x,x) \geqslant 0 \quad \forall x \in \mathcal{X}.$$

Then there exists a unique operator A with

$$\begin{aligned} \operatorname{dom} A &:= \left\{x \in \mathcal{W} \;\middle|\; \exists z(x) \in \mathcal{X} :\ a(x,\psi) = \langle z(x)\,,\,\psi\rangle_{\mathcal{X}}\ \forall\,\psi \in \mathcal{W}\right\}, \\ Ax &:= -z(x) \quad \forall\,x \in \operatorname{dom} A \end{aligned}$$

Furthermore, A is self-adjoint and nonnegative, $\operatorname{dom} A$ *is dense in $\mathcal{W}$, and for every $\lambda > 0$ we have*

$$\operatorname{dom}((\lambda - A)^{\frac{1}{2}}) = \mathcal{W}, \quad \text{and} \quad \langle(-A)^{\frac{1}{2}}x, (-A)^{\frac{1}{2}}y\rangle_{\mathcal{X}} = a(x,y) \quad \forall\,x,y \in \mathcal{W}.$$

A.2. Solutions to inhomogeneous Cauchy problems

Definition A.2.1. Let $A : \operatorname{dom} A \subset \mathcal{X} \to \mathcal{X}$ be the generator of a strongly continuous semigroup $\mathfrak{A}$ on the Banach space $\mathcal{X}$, and denote by $\mathcal{X}_{-1}$ the rigged space defined in Section 2.1. Furthermore, let $x_0 \in \mathcal{X}$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}; \mathcal{X}_{-1})$. A function x is a *strong solution of the Cauchy problem*

$$\dot{x}(t) = Ax(t) + f(t), \qquad x(0) = x_0, \tag{A.3}$$

in $\mathcal{X}$ if and only if $x \in \mathcal{C}(\mathbb{R}_{\geqslant 0}; \mathcal{X})$ and

$$x(t) = x_0 + \int_0^t A|_{\mathcal{X}} x(\tau) + f(\tau)\, \mathrm{d}\tau \quad \forall\, t \geqslant 0,$$

where this Bochner-integral is defined with respect to the norm of $\mathcal{X}_{-1}$.

In view of [Sta05, Definition 3.2.2 (i)], our definition of strong solutions is just a reformulation of [Sta05, Definition 3.8.1]. Hence, we may excerpt the following assertions from [Sta05, Theorem 3.8.2.]:

Lemma A.2.2. *Let $x_0 \in \mathcal{X}$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}; \mathcal{X}_{-1})$.*

(i) The Cauchy problem (A.3) *has at most one strong solution in $\mathcal{X}$. This solution is given by*

$$x(t) = \mathfrak{A}(t)x_0 + \int_0^t \mathfrak{A}(t-s)|_{\mathcal{X}_{-1}} f(s)\, \mathrm{d}s, \qquad t \geqslant 0. \tag{A.4}$$

(ii) If $f \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}; \mathcal{X})$, then the function x defined by (A.4) *is a strong solution of* (A.3) *in $\mathcal{X}$.*

(iii) If $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}; \mathcal{X}_{-1})$, then the function x defined by (A.4) *is a strong solution of* (A.3) *in $\mathcal{X}$.*

Lemma A.2.3. *Let A_{11} and A_{22} generate strongly continuous semigroups on the Hilbert spaces $\mathcal{X}$ and $\mathcal{Z}$, respectively, and let $A_{12} \in \mathcal{B}(\mathcal{Z}; \mathcal{X})$, $A_{21} \in \mathcal{B}(\mathcal{X}; \mathcal{Z})$. Then the operator*

$$A : \ \operatorname{dom} A_{11} \times \operatorname{dom} A_{22} \subset \mathcal{X} \times \mathcal{Z} \to \mathcal{X} \times \mathcal{Z}, \quad A := \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right],$$

generates a strongly continuous semigroup in $\mathcal{X} \times \mathcal{Z}$. Let $[x_1^0\,,\, x_2^0]^\top \in \mathcal{X} \times \mathcal{Z}$ and $[f_1\,,\, f_2]^\top \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}; \mathcal{X} \times \mathcal{Z})$. Then $[x_1\,,\, x_2]^\top \in \mathcal{C}(\mathbb{R}_{\geqslant 0}; \mathcal{X} \times \mathcal{Z})$ is the strong solution of

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}, \tag{A.5}$$

in $\mathcal{X} \times \mathcal{Z}$ if and only if the functions x_1 and x_2 satisfy for all $t \geqslant 0$ the equations

$$x_1(t) = x_1^0 + \int_0^t A_{11}\big|_{\mathcal{X}} x_1(s) + A_{12}x_2(s) + f_1(s)\,\mathrm{d}s, \tag{A.6}$$

$$x_2(t) = x_2^0 + \int_0^t A_{22}\big|_{\mathcal{Z}} x_2(s) + A_{21}x_1(s) + f_2(s)\,\mathrm{d}s, \tag{A.7}$$

where the integrals are computed in $(\operatorname{dom} A_{11}^*)'$ and $(\operatorname{dom} A_{22})'$, respectively. In particular, there exists a unique pair of functions $x_1 \in \mathcal{C}(\mathbb{R}_{\geqslant 0}; \mathcal{X})$, $x_2 \in \mathcal{C}(\mathbb{R}_{\geqslant 0}; \mathcal{Z})$ that fulfills (A.6) and (A.7). If, in addition, A_{11} is bounded, then the component x_1 of this solution is differentiable almost everywhere with respect to the norm of $\mathcal{X}$, and

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + f_1(t) \qquad \text{f.a.a. } t \geqslant 0. \tag{A.8}$$

Proof. We observe that the operator $\operatorname{diag}(A_{11}, A_{22})$ with domain $\operatorname{dom} A_{11} \times \operatorname{dom} A_{22}$ generates a strongly continuous semigroup in $\mathcal{X} \times \mathcal{Z}$. Due to the boundedness of the perturbation $\left[\begin{smallmatrix} 0 & A_{12} \\ A_{21} & 0 \end{smallmatrix}\right] \in \mathcal{B}(\mathcal{X} \times \mathcal{Z})$, the operator A is well-defined and generates a strongly continuous semigroup in $\mathcal{X} \times \mathcal{Z}$. Moreover, an elementary proof shows that the adjoint of A fulfills

$$\operatorname{dom} A^* = (\operatorname{dom} A_{11}^* \times \operatorname{dom} A_{22}^*), \qquad A^* = \left[\begin{smallmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{smallmatrix}\right],$$

and therefore the rigged space corresponding to A is $(\operatorname{dom} A^*)' = (\operatorname{dom} A_{11}^* \times \operatorname{dom} A_{22}^*)'$. By definition and owing to the Uniform boundedness principle, $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ is a strong solution of (A.5) in $\mathcal{X} \times \mathcal{Z}$ if and only if all $\left[\begin{smallmatrix} \varphi_1 \\ \varphi_2 \end{smallmatrix}\right] \in \operatorname{dom} A_{11}^* \times \operatorname{dom} A_{22}^*$ satisfy the equation

$$\begin{aligned} \left\langle \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \right\rangle_{\mathcal{X}\times\mathcal{Z}} &= \left\langle \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}, \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \right\rangle_{\mathcal{X}\times\mathcal{Z}} \\ &\quad + \int_0^t \left\langle \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix}, A^* \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \right\rangle_{\mathcal{X}\times\mathcal{Z}} + \left\langle \begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix}, \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \right\rangle_{\mathcal{X}\times\mathcal{Z}} \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \left\langle x_1^0\,,\,\varphi_1\right\rangle_{\mathcal{X}} + \left\langle x_2^0\,,\,\varphi_2\right\rangle_{\mathcal{Z}} \\
&\quad + \int_0^t \langle x_1(s)\,,\,A_{11}^*\varphi_1 + A_{21}^*\varphi_2\rangle_{\mathcal{X}} + \langle x_2(s)\,,\,A_{12}^*\varphi_1 + A_{22}^*\varphi_2\rangle_{\mathcal{Z}} \\
&\qquad\quad + \langle f_1(s)\,,\,\varphi_1\rangle_{\mathcal{X}} + \langle f_2(s)\,,\,\varphi_2\rangle_{\mathcal{Z}}\,\mathrm{d}s.
\end{aligned}
$$

This holds if and only if, for all $\varphi_1 \in \operatorname{dom} A_{11}^*$ and all $\varphi_2 \in \operatorname{dom} A_{22}^*$, the equations

$$
\begin{aligned}
\langle x_1(t)\,,\,\varphi_1\rangle_{\mathcal{X}} &= \left\langle x_1^0\,,\,\varphi_1\right\rangle_{\mathcal{X}} \\
&\quad + \int_0^t \langle x_1(s)\,,\,A_{11}^*\varphi_1\rangle_{\mathcal{X}} + \langle x_2(s)\,,\,A_{12}^*\varphi_1\rangle_{\mathcal{Z}} + \langle f_1(s)\,,\,\varphi_1\rangle_{\mathcal{X}}\,\mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
\langle x_2(t)\,,\,\varphi_2\rangle_{\mathcal{X}} &= \left\langle x_2^0\,,\,\varphi_2\right\rangle_{\mathcal{Z}} \\
&\quad + \int_0^t \langle x_2(s)\,,\,A_{22}^*\varphi_2\rangle_{\mathcal{Z}} + \langle x_1(s)\,,\,A_{21}^*\varphi_2\rangle_{\mathcal{Z}} + \langle f_2(s)\,,\,\varphi_2\rangle_{\mathcal{Z}}\,\mathrm{d}s
\end{aligned}
$$

hold. Again by the Uniform boundedness principle, these two equations are equivalent to (A.6) and (A.7).

For the additional claim assume now that the operator A_{11} is bounded. The rigged space $(\operatorname{dom} A_{11}^*)'$ then coincides with $\mathcal{X}$, and therefore x_1 satisfies

$$
x_1(t) = x_1^0 + \int_0^t A_{11}x_1(s) + A_{12}x_2(s) + f_1(s)\,\mathrm{d}s \quad \forall\, t \geqslant 0,
$$

where the integration is carried out in $\mathcal{X}$. This equation and Corollary 2 of [HP57, Theorem 3.8.5] imply that, for almost all t, the limit

$$
\lim_{h\to 0} \frac{x_1(t+h) - x_1(t)}{h} = A_{11}x_1(t) + A_{12}x_2(t) + f_1(t)
$$

with respect to $\|\cdot\|_{\mathcal{X}}$ exists. This shows (A.8) and completes the proof of the lemma. □

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List of symbols

Symbols

$\wedge$	the logical “and”-concatenation
$\hookrightarrow$	continuous and dense embedding
$\overline{\mathcal{W}}^{\mathcal{X}}$	the closure of the subset $\mathcal{W}$ with respect to the topology of the normed space $\mathcal{X}$
$\times$	the direct product of normed spaces equipped with the 2-norm
$\oplus$	the internal direct sum of subspaces
$^\top$	the transposed of a matrix
$\mathbb{N}$, $\mathbb{N}_0$	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, resp.
$\mathbb{R}_{\leqslant 0}$, $\mathbb{R}_{<0}$	$[0, \infty)$, $(0, \infty)$, resp.
$\mathbb{R}_{\geqslant 0}$, $\mathbb{R}_{>0}$	$[0, \infty)$, $(0, \infty)$, resp.
$\mathbb{C}_{\geqslant\omega}$, $\mathbb{C}_{>\omega}$	$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega\}$ and $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\}$, resp.
$0_{\mathbb{R}^r}$	the vector $0 \in \mathbb{R}^r$
$\ker A$, $\operatorname{ran} A$	kernel and range of a linear operator A
$\rho(A)$, $\sigma(A)$	resolvent set and spectrum of a linear operator A
$A\vert_{\mathcal{Z}}$	restriction/extension of a mapping $A : \mathcal{X} \to \mathcal{H}$ to the set $\mathcal{Z}$
$\mathfrak{A}\big\vert_{\mathcal{Z}}$	strongly continuous semigroup $t \mapsto \mathfrak{A}(t)\big\vert_{\mathcal{Z}}$ obtained by restriction/extension of each $\mathfrak{A}(t)$ to $\mathcal{Z}$
I	identity mapping

$\mathcal{B}(\mathcal{X};\mathcal{Y})$	space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$
$\mathcal{B}(\mathcal{X})$	$\mathcal{B}(\mathcal{X};\mathcal{X})$
e_λ	the function $\mathrm{e}_\lambda : \mathbb{R} \to \mathbb{C}$ defined by $\mathrm{e}_\lambda(t) := \mathrm{e}^{\lambda t}$
e_n	the n-th unit vector in ℓ^2, $(\delta_{n1}\,, \delta_{n2}\,, \dots)$
f.a.a.	for almost all, i.e. for all up to a set of Lebesgue measure zero
$\pi_{\mathcal{W}}$	the orthogonal projection onto the subspace $\mathcal{W}$ of a Hilbert space $\mathcal{X}$
π_I	the projection of a function $f : \mathbb{R} \to \mathcal{U}$ onto $I \subset \mathbb{R}$
π_+, π_-	the projections onto $\mathbb{R}_{\geqslant 0}$ and $\mathbb{R}_{<0}$, resp.
τ^t, τ	the shift operator $\tau^t x := x(\cdot + t)$ and the shift semigroup $t \mapsto \tau^t$
τ^t_-, τ^t_+	the restricted shift operators $\tau^t\big\|_{L^p_\omega(\mathbb{R}_{\leqslant 0};U)}$ and $\tau^t\big\|_{L^p_\omega(\mathbb{R}_{\geqslant 0};U)}$, resp.
Я	the reflection mapping $x(\cdot) \mapsto x(-\,\cdot)$
$\partial\Omega$	the boundary of the domain Ω
$\mathrm{d}\sigma_\xi$	the Riemann-Lebesgue volume measure on the manifold $\partial\Omega$
$\|\partial\Omega\|$	the surface area of the manifold $\partial\Omega$: $\|\partial\Omega\| = \int_{\partial\Omega} 1\,\mathrm{d}\sigma_\xi$
∂_ν	the directional derivative along the outward normal ν on the boundary $\partial\Omega$ of the domain Ω
$\mathcal{X}_\alpha$	$\alpha \in [-1, 1]$, the rigged and intermediate spaces defined in Sections 2.1 and A.1

Function spaces

ℓ^p	$p \in [1, \infty)$, space of p-summable sequences $\ell^p(\mathbb{N}; \mathbb{C})$
Φ, Φ_{γ_0}	space of admissible funnel boundaries defined in (5.2)
$\mathcal{H}^p_\omega(\mathcal{U})$	Hardy space of analytic functions from $\mathbb{C}_{\geqslant\omega}$ to $\mathcal{U}$ as defined in [Sta05, Section 10.3]
$\mathcal{H}^\infty_\omega(\mathcal{U}; \mathcal{Y})$	space of bounded, analytic functions from $\mathbb{C}_{\geqslant\omega}$ to $\mathcal{B}(\mathcal{U}; \mathcal{Y})$ with the sup-norm
$\mathcal{BUC}(\Omega)$	space of bounded and uniformly continuous functions $f : \Omega \to \mathbb{C}$ with the sup-norm
$\mathcal{C}(\Omega; \mathcal{X})$	space of continuous functions $f : \Omega \to \mathcal{X}$
$L^p(\Omega; \mathcal{X})$,	Lebesgue-Bochner space of functions from $\Omega \subset \mathbb{R}^n$ to the Hilbert space $\mathcal{X}$ as defined in [DU77, Chapter II]
$L^p \cap L^q(\Omega; \mathcal{X})$	$L^p(\Omega; \mathcal{X}) \cap L^q(\Omega; \mathcal{X})$
$L^p(\Omega)$, $\mathcal{C}(\Omega)$	$L^p(\Omega; \mathbb{C})$, $\mathcal{C}(\Omega; \mathbb{C})$, respectively
$L^p_{\mathrm{loc}}(\Omega)$	space of functions $f : \Omega \to \mathbb{C}$ that are locally in L^p
$L^p_{\mathrm{c,loc}}(\mathbb{R}; \mathcal{U})$	the space of all functions in $L^p_{\mathrm{loc}}(\mathbb{R}; \mathcal{U})$ whose support is bounded *to the left.*
$L^p_\omega(I; \mathcal{U})$	$I \subset \mathbb{R}$, space of functions u in $L^p_{\mathrm{loc}}(I; \mathcal{U})$ satisfying $\mathrm{e}_\omega u \in L^p(I; \mathcal{U})$ with norm $\\|\mathrm{e}_{-\omega} \cdot\\|_{L^p(I;\mathcal{U})}$.
$W^{k,p}(\Omega)$	$p \in (1, \infty)$, $k \in \mathbb{R}_{\geqslant 0}$, Sobolev-Slobodeckij space of functions $f : \Omega \to \mathbb{C}$ as defined in [Tri95, Section 4.2.1]
$W^{k,p}(\Omega)$	$p \in \{1, \infty\}$, $k \in \mathbb{N}$, Sobolev space of functions $f : \Omega \to \mathbb{C}$ as defined in [AF03, Chapter 3]
$W^{k,p}_0(\Omega)$	space of functions u in $W^{k,p}(\Omega)$ with $u\|_{\partial\Omega} = 0$
$W^{k,p}_\omega(I; \mathcal{U})$	$k \in \mathbb{N}_0$, $p \in [1, \infty]$, space of functions u with $u^i \in L^p_\omega(\mathbb{R}; \mathcal{U})$ for $i = 0, 1, \ldots, k$
$W^{k,p}_{0,\omega}(I; \mathcal{U})$	$I \in \{\mathbb{R}_{\geqslant 0}, \mathbb{R}_{\leqslant 0}\}$, space of functions u in $W^{k,p}_\omega(I; \mathcal{U})$ with $u(0) = 0$
$W^{2,p}_{\mathrm{c,loc}}(I; \mathcal{U})$	the space of functions that are locally in $W^{2,p}(\mathbb{R}; \mathcal{U})$ and whose support is bounded *from below.*

Index