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Characterization of proper optimal elements with variable ordering structures

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Abstract

In vector optimization with a variable ordering structure the partial ordering defined by a convex cone is replaced by a whole family of convex cones, one associated with each element of the space. As these vector optimization problems are not only of interest in applications but also mathematical challenging, in recent publications it was started to develop a comprehensive theory. In doing that also notions of proper efficiency where generalized to variable ordering structures. In this paper we study the relations between several types of proper optimality notions, among others based on local and global approximations of the considered sets. We give scalarization results based on new functionals defined by elements from the dual cones which allow characterizations also in the nonconvex case.

Key Words: vector optimization, variable ordering structure, proper efficiency, scalarization

Mathematics subject classifications (MSC 2000): 90C29, 90C30, 90C48

1 Introduction

In vector optimization one studies optimization problems with a vector valued objective map. For comparing elements in the objective space, i.e. elements from a set in a linear space, one can assume that a partial ordering, and hence a convex cone introducing this partial ordering, is given. More general concepts allow that preferences vary depending on the current values in the objective space. This means that an individual ordering cone is attached to each element in the objective space which is mathematically modeled by a set-valued map on the objective space, called ordering map, with images being convex cones. Vector optimization problems with such variable ordering structures are the topic of this manuscript.

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Variable ordering structures were already introduced in the seventies [28, 6] and have gained recently more interest due to several applications [2, 27, 16, 25, 26, 10]. Various scalarization approaches have been proposed for these ordering structures, like linear scalarizations based on elements from the dual space [16, 10], and nonlinear scalarizations based on a representation of the images of the ordering map as Bishop-Phelps cones [14], using elements of the augmented dual cones of the images of the ordering map [15] or generalizing a functional known in the literature as Tammer-Weidner functional [11, 24, 1]. Optimality conditions of Fermat and Lagrange type based on scalarization results [14] and also by a more general approach [3] were proposed and also first numerical procedures were presented [19, 12, 5]. Next to optimal and weakly optimal elements, also strongly optimal [14] and properly optimal elements [15] for variable ordering structures were defined. First scalarization results for these proper optimal elements were given in [15] based on functionals defined by elements from the augmented dual cones.

Proper optimal elements for variable ordering structures and their characterization is the main topic of this manuscript. The set of proper optimal elements are a subset of the set of optimal elements. By additional restrictions one tries to eliminate "improper" optimal elements and to allow more satisfactory scalarization results for the proper optimal elements (cf. [23]). By these additional restrictions, those optimal elements are eliminated, which can be interpreted in a finite-dimensional space as having an unbounded trade-off and which are for that reason not of interest in applications. Moreover, it is known that, in case the set is convex, the properly efficient elements of a set in a partially ordered space are completely characterizable by linear scalarization based on elements from the quasi-interior of the dual cone, while for efficient elements, the necessary and the sufficient conditions do not match.

We concentrate in this manuscript on the definitions of properly optimal elements in the sense of Henig [18], Benson [4] and Borwein [7] for variable ordering structures. Thereby, one has to differentiate between the concepts of nondominatedness and of preference w.r.t. a variable ordering structure: based on the ordering map, two different binary relations can be defined leading to two different optimality notions, the minimal and the nondominated elements. In [15] already generalizations of the proper optimality notions known from partially ordered spaces have been suggested for both concepts, the minimal and the nondominated elements. There, also scalarization results based on nonlinear scalar-valued functionals defined by elements from the augmented dual cones are provided. For that, the cones are required to have a bounded base and in the scalarization results it is required that the cones and their ε-conic neighborhoods satisfy some separation property.

In this paper, we study in detail the relation between the introduced proper optimality notions and clarify for they first time their relations and some basic properties. It turns out that most, but not all, results known to hold in partially ordered spaces remain true in case of a variable ordering structure. Moreover, we present a new scalarization approach which is based on elements from the dual cones of the images of the ordering map. This scalarization can be used to characterize optimal elements without convexity assumptions. While the definition of the functional is based on linear functionals from the dual cones, the scalarization functional is in general nonlinear and allows complete characterizations of weakly and properly optimal elements. In contrast to the scalarizations introduced in [15], the assumptions are weaker — and also the proofs are more direct as no theory of augmented dual cones or separation properties for special cones are required. We also give shortly characterization results for minimal elements based on a linear functional. By that we generalize the known linear scalarization results known from partially ordered spaces to vector optimization problems with a variable ordering structure. Based on the proposed characterization results, optimality conditions of Fermat and Lagrange type can be derived as well as numerical solution methods.

In Section 2 we give some preliminary results and collect the definitions of (weakly, strongly) minimal and nondominated elements. The different proper optimality notions and their relations are the topic of Section 3. In Section 4, we present the mentioned scalarization functionals and we give necessary and sufficient conditions for the various optimality notions.

2 Preliminaries

In the following, let $(Y, \|\cdot\|)$ be a real normed space and let 2^Y denote the set of all subsets of Y. For some nonempty set Ω, we denote by $\mathrm{int}(\Omega)$, $\mathrm{cl}(\Omega)$, $\mathrm{cone}(\Omega)$ and $\mathrm{conv}(\Omega)$ the interior of Ω, the closure of Ω, the cone generated by Ω and the convex hull of Ω. For some nonempty set $\Omega \in Y$ and for some element $\bar{y} \in \mathrm{cl}(\Omega)$, $T(\Omega, \bar{y})$ denotes the *contingent cone* (or the *Bouligand tangent cone*) to Ω at $\bar{y}$, i.e.

$$T(\Omega, \bar{y}) := \{h \in Y \mid \ \exists (\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_{++},\ \exists (y_n)_{n\in\mathbb{N}} \subseteq \Omega \text{ such that } \lim_{n\to\infty} y_n = \bar{y} \text{ and } h = \lim_{n\to\infty} \lambda_n (y_n - \bar{y})\}.$$

Thereby, $\mathbb{R}_{++}$ denotes the set of real positive numbers. A nontrivial cone K is a cone K with $K \neq \{0_Y\}$ and a base of a nontrivial cone K is a convex set $B \subseteq K$ such that each element $k \in K \setminus \{0_Y\}$ has a unique representation as $k = \lambda\, b$ with $\lambda > 0$ and $b \in B$.

We assume the variable ordering structure on Y is defined by a set-valued map (also called an *ordering map*) $\mathcal{D}\colon Y \to 2^Y$ with $\mathcal{D}(y)$ a nontrivial convex cone for all $y \in Y$. Let A be a nonempty subset of Y. The following definitions of optimal elements (w.r.t. minimization) are known in the literature for variable ordering structures introduced by a cone-valued map $\mathcal{D}$ [28, 8, 10, 14, 13].

Definition 2.1. *Let $\bar{y} \in A$.*

(a) The element $\bar{y}$ is a nondominated element *of A w.r.t. $\mathcal{D}$ if $\bar{y} \notin \{y\} + \mathcal{D}(y)$ for all $y \in A \setminus \{\bar{y}\}$.*

(b) Supposing that $\mathrm{int}(\mathcal{D}(y)) \neq \emptyset$ *for all $y \in A$, $\bar{y}$ is a* weakly nondominated element *of A w.r.t. $\mathcal{D}$ if $\bar{y} \notin \{y\} + \mathrm{int}(\mathcal{D}(y))$ for all $y \in A$.*

(c) The element $\bar{y}$ is a strongly nondominated element *of A w.r.t. $\mathcal{D}$ if $\bar{y} \in \{y\} - \mathcal{D}(y)$ for all $y \in A$.*

(d) The element $\bar{y}$ is a minimal element *of A w.r.t. $\mathcal{D}$ if $y \notin \{\bar{y}\} - \mathcal{D}(\bar{y})$ for all $y \in A \setminus \{\bar{y}\}$.*

(e) *The element $\bar{y}$ with* $\operatorname{int}(\mathcal{D}(\bar{y})) \neq \emptyset$ *is a* weakly minimal element *of A w.r.t. $\mathcal{D}$ if $y \notin \{\bar{y}\} - \operatorname{int}(\mathcal{D}(\bar{y}))$ for all $y \in A$.*

(f) *The element $\bar{y}$ is a* strongly minimal element *of A w.r.t. $\mathcal{D}$ if $A \subseteq \{\bar{y}\} + \mathcal{D}(\bar{y})$.*

If $\mathcal{D}(y) = K$ for all $y \in Y$ with K some pointed nontrivial convex cone then the definitions of a (weakly/strongly) nondominated element of a set A w.r.t. $\mathcal{D}$ and of a (weakly/strongly) minimal element of a set A w.r.t. $\mathcal{D}$ coincide with the concepts of a (weakly/strongly) optimal element of A in the space Y partially ordered by the convex cone K. We will denote the (weakly/strongly/properly) optimal elements w.r.t. the partial ordering introduced by some convex cone K as *(weakly/strongly/properly) efficient elements*.

Throughout the paper we assume that for the ordering map $\mathcal{D}\colon Y \to 2^Y$ the cones $\mathcal{D}(y)$ are pointed nontrivial convex cones for all $y \in Y$.

3 Proper optimality

Following the definitions for properly efficient elements given by Henig [18], Benson [4] and Borwein [7] in partially ordered space the following generalizations for variable ordering structures were introduced in [15]. Note that we do not require in the definitions that the elements $\bar{y} \in A$ are a nondominated/minimal element of A w.r.t. $\mathcal{D}$ as it was done in [15]. We will show later that the definitions below already imply that $\bar{y}$ is a nondominated or a minimal element of A w.r.t. $\mathcal{D}$, respectively.

Definition 3.1. *Let $\bar{y} \in A$.*

(a) *The element $\bar{y}$ is a* properly nondominated element in the sense of Henig *of A w.r.t. $\mathcal{D}$ if there is a cone-valued map $\mathcal{K}\colon Y \to 2^Y$ with $\mathcal{K}(y)$ a convex cone and $\mathcal{D}(y) \setminus \{0_Y\} \subseteq \operatorname{int}(\mathcal{K}(y))$ for all $y \in Y$ such that $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{K}$, i.e.*

$$\bar{y} \notin \{y\} + \mathcal{K}(y)\ \forall\ y \in A \setminus \{\bar{y}\}.$$

(b) *The element $\bar{y}$ is a* properly nondominated element in the sense of Benson *of A w.r.t. $\mathcal{D}$ if $\bar{y}$ is a nondominated element of the set*

$$\{\bar{y}\} + \operatorname{cl}(\operatorname{cone}(\bigcup_{a\in A} (\{a\} + \mathcal{D}(a)) - \{\bar{y}\}))$$

w.r.t. $\mathcal{D}$.

(c) *The element $\bar{y}$ is a* properly nondominated element in the sense of Borwein *of A w.r.t. $\mathcal{D}$ if $\bar{y}$ is a nondominated element of the set*

$$\{\bar{y}\} + T(\bigcup_{a\in A} (\{a\} + \mathcal{D}(a)), \bar{y})$$

w.r.t. $\mathcal{D}$.

(d) *The element* $\bar{y}$ *is a* properly minimal element in the sense of Henig *of* A *w.r.t.* $\mathcal{D}$ *if there is a cone-valued map* $\mathcal{K}\colon Y \to 2^Y$ *with* $\mathcal{K}(y)$ *a convex cone and* $\mathcal{D}(y) \setminus \{0_Y\} \subseteq \mathrm{int}(\mathcal{K}(y))$ *for all* $y \in Y$ *such that* $\bar{y}$ *is a minimal element of* A *w.r.t.* $\mathcal{K}$*, i.e.*

$$y \notin \{\bar{y}\} - \mathcal{K}(\bar{y}) \ \forall\ y \in A \setminus \{\bar{y}\}.$$

(e) *The element* $\bar{y}$ *is a* properly minimal element in the sense of Benson *of* A *w.r.t.* $\mathcal{D}$ *if* $\bar{y}$ *is a minimal element of the set*

$$\{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(A + \mathcal{D}(\bar{y}) - \{\bar{y}\}))$$

w.r.t. $\mathcal{D}$.

(f) *The element* $\bar{y}$ *is a* properly minimal element in the sense of Borwein *of* A *w.r.t.* $\mathcal{D}$ *if* $\bar{y}$ *is a minimal element of the set*

$$\{\bar{y}\} + T(A + \mathcal{D}(\bar{y}), \bar{y})$$

w.r.t. $\mathcal{D}$.

Note that in the definition of Henig proper optimality one normally requires closed pointed ordering cones (here: $\mathcal{D}(y)$). In case $\mathcal{D}(y)$ is a pointed convex cone with $\mathcal{D}(y) \setminus \{0_Y\}$ an open set for all $y \in Y$, then for $\mathcal{K}(y) := \mathcal{D}(y)$

$$\mathcal{D}(y) \setminus \{0_Y\} = \mathrm{int}(\mathcal{D}(y)) = \mathrm{int}(\mathcal{K}(y)).$$

Hence, in this case, the definitions above for properly minimal/nondominated elements in the sense of Henig coincide with the definitions of minimal/nondominated elements w.r.t. $\mathcal{D}$.

According to [17] we define by $\mathrm{ndGHe}(A, \mathcal{D})$/$\mathrm{ndBe}(A, \mathcal{D})$/$\mathrm{ndBo}(A, \mathcal{D})$ and by $\mathrm{mGHe}(A, \mathcal{D})$/$\mathrm{mBe}(A, \mathcal{D})$/$\mathrm{mBo}(A, \mathcal{D})$ the set of all properly nondominated elements and the set of all properly minimal elements in the sense of Henig/Benson/Borwein of A w.r.t. $\mathcal{D}$, respectively. We will also use the following sets: Let $\bar{y} \in A$,

$$M := \bigcup_{a \in A} (\{a\} + \mathcal{D}(a)) \ \text{ and } \ M_{\bar{y}} := A + \mathcal{D}(\bar{y}) \ . \tag{1}$$

The above definitions imply that $\bar{y}$ is a nondominated or a minimal element of A w.r.t. $\mathcal{D}$, respectively, and hence this requirement, as given in the original definitions in [15], is redundant and can be omitted:

Lemma 3.2. *Let* $\bar{y} \in A$.

(i) *If* $\bar{y} \in \mathrm{ndGHe}(A, \mathcal{D})$ *or* $\bar{y} \in \mathrm{ndBe}(A, \mathcal{D})$ *then* $\bar{y}$ *is a nondominated element of* A *w.r.t.* $\mathcal{D}$.

(ii) *If* $\bar{y} \in \mathrm{mGHe}(A, \mathcal{D})$ *or* $\bar{y} \in \mathrm{mBe}(A; \mathcal{D})$ *then* $\bar{y}$ *is a minimal element of* A *w.r.t.* $\mathcal{D}$.

Proof. The conclusions follow immediately from the definition of a nondominated/minimal element, since $\mathcal{D}(y) \subseteq \mathcal{K}(y)$ for all $y \in Y$, $A \subseteq \{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(M - \{\bar{y}\}))$, and $A \subseteq \{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(M_{\bar{y}} - \{\bar{y}\}))$ with M and $M_{\bar{y}}$ as defined in (1). $\square$

For the analogous results in the case of Borwein we refer to the forthcoming Lemma 3.7(i) and Theorem 3.8(i).

The following results are direct consequences from the fact that for sets Ω and elements $\bar{y} \in \Omega$, $T(\Omega, \bar{y}) \subseteq \mathrm{cl}(\mathrm{cone}(\Omega - \{\bar{y}\}))$, and in case Ω is starshaped w.r.t. $\bar{y}$ even equality holds, see for instance [20, Theorem 3.44 and Corollary 3.46].

Lemma 3.3. *Let $\bar{y} \in A$ and let the sets M and $M_{\bar{y}}$ be defined as in (1). Then the following holds:*

(i) $\bar{y} \in \mathrm{ndBe}(A, \mathcal{D}) \Rightarrow \bar{y} \in \mathrm{ndBo}(A, \mathcal{D})$.

(ii) If the set M is starshaped w.r.t. $\bar{y}$, then $\bar{y} \in \mathrm{ndBe}(A, \mathcal{D}) \Leftrightarrow \bar{y} \in \mathrm{ndBo}(A, \mathcal{D})$.

(iii) $\bar{y} \in \mathrm{mBe}(A, \mathcal{D}) \Rightarrow \bar{y} \in \mathrm{mBo}(A, \mathcal{D})$.

(iv) If the set $M_{\bar{y}}$ is starshaped w.r.t. $\bar{y}$, then $\bar{y} \in \mathrm{mBe}(A, \mathcal{D}) \Leftrightarrow \bar{y} \in \mathrm{mBo}(A, \mathcal{D})$.

For relating properly minimal elements in the sense of Henig w.r.t. $\mathcal{D}$ with properly efficient elements in the sense of Henig in a partially ordered space we need the following lemma. This lemma also gives an alternative way of defining proper minimality in the sense of Henig w.r.t. a variable ordering structure.

Lemma 3.4. *[15, Lemma 4] Let $\bar{y} \in A$. Then $\bar{y} \in \mathrm{mGHe}(A, \mathcal{D})$ if and only if there is a convex cone K with $\mathcal{D}(\bar{y}) \setminus \{0_Y\} \subseteq \mathrm{int}(K)$ such that $y \notin \{\bar{y}\} - K$ for all $y \in A \setminus \{\bar{y}\}$.*

Remark 3.5. *As a direct consequence of the definitions and of Lemma 3.4, $\bar{y}$ is a properly minimal element in the sense of Henig/Benson/Borwein of A w.r.t. $\mathcal{D}$ if and only if it is a properly efficient element in the sense of Henig/Benson/Borwein of A in the space Y partially ordered by the convex cone $K := \mathcal{D}(\bar{y})$.*

We define by $\mathrm{effGHe}(A, K)/\mathrm{effBe}(A, K)/\mathrm{effBo}(A, K)$ the set of all properly efficient elements in the sense of Henig/Benson/Borwein of A w.r.t. a partially ordering introduced by the pointed convex cone K. For the study of the relation of properly nondominated and minimal elements in the sense of Benson and Henig we need the following result.

Lemma 3.6. *[9, Proposition 2.2] Let $P \subseteq Y$ be a weakly closed nontrivial cone and $C \subseteq Y$ be a cone with a weakly compact base such that $P \cap C = \{0_Y\}$. Then there exists a closed pointed convex cone $K \subseteq Y$ which has a closed bounded base such that $C \setminus \{0_Y\} \subseteq \mathrm{int}(K)$ and $P \cap K = \{0_Y\}$.*

This result is used in the proofs of the following lemma and of Theorem 3.8.

Lemma 3.7. *Let $\bar{y} \in A$. Then the following holds:*

(i) If $\bar{y} \in \mathrm{mBo}(A, \mathcal{D})$, then $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$.

(ii) If $\bar{y} \in \mathrm{mGHe}(A, \mathcal{D})$ and the cone $\mathcal{K}(\bar{y})$ in the definition can be chosen to be pointed, then $\bar{y} \in \mathrm{mBe}(A, \mathcal{D})$.

(iii) If $\mathcal{D}(\bar{y})$ has a weakly compact base, then $\bar{y} \in \mathrm{mBe}(A, \mathcal{D}) \Rightarrow \bar{y} \in \mathrm{mGHe}(A, \mathcal{D})$.

Proof.
(i) is a consequence of Remark 3.5 and the fact that $\bar{y} \in \mathrm{effBo}(A, K)$ implies that $\bar{y}$ is an efficient element of A w.r.t. K, see [21, Proposition 3.2].
(ii) follows from Remark 3.5, together with the inclusion $\mathrm{effGHe}(A, K) \subseteq \mathrm{effBe}(A, K)$, given in [17, Theorem 4.2] (and for the finite dimensional case already in [18, Theorem 2.1]) and noting that closedness of K is not required for the proof given there while pointedness of the cone K' which contains $K \setminus \{0_Y\}$ in its interior is needed.
(iii) follows from Remark 3.5, together with the inclusion $\mathrm{effBe}(A, K) \subseteq \mathrm{effGHe}(A, K)$ given in [22, Remark 5.3] based on Lemma 3.6 (see also [17, Theorem 4.2 and p. 9]). □

We give in Lemma 4.8 assumptions under which $\bar{y} \in \mathrm{mGHe}(A, \mathcal{D})$ always implies that the cone $\mathcal{K}(\bar{y})$ in the definition can be chosen to be pointed.

For properly nondominated elements we can prove the following results.

Theorem 3.8. *Let $\bar{y} \in A$. Then the following holds:*

(i) If $\bar{y} \in \mathrm{ndBo}(A, \mathcal{D})$, then $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$.

(ii) If $\mathcal{D}(y)$ has a weakly compact base for all $y \in Y$, then $\bar{y} \in \mathrm{ndBe}(A, \mathcal{D}) \Rightarrow \bar{y} \in \mathrm{ndGHe}(A, \mathcal{D})$.

Proof.
(i) Let $\bar{y} \in \mathrm{ndBo}(A, \mathcal{D})$ and the set M be defined as in (1). Assume that $\bar{y}$ is not a nondominated element of A w.r.t. $\mathcal{D}$. Then there exist $\hat{y} \in A \setminus \{\bar{y}\} \subseteq M \setminus \{\bar{y}\}$ and $d \in \mathcal{D}(\hat{y}) \setminus \{0_Y\}$ such that $\bar{y} = \hat{y} + d$. Let $\lambda_n := n$, $d_n := (1 - \frac{1}{n})d \in \mathcal{D}(\hat{y})$ and $y_n := \hat{y} + d_n \in \{\hat{y}\} + \mathcal{D}(\hat{y}) \subseteq M$ for all $n \in \mathbb{N}$. Then it follows

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} (\hat{y} + d_n) = \lim_{n\to\infty} \left(\bar{y} - \frac{1}{n}d\right) = \bar{y}$$

and

$$\lim_{n\to\infty} \lambda_n (y_n - \bar{y}) = \lim_{n\to\infty} n (\hat{y} + d_n - \bar{y}) = \lim_{n\to\infty} n (d_n - d) = -d \in T(M, \bar{y}).$$

Hence, $\bar{y} = \hat{y} + d \in \{\hat{y}\} + \mathcal{D}(\hat{y})$ and $\hat{y} = \bar{y} - d \in (\{\bar{y}\} + T(M, \bar{y})) \setminus \{\bar{y}\}$ being a contradiction to that $\bar{y} \in \mathrm{ndBo}(A, \mathcal{D})$.
(ii) Let $\bar{y} \in \mathrm{ndBe}(A, \mathcal{D})$ and define $\bar{M}_{\bar{y}} := \{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(M - \{\bar{y}\}))$. Let $y \in A \setminus \{\bar{y}\}$ be arbitrarily chosen. Since $A \subseteq \bar{M}_{\bar{y}}$ and, by definition, $\bar{y} \notin \{z\} + \mathcal{D}(z)$ for all $z \in \bar{M}_{\bar{y}} \setminus \{\bar{y}\}$ we get $y - \bar{y} \notin -\mathcal{D}(y)$. As $\mathcal{D}(y)$ is a cone it follows

$$\mathrm{cl}(\mathrm{cone}(\{y - \bar{y}\})) \cap (-\mathcal{D}(y)) = \{0_Y\}.$$

Using Lemma 3.6 there exists a closed pointed convex cone $\mathcal{K}(y) \subseteq Y$ with

$$\mathcal{D}(y) \setminus \{0_Y\} \subseteq \mathrm{int}(\mathcal{K}(y)) \text{ and } \mathrm{cl}(\mathrm{cone}(\{y - \bar{y}\})) \cap (-\mathcal{K}(y)) = \{0_Y\}.$$

Hence, we obtain $y - \bar{y} \notin -\mathcal{K}(y)$ respectively $\bar{y} \notin \{y\} + \mathcal{K}(y)$, and we are done. □

For the opposite direction in the conclusion of Theorem 3.8(ii) we refer to the following example, which shows that even under strong assumptions $\bar{y} \in \mathrm{ndGHe}(A, \mathcal{D})$ does not imply $\bar{y} \in \mathrm{ndBe}(A, \mathcal{D})$ or $\bar{y} \in \mathrm{ndBo}(A, \mathcal{D})$.

$$
\begin{tikzcd}
\boxed{\bar{y} \in \mathrm{ndGHe}(A,\mathcal{D})} \arrow[d, shift right, "\text{Example 3.9}"', "/" marking] & \boxed{\bar{y} \in \mathrm{mGHe}(A,\mathcal{D})} \arrow[d, shift right, "\mathcal{K}(\bar{y}) \text{ pointed}"'] \\
\boxed{\bar{y} \in \mathrm{ndBe}(A,\mathcal{D})} \arrow[u, shift right, "\mathcal{D}(y) \text{ w.c.b. } \forall y"'] \arrow[d, shift right] & \boxed{\bar{y} \in \mathrm{mBe}(A,\mathcal{D})} \arrow[u, shift right, "\mathcal{D}(\bar{y}) \text{ w.c.b.}"'] \arrow[d, shift right] \\
\boxed{\bar{y} \in \mathrm{ndBo}(A,\mathcal{D})} \arrow[u, shift right, "M \text{ stars. w.r.t. } \bar{y}"'] & \boxed{\bar{y} \in \mathrm{mBo}(A,\mathcal{D})} \arrow[u, shift right, "M_{\bar{y}} \text{ stars. w.r.t. } \bar{y}"']
\end{tikzcd}
$$

Figure 1: Diagram illustrating results of Lemma 3.3, Lemma 3.7, and Theorem 3.8. We use the abbreviations w.c.b. for weakly compact base and stars. for starshaped.

Example 3.9. *Let* $A = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 + y_2 \geq 0,\ y_1 - y_2 \leq 2\}$, $\bar{y} = (0,0)$, *and*

$$\mathcal{D}(y) = \begin{cases} \mathrm{cone}(\mathrm{conv}(\{(0,1),(1,1)\})) & \forall\, y \in A \setminus \{\bar{y}\}, \\ \mathrm{cone}(\mathrm{cone}(\{(-1,1),(1,1)\})) & \forall\, y \in (Y \setminus A) \cup \{\bar{y}\}. \end{cases}$$

Then $\bar{y} \in \mathrm{ndGHe}(A,\mathcal{D})$. *To see this, define (for example) for a small* $\varepsilon > 0$

$$\mathcal{K}(y) = \begin{cases} \mathrm{cone}(\mathrm{conv}(\{(-\varepsilon,1),(1+\varepsilon,1)\})) & \forall\, y \in A \setminus \{\bar{y}\}, \\ \mathrm{cone}(\mathrm{cone}(\{(-1-\varepsilon,1),(1+\varepsilon,1)\})) & \forall\, y \in (Y \setminus A) \cup \{\bar{y}\}. \end{cases}$$

Furthermore, for the set M *defined as in (1) we have* $M = A$ *and for* $z = (2,-2) \in \mathrm{cl}(\mathrm{cone}(M - \{\bar{y}\})) = \{(y_1,y_2) \in \mathbb{R}^2 \mid y_1 + y_2 \geq 0\}$ *we have* $\bar{y} \in \{z\} + \mathcal{D}(z)$. *Hence,* $\bar{y} \notin \mathrm{ndBe}(A,\mathcal{D})$. *Finally, since* M *is a convex set, it follows* $\bar{y} \notin \mathrm{ndBe}(A,\mathcal{D}) = \mathrm{ndBo}(A,\mathcal{D})$ *by using Lemma 3.3(ii).*
The cones $\mathcal{D}(y)$ *are closed pointed convex cones with a compact base and according to [12, Lemma 2.1] the binary relation* $\leq$ *defined by* $y^1 \leq y^2 \;:\Leftrightarrow\; y^2 \in y^1 + \mathcal{D}(y^1)$ *for all* $y^1, y^2 \in Y$ *is even transitive and antisymmetric.*

See Figure 1 for a diagram illustrating the relations between the different proper optimality notions.

4 Scalarization results

In the following $(Y^*, \|\cdot\|_*)$ denotes the topological dual space with the induced norm $\|\cdot\|_*$. To some set $K \subseteq Y$, $K^* := \{\ell \in Y^* \mid \ell(y) \geq 0 \text{ for all } y \in K\}$ denotes the dual cone and $K^\# := \{\ell \in Y^* \mid \ell(y) > 0 \text{ for all } y \in K \setminus \{0_Y\}\}$ denotes the quasi-interior of the dual cone. If K is a closed convex cone, then

$$K = \{y \in Y \mid \ell(y) \geq 0 \text{ for all } \ell \in K^*\}\ , \tag{2}$$

and if K is a convex cone with $\mathrm{int}(K) \neq \emptyset$, then

$$\mathrm{int}(K) = \{y \in Y \mid \ell(y) > 0 \text{ for all } \ell \in K^* \setminus \{0_{Y^*}\}\}, \tag{3}$$

see for instance [20, Lemma 3.21].

4.1 Characterizing nondominated elements

Let a map $\ell\colon Y \to Y^*$ and an element $\bar{y} \in Y$ be given. We consider the functional $\varphi_{\bar{y}}\colon Y \to \mathbb{R}$ with

$$\varphi_{\bar{y}}(y) := \ell(y)(y - \bar{y}) \text{ for all } y \in Y \ .$$

Obviously it holds $\varphi_{\bar{y}}(\bar{y}) = \ell(\bar{y})(\bar{y} - \bar{y}) = \ell(\bar{y})(0_Y) = 0$. Note that while $\ell(y)$ is a continuous linear functional for each y, the functional $\varphi_{\bar{y}}$ is in general nonlinear. This is even the case if $\ell\colon Y \to Y^*$ is a linear map, as the following example demonstrates:

Example 4.1. *Let Y be the Euclidean space $\mathbb{R}^m$ and let $\ell\colon \mathbb{R}^m \to \mathbb{R}^m$ be defined by $\ell(y) = My$ for all $y \in \mathbb{R}^m$ with $M := \mathit{diag}(1, 2, \ldots, m)$. Thus ℓ is linear and each $\ell(y)$ defines a linear map by $z \mapsto (\ell(y))^\top z = y^\top Mz$ for all $z \in \mathbb{R}^m$. Nevertheless, $\varphi_{\bar{y}}(y) = y^\top M(y - \bar{y}) = \sum_{i=1}^m (iy_i^2 - iy_i\bar{y}_i)$ is not a linear map.*

The following lemma gives two sufficient criteria for nondominated elements as well as a complete characterization of weakly and strongly nondominated elements of A w.r.t. $\mathcal{D}$. Note that the necessary condition for weakly nondominated elements is also a necessary condition (in case of $\operatorname{int}(\mathcal{D}(y)) \neq \emptyset$ for all $y \in A$) for nondominated elements.

Lemma 4.2. *Let $\bar{y} \in A$. Then the following holds:*

(i) Let $\ell\colon Y \to Y^*$ be a map with $\ell(y) \in \mathcal{D}(y)^* \setminus \{0_{Y^*}\}$ for all $y \in A \setminus \{\bar{y}\}$. If

$$\varphi_{\bar{y}}(y) > \varphi_{\bar{y}}(\bar{y}) = 0 \ \forall y \in A \setminus \{\bar{y}\}, \tag{4}$$

then $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$.

(ii) Let $\ell\colon Y \to Y^*$ be a map with $\ell(y) \in \mathcal{D}(y)^\#$ for all $y \in A \setminus \{\bar{y}\}$. If

$$\varphi_{\bar{y}}(y) \geq \varphi_{\bar{y}}(\bar{y}) = 0 \ \forall y \in A, \tag{5}$$

then $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$.

(iii) If additionally $\operatorname{int}(\mathcal{D}(y)) \neq \emptyset$ for all $y \in A$, then $\bar{y}$ is a weakly nondominated element of A w.r.t. $\mathcal{D}$ if and only if there is a map $\ell\colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^* \setminus \{0_{Y^*}\}$ for all $y \in A \setminus \{\bar{y}\}$ such that (5) holds.

(vi) If $\mathcal{D}(y)$ is additionally closed for all $y \in A$, then $\bar{y}$ is a strongly nondominated element of A w.r.t. $\mathcal{D}$ if and only if (5) holds for every map $\ell\colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^*$ for all $y \in A \setminus \{\bar{y}\}$.

Proof.
(*i*) Let $\ell\colon Y \to Y^*$ be a map with $\ell(y) \in \mathcal{D}(y)^* \setminus \{0_{Y^*}\}$ and $\varphi_{\bar{y}}(y) > 0$ for all $y \in A \setminus \{\bar{y}\}$. Assume to the contrary that $\bar{y}$ is not a nondominated element of A w.r.t. $\mathcal{D}$. Then there exists $\hat{y} \in A \setminus \{\bar{y}\}$ with $\bar{y} - \hat{y} \in \mathcal{D}(\hat{y}) \setminus \{0_{Y^*}\}$. Since $\ell(\hat{y}) \in \mathcal{D}(\hat{y})^* \setminus \{0_{Y^*}\}$ we obtain $\ell(\hat{y})(\bar{y} - \hat{y}) \geq 0$ and hence $\varphi_{\bar{y}}(\hat{y}) = \ell(\hat{y})(\hat{y} - \bar{y}) \leq 0$ being a contradiction to $\varphi_{\bar{y}}(y) > 0$ for all $y \in A \setminus \{\bar{y}\}$.
(*ii*) Let $\ell\colon Y \to Y^*$ be a map with $\ell(y) \in \mathcal{D}(y)^\#$ and $\varphi_{\bar{y}}(y) \geq 0$ for all $y \in A \setminus \{\bar{y}\}$. Assume to the contrary that $\bar{y}$ is not a nondominated element of A w.r.t. $\mathcal{D}$. Then

there exists $\hat{y} \in A \setminus \{\bar{y}\}$ with $\bar{y} - \hat{y} \in \mathcal{D}(\hat{y}) \setminus \{0_{Y^*}\}$. Since $l(\hat{y}) \in \mathcal{D}(\hat{y})^{\#}$ we obtain $l(\hat{y})(\bar{y} - \hat{y}) > 0$ and hence $l(\hat{y})(\hat{y} - \bar{y}) < 0$ being a contradiction to $\varphi_{\bar{y}}(y) \geq 0$ for all $y \in A$.
(iii) Let $\bar{y}$ be weakly nondominated element of A w.r.t. $\mathcal{D}$ and $y \in A \setminus \{\bar{y}\}$ be arbitrarily chosen. Then

$$\{\bar{y}\} \cap (\{y\} + \operatorname{int}(\mathcal{D}(y))) = \emptyset$$

an by using a separation theorem, see for instance [20, Theorem 3.16], there exist some $l_y \in Y^* \setminus \{0_{Y^*}\}$ and $\alpha \in \mathbb{R}$ with

$$l_y(\bar{y}) \leq \alpha \leq l_y(y + d) \quad \text{for all} \quad d \in \mathcal{D}(y)\ .$$

It follows by standard arguments that $l_y \in \mathcal{D}(y)^* \setminus \{0_{Y^*}\}$, $l_y(\bar{y}) \leq l_y(y)$, and $l_y(y - \bar{y}) \geq 0$. By setting $\ell(y) := l_y$ for all $y \in A \setminus \{\bar{y}\}$ we obtain a map $\ell \colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^* \setminus \{0_{Y^*}\}$ and $\varphi_{\bar{y}}(y) \geq 0$ for all $y \in A \setminus \{\bar{y}\}$ and thus $\varphi_{\bar{y}}(y) \geq 0$ for all $y \in A$.
Next let $\ell \colon Y \to Y^*$ be a map with $\ell(y) \in \mathcal{D}(y)^* \setminus \{0_{Y^*}\}$ and $\varphi_{\bar{y}}(y) \geq 0$ for all $y \in A \setminus \{\bar{y}\}$. Assume to the contrary that $\bar{y}$ is not a weakly nondominated element of A w.r.t. $\mathcal{D}$. Then there exists $\hat{y} \in A \setminus \{\bar{y}\}$ with $\bar{y} - \hat{y} \in \operatorname{int}(\mathcal{D}(\hat{y}))$. Since $\ell(\hat{y}) \in \mathcal{D}(\hat{y})^* \setminus \{0_{Y^*}\}$ it follows by (3) $\ell(\hat{y})(\bar{y} - \hat{y}) > 0$ and hence $\varphi_{\bar{y}}(\hat{y}) = \ell(\hat{y})(\hat{y} - \bar{y}) < 0$ being a contradiction to $\varphi_{\bar{y}}(y) \geq 0$ for all $y \in A \setminus \{\bar{y}\}$.
(iv) If $\bar{y} \in A$ is a strongly nondominated element of A w.r.t. $\mathcal{D}$, then $y - \bar{y} \in \mathcal{D}(y)$ for all $y \in A \setminus \{\bar{y}\}$. Hence, for every $\ell \colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^*$ for all $y \in A \setminus \{\bar{y}\}$ we obtain $\varphi_{\bar{y}}(y) = \ell(y)(y - \bar{y}) \geq 0$ for all $y \in A \setminus \{\bar{y}\}$ and thus for all $y \in A$.
If $\varphi_{\bar{y}}(y) = \ell(y)(y - \bar{y}) \geq 0$ for all $y \in A$ holds for every $\ell \colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^*$ for all $y \in A \setminus \{\bar{y}\}$, then by (2) we obtain $y - \bar{y} \in \mathcal{D}(y)$ for all $y \in A \setminus \{\bar{y}\}$ and we are done. $\square$

For the proof of a sufficient condition for properly nondominated elements in the sense of Henig we need the following lemma which characterizes the interior of special cones.

Lemma 4.3. *Let $\phi \in Y^* \setminus \{0_{Y^*}\}$ and K be the closed convex cone defined by $K := \{u \in Y \mid \phi(u) \geq 0\}$. Then* $\operatorname{int}(K) = \{u \in Y \mid \phi(u) > 0\}$.

Proof. Let $\phi \in Y^* \setminus \{0_{Y^*}\}$. Hence it holds $||\phi||_* > 0$. If $u \in \operatorname{int}(K)$, then there exists an $\varepsilon > 0$ with $\{u\} + B(0_Y, \varepsilon) \subseteq K$. Let $h \in B(0_Y, \varepsilon)$ with $\phi(h) > 0$. Then, as $u - h \in K$, $\phi(u) = \phi(u - h) + \phi(h) > 0$ and thus $\operatorname{int}(K) \subseteq \{u \in Y \mid \phi(u) > 0\}$.
For $u \in Y$ with $\phi(u) > 0$ we get for all $h \in B(0_Y, \varepsilon)$ with $\varepsilon := \frac{\phi(u)}{||\phi||_*} > 0$

$$\phi(u + h) = \phi(u) + \phi(h) \geq \phi(u) - ||\phi||_* \, ||h|| \geq \phi(u) - \varepsilon ||\phi||_* = 0,$$

thus $u + h \in K$ and hence $\{u \in Y \mid \phi(u) > 0\} \subseteq \operatorname{int}(K)$. $\square$

The following theorem gives sufficient and necessary conditions for properly nondominated elements in the sense of Henig/Benson/Borwein of A w.r.t. $\mathcal{D}$. Thereby we get in case of $\mathcal{D}(y)$ is closed for all $y \in A$ a complete characterization of properly nondominated elements in the sense of Henig. Note that the sufficient condition for a properly nondominated element in the sense of Benson of Theorem 3.3(i) is also a sufficient condition for a properly nondominated element in the sense of Borwein and both necessary conditions for a properly nondominated element in the sense of Benson are by Lemma 3.3(*ii*) under the additional assumption that M is starshaped w.r.t. $\bar{y}$ also a necessary conditions for a properly nondominated element in the sense of Borwein.

Theorem 4.4. *Let $\bar{y} \in A$, the set M be defined as in (1), and the sets $\bar{M}_{\bar{y}}$ and $\bar{D}$ be defined by*

$$\bar{M}_{\bar{y}} := \{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(M - \{\bar{y}\})) \text{ and } \bar{D} := \bigcup_{y \in \bar{M}_{\bar{y}}} \mathcal{D}(y).$$

Then the following holds:

(i) If there exists a map $\ell\colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^\#$ for all $y \in A \setminus \{\bar{y}\}$ such that (4) holds, then $\bar{y} \in \mathrm{ndGHe}(A, \mathcal{D})$.

(ii) If additionally $\mathcal{D}(y)$ is closed for all $y \in A$, then $\bar{y} \in \mathrm{ndGHe}(A, \mathcal{D})$ if and only if there is a map $\ell\colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^\#$ for all $y \in A \setminus \{\bar{y}\}$ such that (4) *holds.*

(iii) If there exists a map $\bar{\ell} \in \bar{D}^\#$ with $\bar{\ell}(y - \bar{y}) \geq 0$ for all $y \in A$, i.e. (5) *holds for a map $\ell\colon Y \to Y^*$ with $\ell(y) := \bar{\ell} \in \bar{D}^\#$ for all $y \in A \setminus \{\bar{y}\}$, then $\bar{y} \in \mathrm{ndBe}(A, \mathcal{D})$.*

(iv) If additionally $\mathcal{D}(y)$ is closed for all $y \in A$ and $\bar{y} \in \mathrm{ndBe}(A, \mathcal{D})$, then there exists a map $\ell\colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^* \setminus \{0_{Y^*}\}$ for all $y \in A \setminus \{\bar{y}\}$ such that (4) *holds.*

(v) If additionally the topology gives Y as the topological dual space of Y^*, $\mathcal{D}(y)$ is closed and $\mathrm{int}(\mathcal{D}(y)^*) \neq \emptyset$ for all $y \in A$ and $\bar{y} \in \mathrm{ndBe}(A, \mathcal{D})$, then there exists a map $\ell\colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^\# = \mathrm{int}(\mathcal{D}(y)^*)$ for all $y \in A \setminus \{\bar{y}\}$ such that (5) *holds.*

Proof.
(*i*) Let $\ell\colon Y \to Y^*$ be a map with $\ell(y) \in \mathcal{D}(y)^\# \subseteq \mathcal{D}(y)^* \setminus \{0_{Y^*}\}$ for all $y \in A \setminus \{\bar{y}\}$ such that $\varphi_{\bar{y}}(y) > 0$ for all $y \in A \setminus \{\bar{y}\}$ and let $y \in A \setminus \{\bar{y}\}$ be arbitrarily chosen. We define the set

$$K(y) := \{z \in Y \mid \ell(y)(z) \geq 0\}\ .$$

Obviously $K(y)$ is a closed convex cone. By the assumptions it follows $\ell(y)(w) > 0$ for all $w \in \mathcal{D}(y) \setminus \{0_Y\}$ and $\ell(y)(\bar{y} - y) < 0$. Thus by Lemma 4.3 we obtain $\mathcal{D}(y) \setminus \{0_Y\} \subseteq \mathrm{int}(K(y))$, and also $\bar{y} \notin \{y\} + K(y)$. By setting $\mathcal{K}(y) := K(y)$ for all $y \in A \setminus \{\bar{y}\}$ and $\mathcal{K}(y) := Y$ for all $y \in (Y \setminus A) \cup \{\bar{y}\}$ the assertion follows.
(*ii*) The sufficiency was shown in (i). It remains to show the necessarity of the condition. By definition there is a cone-valued map $\mathcal{K}\colon Y \to 2^Y$ with $\mathcal{K}(y)$ a convex cone and $\mathcal{D}(y) \setminus \{0_Y\} \subseteq \mathrm{int}(\mathcal{K}(y))$ for all $y \in Y$ such that

$$\bar{y} - y \notin \mathcal{K}(y)$$

for all $y \in A \setminus \{\bar{y}\}$. We can assume the cones $\mathcal{K}(y)$ to be closed. Let $y \in A \setminus \{\bar{y}\}$ be arbitrarily chosen. Then by a separation theorem, see for instance [20, Theorem 3.18], there exists some $l_y \in Y^* \setminus \{0_{Y^*}\}$ and some $\alpha \in \mathbb{R}$ such that

$$l_y(\bar{y} - y) < \alpha \leq l_y(k) \;\text{ for all }\; k \in \mathcal{K}(y)$$

By standard arguments we obtain $l_y \in \mathcal{K}(y)^* \setminus \{0_{Y^*}\}$ and $l_y(y - \bar{y}) > 0$. By (3) it holds

$$\mathcal{D}(y) \setminus \{0_Y\} \subseteq \operatorname{int}(\mathcal{K}(y)) = \{z \in Y \mid l(z) > 0 \;\forall\, l \in \mathcal{K}(y)^* \setminus \{0_{Y^*}\}\}$$

and hence $l_y \in \mathcal{D}(y)^\#$. By setting $\ell(y) := l_y$ for all $y \in A \setminus \{\bar{y}\}$ we obtain a map $\ell \colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^\#$ for all $y \in A \setminus \{\bar{y}\}$ and $\varphi_{\bar{y}}(y) > 0$ for all $y \in A \setminus \{\bar{y}\}$.
(iii) Let $\bar{\ell} \in \bar{D}^\#$ be a map with $\bar{\ell}(y - \bar{y}) \geq 0$ for all $y \in A$. Hence $\bar{\ell} \in \mathcal{D}(y)^\#$ for all $y \in \bar{M}_{\bar{y}}$. Let now $y \in \bar{M}_{\bar{y}} \setminus \{\bar{y}\}$ be arbitrarily chosen. Then there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and sequences $(y_n, d_n)_{n\in\mathbb{N}}$ with $y_n \in A$ and $d_n \in \mathcal{D}(y_n)$ for all $n \in \mathbb{N}$ such that

$$y = \bar{y} + h \;\text{ and }\; h := \lim_{n\to\infty} \lambda_n(y_n + d_n - \bar{y}) \;.$$

By the continuity and linearity of $\bar{\ell}$ it follows

$$\bar{\ell}(y - \bar{y}) = \bar{\ell}(h) = \lim_{n\to\infty} \lambda_n \left(\bar{\ell}(y_n - \bar{y}) + \bar{\ell}(d_n)\right) \;.$$

Since $\lambda_n \in \mathbb{R}_+$, $y_n \in A$ and $d_n \in \mathcal{D}(y_n) \subseteq \bar{D}$ for all $n \in \mathbb{N}$ and $\bar{\ell} \in \bar{D}^\#$ we obtain $\bar{\ell}(y - \bar{y}) \geq 0$. By setting $\ell(y) := \bar{\ell}$ for all $y \in \bar{M}_{\bar{y}} \setminus \{\bar{y}\}$ it follows

$$\varphi_{\bar{y}}(y) = \ell(y)(y - \bar{y}) \geq 0$$

and $\ell(y) \in \mathcal{D}(y)^\#$ for all $y \in \bar{M}_{\bar{y}} \setminus \{\bar{y}\}$. Hence, by using Lemma 4.2(ii) $\bar{y}$ is a nondominated element of $\bar{M}_{\bar{y}}$ and thus $\bar{y} \in \operatorname{ndBe}(A, \mathcal{D})$.
(iv) Let $\bar{y} \in \operatorname{ndBe}(A, \mathcal{D})$ and $y = \bar{y} + h \in \{\bar{y}\} + \operatorname{cl}(\operatorname{cone}(M - \{\bar{y}\})) \setminus \{0_Y\}$ be arbitrarily chosen. Then $\bar{y} - y \notin \mathcal{D}(y)$ and by a separation theorem, see for instance [20, Theorem 3.18], there exists some $l_y \in Y^* \setminus \{0_{Y^*}\}$ and some $\alpha \in \mathbb{R}$ such that

$$l_y(\bar{y} - y) < \alpha \leq l_y(d) \;\text{ for all }\; d \in \mathcal{D}(y) \;.$$

By standard arguments we obtain $l_y \in \mathcal{D}(y)^* \setminus \{0_{Y^*}\}$ and $l_y(y - \bar{y}) > 0$. By setting $\ell(y) := l_y$ for all $y \in \{\bar{y}\} + \operatorname{cl}(\operatorname{cone}(M - \{\bar{y}\})) \setminus \{0_Y\}$ we obtain a map $\ell \colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^* \setminus \{0_{Y^*}\}$ and

$$\varphi_{\bar{y}}(y) = \ell(y)(y - \bar{y}) > 0 \;\text{ for all }\; y \in \{\bar{y}\} + \operatorname{cl}(\operatorname{cone}(M - \{\bar{y}\})) \setminus \{0_Y\} \;.$$

Since

$$A - \{\bar{y}\} \subseteq M - \{\bar{y}\} \subseteq \operatorname{cl}(\operatorname{cone}(M - \{\bar{y}\}))$$

the assertion follows.
(v) Let $\bar{y} \in \operatorname{ndBe}(A, \mathcal{D})$ and $y = \bar{y} + h \in \{\bar{y}\} + \operatorname{cl}(\operatorname{cone}(M - \{\bar{y}\})) \setminus \{0_Y\}$ be arbitrarily chosen. Then $\bar{y} - y \notin \mathcal{D}(y)$ and thus $\lambda(y - \bar{y}) \notin -\mathcal{D}(y)$ for all $\lambda > 0$, and hence

$$\operatorname{cone}(\{y - \bar{y}\}) \cap (-\mathcal{D}(y)) = \{0_Y\} \;.$$

By a conic separation theorem, see for instance [20, Theorem 3.22], there exists some $l_y \in Y^* \setminus \{0_{Y^*}\}$ such that

$$l_y(-d) \leq 0 \leq l_y(z) \ \text{ for all } \ d \in \mathcal{D}(y) \ \text{ and all } \ z \in \text{cone}(\{y - \bar{y}\})$$

and $l_y(d) > 0$ for all $d \in \mathcal{D}(y) \setminus \{0_Y\}$. Thus $l_y \in \mathcal{D}(y)^\#$ and $l_y(y-\bar{y}) \geq 0$. By setting $\ell(y) := l_y$ for all $y \in \{\bar{y}\} + \text{cl}(\text{cone}(M - \{\bar{y}\})) \setminus \{0_Y\}$ we obtain a map $\ell \colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(y)^\#$ and

$$\varphi_{\bar{y}}(y) = \ell(y)(y - \bar{y}) \geq 0 \ \text{ for all } \ y \in \{\bar{y}\} + \text{cl}(\text{cone}(M - \{\bar{y}\})) \setminus \{0_Y\} \ .$$

According to [20, Lemma 3.21(d)], $\mathcal{D}(y)^\# = \text{int}(\mathcal{D}(y)^*)$. By using $A - \{\bar{y}\} \subseteq \text{cl}(\text{cone}(M - \{\bar{y}\}))$ we are done. □

Theorem 4.4(i) is also a direct consequence of [15, Theorem 8] by choosing $(\ell_y^\#, \alpha_y^\#) := (\ell(y), 0)$ for all $y \in Y$ as elements from of augmented dual cones of the cones $\mathcal{D}(y)$ in the definition of the scalarization functional used there. For completeness and because of the simplicity of the direct proof, we nevertheless provided the proof above. Theorem 4.4(iii) also follows from [15, Theorem 9] by choosing $\alpha^\# = 0$. Note that $\bar{\ell}(y - \bar{y}) \geq 0$ for all $y \in A$ is equivalent to that $\bar{y}$ is a minimal solution of the scalar-valued optimization problem $\min_{y \in A} \bar{\ell}(y)$. This problem was considered in [10, Theorem 3.2] for some $\bar{\ell} \in \bigcap_{y \in A} (\mathcal{D}(y)^\#)$. There it was shown that a minimal solution of this problem is a nondominated element of A w.r.t. $\mathcal{D}$. Theorem 4.4(ii) and (v) deliver a stronger necessary conditions under weaker assumptions as those proposed in [15, Theorem 10 and 11] and (ii) includes the necessary condition presented in [15, Theorem 10] as a special case.

4.2 Characterizing minimal elements

Let a map $\ell \colon Y \to Y^*$ and an element $\bar{y} \in Y$ be given. We consider additionally the functional $\psi_{\bar{y}} \colon Y \to \mathbb{R}$ with

$$\psi_{\bar{y}}(y) = \ell(\bar{y})(y - \bar{y}) \ \text{ for all } \ y \in Y \ .$$

Obviously it holds $\psi_{\bar{y}}(\bar{y}) = \ell(\bar{y})(\bar{y} - \bar{y}) = \ell(\bar{y})(0_Y) = 0$. We will make use of the following

Remark 4.5. *As a direct consequence of the definitions $\bar{y}$ is a (weakly/strongly) minimal element of A w.r.t. $\mathcal{D}$ if and only if it is a (weakly/strongly) efficient element of A w.r.t. the partial ordering defined by $K := \mathcal{D}(\bar{y})$.*

For (weakly/strongly) efficient elements of A w.r.t. a partially ordering introduced by a pointed convex cone K various scalarization results are well known, see for instance [20, Chapter 5]. Note that some of the results given there are formulated in non-topological spaces using elements from the algebraic dual space, but the results remain true using elements from the topological dual space, i.e. continuous linear functionals (use the separation theorem [20, Theorem 3.16] instead of [20, Theorem 3.14] if necessary in the corresponding proofs). Hence, by using Remark 4.5 we obtain immediately from [20, Theorem 5.18(a)], [20, Theorem 5.18(b)], [20, Theorem 5.4], [20, Theorem 5.28], [20, Corollary 5.29], and [20, Theorem 5.6] the following lemma:

Lemma 4.6. *Let $\bar{y} \in A$ and the set $M_{\bar{y}}$ be defined as in (1). Then the following holds:*

(i) Let $\ell \colon Y \to Y^*$ be a map with $\ell(\bar{y}) \in \mathcal{D}(\bar{y})^*$. If $\psi_{\bar{y}}(y) > \psi_{\bar{y}}(\bar{y}) = 0$ for all $y \in A \setminus \{\bar{y}\}$, then $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$.

(ii) Let $\ell \colon Y \to Y^*$ be a map with $\ell(\bar{y}) \in \mathcal{D}(\bar{y})^{\#}$. If

$$\psi_{\bar{y}}(y) \geq \psi_{\bar{y}}(\bar{y}) = 0 \;\forall y \in A \setminus \{\bar{y}\}, \tag{6}$$

then $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$.

(iii) If $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$ and $M_{\bar{y}}$ is convex with $\mathrm{int}(M_{\bar{y}}) \neq \emptyset$, then there is a map $\ell \colon Y \to Y^*$ with $\ell(\bar{y}) \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ such that (6) holds.

(iv) Let $\ell \colon Y \to Y^*$ be a map with $\ell(\bar{y}) \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ and $\mathrm{int}(\mathcal{D}(\bar{y})) \neq 0$. If (6) holds, then $\bar{y}$ is a weakly minimal element of A w.r.t. $\mathcal{D}$.

(v) If additionally $M_{\bar{y}}$ is convex and $\mathrm{int}(\mathcal{D}(\bar{y})) \neq 0$, then $\bar{y}$ is a weakly minimal element of A w.r.t. $\mathcal{D}$ if and only if there is a map $\ell \colon Y \to Y^*$ with $\ell(\bar{y}) \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ such that (6) holds.

(vi) If $\mathcal{D}(\bar{y})$ is additionally closed, then $\bar{y} \in A$ is a strongly minimal element of A w.r.t. $\mathcal{D}$ if and only if (6) holds for every map $\ell \colon Y \to Y^*$ with $\ell(\bar{y}) \in \mathcal{D}(\bar{y})^*$.

The main drawback of the results above is the required convexity for the necessary conditions. When using the functional $\varphi_{\bar{y}}$ we get the following results for (weakly) minimal elements, where we do not need the convexity of the set $M_{\bar{y}}$.

Lemma 4.7. *Let $\bar{y} \in A$. Then the following holds:*

(i) If additionally $\mathcal{D}(\bar{y})$ is closed, then $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$ if and only if there is a map $\ell \colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ for all $y \in A \setminus \{\bar{y}\}$ such that (4) holds.

(ii) If additionally $\mathrm{int}(\mathcal{D}(\bar{y})) \neq \emptyset$, then $\bar{y}$ is a weakly minimal element of A w.r.t. $\mathcal{D}$ if and only if there is a map $\ell \colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ for all $y \in A \setminus \{\bar{y}\}$ such that (5) holds.

Proof.
(i) The conclusion follows immediately from the Remark 4.5 and [20, Theorem 5.5].
(ii) Let $\bar{y}$ be a weakly minimal element of A w.r.t. $\mathcal{D}$, then

$$y \notin \{\bar{y}\} - \mathrm{int}(\mathcal{D}(\bar{y})) \text{ for all } \; y \in A \setminus \{\bar{y}\} \; .$$

Since $\mathrm{int}(\mathcal{D}(\bar{y})) \neq \emptyset$ it also holds $\mathrm{int}(\{\bar{y}\} - \mathcal{D}(\bar{y})) \neq \emptyset$. Furthermore the set $\{\bar{y}\} - \mathcal{D}(\bar{y})$ is convex. Using a separation theorem, see for instance [20, Theorem 3.16], there exists for each $y \in A \setminus \{\bar{y}\}$ some $l_y \in Y^* \setminus \{0_{Y^*}\}$ and some $\alpha \in \mathbb{R}$ with

$$l_y(\bar{y}) - l_y(k) \leq \alpha \leq l_y(y) \;\text{ for all }\; k \in \mathcal{D}(\bar{y}) \; .$$

By standard arguments we obtain $l_y \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ and $l_y(y - \bar{y}) \geq 0$. Hence, by setting $\ell(y) := l_y$ for each $y \in A \setminus \{\bar{y}\}$ we obtain a map $\ell \colon Y \to Y^*$ with

$\ell(y) \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ and $\varphi_{\bar{y}}(y) \geq 0$ for all $y \in A \setminus \{\bar{y}\}$.
If $\operatorname{int}(\mathcal{D}(\bar{y})) \neq \emptyset$ and $\ell\colon Y \to Y^*$ is a map with $\ell(y) \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ and $\varphi_{\bar{y}}(y) = \ell(y)(y - \bar{y}) \geq 0$ for all $y \in A \setminus \{\bar{y}\}$, then by using (3) we obtain $y \notin \{\bar{y}\} - \operatorname{int}(\mathcal{D}(\bar{y}))$ for all $y \in A$ and we are done. □

Note that in Lemma 4.2 we assume ℓ to be a map with $\ell(y) \in \mathcal{D}(y)^* \setminus \{0_{Y^*}\}$ and $\ell(y) \in \mathcal{D}(y)^\#$ for all $y \in A \setminus \{\bar{y}\}$ while in the lemma above we assume $\ell(y) \in \mathcal{D}(\bar{y})^* \setminus \{0_{Y^*}\}$ and $\ell(y) \in \mathcal{D}(\bar{y})^\#$ for all $y \in A \setminus \{\bar{y}\}$, respectively. If $\mathcal{D}(\bar{y}) \subseteq \mathcal{D}(y)$ holds for a nondominated element $\bar{y}$ w.r.t. $\mathcal{D}$ and all $y \in A \setminus \{\bar{y}\}$, and thus $\mathcal{D}(y)^* \subseteq \mathcal{D}(\bar{y})^*$, then it is known that $\bar{y}$ is also a minimal element of A w.r.t. $\mathcal{D}$, cf. [10, Remark 2.1]. Analogously for minimal elements in case $\mathcal{D}(y) \subseteq \mathcal{D}(\bar{y})$ for all $y \in A \setminus \{\bar{y}\}$.

The next lemma gives conditions under which $\bar{y} \in \operatorname{mGHe}(A, \mathcal{D})$ always implies that the cone $\mathcal{K}(\bar{y})$ in the definition can be chosen to be pointed. We need this lemma for the proof of the necessary condition in Lemma 4.9(*iv*).

Lemma 4.8. *Let the topology give Y as the topological dual space of Y^*, $\bar{y} \in \operatorname{mGHe}(A, \mathcal{D})$, $\mathcal{D}(\bar{y})$ be additional closed and $\operatorname{int}(\mathcal{D}(\bar{y})^*) \neq \emptyset$. Then there is a pointed convex cone $\tilde{K}$ with $\mathcal{D}(\bar{y}) \setminus \{0_Y\} \subseteq \operatorname{int}(\tilde{K})$ such that $y \notin \{\bar{y}\} - \tilde{K}$ for all $y \in A \setminus \{\bar{y}\}$, i.e. the cone $\mathcal{K}(\bar{y})$ in the definition of Henig proper minimal can be chosen to be pointed.*

Proof. By Lemma 3.4, $\bar{y} \in \operatorname{mGHe}(A, \mathcal{D})$ if and only if there exists a convex cone K with $\mathcal{D}(\bar{y}) \setminus \{0_Y\} \subseteq \operatorname{int}(K)$ such that $y \notin \{\bar{y}\} - K$ for all $y \in A \setminus \{\bar{y}\}$. If K is pointed, then we are done. Otherwise, there exists some element $k \neq 0_Y$ with $k \in K \cap (-K)$. As $\mathcal{D}(\bar{y})$ is pointed, at least k or $-k \notin \mathcal{D}(\bar{y})$. Without loss of generality let $k \notin -\mathcal{D}(\bar{y})$, i.e.

$$\operatorname{cone}(\{k\}) \cap (-\mathcal{D}(\bar{y})) = \{0_Y\} \ .$$

By a separation theorem for closed convex cones, see for instance [20, Theorem 3.22], there exists $l \in Y^* \setminus \{0_{Y^*}\}$ with

$$l(x) \leq 0 \leq l(y) \quad \text{for all} \quad x \in -\mathcal{D}(\bar{y}) \quad \text{and all} \quad y \in \operatorname{cone}(\{k\})$$

and $l(x) > 0$ for all $x \in \mathcal{D}(\bar{y}) \setminus \{0_Y\}$. Thus $l \in \mathcal{D}(\bar{y})^\#$. We define the pointed convex cones

$$C := \{y \in Y \mid l(y) > 0\} \cup \{0_Y\}$$

and

$$\tilde{K} := K \cap C \subseteq K \ .$$

It holds $\mathcal{D}(\bar{y}) \subseteq C$. Moreover,

$$\operatorname{int}(\tilde{K}) = \operatorname{int}(K) \cap \operatorname{int}(C) = \operatorname{int}(K) \cap (C \setminus \{0_Y\}) \ .$$

Thus $\mathcal{D}(\bar{y}) \setminus \{0_Y\} \subseteq \operatorname{int}(\tilde{K})$ and we are done. □

Lemma 4.9. *Let $\bar{y} \in A$ and the set $M_{\bar{y}}$ be defined as in (1). Then the following holds:*

(i) If there exists a map $\ell\colon Y \to Y^*$ with $\ell(\bar{y}) \in \mathcal{D}(\bar{y})^\#$ such that (6) holds, then $\bar{y} \in \operatorname{mBe}(A, \mathcal{D})$ and $\bar{y} \in \operatorname{mBo}(A, \mathcal{D})$.

(ii) Let additionally $\mathcal{D}(\bar{y})$ have a weakly compact base and let $\ell\colon Y \to Y^*$ be a map with $\ell(\bar{y}) \in \mathcal{D}(\bar{y})^{\#}$. If (6) *holds, then $\bar{y} \in \mathrm{mGHe}(A, \mathcal{D})$.*

(iii) If additionally the topology gives Y as the topological dual space of Y^*, $\mathcal{D}(\bar{y})$ is closed with $\mathrm{int}(\mathcal{D}(\bar{y})^*) \neq \emptyset$, and $M_{\bar{y}}$ is convex, then $\bar{y} \in \mathrm{mBo}(A, \mathcal{D}) = \mathrm{mBe}(A, \mathcal{D})$ if and only if there is a map $\ell\colon Y \to Y^*$ with $\ell(\bar{y}) \in \mathcal{D}(\bar{y})^{\#} = \mathrm{int}(\mathcal{D}(\bar{y})^*)$ such that (6) holds.

(iv) Let the topology give Y as the topological dual space of Y^*, $\mathcal{D}(\bar{y})$ be closed with $\mathrm{int}(\mathcal{D}(\bar{y})^*) \neq \emptyset$, and $M_{\bar{y}}$ be convex. If $\bar{y} \in \mathrm{mGHe}(A, \mathcal{D})$, then there is a map $\ell\colon Y \to Y^*$ with $\ell(\bar{y}) \in \mathcal{D}(\bar{y})^{\#} = \mathrm{int}(\mathcal{D}(\bar{y})^*)$ such that (6) holds.

Proof.
(i) follows immediately from the Remark 4.5, [20, Theorem 5.21], and Lemma 3.3(iii). Note that the proof of [20, Theorem 5.21] is given for properly efficient elements in the sense of Borwein but works analogously for properly efficient elements in the sense of Benson.
(ii) follows immediately from (i) together with Lemma 3.7(iii). Note that the conclusion can also be proven directly by using Lemma 3.4 and Lemma 3.6.
(iii) follows immediately from the Remark 4.5, [20, Corollary 5.22], Lemma 3.3(iv), and [20, Lemma 3.21(d)].
(iv) follows immediately from (iii) together with Lemma 3.7(ii) and Lemma 4.8. ☐

Lemma 4.9(i) is also a direct consequence of [15, Theorem 5] by choosing $(\ell^{\#}_{\bar{y}}, \alpha^{\#}_{\bar{y}}) := (\ell(\bar{y}), 0)$ as element of the augmented dual cone of the cone $\mathcal{D}(\bar{y})$ in the definition of the scalarization functional used there. Lemma 4.9(ii) gives the same sufficient condition as in [15, Theorem 6(iii)] by choosing $(\ell^{\#}_{\bar{y}}, \alpha^{\#}_{\bar{y}}) := (\ell(\bar{y}), 0)$. But in [15], the assumption of a reflexive space as well as that the cones $\mathcal{D}(y)$ have ε-conic neighborhoods with which the cones satisfy a separation property are needed.

However, the necessary conditions for properly minimal elements in the sense of Benson/Borwein/Henig presented in [15, Theorem 6(i),(ii) and Theorem 7] do not require convexity of the set $M_{\bar{y}}$ as assumed here in Lemma 4.9(iii) and (iv). When we use the functional $\varphi_{\bar{y}}$ instead of $\psi_{\bar{y}}$, we can also give necessary conditions without this convexity assumption. Note that the necessary condition in (i) for a proper minimal element in the sense of Henig is also a necessary condition for a proper minimal element in the sense of Benson in case the cone $\mathcal{D}(\bar{y})$ has a weakly compact base. Moreover, in case the set $M_{\bar{y}}$ is starshaped w.r.t. $\bar{y}$, then (ii) also gives a necessary condition for a Borwein proper minimal element.

Theorem 4.10. *Let $\bar{y} \in A$ and the set $M_{\bar{y}}$ be defined as in (1). If $\mathcal{D}(\bar{y})$ is closed, then the following holds:*

(i) If $\bar{y} \in \mathrm{mGHe}(A, \mathcal{D})$, then there exists a map $\ell\colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(\bar{y})^{\#}$ for all $y \in A \setminus \{\bar{y}\}$ such that (4) *holds.*

(ii) If additionally the topology gives Y as the topological dual space of Y^*, $\mathrm{int}(\mathcal{D}(\bar{y})^*) \neq \emptyset$ and $\bar{y} \in \mathrm{mBe}(A, \mathcal{D})$, then there exists a map $\ell\colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(\bar{y})^{\#} = \mathrm{int}(\mathcal{D}(\bar{y})^*)$ for all $y \in A \setminus \{\bar{y}\}$ such that (5) *holds.*

Proof.
(i) By Lemma 3.4 there is a convex cone K with $\mathcal{D}(\bar{y}) \setminus \{0_Y\} \subseteq \mathrm{int}(K)$ such that $\bar{y} - y \notin K$ for all $y \in A \setminus \{\bar{y}\}$. We can assume the cone K to be closed. Let $y \in A \setminus \{\bar{y}\}$ be arbitrarily chosen. Then by a separation theorem, see for instance [20, Theorem 3.18], there exists some $l_y \in Y^* \setminus \{0_{Y^*}\}$ and some $\alpha \in \mathbb{R}$ such that

$$l_y(\bar{y} - y) < \alpha \leq l_y(k) \ \text{ for all } \ k \in K \ .$$

By standard arguments we obtain $l_y \in K^* \setminus \{0_{Y^*}\}$ and $l_y(y - \bar{y}) > 0$. By (3) it holds

$$\mathcal{D}(\bar{y}) \setminus \{0_Y\} \subseteq \mathrm{int}(K) = \{z \in Y \mid l(z) > 0 \ \forall \ l \in K^* \setminus \{0_{Y^*}\}\}$$

and hence $l_y \in \mathcal{D}(\bar{y})^{\#}$. By setting $\ell(y) := l_y$ for all $y \in A \setminus \{\bar{y}\}$ we obtain a map $\ell\colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(\bar{y})^{\#}$ for all $y \in A \setminus \{\bar{y}\}$ and $\varphi_{\bar{y}}(y) > 0$ for all $y \in A \setminus \{\bar{y}\}$.
(ii) Let $\bar{y} \in \mathrm{mBe}(A, \mathcal{D})$ and $y = \bar{y} + h \in \{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(M_{\bar{y}} - \{\bar{y}\})) \setminus \{0_Y\}$ be arbitrarily chosen. Then $\bar{y} - y \notin \mathcal{D}(\bar{y})$ and thus

$$\mathrm{cone}(\{y - \bar{y}\}) \cap (-\mathcal{D}(\bar{y})) = \{0_Y\} \ .$$

By a conic separation theorem, see for instance [20, Theorem 3.22], there exists some $l_y \in Y^* \setminus \{0_{Y^*}\}$ such that

$$l_y(-d) \leq 0 \leq l_y(z) \ \text{ for all } \ d \in \mathcal{D}(\bar{y}) \ \text{ and all } \ z \in \mathrm{cone}(\{y - \bar{y}\})$$

and $l_y(d) > 0$ for all $d \in \mathcal{D}(\bar{y}) \setminus \{0_Y\}$. Thus $l_y \in \mathcal{D}(\bar{y})^{\#}$ and $l_y(y - \bar{y}) \geq 0$. By setting $\ell(y) := l_y$ for all $y \in \{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(M_{\bar{y}} - \{\bar{y}\})) \setminus \{0_Y\}$ we obtain a map $\ell\colon Y \to Y^*$ with $\ell(y) \in \mathcal{D}(\bar{y})^{\#}$ and

$$\varphi_{\bar{y}}(y) = \ell(y)(y - \bar{y}) \geq 0 \ \text{ for all } \ y \in \{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(M_{\bar{y}} - \{\bar{y}\})) \setminus \{0_Y\} \ .$$

According to [20, Lemma 3.21(d)], $\mathcal{D}(y)^{\#} = \mathrm{int}(\mathcal{D}(y)^*)$. By using

$$A - \{\bar{y}\} \subseteq \mathrm{cl}(\mathrm{cone}(M_{\bar{y}} - \{\bar{y}\}))$$

we are done. □

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