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UNCERTAINTY IN FUZZY SCALES BASED MEASUREMENTS

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Abstract - Fuzzy scales were introduced as a transition between weak scales and strong scales. Preceding studies on fuzzy scales considered only ideal exact measurement without any consideration of uncertainty. The goal of this paper is to present a general approach for the management of uncertainty within the context of fuzzy scale based meaurements.

Keywords uncertainty, fuzzy scale, measurement science.

1. INTRODUCTION

The concept of fuzzy scale was introduced in order to add properties to nominal scales and ordinal scales, and by the way, to establish a link between strongly defined measurements used for example in physical sciences and weakly defined measurements used in behavioural sciences as for example in psychophysics [1]. Actually, except in Physics, the Measurement Science is more and more concerned by weakly defined measurements throw for example Psychophysics or Economy but also with quantities derived from heterogeneous sources.

Since their first introduction, fuzzy scales were exclusively used to perform a precise representation of a unique property manifestation. In a measurement context, it is now admitted that the representation of a property manifestation needs to expose the imprecision and the uncertainty related to the definition of the property and to the measurement process. A first stage of a measuring process using a ratio scale or an interval scale is able to produce a set of numerical values. This set of values is usually processed to perform a type A measurement, or to characterize the measuring system in order to perform a posterior type B measurements. Within the context of fuzzy scales, a second stage using a fuzzy scale performs a translation from a numerical representation to a fuzzy lexical one. The purpose of this paper is to present how this representation can be used to manage uncertainty. It first introduces the concept of fuzzy scale, and the existing approaches to manage uncertainty. The general approach of evidence theory is recalled then applied to build a probabilistic and an possibilistic representation of the same uncertainty.

2. FUZZY SCALES

Fuzzy scale are characterized by their capability to represent a similarity relation between manifestations with a fuzzy equivalence relation between quantity values [2]. The quantity values are expressed as Lexical Fuzzy Subset (LFS), i.e. as fuzzy subsets of lexical terms.

This new family presently includes the weakest fuzzy scales: the fuzzy nominal scales, and metrical fuzzy scales [3]. The last ones are fuzzy nominal scale enhanced with the definition of a distance. The fuzzy ordinal scales are still under studies. Expressing quantity values with LFSs can be performed by measurements with physical sensors, or with psychophysics experiments. For the first case, quantity values are first expressed on a conventional numerical space with a ratio scale or at least with an interval scale. At this step, empirical relations are mapped into numerical relations. Then a second translation maps the numerical expression of quantity values into a fuzzy lexical expression. The second mapping, based on a fuzzy scale, transfers only the fuzzy equivalence relation from the empirical relational set to the representational relational set in the case of a fuzzy nominal scale, and transfer a distance operator in case of a metrical scale.

Fuzzy scales are defined by fuzzy *symbolisms* $<E,T,R>$ where:

- E is the set of the manifestations of a quantity,
- T is a set of terms used to qualify measurements,
- R is a fuzzy relation on ExT.

Let denote $FS(T)$ the set of fuzzy subsets of T, and D the injective mapping defined as

$$\begin{aligned} &D: E \rightarrow FS(T) \\ &\forall e \in E \qquad D(e) = \{t \in T | eRt\} \end{aligned} \tag{1}$$

From a mathematical point of view, the measurement values, expressed on $FS(T)$, are elements of a subset of a $|T|$dimensional cube where $|T|$ is the cardinality of T.

3. MODELLING UNCERTAINTY

3.1. Model defined on a numerical space

In a first trivial approach, the set of the numerical expression of the observations, i.e. the set of quantity values expressed in a numerical space, is used to produce a measurement result defined on this numerical set. The usual way is to define a probability density function (PDF) that models the set of quantity values (see GUM). An approximation of the probability density function by a parametric function allows analytical computation on PDFs. Another way also compatible with GUM is to model the set of quantity values with a possibility function (PF) that supports also the approximation with parametric functions [4]. The second stage is to perform a projection of this function on the set of LFSs. The result is a PDF (or a possibility function) on LFSs.

The advantage of this approach is the possibility to use the existing mathematical tools to create the PDFs or the possibility functions. The disadvantage is that the fuzzy scales need a lot of parameters to be defined [3]. Then a function with few parameters is mapped into a function with a lots of parameters, and the parametrization becomes useless. Another disadvantage is the impossibility to use this method for psychophysical measurements due to the fact that the numerical stage is avoided.

3.2. Models defined on LFSs

In a second approach, each observation is first expressed as a Lexical Fuzzy Subset of terms. Then a parametric function is created on LFSs to represent the set of quantity values. As mentioned in previous studies, the relations and operators that can be used to define the parametric functions are limited by the nature of the scale. For fuzzy nominal scales, the available relation is a fuzzy equivalence relation ~. The metrical fuzzy scale defines a distance between LFSs. Let first define a general family of parametric functions on LFSs when the measurement is performed with a fuzzy nominal scale.

Definition 1:

- Let $m \in FS(T)$ be a parameter.
- Let h be a monotonic decreasing function with domain [0,1] and codomain [0, +∞[.
- Let ~ be the fuzzy equivalence relation on $FS(T)$ that characterize a fuzzy nominal scale
- Let $f_{h,m}$ be a parametric function with domain $FS(T)$, codomain [0, +∞[and defined by $f_{h,m}(x) = h(1 - (\sim)(x,m))$, where $(\sim)(a,b)$ gives the membership function of the relation $a \sim b$. (for any a in $FS(T)$, $(\sim)(a,a) = 1$).

If $f_{h,m}$ is such that

$$\sum_{s \in FS(T)} f_{h,m}(s) = 1 \quad (2)$$

then $f_{h,m}$ is equivalent to a probability density function on $FS(T)$ with mean m.

If h is such that $h(0) = 1$, then $f_{h,m}$ verifies

$$max(f_{h,m}(s) | s \in FS(T)) = 1 \quad (3)$$

and is $f_{h,m}$ is equivalent to a possibility function which kernel includes m.

We can simply verify that $d(a,b) = (1 - (\sim)(a,b))$ is actually a distance with a saturation effect. At this step ,we may introduce the next assumption:

$\sigma << 1$

where σ is a standard deviation defined on $FS(T)$.

With this assumption, the saturation of d doesn't concern our problem. But the standard deviation on $FS(T)$ need to be defined.

The definition 1 can be extended to the case of metrical fuzzy scales:

Definition 2:

- Let $m \in FS(T)$ be a parameter.
- Let h be a monotonic decreasing function with domain [0,+∞] and codomain [0, +∞[.
- Let d be the distance on $FS(T)$ that characterizes a metrical fuzzy scale.
- Let $f_{h,m}$ be a parametric function with domain $FS(T)$, codomain [0, +∞[and defined by $f_{h,m}(x) = h(d(x,m))$

If $f_{h,m}$ verifies Eq. (2) or Eq. (3), $f_{h,m}$ is respectively equivalent to a probability distribution or a possibility distribution.

At this step, we can conclude that with a distance, or a fuzzy equivalence between LFSs, it is possible to define parametric PDFs or PFs centred on a LFS.

The available methods to define the parameters are similar to the well known methods used on numerical spaces. Concerning the translations between PDFs and possibility functions, a general approach based on the Dempter-Shafer evidence theory [5] is directly applicable.

3.3. Evidence theory

The Dempster-Shafer evidence theory, also named the transferable belief model (TBM) is a general approach for uncertainty management based on the assignment of probabilities to sets. It generalises both the probability theory and the possibility theory [6]. The main element of the TBM is the ***frame of discernment*** denoted Ω and presented as a set of elementary events, or as a discourse set for the expression of the belief. The knowledge is then expressed with a mapping denoted m and called ***belief function*** from the set of subsets of Ω, denoted 2^{Ω}, to the set [0,1]. This mapping represents the distribution of a ***weight of belief*** over the elements of 2^{Ω}. It verifies:

$$m(\varnothing) = 0$$

$$\sum_{A \in 2^{\Omega}} m(A) = 1 \quad (4)$$

The set $\{A | m(A) > 0\}$ is called the set of ***focal elements*** of m, and the couple $(\{A | m(A) > 0\}, m)$ is called a ***body of evidence*** of a variable with values on Ω. Two measures Bel and Pl are defined on the subsets of Ω.

$$Bel(A) = \sum_{B \subseteq A} m(B) \quad (5)$$

$$Pl(A) = \sum_{B \cap A \neq \varnothing} m(B) \quad (6)$$

When the focal elements are singletons, then the belief function is qualified as Bayesian. In this case, the two measures *Bel* and *Pl* are equal to a probabilistic measure *P*. The belief function *m* is then equivalent to an assignment of probabilities to individual values i.e. equivalent to a PDF.

When the focal elements are sets ordered by the inclusion operator, called nested sets or consonant sets, *Bel* and *Pl* are respectively the necessity measure *N* and the possibility measure Π of a possibility distribution derived from *m*.

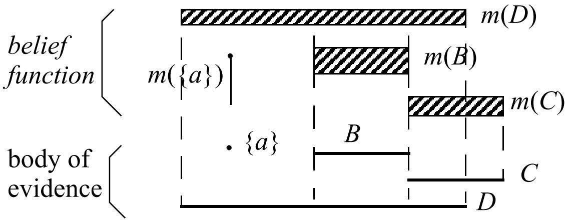

Fig. 1. Sematic use in this paper to represent belief functions. As a mass function, the belief function is represented by a surface

In [5] Yamada proposed a probability-possibility transformation based on the evidence theory. The principle is to identify a set of disjoint subsets of Ω computed from a PDF, then to compute a new body of evidence made of consonant subsets. We propose in this paper to build a parametric set of disjoint subsets in order to model a PDF within the framework of evidence theory. Then the Yamada's approach is applied to build the corresponding possibility distribution.

Within the context of metrology, the frame of discernment is the set all individual measurement values, and a measurement result is interpreted as a body of evidence of a quantity. It expresses the known representational information about a quantity.

3.4. Build method for PDFs and PFs

Let $S \subset FS(T)$ a finite set of measurement values, being the frame of discernment. Let $s_0 \in S$ being a measurement value that will play the role of mean value.

Let $F = \{S_i \mid 1 \leq i \leq n\}$ being an ordered family of *n* non-empty subsets of *S* defined by
$S_i = \{s \mid d_i \leq d(s,s_0) < d_{i+1}\}$.

Where *d* is a distance on *S* as defined in Chap. 3.2. and $d_1 = 0$.

A analogous definition on an euclidian plane will define a family of rings, all centred on s_0.

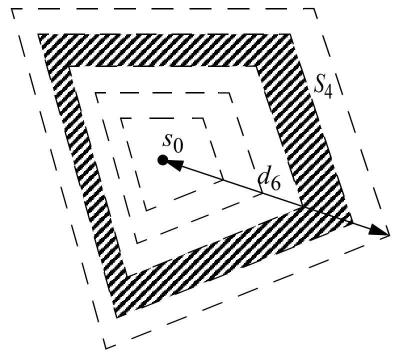

Fig. 2. Example of an ordered set family centred on s_0 and built with a non-euclidian distance.

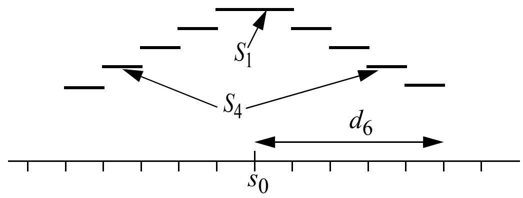

Fig. 3. Another example on a mono-dimentional space

Let F^P a set of singletons defined by
$F^P = \{\{s\} \mid s \in S_1 \cup S_2 \cup _{...} \cup S_n\}$.

Let m_p denotes a Bayesian belief function defined by: $m_p(\{s\} \mid s \in S_i) = h(d_i)$ where *h* is a monotonic decreasing function with domain $[0,+\infty]$ and codomain $[0, +\infty[$ that verifies

$$\sum_{i=1}^{n} h(d_i) \cdot |S_i| = 1 , \qquad (7)$$

and $h(d_{n+1})=0$,
where $|S_i|$ denotes the cardinality of S_i.

We can remark that (F^P,m_p) is a body of evidence defining a probability distribution.

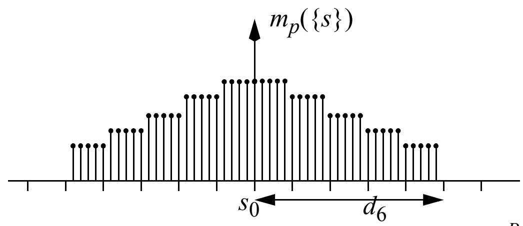

Fig. 4. Belief function m_p defined on the singleton set F^P

Let now follow the Yamada's approach. A new family of sets is built from the family *F*.

$$F^{\pi} = \{F_k^{\pi} \mid 1 \leq k \leq n\} \qquad (8)$$

$$\text{where } F_k^{\pi} = \bigcup_{h=1}^{k} S_h \qquad (9)$$

By definition these sets are nested. It is then possible to define a belief function m_{π} which plausibility measure Pl is a possibility measure Π.

$$m_\pi(F_k^\pi) = \left|F_k^\pi\right|(h(d_k) - h(d_{k+1})) = \left(\sum_{h=1}^{k} |S_h|\right)(h(d_k) - h(d_{k+1})) \qquad (10)$$

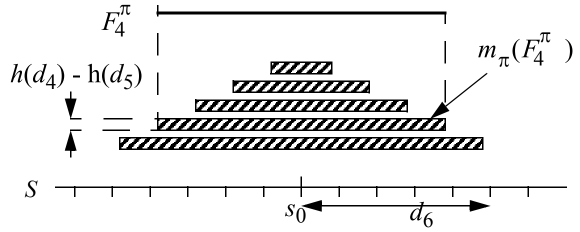

Fig. 5. Belief function m_π defined on the singleton set F^π

The possibility distribution is directly deduced from the belief function m_π.

$$\pi(s) = \Pi(\{s\}) = Pl(\{s\}) = \sum_{B \cap \{s\} \neq \varnothing} m_\pi(B) \qquad (11)$$

$$\forall s \in S_k,\ \pi(s) = \sum_{h=k}^{n} m_\pi(F_h^\pi) = \left|F_k^\pi\right| h(d_k) + \sum_{j=k+1}^{n} |S_j| h(d_j \qquad (12)$$

3.5. Definition of the parameters

With the chosen approach, all information about the probabilistic or the possibilistic representation of uncertainty on the space of LFSs is included into the ordered family $S_i = \{s \mid d_i \leq d(s,s_0) < d_{i+1}\}$ and on the mapping h. The procedure to define this information is

- to acquire a set of statistical data
- to translate this data into LFSs
- to extract the mean s_0 from the set of data
- to build a set family F
- to create an histogram from data mapped into F
- to compute the mapping h

The value s_0 has the same semantic than the mean in usual PDFs defined on interval scales or ratio scales. Due to the scales used in our case, the addition operator is not allowed for computations in the set of LFSs. So the mean cannot be computed with the usual formula. The value s_0 is then obtain by minimization of the cumulative distance to the data.

The set family F is also characterized by a distance d on LFSs and a set $\{d_i \mid 1 \leq i \leq n+1\}$. The distance d is induced by the scale and cannot be considered as a parameter except for a calibration process.

The set $\{d_i \mid 1 \leq i \leq n+1\}$ is related to discrete aspect of the method. The influence of this information the PDF or the PF is small and, in a first approximation, it can be reduced to the parameters Δd and n. The set become $\{d_i = (i - 1)\Delta d \mid 1 \leq i \leq n+1\}$.

Finally, h is the main source of parametrization. It can be defined as a continuous parametric function, or as a discrete set of n values. The usual statistical methods are then applied to estimate h parameters.

4. DISCUSSION

This paper shows that uncertainty can be managed in the case of measurements performed with fuzzy nominal scales and metrical fuzzy scales. It provide a method to represent uncertainty on fuzzy lexical sets with the two main framework that are the probability theory and the possibility theory. The Dempster-Shafer theory is used mainly to build a possibility distribution from a probability distribution.

The capability to manage uncertainty directly on fuzzy lexical sets is an important advance for behavioural sciences that cannot express their measurement result on numerical spaces.

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