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Abstract
We prove a new lower bound on the independence number of a simple connected graph in terms of its degrees.

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1 Introduction

We consider finite, simple, and undirected graphs $G$ with vertex set $V$. For a graph $G$, we denote its order by $n$ and its size by $m$, respectively. The degree of $u$ in $G$ is denoted by $d(u)$ and $\Delta$ is the maximum degree of $G$. A set of vertices $I \subseteq V$ in a graph $G$ is independent, if no two vertices in $I$ are adjacent. The independence number $\alpha$ of $G$ is the maximum cardinality of an independent set of $G$.

The independence number is one of the most fundamental and well-studied graph parameters [6]. In view of its computational hardness [5] various bounds on the independence number have been proposed. The following classical bound holds for every graph $G$ and is due to Caro and Wei [1,7]

$$\alpha \geq \sum_{u \in V} \frac{1}{d(u) + 1}. \quad (1)$$

Since the only graphs for which (1) is best-possible are the disjoint unions of cliques, additional structural assumptions excluding these graphs allow improvements of (1). A natural candidate for such assumptions is connectivity.

For connected graphs, Harant and Rautenbach proved [2] (cf. also [3] and [4])

**Theorem 1** If $G$ is a connected graph, then there exist a positive integer $k \in \mathbb{N}$ and a function $\phi : V \to \mathbb{N}_0$ with non-negative integer values such that $\phi(u) \leq d(u)$ for all $u \in V$,

$$\alpha \geq k \geq \sum_{u \in V} \frac{1}{d(u) + 1 - \phi(u)}, \quad (2)$$

and

$$\sum_{u \in V} \phi(u) \geq 2(k - 1). \quad (3)$$
Note that Theorem 1 is best-possible for the connected graphs which arise by adding bridges to disjoint unions of cliques, i.e. it is best-possible for the intuitively most natural candidate of a connected graph with small independence number. In [3], a weaker version of Theorem 1 is proved. This result is obtained from Theorem 1 by replacing the inequality (3) by \( \sum u \in V \phi(u) \geq k - 1 \).

For an integer \( l \) with \( 0 \leq l \leq 2m \) let \( f(l) = \min \sum u \in V \frac{1}{d(u)+1-\phi(u)} \), where the minimum is taken over all integers \( 0 \leq \phi(u) \leq d(u) \) with \( \sum u \in V \phi(u) = l \).

Obviously, \( f \) is strictly increasing. With this function \( f \), it follows the existence of positive integers \( k_1 \) and \( k_2 \) such that \( \alpha \geq k_1 + 1 \geq f(k_1) \) (put \( k_1 = k - 1 \) and use the result in [3]) and \( \alpha \geq \frac{k_2}{2} + 1 \geq f(k_2) \) (with \( k_2 = 2(k - 1) \) and Theorem 1). After extending \( f \) to real arguments, in [4], it is proved that the function \( 1 - f(l) \) is continuous and strictly increasing and that \( k_1 \) is at least the unique zero \( k_0 \) of this function. Finally, \( \alpha \geq k_0 + 1 \) is the main result in [4].

Here we will show that the continuous function \( \frac{1}{2} + 1 - f(l) \) is also strictly increasing. If we assume that \( \sum u \in V \frac{1}{d(u)+1} \geq 2 \) for the graph in question then \( f(2) > f(0) = \sum u \in V \frac{1}{d(u)+1} \geq 2 \). It will be proved that there is a unique solution \( l_0 \) of the equation \( \frac{1}{2} + 1 = f(l) \) and, because \( \frac{1}{2} + 1 - f(l) \) is strictly increasing and \( \frac{1}{2} + 1 - f(2) < 0 \), it follows \( l_0 > 2 \). Consequently, \( \frac{l_0}{2} + 1 = f(l_0) > f(\frac{l_0}{2} + 1) \), since \( f \) is strictly increasing, hence, \( \frac{l_0}{2} > k_0 \).

The inequality \( \alpha \geq \frac{l_0}{2} + 1 \) is the content of the following Theorem 2.

In case \( \sum u \in V \frac{1}{d(u)+1} \geq 2 \), Remark 2 gives a lower bound on the improvement

\[
\left( \frac{l_0}{2} + 1 \right) - (k_0 + 1) = f(l_0) - f(k_0).
\]

**Theorem 2** Let \( G \) be a finite, simple, connected, and non-complete graph on \( n \geq 3 \) vertices of size \( m \geq n \). Moreover, let \( \alpha \leq \frac{n}{2} \) be the independence number, \( \Delta \) be the maximum degree of \( G \), \( n_j \) be the number of vertices of degree \( j \) in \( G \), and

\[
x(j) = \frac{j(j+1)}{j(j+1) - 2} \left( \frac{2}{j+1} - (\Delta - j) \right) n_\Delta + \ldots + \left( \frac{2}{j+1} - 1 \right) n_{j+1} + \frac{2n_j}{j+1} + \ldots + \frac{2n_1}{2} - 2
\]

for \( j \in \{2, \ldots, \Delta\} \).

Then

(i) there is a unique \( j_0 \in \{2, \ldots, \Delta\} \) such that \( 0 \leq x(j_0) < n_\Delta + \ldots + n_{j_0} \) and

(ii)

\[
\alpha \geq \left( \sum u \in V \frac{1}{d(u)+1} \right) + \frac{n_\Delta}{\Delta(\Delta + 1)} + \frac{n_\Delta + n_{\Delta-1}}{(\Delta - 1)\Delta} + \ldots + \frac{n_\Delta + \ldots + n_{j_0+1}}{(j_0 + 2)(j_0 + 1)} + \frac{x(j_0)}{(j_0 + 1)j_0}
\]

\[
= x(j_0) + \frac{n_{j_0+1}}{2} + 2n_{j_0+2} + \ldots + (\Delta - j_0)n_\Delta + 1.
\]

# 2 Proof of Theorem 2

In the sequel let \( k \) be the lower bound on \( \alpha \) of Theorem 1.

By Theorem 1, it follows

**Lemma 1** \( k \geq f(2(k - 1)) \).
For a finite family \( F \) of integers let \( \text{max}(F) \) be a maximum member of \( F \). Note that a member of a family may occur more than once. If for instance \( F = \{1, 2, 2\} \) then \( (F \setminus \{\text{max}(F)\}) \cup \{\text{max}(F) - 1\} = \{1, 1, 2\} \). The following Lemma 2, Lemma 3, and Lemma 4 are proved in [3] and [4].

**Lemma 2** Given an integer \( l \) with \( 0 \leq l \leq 2m \), the following algorithm calculates \( f(l) \):

Input: The family \( F = \{d(u) \mid u \in V\} \).

\( j := 0 \),

while \( j < l \) do begin \( F := (F \setminus \{\text{max}(F)\}) \cup \{\text{max}(F) - 1\} \); \( j := j + 1 \) end

Output: \( f(l) = \sum_{m \in F} \frac{1}{m + 1} \).

Lemma 3 is a consequence of Lemma 2.

**Lemma 3** Given an integer \( 0 \leq l \leq 2m \),

(i) there are unique integers \( j \) and \( x \) with \( j \in \{1, \ldots, \Delta\} \) and \( x \in \{0, \ldots, n_{\Delta} + \ldots + n_j - 1\} \) such that

\[ l = n_{\Delta} + (n_{\Delta} + n_{\Delta - 1}) + \ldots + (n_{\Delta} + n_{\Delta - 1} + \ldots + n_{j+1}) + x \]

and

\[ f(l) = (n_{\Delta} + \ldots + n_j - x) \frac{1}{j+1} + \frac{x}{j} + \frac{n_{j-1}}{j} + \ldots + \frac{n_1}{2} \]

(ii) \( f(l) = (n_{\Delta} + \ldots + n_j - x) \frac{1}{j+1} + \frac{x}{j} + \frac{n_{j-1}}{j} + \ldots + \frac{n_1}{2} \)

By Lemma 3, it follows

**Lemma 4** If \( l = x + n_{j+1} + 2n_{j+2} + \ldots + (\Delta - j)n_{\Delta} \) with \( j \in \{1, \ldots, \Delta\} \) and \( x \in \{0, \ldots, n_{\Delta} + \ldots + n_j - 1\} \) then \( f(l+1) - f(l) = 0 \cdot \frac{1}{j+1} \).

Using Lemma 3, the calculation of \( f(l) \) is possible without taking a minimum and without using the algorithm above. We will now define the function \( f \) for real \( l \in [1, m] \). For given \( j \in \{1, \ldots, \Delta\} \) and a real number \( x \) with \( 0 \leq x < n_{\Delta} + \ldots + n_j \) let the real numbers \( l \) and \( f(l) \) (implicitly) be defined as

\[ l = x + n_{j+1} + 2n_{j+2} + \ldots + (\Delta - j)n_{\Delta} \] \( f(l) = (n_{\Delta} + \ldots + n_j) \frac{1}{j+1} + \frac{x}{j} + \frac{n_{j-1}}{j} + \ldots + \frac{n_1}{2} \)

We will prove Lemma 5.

**Lemma 5** The function \( g \) with \( g(l) = \frac{l}{2} + 1 - f(l) \) is continuous and strictly increasing on \([1, n]\).

**Proof.** Consider \( l \in [1, n] \). Then there are \( j \in \{1, \ldots, \Delta\} \) and \( x \) with \( 0 \leq x < n_{\Delta} + \ldots + n_j \) such that

\[ l = x + n_{j+1} + 2n_{j+2} + \ldots + (\Delta - j)n_{\Delta} \]

If \( j = 1 \) then \( n_{\Delta} + n_2 + n_3 + \ldots + (\Delta - 1)n_{\Delta} = 2m - n \), a contradiction to \( n \leq m \). Hence, \( j \geq 2 \), and \( l \) belongs to the interval

\[ I(j) = [n_{j+1} + 2n_{j+2} + \ldots + (\Delta - j)n_{\Delta}, n_j + 2n_{j+1} + \ldots + (\Delta - j + 1)n_{\Delta}] \]

By Lemma 3, \( g(l + \epsilon) - g(l) = \epsilon(\frac{1}{2} - \frac{1}{j+1}) \) and, consequently, \( g(l) \) is continuous and, because \( j \geq 2 \), strictly increasing on \( I(j) \).

Note that \( I(j) \cap I(j') = \emptyset \) if \( j \neq j' \) and that \( I(1) \cup \ldots \cup I(\Delta) = [1, 2m - n] \supseteq [1, n] \).

It is easy to see that \( g \) is also continuous in \( l = n_{j+1} + 2n_{j+2} + \ldots + (\Delta - j)n_{\Delta} \) for \( j \in \{2, \ldots, \Delta - 1\} \).

Since the classical bound due to Caro and Wei is tight only for complete graphs, it follows

\[ g(0) = 1 - \sum_{u \in V} \frac{1}{d(u) + 1} < 0 \]

and, by Lemma 1, \( g(2(k-1)) \geq 0 \). Using Lemma 5, there is a unique zero \( l_0 = x(j_0) + n_{j_0+1} + 2n_{j_0+2} + \ldots + (\Delta - j_0)n_{\Delta} \) of \( g \) with \( 1 < l_0 \leq 2(k-1) \leq 2(\alpha - 1) < n \) and \( 0 \leq x(j_0) < n_{\Delta} + \ldots + n_{j_0} \). It follows Lemma 6.

**Lemma 6** \( \alpha \geq k \geq \frac{\alpha}{2} + 1 \), where \( l_0 \in (0, n] \) is the unique solution of \( \frac{l}{2} + 1 = f(l) \).
Considering the equation \( \frac{\nu}{2} + 1 = f(l_0) \), i.e.,
\[
2 + x(j_0) + n_{j_0+1} + 2n_{j_0+2} + \ldots + (\Delta - j_0)n_{\Delta} = 2((\Delta + \ldots + n_{j_0})\frac{1}{j_0+1} + \frac{x(j_0)}{j_0(j_0+1)} + \frac{n_{j_0-1}}{j_0} + \ldots + \frac{n_0}{2})
\]
it follows
\[
x_0 = \frac{j_0(j_0+1)}{j_0(j_0+1)-2} \left( \frac{2}{j_0+1} - (\Delta - j_0) \right)n_\Delta + \ldots + \left( \frac{2}{j_0+1} - 1 \right)n_{j_0+1} + \frac{2n_{j_0}}{j_0+1} + \ldots + \frac{2n_1}{2} - 2 \right).
\]
We obtain Lemma 7.

**Lemma 7** If \( j \in \{2, \ldots, \Delta\} \) and \( l = x + n_{j+1} + 2n_{j+2} + \ldots + (\Delta - j)n_\Delta \) with
\[
0 \leq x < n_\Delta + \ldots + n_j, \text{ then } \frac{l}{2} = f(l) \text{ if and only if }
\]
\[
x = \frac{j(j+1)}{J(J+1)-2} \left( \frac{2}{J+1} - (\Delta - j) \right)n_\Delta + \ldots + \left( \frac{2}{J+1} - 1 \right)n_{j+1} + \frac{2n_{j-1}}{J+1} + \ldots + \frac{2n_1}{2} - 2 \right).
\]

Now we complete the proof of Theorem 2. By Lemma 4 and Lemma 6,
\[
\alpha \geq k \geq f(l_0) = f(0) + (f(1) - f(0)) + \ldots + \left( f(\lfloor l_0 \rfloor) - f(\lfloor l_0 \rfloor - 1) \right) + (f(l_0) - f(\lfloor l_0 \rfloor))
\]
\[
= \left( \sum_{u \in V} \frac{1}{d(u)+1} + \frac{n_\Delta}{\Delta(J+1)} + \frac{n_\Delta+1}{(\Delta+1)J+1} + \ldots + \frac{n_{\Delta+n_\Delta-1}}{(J+2)(J+1)} + \frac{x(j_0)}{j_0(j_0+1)} \right)
\]
because
\[
l_0 = x(j_0) + n_{j_0+1} + 2n_{j_0+2} + \ldots + (\Delta - j_0)n_\Delta
\]
\[
= n_\Delta + (n_\Delta + n_\Delta - 1) + \ldots + (n_\Delta + n_\Delta - 1 + \ldots + n_{j_0+1}) + x(j_0).
\]

With \( f(l_0) = \frac{\nu}{2} + 1 = \frac{x(j_0)+n_{j_0+1}+2n_{j_0+2}+\ldots+(\Delta-j_0)n_\Delta}{2} + 1 \) Theorem 2 is proved.

\[\square\]

### 3 Remarks

The following Remark 1 is proved in the introduction.

**Remark 1** If \( \sum_{u \in V} \frac{1}{d(u)+1} \geq 2 \) then \( \frac{\nu}{2} > k_0 \).

Remark 2 compares the lower bound \( \frac{\nu}{2} + 1 \) on \( \alpha \) in Theorem 2 to the lower bound \( k_0 + 1 \) on \( \alpha \) in the main result in [4].

**Remark 2** If \( \sum_{u \in V} \frac{1}{d(u)+1} \geq 2 \) and
\[
k_0 = n_\Delta + (n_\Delta + n_\Delta - 1) + \ldots + (n_\Delta + n_\Delta - 1 + \ldots + n_{j+1}) + x
\]
with \( 0 \leq x < n_\Delta + \ldots + n_j \) then \( \frac{\nu}{2} - k_0 \geq \frac{k_0}{J(J+1)} \).

**Proof.** Remark 1 implies \( \frac{\nu}{2} - k_0 = f(l_0) - f(k_0) > f(2k_0) - f(k_0) \).

According to Lemma 2, the family \( F \) contains the member 1, the member 2 \ldots, and the member \( \Delta \) exactly \( n_1 \) times, \( n_2 \) times, \ldots, \( n_\Delta \) times, respectively.

Therefore, let the output \( f(l) \) of the algorithm in Lemma 2 be denoted by \( f_{n_1,\ldots,n_\Delta}(l) \).

With this notation, for example \( f_{n_1,\ldots,n_\Delta}(1) = f_{n_1,\ldots,n_\Delta-1+1,n_\Delta-1}(0) \).

Using Lemma 3 (ii), it follows \( f_{n_1,\ldots,n_\Delta}(k_0) = f_{n_1,\ldots,n_{j_2},n_{j_2+1},n_{\Delta+1}+\ldots+n_{j-1}+x}(0) \) and
\[
f_{n_1,\ldots,n_\Delta}(2k_0) = f_{n_1,\ldots,n_{j_2},n_{j_2+1},n_{\Delta+1}+\ldots+n_{j-1}+x}(k_0).
\]
Consequently,
\[
f_{n_1,\ldots,n_\Delta}(2k_0) - f_{n_1,\ldots,n_\Delta}(k_0) = f_{n_1,\ldots,n_{j_2},n_{j_2+1},n_{\Delta+1}+\ldots+n_{j-1}+x}(k_0) - f_{n_1,\ldots,n_{j_2},n_{j_2+1},n_{\Delta+1}+\ldots+n_{j-1}+x}(0)
\]
\[
= \left( f_{n_1,\ldots,n_{j_2},n_{j_2+1},n_{\Delta+1}+\ldots+n_{j-1}+x}(k_0) - f_{n_1,\ldots,n_{j_2},n_{j_2+1},n_{\Delta+1}+\ldots+n_{j-1}+x}(0) \right) \left( k_0 - 1 \right)
\]
\[
+ \left( f_{n_1,\ldots,n_{j_2},n_{j_2+1},n_{\Delta+1}+\ldots+n_{j-1}+x}(k_0 - 2) \right) \left( k_0 - 1 \right)
\]
\[
\ldots + \left( f_{n_1,\ldots,n_{j_2},n_{j_2+1},n_{\Delta+1}+\ldots+n_{j-1}+x}(0) \right) \left( k_0 - 1 \right)
\]
\[
= \left( f_{n_1,\ldots,n_{j_2},n_{j_2+1},n_{\Delta+1}+\ldots+n_{j-1}+x}(0) \right) \left( k_0 - 1 \right) \left( k_0 \right).
\]

Note that the expressions \( f_{n_1,\ldots,n_{j_2},n_{j_2+1},n_{\Delta+1}+\ldots+n_{j-1}+x}(s) - f_{n_1,\ldots,n_{j_2},n_{j_2+1},n_{\Delta+1}+\ldots+n_{j-1}+x}(s-1) \) equal fractions of type \( \frac{1}{(s+1)} \) (see Lemma 3 and Lemma 4) with \( s \leq j \) for \( s = 1, \ldots, k_0 \).

Thus, \( f_{n_1,\ldots,n_\Delta}(2k_0) - f_{n_1,\ldots,n_\Delta}(k_0) \geq \frac{k_0}{J(J+1)} \).
For integers \( r \geq 2 \) and \( s \geq 2 \), consider the graph \( G_{r,s} \) obtained from \( s \) copies of the clique \( K_r \) on \( r \) vertices and adding \( s-1 \) mutually independent edges between these cliques such that \( G_{r,s} \) is connected. It follows \( \Delta = r \), \( n_j = 0 \) for \( j < r-1 \), \( n_{r-1} = sr - 2(s-1) \), \( n_r = 2(s-1) \), and \( \alpha = s \) for \( G_{r,s} \).

Using Theorem 2, we obtain
\[
x(r-1) = \frac{(r-1)r}{(r-1)r-2} \left( \frac{2}{r} - 1 \right) n_r + \frac{2n_{r-1}}{r} - 2 = \frac{(r-1)r}{(r-1)r-2} \left( \frac{2}{r} - 1 \right) 2(s-1) + \frac{2sr-4(s-1)}{r} - 2 = 0.
\]

Hence, \( j_0 = r-1 \),
\[
x(j_0)+n_{j_0+1}+2n_{j_0+2}+...+(\Delta-j_0)n_{\Delta} + 1 = \frac{n_r}{2} + 1 = s = \alpha, \text{ and Remark 3 follows.}
\]

**Remark 3** There are infinitely many graphs \( G \) such that the lower bound on \( \alpha \) of Theorem 2 is tight.

**References**


