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Small deviation probabilities of some stochastic processes

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Einleitung

Die vorliegende Arbeit behandelt das sogenannte Problem der kleinen Abweichungen für verschiedene Arten von stochastischen Prozessen. Hierbei handelt es sich um die Frage nach der Wahrscheinlichkeit, dass ein stochastischer Prozess X mit Werten in einem normierten Raum $(E, \|.\|)$ eine Norm kleiner als eine positive Zahl ε hat. D.h., wir untersuchen die Größe

$$\mathbb{P}\left(\|X\| \leq \varepsilon\right), \qquad \text{für } \varepsilon \to 0+. \tag{SD}$$

In Abschnitt 1.1 wird ein kurzer Überblick über die Bedeutung und die verschiedenen Anwendungen des Problems der kleinen Abweichungen gegeben.

Ein klassisches Beispiel ist das Verhalten der kleinen Abweichungen für eine Brownsche Bewegung $B = (B_t)$, für die gilt

$$\mathbb{P}\left(\sup_{0\leq t\leq 1} |B_t| \leq \varepsilon\right) \sim \frac{4}{\pi}\, e^{-\frac{\pi^2}{8}\,\varepsilon^{-2}}$$

und, für alle $1 \leq p < \infty$ und gewisse Konstanten $c_p, k_p > 0$,

$$\mathbb{P}\left(\int_0^1 |B_t|^p\, dt \leq \varepsilon^p\right) \sim c_p\, \varepsilon\, e^{-k_p \varepsilon^{-2}}.$$

Historische Anmerkungen zu diesen Ergebnissen finden sich in Abschnitt 7.3 in [30]. Die in dieser Arbeit benutzte Notation ist in Abschnitt 1.2 definiert.

Für die meisten anderen Beispiele von Prozessen ist es nur möglich, die Ordnung des Logarithmus' der Größe in (SD) zu bestimmen. Diese Ordnung heißt auch ‚Rate der kleinen Abweichungen' (im Falle der Brownschen Bewegung beträgt diese ε^{-2}). Wenn der Quotient aus dem Logarithmus der Größe in (SD) und der Rate der kleinen Abweichungen gegen eine Konstante konvergiert, nennt man diese die ‚Konstante der kleinen Abweichungen' (für die Brownsche Bewegung ist dies k_p). Selbst in den Fällen, in denen die Konstante existiert, ist im allgemeinen wenig über ihren exakten Wert bekannt (für die Brownsche Bewegung weiß man z.B., dass $k_\infty = \pi^2/8$ und $k_2 = 1/8$).

Es stellt sich die Aufgabe, Größen zu finden, die mit dem Prozess X und dem Raum $(E, \|.\|)$ zusammenhängen und die Rate der kleinen Abweichungen und die Konstante der kleinen Abweichungen – falls diese existiert – bestimmen helfen. Im Falle Gaußscher Prozesse ist es möglich, diese Frage auf ein völlig funktionalanalytisches Problem zurückzuführen, nämlich die Untersuchung der Entropieeigenschaften eines mit dem Prozess X verwandten Operators (siehe genauere Erklärung in Kapitel 3). Es ist ein interessantes Problem, zu untersuchen, ob ähnliche Zusammenhänge auch für andere Prozesse existieren. Hier bieten sich als erstes stabile Prozesse an.

Diese Arbeit beschäftigt sich mit relativ einfachen stochastischen Prozessen. Die Frage nach den kleinen Abweichungen ist von eigener Bedeutung und wurde in der Literatur tiefgründig studiert. Trotzdem führt uns die Lösung der Frage für diese einfachen Prozesse im Falle α-stabiler Zufallsvariablen zur Antwort auf das obige Problem. Es zeigt sich nämlich, dass im stabilen Fall ein Zusammenhang zwischen der Rate der kleinen Abweichungen und den Entropieeigenschaften des verwandten Operators nicht existieren kann.

Dieser Zusammenhang wird das Anliegen des Kapitels 3 sein. Wenden wir uns nun aber dem Hauptproblem dieser Arbeit zu, welches ebenfalls überraschende Aspekte bietet.

Sei $\theta, \theta_1, \theta_2, \ldots$ eine Folge von unabhängigen, identisch verteilten Zufallsvariablen, die nicht in einem Punkt konzentriert sind. Weiterhin sei (σ_n) eine Folge positiver Zahlen, die fallend ist und gegen Null konvergiert. Wir betrachten Folgen der Form $(\sigma_n\theta_n) = (\sigma_1\theta_1, \sigma_2\theta_2, \ldots)$ in l_p, also in Bezug auf die (Quasi-)Norm

$$\|(\sigma_n\theta_n)\|_p := \begin{cases} \sup_{n\geq 1} |\sigma_n\theta_n| & p = \infty, \\ \left(\sum_{n\geq 1} |\sigma_n\theta_n|^p\right)^{1/p} & 0 < p < \infty. \end{cases}$$

Die Hauptfrage dieser Arbeit ist die nach den kleinen Abweichungen des Vektors $(\sigma_n\theta_n) \in l_p$, also dem Verhalten des Ausdrucks

$$\log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \leq \varepsilon\right), \qquad \text{für } \varepsilon \to 0+. \tag{P}$$

Dieses Problem war ebenfalls das Thema des Artikels [3], welcher Teile der Ergebnisse dieser Arbeit enthält.

Es stellt sich sofort die Frage, wann das Problem (P) überhaupt sinnvoll ist, d.h. es sollte untersucht werden, in welchen Fällen

$$\mathbb{P}\left(\|(\sigma_n\theta_n)\|_p < \infty\right) = 1 \tag{0.1}$$

gilt. Man könnte erwarten, dass dies stark vom Verhalten der Verteilung der Zufallsvariable θ abhängt. Dies ist jedoch nicht der Fall. Den Haupteinfluss hat die Ordnung der Folge (σ_n), sowie ihr Verhältnis zu den *großen* Abweichungen der Verteilung von θ. Dies ist vielleicht nicht selbstverständlich, da wir eigentlich ein Problem der *kleinen* Abweichungen (P) untersuchen. In Abschnitt 1.4 werden allgemeine notwendige und hinreichende Bedingungen für (0.1) diskutiert.

Bevor wir jedoch dazu kommen, erinnern wir an bekannte Techniken, die das Verhalten der kleinen Abweichungen mit dem der Laplace-Transformierten verbinden. Diese Ergebnisse werden gemeinhin als Taubersätze bezeichnet. Einige Resultate in dieser Richtung werden in Abschnitt 1.3 diskutiert. Der Wechsel von der Betrachtung der kleinen Abweichungen zur Laplace-Transformierten ist die Hauptmethode in den Beweisen der Ergebnisse dieser Arbeit für $p < \infty$. Dies ist nicht überraschend und wurde bereits zuvor beim Studium des Problems (P) angewandt, da es die Benutzung der Unabhängigkeit erlaubt.

Kapitel 2 beschäftigt sich mit der Lösung des Problems (P) in den Fällen, in denen es sinnvoll gestellt ist. Zunächst werden in Abschnitt 2.1 die Ergebnisse aus der Literatur diskutiert. Später (Abschnitt 2.5.3) können wir diese mit den neuen Resultaten dieser Arbeit vergleichen. Abschnitt 2.2 enthält dagegen die Hauptidee der Herangehensweise in dieser Arbeit. Wir beweisen hier allgemeine Resultate für die Fälle $p = \infty$ bzw. $p < \infty$. Dies führt zu allgemeinen oberen Schranken in den Fällen, in denen die Folge (σ_n) eine weitere Einschränkung erfüllt. Dies ist in Abschnitt 2.3 ausgeführt.

Für die untere Schranke (Abschnitt 2.4) müssen wir uns auf das Beispiel von im Wesentlichen polynomiellen Verhalten beschränken, d.h. auf den Fall

$$\sigma_n \sim n^{-\mu}(1 + \log n)^{-\nu}, \qquad \mu > 0,\ \nu \in \mathbb{R}.$$

Es erscheint intuitiv klar, dass die Rate der kleinen Abweichungen und die Existenz der Konstante der kleinen Abweichungen nur vom Verhalten der Verteilung von θ bei Null bzw. bei Unendlich abhängen kann. Das Hauptresultat dieser Arbeit, Theorem 2.9, drückt genau diese Tatsache aus. Zum einen stellen wir eine (nicht sehr einschränkende) Bedingung an die Verteiltung von θ bei Null, zum anderen benötigen wir Bedingungen an das Verhalten der großen Abweichungen von θ.

Die Resultate aus der vorhandenen Literatur haben einen anderen Blickwinkel. Im allgemeinen werden hier sehr starke Bedingungen an kleine und große Abweichungen von θ gestellt. Dafür sind die Resultate auch sehr präzise. So ist es möglich, nicht nur das Problem (P) zu lösen, sondern sogar die Ordnung von (SD) zu bestimmen. Die vorliegende Arbeit hat ein anderes Ziel: Wir lösen das einfachere Problem (P) unter weniger starken Einschränkungen. Insbesondere genügt es, die Ordnung der Folge (σ_n) in starker Asymptotik zu kennen (siehe Abschnitt 1.5).

Das Hauptresultat aus Abschnitt 2.5.2 kann wie folgt zusammengefasst werden: Falls $(\sigma_n\theta_n) \in l_p$ mit $\sigma_n \sim n^{-\mu}(1 + \log n)^{-\nu}$, dann ist die Rate der kleinen Abweichungen gegeben durch

$$\lim_{\varepsilon \to 0+} \varepsilon^{\frac{1}{\mu - 1/p}} (-\log \varepsilon)^{\frac{\nu}{\mu - 1/p}} \log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \leq \varepsilon \right) = -C_p,$$

mit der explizit berechenbaren Konstanten C_p, die genau dann endlich ist, wenn $\mathbb{E}|\theta|^{1/\mu} < \infty$ und $\mu > 1/p$.

In den anderen Fällen, für die $\mathbb{P}\left(\|(\sigma_n\theta_n)\|_p < \infty\right) = 1$ gilt, ist die Rate der kleinen Abweichungen anders. Wir bestimmen diese Rate in einigen dieser Grenzfälle und schätzen sie in den anderen ab.

In Abschnitt 2.6 werden die Resultate auf konkrete Beispiele für Zufallsvariablen angewandt. Insbesondere zeigen wir, dass sie mit den bereits vorhandenen Resultaten im Gaußfall übereinstimmen. Die Resultate für stabile bzw. allgemeine Gamma-verteilte Zufallsvariablen scheinen neu zu sein.

Abgesehen von diesen Beispielen, zeigen wir in Abschnitt 2.6.4 eine Anwendung unserer Ergebnisse für zufällige Fourierreihen auf. Ist (f_n) eine Orthonormalbasis des Hilbertraumes $L_2[0,1]$, können wir den Prozess

$$X_t = \sum_{n=1}^{\infty} \sigma_n\theta_n f_n(t), \qquad t \in [0,1],$$

betrachten, mit (σ_n) und (θ_n) wie oben beschrieben. Natürlich gilt

$$\|X\|_{L_2[0,1]} = \|(\sigma_n\theta_n)\|_2 .$$

Insbesondere falls (f_n) ein System trigonometrischer Funktionen ist, nennt man X zufällige Fourierreihe. Das Studium der kleinen Abweichungen zufälliger Fourierreihen in $L_2[0,1]$ ist also bereits im Problem (P) enthalten.

Am Anfang der Einleitung wurde der Zusammenhang zwischen Entropie und kleinen Abweichungen thematisiert, welches der Hauptbestandteil des Kapitels 3 ist. Dort wird gezeigt, dass es einen allgemeinen Zusammenhang im Falle stabiler Prozesse nicht geben kann. Trotzdem ist es für bestimmte Beispiele möglich, Entropieresultate zur Bestimmung von oberen Schranken für kleine Abweichungen zu nutzen. Dies wird am Beispiel Stabiler Faltungen im Abschnitt 3.3 demonstriert. Hierbei handelt es sich um Prozesse der Form

$$X_t = \int_0^t f(t-s)\, dZ_s, \qquad t \in [0,1],$$

wobei f eine glatte Funktion und Z ein symmetrischer α-stabiler Lévyprozess ist. Dieses Thema ist durch gemeinsame Arbeit mit Dr. Thomas Simon entstanden und Teil der Untersuchungen in [5].

Schlussendlich werden in Kapitel 4 einige offene Probleme aufgelistet.

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Introduction

This thesis discusses the so-called small deviation problem for certain types of stochastic processes. The small deviation problem represents the question for the probability that a stochastic process X with values in some normed space $(E, \|.\|)$ has norm less than a positive number ε, i.e. the quantity

$$\mathbb{P}\left(\|X\| \leq \varepsilon\right), \qquad \text{as } \varepsilon \to 0+, \tag{SD}$$

is investigated. We give a brief overview on the significance of the small deviation problem – also called small ball or lower tail probability problem – in Section 1.1.

A classical example is the small deviation behaviour of a Brownian motion $B = (B_t)$, where

$$\mathbb{P}\left(\sup_{0\leq t\leq 1} |B_t| \leq \varepsilon\right) \sim \frac{4}{\pi}\, e^{-\frac{\pi^2}{8}\,\varepsilon^{-2}}$$

and, for $1 \leq p < \infty$ and some constants $c_p, k_p > 0$,

$$\mathbb{P}\left(\int_0^1 |B_t|^p\, dt \leq \varepsilon^p\right) \sim c_p\, \varepsilon\, e^{-k_p \varepsilon^{-2}}.$$

We refer to Section 7.3 in [30] for historical references for this result. For an explanation of further notation see Section 1.2.

For most other examples of processes it is only possible to determine the rate of the logarithm of the expression in (SD), which is called 'small deviation function' or 'small ball function'. Its rate is called 'small deviation rate' (for the Brownian motion ε^{-2}). If the quotient of the small deviation function and the small deviation rate tends to a constant, the latter is called 'small deviation constant' (for the Brownian motion this is k_p). Even if the small deviation constant exists, usually little is known about its exact numerical value (for the Brownian motion $k_\infty = \pi^2/8$ and $k_2 = 1/8$).

The question arises to find quantities related to the process X and the space $(E, \|.\|)$ that help to determine the rate of the small deviation function and – if it exists – the small deviation constant. In the case of Gaussian processes it is possible to relate the question for the rate to a purely functional analytic problem, namely, the investigation of the metric entropy of an operator related to the process X (cf. the detailed explanation in Chapter 3). It seems challenging to ask whether similar relations also hold for other classes of processes, e.g. stable processes.

This thesis is concerned with a particularly simple form of a process. The question for the small deviation probability of these processes is of importance itself and has been studied thoroughly in literature. However, solving it in the case of α-stable processes shows that the small deviation rate of a process and the metric entropy of the operator related to the process do not coincide in general in the non-Gaussian case.

We come back to this question in detail in Chapter 3. For the time being, let us outline this thesis' main problem, which has some other surprising features.

Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables that are not concentrated at a single point. Furthermore, let (σ_n) be a sequence of positive numbers that is strictly decreasing and tends to zero. We consider sequences of the form $(\sigma_n\theta_n) = (\sigma_1\theta_1, \sigma_2\theta_2, \ldots)$ in l_p, i.e. w.r.t. the (quasi-)norm

$$\|(\sigma_n\theta_n)\|_p := \begin{cases} \sup_{n\geq 1} |\sigma_n\theta_n| & p = \infty, \\ \left(\sum_{n\geq 1} |\sigma_n\theta_n|^p\right)^{1/p} & 0 < p < \infty. \end{cases}$$

The question that is addressed in this thesis is the small deviation probability of the vector $(\sigma_n\theta_n) \in l_p$, i.e. the behaviour of the quantity

$$\log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \leq \varepsilon\right), \qquad \text{as } \varepsilon \to 0+. \tag{P}$$

This problem is the subject of investigation in the study [3], which contains parts of the results of this thesis.

Immediately, the question arises when problem (P) is well-posed, i.e. it should be the first goal to determine those cases when

$$\mathbb{P}\left(\|(\sigma_n\theta_n)\|_p < \infty\right) = 1. \tag{0.2}$$

One might expect that this depends heavily on the distribution of the random variable θ. However, this is not true. In fact, the main influence comes from the rate of decrease of the sequence (σ_n) and its relation to the *upper* tail behaviour of the distribution of θ. This may come as a surprise, since we are studying a *lower* tail problem (P). General necessary and sufficient conditions for (0.2) are given in Section 1.4.

However, before coming to that, we have to collect some results that connect the small deviation probability to the behaviour of the respective Laplace transform. These results are usually called Tauberian theorems, and we discuss some results of further interest in Section 1.3. Passing over from small deviation quantities to Laplace transforms is the main method in our proofs (of both necessary and sufficient conditions) when $p < \infty$. This is not surprising and was used very often in the study of the problem (P), since it allows us to use the independence property.

Chapter 2 is devoted to the solution of the problem (P) for those cases when the problem makes sense. We start by reviewing the results that are known for this problem from literature in Section 2.1. Later on (Section 2.5.3), we can compare them to the new results from the current approach. In Section 2.2, the main idea is presented. We prove general results in the cases $p = \infty$ and $p < \infty$. These results lead to general upper bounds in the case that an additional restriction is imposed on the sequence (σ_n), which is investigated in Section 2.3.

For the lower bound, we concentrate on the example of polynomical behaviour with logarithmic correction, i.e. the example

$$\sigma_n \sim n^{-\mu}(1 + \log n)^{-\nu},$$

for some $\mu > 0$ and $\nu \in \mathbb{R}$. This case is investigated in Section 2.4.

It seems intuitively clear that the rate of the small deviation probabilities and the existence of the small deviation constant only depend on the behaviour of the distribution of θ at the origin and the tail behaviour. The main result of this thesis, Theorem 2.9, expresses exactly this fact. On the one hand, we impose a (very mild) condition on the behaviour of the distribution of θ at the origin (we assume that the distribution of θ satisfies condition (O) with $r > \mu$, cf. Definition 1.1 below). On the other hand, we have to make an assumption for the tail behaviour of θ.

The results that are available in the respective literature have a different focus. There, usually strong conditions for both lower and upper tail behaviour of the distribution of θ are imposed. In turn, the results are very precise. In many cases, it is possible to solve not only problem (P) but, in fact, to estimate the order of (SD) itself. The focus of this thesis is rather different. We solve the simpler problem (P) under much milder assumptions. In particular, we only require to know the sequence (σ_n) up to strong asymptotics, cf. Section 1.5.

The main result can be outlined as follows: If $(\sigma_n \theta_n) \in l_p$ with $\sigma_n \sim n^{-\mu}(1 + \log n)^{-\nu}$, then the small deviation probability satisfies:

$$\lim_{\varepsilon \to 0+} \varepsilon^{\frac{1}{\mu - 1/p}} (-\log \varepsilon)^{\frac{\nu}{\mu - 1/p}} \log \mathbb{P}\left(\|(\sigma_n \theta_n)\|_p \le \varepsilon \right) = -C_p,$$

with the explicitly computable small deviation constant C_p, which is finite if and only if $\mathbb{E}|\theta|^{1/\mu} < \infty$ and $\mu > 1/p$. This result is proved in Section 2.5.2.

In the other cases when $\mathbb{P}\left(\|(\sigma_n \theta_n)\|_p < \infty \right) = 1$, the rate of the small deviation probability is different. We determine the rate function in some of those borderline cases and estimate it for the others.

The results are applied to some concrete cases of random variables in Section 2.6. In particular, we show that the results coincide with those previously obtained in the Gaussian case. The results seem to be new for stable random variables and e.g. Gamma-distributed random variables.

Furthermore, we give an application of the results to random Fourier series in Section 2.6.4. If (f_n) is a orthonormal basis of the Hilbert space $L_2[0,1]$, we can consider the process

$$X_t = \sum_{n=1}^{\infty} \sigma_n \theta_n f_n(t), \qquad t \in [0,1],$$

where (σ_n) and (θ_n) are arbitrary sequences as above. Clearly,

$$\|X\|_{L_2[0,1]} = \|(\sigma_n \theta_n)\|_2 .$$

In particular, if (f_n) is a system of trigonometric functions, one calls X a random Fourier series. Thus, studying the small deviation problem for random Fourier series in $L_2[0,1]$ is fully covered by problem (P).

At the beginning of the introduction, the connection between metric entropy and small deviations was discussed. This is the main topic of Chapter 3. It will be shown that a general relation does not exist in the case of stable processes. Still, it is possible to use entropy results in order to obtain upper bounds for the small deviation rate in some examples. This is demonstrated for so-called stable convolutions in Section 3.3. These are processes of the form

$$X_t = \int_0^t f(t-s)\, dZ_s, \qquad t \in [0,1],$$

where f is a smooth function and Z is a symmetric α-stable Lévy process. In this example, it is possible to prove the correct upper bound for the small deviation rate with the help of metric entropy techniques, which is impossible for our main problem (P). This investigation of stable convolutions originates in joint work with Dr. Thomas Simon and is the topic of the paper [5].

Finally, in Chapter 4, some open problems are listed.

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Chapter 1

Preliminaries

1.1 The small deviation problem

Let us review the task, the applications, and the connections to other fields of the so-called small deviation problem. For a stochastic process X with values in some normed space $(E, \|.\|)$ we study the quantity

$$\mathbb{P}\left(\|X\| \leq \varepsilon\right), \qquad \text{as } \varepsilon \to 0+.$$

In particular, one is interested in processes indexed by a subset of the real line, e.g. $(X_t)_{t\geq 0}$. In this case, problems of small deviations are considered in various function spaces, e.g.

$$\mathbb{P}\left(\sup_{0\leq t\leq 1} |X_t| \leq \varepsilon\right) \tag{1.1}$$

or

$$\mathbb{P}\left(\int_0^1 |X_t|^p \, d\rho(t) \leq \varepsilon^p\right), \tag{1.2}$$

for $\varepsilon \to 0+$, where ρ is some measure. Analogous problems are studied equally in higher dimensions, i.e. processes indexed by subsets of $\mathbb{R}^d$ or processes with values in $\mathbb{R}^d$.

A detailed study of small deviation probability problems in the framework of *Gaussian* processes can be found in the survey articles [29] and [27]. In the latter article, various applications of small deviation problems (such as laws of iterated logarithm of Chung type, Strassen laws, and many others) are outlined, most of which are also important for non-Gaussian processes.

Another survey of results can be found in [31]. Here, some important examples of stable processes are studied under a wide range of norms.

In the Gaussian case, the problem of small deviations is equivalent to the question of the entropy of certain embedding operators in functional analysis, cf. [18] and [25]. Similar results can be obtained in the non-Gaussian stable case, cf. [26]. However, the results of this thesis show that the connection between entropy and small deviations in the stable case cannot be as tight as in the Gaussian one. This point is explained in detail in Chapter 3.

Furthermore, let us mention that in the Gaussian case the problem (1.2) for $p = 2$, is fully equivalent to problem (P) which comprises the main problem of this thesis. We refer to [14] and [35] for recent results in the L_2 case.

However, even in other cases (non-Gaussian, p different to 2) the investigation of the problem with countable parameter set – i.e. our problem (P) – can *help* to understand problems with infinite dimensional parameter set like (1.1) or (1.2). This idea was successfully applied, for example, in [39].

Apart from the applications mentioned in [27], a wide field for the application of small deviation results is represented by the area of so-called high resolution quantization and randomly centred balls. For a recent survey we refer to [10] and [11].

Finally, let us mention that small deviation results also give an idea about the regularity of the process X. If the quantity in (1.1) is large (i.e. tends to zero slowly) then the process should be rather regular. And, on the other hand, fast convergence indicates that the process is not likely to stay close to its past values, which indicates many perturbations and thus little smoothness. The idea that small deviation probabilities mimic the local smoothness properties of the process has been used in many cases, cf. e.g. Section 6 in [31].

Our particular problem (P), however, can not only help to understand the more complicated cases, but is of importance itself. It is a main tool to understand rates of escape of certain processes, cf. [13] or [9]. In general, provided the process X satisfies certain scaling properties, small deviation problems as (1.1) can be reformulated as problems for exit times.

Another application of our results is introduced in Section 2.6.4. We study random Fourier series in $L_2[0, 1]$. Here, the norm of the stochastic process can be calculated as a sum of weighted i.d.d. random variables. The present results provide a tool to study in particular stable random Fourier series.

1.2 Notation and elementary facts

Let us fix the notation used in this thesis. Throughout, we denote by $\theta, \theta_1, \theta_2, \ldots$ a sequence of i.i.d. random variables, that are not concentrated at a single point. Furthermore, (σ_n) denotes a sequence of positive numbers that is strictly decreasing and tends to zero.

We consider (deterministic or random) sequences (a_n) in l_p, i.e. w.r.t. the norm

$$\|(a_n)\|_p := \begin{cases} \sup_{n\geq 1} |a_n| & p = \infty, \\ \left(\sum_{n\geq 1} |a_n|^p\right)^{1/p} & 0 < p < \infty. \end{cases}$$

Sometimes it is convenient to consider the truncated sequence $(a_n)_{n=1}^N := (a_1, \ldots, a_N, 0, 0, \ldots)$, where the norm of which is defined as above. For the sake of readibility, we also sometimes drop the subscripts of sequences, suprema, or sums.

The main idea for obtaining the sufficient conditions and the rate of the small deviation problem is to interpolate the sequence (σ_n) by a smooth function S. Since (σ_n) is strictly decreasing and tends to zero, we can find a function $S : [1, \infty[\to]0, \sigma_1]$ which interpolates the σ_n, i.e. $S(n) = \sigma_n$, for all $n = 1, 2, \ldots$, is twice continuously differentiable and strictly decreasing. From the fact that (σ_n) tends to zero and the continuity of S it follows that $S(x) \to 0$, as $x \to \infty$. We shall denote by S^{-1} the inverse of S, which is defined on $]0, \sigma_1]$. However, even though it has no concrete meaning, it is sometimes convenient to consider S^{-1} on $]0, \infty[$. This is possible, since the function can be continuously extended by setting $S^{-1}(x) := 1$, for all $x > \sigma_1$.

Throughout we use the following notation for weak and strong asymptotics. For two functions f and g, $f(x) \sim g(x)$, as $x \to \infty$, means that $f(x)/g(x) \to 1$, as $x \to \infty$. On the other hand, we use the notation $f(x) \preccurlyeq g(x)$, as $x \to \infty$, if there is a $0 < C < \infty$ such that $f(x) \leq Cg(x)$, for all large enough x. We also write $g(x) \succcurlyeq f(x)$ in this case. Furthermore, we write $f(x) \approx g(x)$, as $x \to \infty$, if $f(x) \preccurlyeq g(x)$ and $g(x) \preccurlyeq f(x)$. The notation is defined analogously when $x \to 0$ and for sequences.

As usual, for given $q \geq 1$, we denote by q' the number such that $1/q + 1/q' = 1$. Note that this definition also makes sense for $q = 1$ and $q = \infty$, via the convenient notation $1/0 = \infty$ and $1/\infty = 0$. Let us also mention that we use $\mathbb{I}_A$ for the indicator function of the set A.

At some point, we make an assumption for the distribution of θ in the neighbourhood of the origin. Roughly speaking, we assume that the distribution of θ does have some mass near the origin. More precisely, we use the following notation.

Definition 1.1. *We say that the distribution of the random variable θ satisfies condition (O) with $r > 0$ if there exists a constant $C_1 > 0$ such that*

$$\mathbb{P}(|\theta| \leq t) \geq e^{-C_1 t^{-1/r}}, \qquad \textit{for all } 0 < t \leq 1. \tag{O}$$

In connection with this notation, we require the following elementary fact.

Lemma 1.1. *If the distribution of θ satisfies condition (O) with $r > 0$, so does the distribution of $\left\|(\sigma_n\theta_n)_{n=1}^N\right\|_p$. I.e., for all $0 < p \leq \infty$ and for all decreasing sequences (σ_n),*

$$\mathbb{P}\left(\left\|(\sigma_n\theta_n)_{n=1}^N\right\|_p \leq t\right) \geq e^{-C_N t^{-1/r}}, \qquad \textit{for all } 0 < t \leq 1,$$

with some constant $C_N > 0$ depending on N and σ_1 only.

Proof: We have, using the independence of the (θ_n) and the inequality $\|.\|_{l_p^N} \leq N^{1/p} \|.\|_{l_\infty^N}$, for $N \geq 1$,

$$\begin{aligned} \mathbb{P}\left(\left\|(\sigma_n\theta_n)_{n=1}^N\right\|_p \leq \varepsilon\right) &\geq \mathbb{P}\left(N^{1/p}\left\|(\sigma_n\theta_n)_{n=1}^N\right\|_\infty \leq \varepsilon\right) \\ &= \prod_{n=1}^N \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{N^{1/p}\sigma_n}\right) \geq \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{N^{1/p}\sigma_1}\right)^N \end{aligned}$$

Thus,

$$\log \mathbb{P}\left(\left\|(\sigma_n\theta_n)_{n=1}^N\right\|_p \leq \varepsilon\right) \geq N(-C_1)\left(\frac{\varepsilon}{N^{1/p}\sigma_1}\right)^{-\frac{1}{r}} = -C_1\sigma_1^{\frac{1}{r}} N^{1+\frac{1}{rp}} \varepsilon^{-\frac{1}{r}}.$$

■

Furthermore, we use the following simple fact frequently in our considerations.

Lemma 1.2. *Let $z_0 > 0$. Then there is a constant $C = C(z_0) > 0$ such that*

$$C(z-1) \leq \log z \leq z - 1, \qquad \textit{for all } z_0 \leq z \leq 1.$$

Finally, let us recall the notation of regularly and slowly varying functions. We refer to [6] for a detailed study of properties of these functions. A function $L :]0,1[\to]0,\infty[$ is called slowly varying at zero if

$$\frac{L(xt)}{L(t)} \to 1, \qquad \text{as } t \to 0, \text{ for all } x > 0.$$

A function T is called regularly varying at zero with exponent $\gamma \in \mathbb{R}$ if T can be written as $T(t) = t^\gamma L(t)$, with some slowly varying function L. We say that L is slowly varying at infinity if $L(1/.)$ is slowly varying at zero; analogously the notation is used for regular variation.

1.3 Tauberian theorems

A main tool in the forthcoming considerations is the use of Tauberian theorems for Laplace transforms. In this section, we collect some results in this direction that we need in the following.

If V is a non-negative random variable, one can consider its Laplace transform:

$$\mathbb{E}e^{-\lambda V}, \qquad \lambda > 0.$$

It is evident that the small deviation behaviour of V is connected to the behaviour of its Laplace transform, as $\lambda \to \infty$. Similarly, the tail behaviour of V is encoded by the behaviour of the Laplace transform, as $\lambda \to 0$.

The question is not only important for small deviation and Laplace transform itself but as well for the logarithm of which.

We shall quantify the behaviour by comparing it to a polynomial-logarithmic rate function. This can be done by strong ($\sim$) or weak ($\approx$) asymptotics.

The main source for the quoted results is [6], where the question was studied for regularly varying functions.

Let us start with the most commonly used Tauberian theorem in the field of small deviations – the so-called de Bruijn Tauberian Theorem, a sketch of the proof of which can be found in Section 4.12, §5 in [6] (Theorem 4.12.9). The formulation from Theorem 3.5 in [27] is more convenient; however, it contains a small mistake. In [27], also some applications of the theorem to small deviation problems are given. From Lemma 1.5 below, it can be seen that the theorem is also valid for the case $K = \infty$.

Lemma 1.3. *Let $0 < \gamma < 1$, $\delta \in \mathbb{R}$, and $K \in]0, \infty]$. Then*

$$\log \mathbb{E}e^{-\lambda V} \sim -K\lambda^{\gamma}(\log \lambda)^{\delta}, \qquad \textit{as } \lambda \to \infty,$$

if and only if

$$\log \mathbb{P}\left(V \le \varepsilon\right) \sim -\left[K\gamma^{\gamma}(1-\gamma)^{1-\gamma-\delta}\varepsilon^{-\gamma}(-\log \varepsilon)^{\delta}\right]^{\frac{1}{1-\gamma}}, \qquad \textit{as } \varepsilon \to 0+.$$

We need this precise form in order to determine the small deviation constant in our small ball problem.

The respective result for the Laplace transform itself, as $\lambda \to 0+$, looks as follows. Here, we quote Corollary 8.1.7 and the considerations at the beginning of Section 8.1, §3 in [6].

Lemma 1.4. *Let L be a slowly varying function.*

- *For $0 \le \gamma < 1$, the following are equivalent:*

$$\begin{aligned} 1 - \mathbb{E}e^{-\lambda V} &\sim \lambda^\gamma L(1/\lambda), \quad \text{as } \lambda \to 0+. \\ \mathbb{P}\left(V > t\right) &\sim \tfrac{t^{-\gamma}L(t)}{\Gamma(1-\gamma)}, \qquad \text{as } t \to \infty. \end{aligned}$$

- *Let $F(t) := \mathbb{P}\left(V \le t\right)$. We have*

$$\begin{cases} \int_{[0,x]} t\, dF(t) \sim L(x) & \text{as } x \to \infty, \\ \int_0^x 1 - F(t)\, dt \sim L(x) & \text{as } x \to \infty, \end{cases}$$

  if and only if

$$1 - \mathbb{E}e^{-\lambda V} \sim \lambda L(1/\lambda), \qquad \text{as } \lambda \to 0+.$$

- *If $\mathbb{E}V < \infty$, we have*

$$1 - \mathbb{E}e^{-\lambda V} \sim \lambda \mathbb{E}V, \qquad \text{as } \lambda \to 0+.$$

A typical example for the above conditions are stable random variables. It is well-known (cf. Chapter 1.2 in [40]) that if θ is an α-stable random variable then $V = |\theta|$ has the tail behaviour as given in the first case of the above lemma when $0 < \alpha < 1$ with $\gamma = \alpha$ and L being constant. The conditions in the second case of the above lemma are satisfied for $\alpha = 1$ and $L(t) = C \log t$. Finally, for $\alpha > 1$, α-stable random variables possess a finite first moment, which corresponds to the third case of the above lemma.

Now, we come to a fact for the logarithmic small deviation / Laplace transform behaviour ($\lambda \to \infty$) for *weak* asymptotics. Unfortunately, this result does not seem to be in literature in the form we require it. Therefore, we shall give its proof. Note that it does not follow immediately from Lemma 1.3.

Lemma 1.5. *Let $0 < \gamma < 1$ and $\delta \in \mathbb{R}$. Then*

$$\log \mathbb{E}e^{-\lambda V} \approx -\lambda^\gamma (\log \lambda)^\delta, \qquad \text{as } \lambda \to \infty,$$

if and only if

$$\log \mathbb{P}\left(V \le \varepsilon\right) \approx -\left[\varepsilon^{-\gamma}(-\log \varepsilon)^\delta\right]^{\frac{1}{1-\gamma}}, \qquad \text{as } \varepsilon \to 0+.$$

Proof: Let us start with the *crucial inequalities.* On the one hand, for all $\varepsilon > 0$ and $\lambda > 0$, by Chebyshev's Inequality,

$$\log \mathbb{P}(V \le \varepsilon) = \log \mathbb{P}\left(e^{-\lambda V} \ge e^{-\lambda\varepsilon}\right) \le \lambda\varepsilon + \log \mathbb{E}e^{-\lambda V}. \tag{1.3}$$

On the other hand, for all $\varepsilon > 0$ and $\lambda > 0$,

$$\mathbb{E}e^{-\lambda V} = \int_0^1 \mathbb{P}\left(e^{-\lambda V} \ge t\right)\, dt \le \int_0^{e^{-\lambda\varepsilon}} 1\, dt + \int_{e^{-\lambda\varepsilon}}^1 \mathbb{P}\left(V \le -\frac{1}{\lambda}\log t\right)\, dt.$$

In the second integral $t \ge e^{-\lambda\varepsilon}$ and thus $-\lambda^{-1}\log t \le \varepsilon$. Therefore,

$$\mathbb{E}e^{-\lambda V} \le e^{-\lambda\varepsilon} + (1 - e^{-\lambda\varepsilon})\mathbb{P}(V \le \varepsilon). \tag{1.4}$$

Let us show the *'only if' part.* On the one hand, let us assume that

$$\log \mathbb{E}e^{-\lambda V} \le -C\lambda^\gamma(\log\lambda)^\delta,$$

for $\lambda > \lambda_1$ and some $C > 0$. We use $\lambda := K\varepsilon^{-\frac{1}{1-\gamma}}(-\log\varepsilon)^{\frac{\delta}{1-\gamma}}$, where K will be chosen later. Then inequality (1.3) yields

$$\begin{aligned}
\log \mathbb{P}(V \le \varepsilon) &\le K\varepsilon^{-\frac{\gamma}{1-\gamma}}(-\log\varepsilon)^{\frac{\delta}{1-\gamma}} - CK^\gamma\varepsilon^{-\frac{\gamma}{1-\gamma}}(-\log\varepsilon)^{\frac{\gamma\delta}{1-\gamma}}D(-\log\varepsilon)^\delta \\
&= -\varepsilon^{-\frac{\gamma}{1-\gamma}}(-\log\varepsilon)^{\frac{\delta}{1-\gamma}}(-K + CDK^\gamma),
\end{aligned}$$

for some constant $D > 0$ and ε small enough. By the fact that $0 < \gamma < 1$, it is possible to choose K sufficiently small such that $-K + CDK^\gamma = C' > 0$. This shows that, for small enough ε,

$$\log \mathbb{P}(V \le \varepsilon) \le -C'\varepsilon^{-\frac{\gamma}{1-\gamma}}(-\log\varepsilon)^{\frac{\delta}{1-\gamma}}.$$

On the other hand, let us assume that

$$\log \mathbb{E}e^{-\lambda V} \ge -C\lambda^\gamma(\log\lambda)^\delta,$$

for $\lambda > \lambda_1$. Then (1.4) yields

$$e^{-C\lambda^\gamma(\log\lambda)^\delta} \le e^{-\lambda\varepsilon} + (1 - e^{-\lambda\varepsilon})\mathbb{P}(V \le \varepsilon).$$

We can use $\lambda := K\varepsilon^{-\frac{1}{1-\gamma}}(-\log\varepsilon)^{\frac{\delta}{1-\gamma}}$, with $K > 0$ to be chosen later. This gives us $\lambda\varepsilon = K\varepsilon^{-\frac{\gamma}{1-\gamma}}(-\log\varepsilon)^{\frac{\delta}{1-\gamma}}$. Therefore, with the last inequality, for some $D > 0$ and small enough ε,

$$\frac{e^{-CDK^\gamma\varepsilon^{-\frac{\gamma}{1-\gamma}}(-\log\varepsilon)^{\frac{\delta}{1-\gamma}}} - e^{-K\varepsilon^{-\frac{\gamma}{1-\gamma}}(-\log\varepsilon)^{\frac{\delta}{1-\gamma}}}}{1 - e^{-K\varepsilon^{-\frac{\gamma}{1-\gamma}}(-\log\varepsilon)^{\frac{\delta}{1-\gamma}}}} \le \mathbb{P}(V \le \varepsilon).$$

Note that the denominator is bounded from above. This yields, for small enough ε,

$$\frac{1}{2}\, e^{-CDK^\gamma \varepsilon^{-\frac{\gamma}{1-\gamma}}(-\log\varepsilon)^{\frac{\delta}{1-\gamma}}} \left(1 - e^{(CDK^\gamma - K)\varepsilon^{-\frac{\gamma}{1-\gamma}}(-\log\varepsilon)^{\frac{\delta}{1-\gamma}}}\right) \le \mathbb{P}\left(V \le \varepsilon\right).$$

By the fact that $0 < \gamma < 1$, there is a $K > 0$ (sufficiently large) such that $CDK^\gamma - K < 0$. This shows that, for some $C' > 0$ and sufficiently small ε,

$$-C'\varepsilon^{-\frac{\gamma}{1-\gamma}}(-\log\varepsilon)^{\frac{\delta}{1-\gamma}} \le \log \mathbb{P}\left(V \le \varepsilon\right).$$

The *'if' part* goes along the same lines. Also, we do not use the implication in the following; we therefore omit this part of the proof. ■

The above theorem excludes the case $\gamma = 1$. Let us study this case a bit closer. Note that, by Jensen's Inequality, for $\lambda \ge 1$

$$\frac{1}{\lambda}\log \mathbb{E}e^{-\lambda V} = \log(\mathbb{E}e^{-\lambda V})^{\frac{1}{\lambda}} \ge \log \mathbb{E}e^{-V} =: -C > -\infty,$$

if V is an (a.s. finite) random variable. Therefore, for all $\lambda \ge 1$, we have

$$\log \mathbb{E}e^{-\lambda V} \ge -C\lambda.$$

If, on the other hand, also $\log \mathbb{E}e^{-\lambda V} \le -C\lambda$, for large λ and some $C > 0$, then Inequality (1.3) yields that $\mathbb{P}\left(V \le \varepsilon\right) = 0$, for sufficiently small ε. This argument shows that only the case $\gamma = 1$ and $\delta < 0$ is a possible and interesting behaviour of the Laplace transform. The following Tauberian theorem gives bounds for the small deviation probability in this case. It is of lesser importance, but it helps us to clarify some borderline cases in the later considerations; even though the bounds it offers are very rough.

Lemma 1.6. *Let $\delta < 0$. Then*

$$\log \mathbb{E}e^{-\lambda V} \approx -\lambda(\log\lambda)^\delta, \qquad \text{as } \lambda \to \infty,$$

implies that

$$\log(-\log \mathbb{P}\left(V \le \varepsilon\right)) \approx \varepsilon^{1/\delta}, \qquad \text{as } \varepsilon \to 0+.$$

The proof is similar to the one of Lemma 1.5.

1.4 The necessary and sufficient conditions for boundedness

In this section, we determine necessary and sufficient conditions for the sequence $(\sigma_n\theta_n)$ to be a.s. in l_p. Certainly, this is a necessary condition for a solution of our problem (P). Furthermore, we ask for those cases when almost surely $\lim_{n\to\infty}\sigma_n\theta_n = 0$. This is a necessary condition as well. Note that the latter is implied by the first for $p<\infty$. However, for $p=\infty$, it is exactly the other way round, which can cause a difference for the small deviation problem. We shall come back to this presently.

The case $p = \infty$. The following result clarifies the above questions.

Theorem 1.1. *Let (σ_n), (θ_n), and S be as mentioned in Section 1.2. Then*

$$\sup_n |\sigma_n\theta_n| < \infty \qquad a.s. \tag{1.5}$$

if and only if

$$\mathbb{E}S^{-1}\left(\frac{c_0}{|\theta|}\right) < \infty, \tag{1.6}$$

for some $c_0 > 0$. On the other hand,

$$\lim_{n\to\infty}\sigma_n\theta_n = 0 \qquad a.s. \tag{1.7}$$

if and only (1.6) holds for all $c_0 > 0$.

Proof: Clearly,

$$\left\{\sup_n |\sigma_n\theta_n| < \infty\right\} = \bigcup_{c>0}\{|\sigma_n\theta_n| < c \text{ eventually}\}.$$

Thus, by Kolmogorov's Zero-One-Law, the sequence $(\sigma_n\theta_n)$ is almost surely bounded if and only if

$$0 = \mathbb{P}\left(\bigcap_{c>0}\{|\sigma_n\theta_n| > c \text{ infinitely often}\}\right) = \lim_{c\to\infty}\mathbb{P}\left(|\sigma_n\theta_n| > c \text{ i.o.}\right).$$

However, by the Borel and Cantelli Lemma this is true if and only if there exists a $c_0 > 0$ such that $\sum_n \mathbb{P}(|\sigma_n\theta_n| \geq c_0) < \infty$. The latter holds if and only if, for some $c_0 > 0$,

$$\sum_n \mathbb{P}\left(\frac{c_0}{|\theta|} \leq \sigma_n\right) < \infty.$$

By the definition of S^{-1}, the latter is true if and only if

$$\sum_n \mathbb{P}\left(S^{-1}\left(\frac{c_0}{|\theta|}\right) \geq n\right) < \infty,$$

for some $c_0 > 0$. It is elementary to check that this is valid if and only if $\mathbb{E}S^{-1}(c_0/|\theta|) < \infty$.

The proof of the second part uses exactly the same method; and we therefore omit it. ■

Remark on so-called empty balls. Certainly, it is necessary that condition (1.5) is satisfied in order to have a solution of problem (P). However, it might be that

$$\mathbb{P}\left(\|(\sigma_n\theta_n)\|_\infty < \infty\right) = 1, \qquad \text{but} \qquad \mathbb{P}\left(\|(\sigma_n\theta_n)\|_\infty \leq \varepsilon\right) = 0,$$

for small enough $\varepsilon > 0$, which happens exactly if (1.7) fails to hold, since, in any case, $\overline{\lim}_{n\to\infty} \sigma_n\theta_n$ is constant a.s.

This phenomenon is called "empty balls". It occurs, for example, in the case of Gaussian random variables for certain cases of sequences, cf. [23, pp. 60]. It corresponds to the borderline case of the above theorem: It is well-known that, for a standard Gaussian random variable θ, we have

$$\mathbb{E}e^{(\theta/c_0)^2} < \infty \qquad \text{if and only if} \qquad c_0 > \sqrt{2}.$$

If we take $\sigma_n = (1 + \log n)^{-1/2}$, the latter fact, by Theorem 1.1, implies that $\sup_n |\sigma_n\theta_n| < \infty$ almost surely, but (1.7) does not hold. Indeed, we have

$$\mathbb{P}\left(\sup_n |\sigma_n\theta_n| < \varepsilon\right) = 0, \qquad \text{for } \varepsilon < \sqrt{2},$$

as can be seen from Formula (3.7) in [23].

However, in the case of α-stable (non-Gaussian) random variables – which are of main interest in Chapter 3 – this effect cannot occur. This can be seen as follows.

Let $c > 0$. Then (1.6) holds if and only if $\int_0^\infty \mathbb{P}\left(S^{-1}\left(\frac{c}{|\theta|}\right) > t\right) dt < \infty$. This is true if and only if $\int_1^\infty \mathbb{P}\left(|\theta| > \frac{c}{S(t)}\right) dt < \infty$. Using the well-known tail behaviour of α-stable (non-Gaussian) random variables (cf. Chapter 1.2 in [40]), one can see that this is true if and only if $\int_1^\infty \left(\frac{c}{S(t)}\right)^{-\alpha} dt < \infty$, which clearly (recall that $\sigma_n = S(n)$ and that S is decreasing) holds if and only if $\sum_n \sigma_n^\alpha < \infty$. Since this condition does not depend on c, we can conclude that (1.5) holds if and only if (1.7) does. Even though we stated this for stable non-Gaussian random variables, in fact, only the tail behaviour was used, not the stability property.

The case $p < \infty$. In the case $p < \infty$, the aforementioned problem cannot occur, since every summable sequence is automatically decreasing.

We have to distinguish three subcases, which correspond to the subcases of Lemma 1.4. In particular, we have to assume that the tail behaviour of the distribution is regularly varying in certain cases. This is clearly a restriction. However, since it is satisfied for all distributions that are usually considered it does not seem so pressing to search for a different proof. In particular, the assumptions are satisfied for certain α-stable distributions, as mentioned above in the discussion following Lemma 1.4.

A different method to prove the following theorem is offered by Kolmogorov's Three-Series-Theorem. The result is as follows.

Theorem 1.2. *Let (σ_n) and (θ_n) be as above. Let L be a slowly varying function.*

- *If $\mathbb{P}(|\theta|^p > t) \sim t^{-\alpha/p} L(t)$, as $t \to \infty$, for some $0 < \alpha < p$, we have*
$$\sum_n |\sigma_n \theta_n|^p < \infty \text{ a.s.} \qquad \text{if and only if} \qquad \sum_n \sigma_n^\alpha L(\sigma_n^{-p}) < \infty.$$
- *Let $F(t) := \mathbb{P}(|\theta|^p > t)$. If*
$$\begin{cases} \int_{[0,x]} t\, dF(t) \sim L(x) & \text{as } x \to \infty, \\ \int_0^x 1 - F(t)\, dt \sim L(x) & \text{as } x \to \infty, \end{cases}$$
we have
$$\sum_n |\sigma_n \theta_n|^p < \infty \text{ a.s.} \qquad \text{if and only if} \qquad \sum_n \sigma_n^p L(\sigma_n^{-p}) < \infty.$$
- *If $\mathbb{E}|\theta|^p < \infty$, we have*
$$\sum_n |\sigma_n \theta_n|^p < \infty \text{ a.s.} \qquad \text{if and only if} \qquad \sum_n \sigma_n^p < \infty.$$

Proof: Clearly, $\sum_n |\sigma_n \theta_n|^p < \infty$ a.s. if and only if
$$-\infty < \log \mathbb{E} e^{-\sum_n |\sigma_n \theta_n|^p} = \sum_n \log \mathbb{E} e^{-\sigma_n^p |\theta|^p}.$$

By Lemma 1.2 and the fact that $\mathbb{E} e^{-\sigma_n^p |\theta|^p} \geq \mathbb{E} e^{-\sigma_1^p |\theta|^p} > 0$, the last term can be estimated from above and below with positive constants by
$$\sum_n \mathbb{E} e^{-\sigma_n^p |\theta|^p} - 1 = -\sum_n \mathbb{E}(1 - e^{-\sigma_n^p |\theta|^p}). \tag{1.8}$$

For the convergence of the latter sum it is crucial to know the behaviour of the Laplace transform of $|\theta|^p$ at the origin, since $\sigma_n^p \to 0$.

Let us assume first that $\mathbb{E}|\theta|^p < \infty$. Then, by Lemma 1.4, the last term in (1.8) is bounded from below and above with positive constants by

$$-\sum_n \mathbb{E}|\theta|^p \sigma_n^p = -\mathbb{E}|\theta|^p \sum_n \sigma_n^p,$$

which already shows the assertion in this case.

Assume now that $\mathbb{P}(|\theta|^p > t) \sim t^{-\alpha/p} L(t)$, for some $0 < \alpha < p$ and some slowly varying function L. Then, by Lemma 1.4, the last term in (1.8) is bounded from below and above with positive constants by $-\sum_n \sigma_n^\alpha L(\sigma_n^{-p})$, which again shows the assertion in this case.

The proof of the second case is identical; we have to use the second part of Lemma 1.4. ■

Notice that the method used in the above proof is not sufficient to determine the rate of $\log \mathbb{E} e^{-\lambda \sum_n |\sigma_n \theta_n|^p}$, as $\lambda \to \infty$. We shall employ other methods to determine this rate, which, in turn, gives us the rate of the small deviation of $\sum_n |\sigma_n \theta_n|^p$, by Lemma 1.3.

1.5 Modification of the sequence (σ_n)

At this point, we investigate what happens if we modify the sequence (σ_n) without change in the sense of $\sim$. On the one hand, this is necessary since we later need to use the inverse of the function S interpolating the sequence (σ_n); and this inverse cannot be computed explicitly in many cases. On the other hand, it lends greater generality to our results. Namely, we shall see that only the behaviour of the sequence in the sense of $\sim$ matters if we calculate the probability on the log-level, as long as the random variables have 'some' mass close to the origin (in the sense of condition (O)). Thus, we have to know only the order of (σ_n) as n tends to infinity.

Note that we require condition (O) at this point. This condition limits the generality slightly; it could be softened by adding an extra slowly varying term to it. However, we shall not do so in order to avoid technical complications. This condition is still much softer than the usual assumption that $t \mapsto \mathbb{P}(|\theta| \le t)$ is regularly varying at the origin.

Lemma 1.7. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables, (σ_n) and $(\tilde{\sigma}_n)$ be two decreasing sequences of positive numbers that tend to zero and that satisfy $\sigma_n \sim \tilde{\sigma}_n$. Let $C \in [0, \infty]$ and $T :]0,1] \to]0,\infty[$ be a regularly* *varying function at zero with exponent $\gamma > 0$; and let us furthermore assume that the distribution of θ satisfies condition (O) with $r > 1/\gamma$.*

Then

$$\lim_{\varepsilon\to 0+} T(\varepsilon) \log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon \right) = -C$$

if and only if

$$\lim_{\varepsilon\to 0+} T(\varepsilon) \log \mathbb{P}\left(\|(\tilde{\sigma}_n\theta_n)\|_p \le \varepsilon \right) = -C.$$

Proof: Note that T can be written as $T(x) = x^\gamma L(x)$, with a slowly varying function L. Let us consider the case $p = \infty$ first. Let $0 < \delta < 1$. Then there is an n_0 such that, for all $n > n_0$, $1 - \delta \le \sigma_n/\tilde{\sigma}_n \le 1 + \delta$. This implies that

$$\begin{aligned} \|(\sigma_n\theta_n)\|_\infty &= \max\left(\sup_{1\le n\le n_0} |\sigma_n\theta_n|, \sup_{n>n_0} |\sigma_n\theta_n| \right) \\ &\le \max\left(\sup_{1\le n\le n_0} |\sigma_n\theta_n|, (1+\delta) \sup_{n>n_0} |\tilde{\sigma}_n\theta_n| \right). \end{aligned}$$

Thus, by the independence of the (θ_n),

$$\begin{aligned} \mathbb{P}\left(\|(\sigma_n\theta_n)\|_\infty \le \varepsilon\right) &\ge \mathbb{P}\left(\|(\sigma_n\theta_n)_1^{n_0}\|_\infty \le \varepsilon\right) \mathbb{P}\left(\left\|(\tilde{\sigma}_n\theta_n)_{n_0+1}^\infty\right\|_\infty \le \frac{\varepsilon}{1+\delta} \right) \\ &\ge \mathbb{P}\left(\|(\sigma_n\theta_n)_{n=1}^{n_0}\|_\infty \le \varepsilon\right) \mathbb{P}\left(\|(\tilde{\sigma}_n\theta_n)\|_\infty \le \frac{\varepsilon}{1+\delta} \right). \end{aligned}$$

Taking the logarithm of both sides, applying Lemma 1.1 to the first term, and multiplying by $\varepsilon^\gamma L(\varepsilon)$, we get

$$\begin{aligned} &\varepsilon^\gamma L(\varepsilon) \log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_\infty \le \varepsilon\right) \\ &\qquad \ge -\varepsilon^\gamma L(\varepsilon) C_{n_0} \varepsilon^{-1/r} + \varepsilon^\gamma L(\varepsilon) \log \mathbb{P}\left(\|(\tilde{\sigma}_n\theta_n)\|_\infty \le \varepsilon/(1+\delta)\right). \end{aligned}$$

Letting ε tend to zero gives us, since by assumption $\gamma > 1/r$,

$$\begin{aligned} -C &\ge \overline{\lim_{\varepsilon\to 0+}} (1+\delta)^\gamma \varepsilon^\gamma L((1+\delta)\varepsilon) \log \mathbb{P}\left(\|(\tilde{\sigma}_n\theta_n)\|_\infty \le \varepsilon\right) \\ &= (1+\delta)^\gamma \overline{\lim_{\varepsilon\to 0+}} \varepsilon^\gamma L(\varepsilon) \log \mathbb{P}\left(\|(\tilde{\sigma}_n\theta_n)\|_\infty \le \varepsilon\right). \end{aligned}$$

Letting δ tend to zero gives one side of the assertion. Exchanging the role of (σ_n) and $(\tilde{\sigma}_n)$ gives the other side.

The case $p < \infty$ contains exactly the same argument. We can even assume w.l.o.g. that $p = 1$. Then we can use

$$\begin{aligned} \|(\sigma_n\theta_n)\|_1 &\le \|(\sigma_n\theta_n)_{n=1}^{n_0}\|_1 + \left\|(\sigma_n\theta_n)_{n=n_0+1}^\infty\right\|_1 \\ &\le \|(\sigma_n\theta_n)_{n=1}^{n_0}\|_1 + (1+\delta)\left\|(\tilde{\sigma}_n\theta_n)_{n=n_0+1}^\infty\right\|_1. \end{aligned}$$

As above, by the independence, this implies, for all $0 < \rho < 1$,

$$\begin{aligned}
& \mathbb{P}\left(\|(\sigma_n\theta_n)\|_1 \le \varepsilon\right) \\
\ge\; & \mathbb{P}\left(\|(\sigma_n\theta_n)_{n=1}^{n_0}\|_1 \le \rho\varepsilon\right) \mathbb{P}\left(\left\|(\tilde{\sigma}_n\theta_n)_{n=n_0+1}^{\infty}\right\|_1 \le \frac{(1-\rho)\varepsilon}{1+\delta}\right) \\
\ge\; & \mathbb{P}\left(\|(\sigma_n\theta_n)_{n=1}^{n_0}\|_1 \le \rho\varepsilon\right) \mathbb{P}\left(\|(\tilde{\sigma}_n\theta_n)_{n=1}^{\infty}\|_1 \le \frac{(1-\rho)\varepsilon}{1+\delta}\right).
\end{aligned}$$

We reason as above and thus obtain

$$\begin{aligned}
-C \;\ge\; & \overline{\lim_{\varepsilon\to 0+}} \left(\frac{1+\delta}{1-\rho}\,\varepsilon\right)^{\gamma} L\left(\frac{1+\delta}{1-\rho}\,\varepsilon\right) \log \mathbb{P}\left(\|(\tilde{\sigma}_n\theta_n)\|_1 \le \varepsilon\right) \\
=\; & \left(\frac{1+\delta}{1-\rho}\right)^{\gamma} \overline{\lim_{\varepsilon\to 0+}} \varepsilon^{\gamma} L(\varepsilon) \log \mathbb{P}\left(\|(\tilde{\sigma}_n\theta_n)\|_1 \le \varepsilon\right).
\end{aligned}$$

Letting δ and ρ tend to zero gives one side of the assertion; exchanging the role of (σ_n) and $(\tilde{\sigma}_n)$ the other. ■

Note that the assertion of the last lemma is an easy consequence of the main result of [15] under the stronger assumptions of that work. However, since we would like to keep the assumptions to a minimum (i.e. we only make the significantly milder assumption (O)), Lemma 1.7 offers a different proof.

Chapter 2

Determination of the rate

2.1 Known results

Question (P) has been studied by many authors. In particular, for $0 < p < \infty$, the problem of sums of independent random variables was under thorough investigation in [28], [12], and [36]. These works are based on the treatment in [28]. However, e.g. [15] is written in the same spirit. The results are more precise than those in the present thesis. However, the approach has the great disadvantage that only random variables possessing variance are considered. This, however, is not a natural assumption. As shown by the necessary and sufficient conditions for boundedness in Section 1.4, the moment condition (or tail condition) for θ depends on the speed of decrease of the sequence (σ_n). This is a rather natural relation, contrary to a general moment assumption.

Let us examine the results in [12]; we quote Proposition 4.1 in [12], which exactly treats the case $\sigma_n = n^{-\mu}$ and $p = 1$ that corresponds to the main result of the present treatment, Theorem 2.9.

Proposition 2.1. *Let $\mu > 1$ and let $\theta, \theta_1, \theta_2, \ldots$ be i.i.d. random variables with finite second moment. Assume furthermore that $\mathbb{P}_\theta$ satisfies condition (I) from [12] and that $\mathbb{P}(|\theta| \le t) \sim t^\rho L(t)$, as $t \to 0$, with a slowly varying function L and some $\rho > 0$. Then*

$$\mathbb{P}\left(\sum_{n=1}^{\infty} n^{-\mu}|\theta_n| \le \varepsilon\right) \sim \sqrt{\frac{a\mu^{a(1-\rho)}}{(2\pi)^{1-\rho\mu}\Gamma(1+\rho)K^{a(1-\rho)}}\,\frac{\varepsilon^{\frac{1-\rho\mu}{\mu-1}}}{L(\varepsilon^a)}}\,\exp\left(-\frac{(\mu-1)K^a}{\mu^a}\,\varepsilon^{-\frac{1}{\mu-1}}\right),$$

where $a = \mu/(\mu-1)$ and $K = -\int_0^\infty t^{-1/\mu}\frac{d}{dt}\left[\log \mathbb{E}e^{-t|\theta|}\right]\,dt$.

Let us re-write the small deviation constant K.

Lemma 2.1. *Under the assumption of Proposition 2.1,*

$$K = -\int_0^\infty \log \mathbb{E} e^{-\lambda^{-\mu}|\theta|} \, d\lambda.$$

Proof: Integration by parts yields

$$K = \left[-t^{-1/\mu} \log \mathbb{E} e^{-t|\theta|}\right]_{t=0}^\infty - \int_0^\infty \left(-\frac{1}{\mu}\right) t^{-\frac{1}{\mu}-1} \log \mathbb{E} e^{-t|\theta|} \, dt.$$

On the one hand, by Lemma 1.4 and Lemma 1.2, $\log \mathbb{E} e^{-t|\theta|} \approx t\mathbb{E}|\theta|$, as $t \to 0$. Thus, we have $-t^{-1/\mu} \log \mathbb{E} e^{-t|\theta|} \to 0$, as $t \to 0$, since $\mu > 1$. On the other hand, Inequality (1.3) on page 15 with $\varepsilon := 1/\lambda := 1/t$ and the assumption for the lower tail of $|\theta|$ give

$$0 \geq \log \mathbb{E} e^{-t|\theta|} \geq 1 + \log \left(C t^{-\rho} L(1/t) \right),$$

which shows that $-t^{-1/\mu} \log \mathbb{E} e^{-t|\theta|} \to 0$, as $t \to \infty$. ∎

The last lemma makes it easy to compare the results obtained by the means presented in this thesis. Theorem 2.9 below shows that the small deviation rate and the small deviation constant given in the above proposition are also the correct ones under significantly milder assumptions.

In particular, we do not assume that θ has finite second moment. Instead, we distinguish the cases suggested by the necessary and sufficient conditions in Section 1.4. Furthermore, we replace the assumption that $\mathbb{P}(|\theta| \leq t)$ is regularly varying, as $t \to 0$, by the significantly milder condition (O). Also, we drop condition (I) from [12] totally.

Additionally, we obtain yet greater generality in Theorem 2.9 since we only require the order (in the sense of $\sim$) of the sequence (σ_n) not the particular form. This question was also studied in [15], where a (much better) solution is given under much stronger conditions.

Furthermore, the present considerations bring to light many similarities between the small deviation of sums of independent random variables and the somewhat simpler case of the supremum.

Of course, it has to be stated clearly that the present treatment studies the *logarithmic* small deviation problem, whereas [28] and the works based on it determine the rate of the small deviation probability itself. This is a much more difficult problem.

Certainly, the questions have been studied in detail for particular examples of sequences. In the case of Gaussian random variables, Proposition 2.1

applies and leads to very precise bounds. However, the small deviation problem was investigated systematically much earlier by other methods, cf. [24], [9, Theorem 4.1], and [16], where the latter also consider the case $p = \infty$. We come back to the Gaussian case in Section 2.6.1, where we compare the results of the present approach to those previously obtained for Gaussian random variables.

2.2 Main lemma

From now on, we investigate problem (P). We start with the main idea that this treatment is based on. First, let us consider the case $p = \infty$.

Theorem 2.1. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables, (σ_n) a sequence of positive numbers that is strictly decreasing to zero, and S a function as constructed in Section 1.2. Then, for all $\varepsilon > 0$,*

$$\log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{\sigma_1}\right) - \int_0^{\frac{\sigma_1}{\varepsilon}} \log \mathbb{P}\left(|\theta| \leq \frac{1}{y}\right) \frac{d}{dy}\left[S^{-1}(\varepsilon y)\right] dy$$
$$\leq \log \mathbb{P}\left(\sup_n |\sigma_n \theta_n| \leq \varepsilon\right) \leq - \int_0^{\frac{\sigma_1}{\varepsilon}} \log \mathbb{P}\left(|\theta| \leq \frac{1}{y}\right) \frac{d}{dy}\left[S^{-1}(\varepsilon y)\right] dy.$$

Proof: Because of the independence of the (θ_n), we can write

$$\log \mathbb{P}\left(\sup_n |\sigma_n \theta_n| \leq \varepsilon\right) = \log \prod_{n=1}^{\infty} \mathbb{P}\left(|\sigma_n \theta_n| \leq \varepsilon\right) = \sum_{n=1}^{\infty} \log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{\sigma_n}\right).$$

Let us define the function

$$F_\varepsilon(x) := \log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{S(x)}\right), \qquad x \geq 1.$$

In this notation, we have just seen that the term we are interested in satisfies

$$\log \mathbb{P}\left(\sup_n |\sigma_n \theta_n| \leq \varepsilon\right) = \sum_{n=1}^{\infty} F_\varepsilon(n).$$

Using the properties of S, we note that the function F_ε is non-positive, increasing, and tends to zero as $x \to \infty$. A simple comparison of sum and integral shows that, for all $\varepsilon > 0$,

$$F_\varepsilon(1) + \int_1^\infty F_\varepsilon(x)\, dx \leq \sum_{n=1}^{\infty} F_\varepsilon(n) \leq \int_1^\infty F_\varepsilon(x)\, dx.$$

We transform the integral setting $S(x) = \varepsilon y$ in order to separate the distribution of θ from ε. This exactly leads to the asserted inequalities. ■

This is a rather general result. For a given sequence (σ_n) one can construct an appropriate function S with the properties mentioned above and calculate both – the integral and the remaining probability term on the left-hand side. Since ε does not appear in connection with the distribution in the integrand, one can – at least asymptotically – separate ε from the integral. This is demonstrated in the case that S can be chosen to be a regularly varying function at infinity in Section 2.3.

Now let us come to the case $p < \infty$. Let us consider a smooth function S that interpolates the sequence (σ_n), as above. Again, the decisive idea is to express the desired term as a sum, replace that sum by an integral, and use an integral transformation that separates the distribution of θ from ε. However, in the case $p < \infty$, a different approach is needed.

Recall that we are interested in the behaviour of the quantity

$$\log \mathbb{P}\left(\sum_{n=1}^{\infty} |\sigma_n \theta_n|^p \leq \varepsilon^p\right).$$

In order to study this, we switch to the logarithmic Laplace transform of the random variable $\sum_{n=1}^{\infty} |\sigma_n \theta_n|^p$, as suggested by Lemma 1.3.

The result is as follows.

Theorem 2.2. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables, (σ_n) a sequence of positive numbers that is strictly decreasing to zero, and S a function as constructed in Section 1.2. Then, for all $\lambda > 0$,*

$$\log \mathbb{E} e^{-\lambda|\sigma_1\theta|^p} - \int_0^{\sigma_1 \lambda^{1/p}} \log \mathbb{E} e^{-|y\theta|^p} \frac{d}{dy}\left[S^{-1}(y\lambda^{-1/p})\right]\, dy$$
$$\leq \log \mathbb{E} e^{-\lambda \sum_n |\sigma_n\theta_n|^p} \leq -\int_0^{\sigma_1 \lambda^{1/p}} \log \mathbb{E} e^{-|y\theta|^p} \frac{d}{dy}\left[S^{-1}(y\lambda^{-1/p})\right]\, dy.$$

Proof: Let us consider

$$\log \mathbb{E} e^{-\lambda \sum_n |\sigma_n\theta_n|^p} = \log \prod_{n=1}^{\infty} \mathbb{E} e^{-\lambda|\sigma_n\theta_n|^p} = \sum_{n=1}^{\infty} \log \mathbb{E} e^{-\lambda|\sigma_n\theta|^p}.$$

For fixed $\lambda > 0$, we define the function

$$G_\lambda(x) := \log \mathbb{E} e^{-\lambda|S(x)\theta|^p}, \qquad x \geq 1.$$

Using this definition, the Laplace transform in question satisfies

$$\log \mathbb{E} e^{-\lambda \sum_n |\sigma_n\theta_n|^p} = \sum_{n=1}^{\infty} G_\lambda(n).$$

Note that, using the properties of S, we see that the function G_λ is non-positive, increasing, and tends to zero, as x tends to infinity. A comparison between integral and sum shows that, for all $\lambda > 0$,

$$G_\lambda(1) + \int_1^\infty G_\lambda(x)\,dx \leq \sum_{n=1}^\infty G_\lambda(n) \leq \int_1^\infty G_\lambda(x)\,dx.$$

Finally, we substitute $\lambda S(x)^p = y^p$, which gives us the assertion. ■

Again this is a very general result, which enables us to obtain bounds for $\log \mathbb{E} e^{-\lambda \sum_n |\sigma_n \theta_n|^p}$ for a given sequence (σ_n). By virtue of the above mentioned Tauberian theorem, this implies bounds for the small deviation probability of $\sum_n |\sigma_n \theta_n|^p$ and so finally of $(\sum_n |\sigma_n \theta_n|^p)^{1/p}$. This is demonstrated in the next section.

Note that Theorem 2.1 and Theorem 2.2 are identical if one replaces $\mathbb{E} e^{-|.|^p}$ by $\mathbb{P}(|.| \leq 1)$ and $\lambda^{-1/p}$ by ε.

2.3 Upper bound for regularly varying functions

In this section, we investigate the question of the lower tail probabilities in the case that the function S interpolating the sequence (σ_n) can be chosen to be a regularly varying function (for the notation see Section 1.2).

We shall see that the behaviour of the integrals in Theorems 2.1 and 2.2 can be quantified in a very precise way if certain assumptions for S are made.

For the remaining part of Section 2.3, let us assume that S can be chosen such that it is a regularly varying function with exponent $-\gamma < 0$ at infinity and its derivative S' is increasing (i.e. $S'' > 0$) for large enough arguments.

It follows from the theory of regularly varying functions that the inverse $S^{-1} :]0, \sigma_1] \to \mathbb{R}_{>0}$ is regularly varying at zero with exponent $-1/\gamma$ (cf. Theorem 1.5.12 in [6]). Furthermore, note that S^{-1} is strictly decreasing.

Let us start with the case $p = \infty$.

Theorem 2.3. *Let $\theta, \theta_1, \theta_2, \dots$ be a sequence of i.i.d. random variables, (σ_n) a sequence of positive numbers that is decreasing to zero, and S a function satisfying the above assumptions. Then*

$$\overline{\lim_{\varepsilon \to 0+}} \frac{\log \mathbb{P}(\sup_n |\sigma_n \theta_n| \leq \varepsilon)}{S^{-1}(\varepsilon)} \leq \int_0^\infty \log \mathbb{P}(|\theta| \leq z^\gamma)\,dz.$$

Proof: Theorem 2.1 suggests to investigate the quantity

$$I_\infty(\varepsilon) := \int_0^{\frac{\sigma_1}{\varepsilon}} \log \mathbb{P}\left(|\theta| \leq \frac{1}{y}\right) \frac{d}{dy}\left[S^{-1}(\varepsilon y)\right] dy, \qquad \text{as } \varepsilon \to 0+. \tag{2.1}$$

Note that I_∞ is positive, since S^{-1} is decreasing. Let us consider

$$\lim_{\varepsilon\to 0+} \frac{I_\infty(\varepsilon)}{S^{-1}(\varepsilon)} = \lim_{\varepsilon\to 0+} \int_0^{\frac{\sigma_1}{\varepsilon}} \log \mathbb{P}\left(|\theta| \leq \frac{1}{y}\right) \frac{d}{dy}\left[\frac{S^{-1}(\varepsilon y)}{S^{-1}(\varepsilon)}\right] dy. \tag{2.2}$$

As $\varepsilon \to 0+$, we have

$$\frac{d}{dy}\left[\frac{S^{-1}(\varepsilon y)}{S^{-1}(\varepsilon)}\right] = \frac{(S^{-1})'(\varepsilon y)\varepsilon}{S^{-1}(\varepsilon)} = \frac{1}{y}\frac{(S^{-1})'(\varepsilon y)(\varepsilon y)}{S^{-1}(\varepsilon y)}\,\frac{S^{-1}(\varepsilon y)}{S^{-1}(\varepsilon)} \to \frac{1}{y}\left(-\frac{1}{\gamma}\right) y^{-\frac{1}{\gamma}}, \tag{2.3}$$

by the fact that S^{-1} is regularly varying and a result on regularly varying functions (cf. problem 13 on page 59 in [6] or reference therein), where we need the assumption on S' for large enough arguments.

Therefore, by Fatou's Lemma,

$$\begin{aligned}\underline{\lim}_{\varepsilon\to 0+} \frac{I_\infty(\varepsilon)}{S^{-1}(\varepsilon)} &\geq \int_0^\infty \log \mathbb{P}\left(|\theta| \leq \frac{1}{y}\right) \underline{\lim}_{\varepsilon\to 0+} \frac{d}{dy}\left[\frac{S^{-1}(\varepsilon y)}{S^{-1}(\varepsilon)}\right] dy \\ &= \int_0^\infty \log \mathbb{P}\left(|\theta| \leq \frac{1}{y}\right) \frac{1}{y}\left(-\frac{1}{\gamma}\right) y^{-\frac{1}{\gamma}}\, dy = -\int_0^\infty \log \mathbb{P}\left(|\theta| \leq z^\gamma\right) dz,\end{aligned}$$

where we used (2.3) in the second step. Using Theorem 2.1 gives the assertion. ∎

Note that we have not imposed *any* restriction on the distribution of θ yet. This is represented by the finiteness of the expression on the right-hand side in the theorem. We investigate conditions for the finiteness of this integral in Section 2.5.1.

We saw that under the assumptions for S mentioned above, it can be proved that I_∞ increases at least as S^{-1}, as $\varepsilon \to 0+$. Unfortunately, it does not seem to be possible to prove the reverse inequality under the same assumptions. It is not clear that the limit in (2.2) exists in general. Nevertheless, the examples in Section 2.4 make it seem likely that this is true – possibly under additional assumptions. Cf. the remark on p. 69.

The case of the sum of independent random variables is entirely similar.

Theorem 2.4. *Let $\theta, \theta_1, \theta_2, \dots$ be a sequence of i.i.d. random variables, (σ_n) a sequence of positive numbers that is decreasing to zero, and S a function satisfying the above assumptions. Then*

$$\overline{\lim}_{\lambda\to\infty} \frac{\log \mathbb{E}\exp\left(-\lambda \sum_n |\sigma_n \theta_n|^p\right)}{S^{-1}(\lambda^{-1/p})} \leq \int_0^\infty \log \mathbb{E} e^{-|z^{-\gamma}\theta|^p}\, dz.$$

Proof: Theorem 2.2 suggests the investigation of the quantity

$$I_p(\lambda) := \int_0^{\sigma_1 \lambda^{\frac{1}{p}}} \log \mathbb{E} e^{-|y\theta|^p} \frac{d}{dy} \left[S^{-1}(\lambda^{-\frac{1}{p}} y) \right] dy, \qquad \text{as } \lambda \to \infty. \tag{2.4}$$

Note the correspondence to the case $p = \infty$, for $\varepsilon = \lambda^{-1/p}$. Thus, as in (2.3), we see that

$$\frac{d}{dy} \left[\frac{S^{-1}(\lambda^{-\frac{1}{p}} y)}{S^{-1}(\lambda^{-\frac{1}{p}})} \right] \to \frac{1}{y} \left(-\frac{1}{\gamma} \right) y^{-\frac{1}{\gamma}} = -\frac{1}{\gamma} y^{-\frac{1}{\gamma}-1},$$

as $\lambda \to \infty$. Therefore, by Fatou's Lemma,

$$\begin{aligned} \varliminf_{\lambda \to \infty} \frac{I_p(\lambda)}{S^{-1}(\lambda^{-\frac{1}{p}})} &\geq \int_0^\infty \log \mathbb{E} e^{-|y\theta|^p} \left(-\frac{y^{-\frac{1}{\gamma}-1}}{\gamma} \right) dy \\ &= -\int_0^\infty \log \mathbb{E} e^{-|z^{-\gamma}\theta|^p} \, dz. \end{aligned} \tag{2.5}$$

Using Theorem 2.2 finishes the proof. ■

Using a stronger Tauberian theorem than those presented in Section 1.3 (namely, Theorem 4.12.9 in [6]), the last theorem gives an upper bound for the small deviation probability of $\left(\sum_n |\sigma_n \theta_n|^p\right)^{1/p}$. However, we omit stating it here, since an explicit expression for the rate function is not possible in this general case.

2.4 Lower bound

In this section, we prove the respective lower bound of our problem (P) when

$$\sigma_n \sim n^{-\mu}(1 + \log n)^{-\nu}, \qquad \mu > 0, \nu \in \mathbb{R}.$$

First, let us consider the case $\sigma_n \sim n^{-\mu}$, with $\mu > 0$, i.e. the case $\nu = 0$. It is significantly simpler and demonstrates the method of the proof very well.

Theorem 2.5. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables and let us assume $\sigma_n \sim n^{-\mu}$, with $\mu > 0$. Then, if the distribution of θ satisfies condition (O) with $r > \mu$, we have*

$$\lim_{\varepsilon \to 0+} \varepsilon^{1/\mu} \log \mathbb{P} \left(\sup_n |\sigma_n \theta_n| \leq \varepsilon \right) = \int_0^\infty \log \mathbb{P} \left(|\theta| \leq z^\mu \right) dz. \tag{2.6}$$

Proof: By Lemma 1.7, it is sufficient to consider the case $\sigma_n = n^{-\mu}$. As an interpolating function, we can choose $S(x) = x^{-\mu}$, $x \geq 1$. If the distribution of θ satisfies condition (O) with $r > \mu$ we can estimate the remaining probability term on the left-hand side in Theorem 2.1 as follows:

$$\log \mathbb{P}\left(|\theta| \leq \varepsilon/\sigma_1\right) \geq -C_1(\varepsilon/\sigma_1)^{-1/r} = -C_1' \varepsilon^{-1/r}.$$

Thus, since $r > \mu$,

$$\varliminf_{\varepsilon \to 0+} \frac{\log \mathbb{P}\left(|\theta| \leq \varepsilon/\sigma_1\right)}{S^{-1}(\varepsilon)} \geq -C_1' \varlimsup_{\varepsilon \to 0+} \varepsilon^{1/\mu - 1/r} = 0.$$

On the other hand, for $I_\infty(\varepsilon)$ as defined in (2.1), we obtain precisely

$$I_\infty(\varepsilon) = \left(\int_0^{1/\varepsilon} \log \mathbb{P}\left(|\theta| \leq \frac{1}{y}\right) \left(-\frac{1}{\mu}\right) y^{-\frac{1}{\mu}-1}\, dy \right) \varepsilon^{-\frac{1}{\mu}}.$$

This shows the assertion, by Theorem 2.1. ■

Fully analogously, we obtain the result for $p < \infty$.

Theorem 2.6. *Let $\theta, \theta_1, \theta_2, \dots$ be a sequence of i.i.d. random variables and let us assume $\sigma_n \sim n^{-\mu}$, with $\mu > 1/p$. Then, if the distribution of θ satisfies condition (O) with $r > \mu$, we have*

$$\lim_{\varepsilon \to 0+} \varepsilon^{\frac{1}{\mu - 1/p}} \log \mathbb{P}\left(\sum_n |\sigma_n \theta_n|^p \leq \varepsilon^p \right) = -(\mu - 1/p) \left[\frac{K^\mu}{p^{1/p} \mu^\mu} \right]^{\frac{1}{\mu - 1/p}}, \quad (2.7)$$

where

$$K := -\int_0^\infty \log \mathbb{E} e^{-|z^{-\mu}\theta|^p}\, dz. \quad (2.8)$$

Proof: Again, by Lemma 1.7, it is sufficient to consider $\sigma_n = n^{-\mu}$; so we can use $S(x) = x^{-\mu}$, for $x \geq 1$. Condition (O) helps us to take care of the remaining term on the left-hand side in Theorem 2.2, since

$$\log \mathbb{E} e^{-\lambda|\sigma_1\theta|^p} \geq \log \mathbb{E} e^{-\lambda|\sigma_1\theta|^p}\, \mathbb{I}_{\{-\lambda|\sigma_1\theta|^p \geq -\sigma_1^p\}} \geq \log \left(e^{-\sigma_1^p} \mathbb{P}\left(|\theta| \leq \lambda^{-1/p}\right)\right),$$

which is greater or equal to $-\sigma_1^p - C_1 \lambda^{\frac{1}{pr}}$. Thus

$$\varliminf_{\lambda \to \infty} \frac{\log \mathbb{E} e^{-\lambda|\sigma_1\theta|^p}}{S^{-1}(\lambda^{-1/p})} = 0.$$

On the other hand, I_p as defined in (2.4) turns out to be

$$I_p(\lambda) = \lambda^{\frac{1}{\mu p}} \int_0^{\sigma_1 \lambda^{1/p}} \log \mathbb{E} e^{-|y\theta|^p} \frac{d}{dy} (y^{-1/\mu})\, y.$$

Using Theorem 2.2, this shows that the order of $\log \mathbb{E} \exp(-\lambda \sum_n |\sigma_n \theta_n|^p)$ is $S^{-1}(\lambda^{-1/p}) = \lambda^{\frac{1}{\mu p}}$. Additionally, we can deduce that the quotient of the latter quantities tends to the constant $-K$. Using Lemma 1.3, one obtains the small deviation rate and the small deviation constant. ■

If we apply Theorem 2.5 and Theorem 2.6 we have to check the finiteness of the small deviation constants in (2.6) and (2.8). This question is investigated in Section 2.5.1. However, the equality in (2.6) and (2.7) is to be understood in the sense that either both sides are finite and equal or both sides are $-\infty$.

Let us finally consider the case $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$, with $\mu > 0$ and $\nu \in \mathbb{R} \setminus \{0\}$. In the case $p = \infty$, we obtain the following result.

Theorem 2.7. *Let $\theta, \theta_1, \theta_2, \dots$ be a sequence of i.i.d. random variables and let us assume that $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$, with $\mu > 0$ and $\nu \in \mathbb{R} \setminus \{0\}$. Let us assume that the distribution of θ satisfies condition (O) with $r > \mu$. Furthermore, if $\nu < 0$ let us assume that*

$$\int_0^1 -\log \mathbb{P}\left(|\theta| \le \frac{1}{y}\right) y^{-\frac{1}{\mu}-1} (1-\log y)^{\frac{-\nu}{\mu}}\, dy < \infty. \tag{2.9}$$

Then we have

$$\lim_{\varepsilon \to 0+} \varepsilon^{\frac{1}{\mu}} (-\log \varepsilon)^{\frac{\nu}{\mu}} \log \mathbb{P}\left(\sup_n |\sigma_n \theta_n| \le \varepsilon\right) = \mu^{\frac{\nu}{\mu}} \int_0^\infty \log \mathbb{P}\left(|\theta| \le z^{\mu}\right) dz. \tag{2.10}$$

Again, the formulation of the result for $p < \infty$ is slightly more complicated. Nevertheless, the proof is essentially the same.

Theorem 2.8. *Let $\theta, \theta_1, \theta_2, \dots$ be a sequence of i.i.d. random variables and let us assume that $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$, with $\mu > 0$ and $\nu \in \mathbb{R} \setminus \{0\}$. Let us assume that the distribution of θ satisfies condition (O) with $r > \mu$. Furthermore, if $\nu < 0$ let us assume*

$$\int_0^1 -\log \mathbb{E} e^{-|y\theta|^p} y^{-\frac{1}{\mu}-1} (1-\log y)^{\frac{-\nu}{\mu}}\, dy < \infty. \tag{2.11}$$

Then we have

$$\lim_{\varepsilon\to 0+} \varepsilon^{\frac{1}{\mu-1/p}} (-\log\varepsilon)^{\frac{\nu}{\mu-1/p}} \log \mathbb{P}\left(\sum_n |\sigma_n\theta_n|^p \le \varepsilon^p\right)$$

$$= -\left[\frac{(\mu-1/p)^{\mu-1/p+\nu}}{\mu^\mu p^{1/p}} \left(-\int_0^\infty \log \mathbb{E} e^{-|z^{-\mu}\theta|^p}\, dz\right)^\mu\right]^{\frac{1}{\mu-1/p}}. \quad (2.12)$$

Proof of Theorem 2.7 and Theorem 2.8: By Lemma 1.7, it is sufficient to deal with the sequence $\tilde{\sigma}_n = n^{-\mu}(1+\log n)^{-\nu}$, with $\mu > 0$ and $\nu \in \mathbb{R}\setminus\{0\}$. The logical choice for the interpolation function is

$$\tilde{S}(y) = y^{-\mu}(1+\log y)^{-\nu}, \qquad y \ge 1.$$

However, since we cannot compute the inverse of $\tilde{S}$ explicitly, we switch to the sequence (σ_n) given by $\sigma_n := S(n)$, where S is the inverse of

$$S^{-1}(x) := \mu^{\nu/\mu} x^{-1/\mu} (1-\log x)^{-\nu/\mu}, \qquad x \le 1.$$

According to Lemma 1.7, we have to check whether $\lim_{n\to\infty} \sigma_n/\tilde{\sigma}_n \to 1$. This is true since

$$\lim_{y\to\infty} \frac{S(y)}{\tilde{S}(y)} = \lim_{x\to 0+} \frac{S(S^{-1}(x))}{\tilde{S}(S^{-1}(x))} = \lim_{x\to 0+} \frac{x}{\tilde{S}(S^{-1}(x))} = 1,$$

which can be seen as follows:

$$\begin{aligned}
& \lim_{x\to 0+} x \left(\mu^{\frac{\nu}{\mu}} x^{-\frac{1}{\mu}} \left(-\log\frac{x}{e}\right)^{-\frac{\nu}{\mu}}\right)^\mu \left[1+\log\left(\mu^{\frac{\nu}{\mu}} x^{-\frac{1}{\mu}} \left(-\log\frac{x}{e}\right)^{-\frac{\nu}{\mu}}\right)\right]^\nu \\
= \;& \lim_{x\to 0+} \mu^\nu \left(-\log\frac{x}{e}\right)^{-\nu} \left[1-\frac{1}{\mu}\log\left(\mu^{-\nu} x \left(-\log\frac{x}{e}\right)^\nu\right)\right]^\nu \\
= \;& \lim_{x\to 0+} \left(-\log\frac{x}{e}\right)^{-\nu} \left[\mu-\log\left(\mu^{-\nu} e\frac{x}{e} \left(-\log\frac{x}{e}\right)^\nu\right)\right]^\nu \\
= \;& \lim_{x\to 0+} \left[-\frac{\mu-\log(\mu^{-\nu}e)}{\log x/e} + 1 + \frac{\nu\log\log e/x}{\log x/e}\right]^\nu \\
= \;& 1,
\end{aligned}$$

as required. Thus, by Lemma 1.7, it is sufficient to show the assertion for the modified sequence (σ_n) instead of the original $(\tilde{\sigma}_n)$.

Theorem 2.3 and Theorem 2.4 already give the correct upper bounds for the desired small deviation probabilities. However, since we would like to have lower bounds as well, we have to prove that the corresponding upper limits in (2.2) and (2.5) tend to the same constants.

It is easy to calculate that in our concrete example

$$(S^{-1})'(x) = \mu^{\frac{\nu}{\mu}} \left(-\frac{1}{\mu}\right) x^{-\frac{1}{\mu}-1}(-\log(x/e))^{-\frac{\nu}{\mu}} \left(1 + \frac{\nu}{\log(x/e)}\right). \tag{2.13}$$

Therefore,

$$\frac{d}{dy}\left[\frac{S^{-1}(\varepsilon y)}{S^{-1}(\varepsilon)}\right] = \frac{(S^{-1})'(\varepsilon y)\varepsilon}{S^{-1}(\varepsilon)} = \frac{y^{-\frac{1}{\mu}-1}}{-\mu}\left(1+\frac{\log y}{\log\frac{\varepsilon}{e}}\right)^{-\frac{\nu}{\mu}}\left(1+\frac{\nu}{\log\frac{y\varepsilon}{e}}\right).$$

For simplicity, let us define

$$P(y) := \begin{cases} \left(-\frac{1}{\mu}\, y^{-\frac{1}{\mu}-1}\right)\log \mathbb{P}\left(|\theta| \le \frac{1}{y}\right) & p=\infty, \\ \left(-\frac{1}{\mu}\, y^{-\frac{1}{\mu}-1}\right)\log \mathbb{E}e^{-|y\theta|^p} & p<\infty. \end{cases}$$

Note that the function P is non-negative.

Let $0 < p \le \infty$. We show that (recall the definition of I_p from (2.1) and (2.4), respectively)

$$\begin{aligned} \overline{\lim_{\varepsilon\to 0+}} \frac{I_p(\varepsilon)}{S^{-1}(\varepsilon)} &= \overline{\lim_{\varepsilon\to 0+}} \int_0^{1/\varepsilon} P(y)\left(1+\frac{\log y}{\log(\varepsilon/e)}\right)^{-\frac{\nu}{\mu}}\left(1+\frac{\nu}{\log(y\varepsilon/e)}\right) dy \\ &\le \int_0^\infty P(y)\, dy, \end{aligned} \tag{2.14}$$

provided the right-hand side is finite. If it is not finite, Theorem 2.3 and Theorem 2.4 already give the assertion. From (2.14) we can deduce by Theorems 2.3 and 2.4, respectively, that even

$$\frac{I_p(\varepsilon)}{S^{-1}(\varepsilon)} \to \int_0^\infty P(y)\, dy, \qquad \text{as } \varepsilon \to 0+. \tag{2.15}$$

We prove (2.14) for the integrals from 0 to 1, and from 1 to ∞ separately. *Integral from* 0 *to* 1, *case* $\nu > 0$. Note that, if $0 \le y \le 1$ and $\nu > 0$, we have, for $\varepsilon < e$, $1 + \log y/(\log(\varepsilon/e)) \ge 1$ and $1 + \nu/(\log(\varepsilon y/e)) \le 1$. Thus,

$$\overline{\lim_{\varepsilon\to 0+}} \int_0^1 P(y)\left(1+\frac{\log y}{\log(\varepsilon/e)}\right)^{-\frac{\nu}{\mu}}\left(1+\frac{\nu}{\log(y\varepsilon/e)}\right) dy \le \int_0^1 P(y)\, dy$$

which implies the respective part of (2.14).

Integral from 0 *to* 1*, case* $\nu < 0$. In this case, for $\varepsilon \le \min(1, e^{1-2\nu})$, we have

$$\left(1+\frac{\log y}{\log(\varepsilon/e)}\right)^{-\frac{\nu}{\mu}} \le \left(1+\frac{\log y}{\log(1/e)}\right)^{-\frac{\nu}{\mu}} = (1-\log y)^{-\nu/\mu}$$

and

$$1+\frac{\nu}{\log(y\varepsilon/e)} \le 1+\frac{\nu}{\log e^{-2\nu}} = \frac{1}{2}.$$

Assumption (2.9) and (2.11), respectively, state that $P(y)(1-\log y)^{-\nu/\mu}$ is integrable over $[0,1]$, we have an integrable function majorising the integrand. By Lebesgue's Theorem, we have proved the existence of the respective part of (2.14).

Integral from 1 *to infinity, case* $\nu < 0$. Note that, in this case, for $\varepsilon < e$, $1+\log y/(\log(\varepsilon/e)) \le 1$ and so

$$\left(1+\frac{\log y}{\log(\varepsilon/e)}\right)^{-\frac{\nu}{\mu}} \le 1.$$

On the other hand, for $y \le 1/\varepsilon$,

$$1+\frac{\nu}{\log(y\varepsilon/e)} \le 1+\frac{\nu}{\log(1/e)} = 1-\nu,$$

which is a positive constant. Thus,

$$P(y)\left(1+\frac{\log y}{\log(\varepsilon/e)}\right)^{-\frac{\nu}{\mu}}\left(1+\frac{\nu}{\log(y\varepsilon/e)}\right)\mathbb{I}_{\{y\le 1/\varepsilon\}} \le P(y)(1-\nu),$$

which is integrable. By Lebesgue's Theorem, we obtain (2.14).

Integral from 1 *to infinity, case* $\nu > 0$. Since $y < 1/\varepsilon$, we have

$$1+\frac{\nu}{\log(y\varepsilon/e)} \le 1.$$

On the other hand,

$$\left(1+\frac{\log y}{\log(\varepsilon/e)}\right)^{-\frac{\nu}{\mu}} \le (1+\log y)^{\nu/\mu},$$

for $y \le 1/\varepsilon$. Note that, because condition (O) with $r > \mu$ holds, $P(y)(1+\log y)^{\nu/\mu}$ is an integrable function majorising the integrand in (2.14). We can deduce (2.14), by Lebesgue's Theorem.

In order to apply Theorems 2.1 and 2.2, we finally have to take care of the remaining terms on the left-hand side in those theorems. This is handled the same way as in the case $\nu = 0$ with the help of condition (O).

For $p = \infty$, the proof is finished with (2.15), whereas, for $p < \infty$, we have to use Lemma 1.3 in order to return from the logarithmic Laplace transform to the small deviation rate. ■

Similarly to the case $\nu = 0$, it is desirable to check the finiteness of the small deviation constants in (2.10) and (2.12). This question is investigated in the next section. However, independent of their finiteness, the equalities in (2.10) and (2.12) are to be understood in the sense that either both sides are finite and equal or both sides are $-\infty$.

2.5 The rate

2.5.1 The small deviation constant

In many cases we would like to check the finiteness of the small deviation constant. The crucial part of this is represented by

$$K := \begin{cases} -\int_0^\infty \log \mathbb{P}\left(|\theta| \leq z^\mu\right) dz & p = \infty, \\ -\int_0^\infty \log \mathbb{E} e^{-|z^{-\mu}\theta|^p} dz & p < \infty. \end{cases}$$

It is not immediately transparent when this expression is finite. Nevertheless, it is clear that only the behaviour of the distribution near zero and infinity matters. First of all, we shall see that the finiteness of the integral from zero to one is always ensured under condition (O).

Lemma 2.2. *Let $\mu > 0$ and assume that the distribution of θ satisfies condition (O) with $r > \mu$. Then $K_1 < \infty$, where*

$$K_1 := \begin{cases} -\int_0^1 \log \mathbb{P}\left(|\theta| \leq z^\mu\right) dz & p = \infty, \\ -\int_0^1 \log \mathbb{E} e^{-|z^{-\mu}\theta|^p} dz & p < \infty. \end{cases}$$

Proof: This is clear if $p = \infty$. In the case $p < \infty$ it follows from the calculation

$$\log \mathbb{E} e^{-|z^{-\mu}\theta|^p} \geq \log \mathbb{E} e^{-|z^{-\mu}\theta|^p} \mathbb{I}_{\{|z^{-\mu}\theta| \leq 1\}} \geq \log\left(e^{-1}\mathbb{P}\left(|\theta| \leq z^\mu\right)\right). \quad (2.16)$$

■

Thus, as long as condition (O) is satisfied with sufficiently large r, no additional restriction to ensure the convergence of the integral defining K at zero is necessary. For the remaining integral we have to distinguish the cases $p = \infty$ and $p < \infty$. For $p < \infty$, the following simple necessary and sufficient condition holds.

Lemma 2.3. *We have*

$$-\int_1^\infty \log \mathbb{E} e^{-|z^{-\mu}\theta|^p}\, dz < \infty \tag{2.17}$$

if and only if $\mathbb{E}|\theta|^{1/\mu} < \infty$ *and* $\mu > 1/p$.

Proof: For $z \geq 1$, we have $\mathbb{E}e^{-|z^{-\mu}\theta|^p} \geq \mathbb{E}e^{-|\theta|^p} > 0$. Therefore, by Lemma 1.2, for some $C > 0$,

$$\int_1^\infty \log \mathbb{E} e^{-|z^{-\mu}\theta|^p}\, dz \geq \int_1^\infty C\left(\mathbb{E}e^{-|z^{-\mu}\theta|^p} - 1\right)\, dz.$$

On the other hand, the reverse inequality is true with $C = 1$. Thus, (2.17) holds if and only if

$$\int_1^\infty \mathbb{E}\left(1 - e^{-|z^{-\mu}\theta|^p}\right)\, dz < \infty. \tag{2.18}$$

It is clear that $0 \leq \int_0^1 \mathbb{E}\left(1 - e^{-|z^{-\mu}\theta|^p}\right)\, dz \leq 1$. Thus, (2.18) is true if and only if $\int_0^\infty \mathbb{E}\left(1 - e^{-|z^{-\mu}\theta|^p}\right)\, dz < \infty$. To finish the proof we observe that, by Fubini's Theorem,

$$\begin{aligned}
\int_0^\infty \mathbb{E}\left(1 - e^{-|z^{-\mu}\theta|^p}\right)\, dz &= \mathbb{E}\int_0^\infty \left(1 - e^{-|z^{-\mu}\theta|^p}\right)\, dz \\
&= \mathbb{E}|\theta|^{1/\mu} \int_0^\infty \left(1 - e^{-y^{-p\mu}}\right)\, dy.
\end{aligned}$$

It is elementary to check that $\int_0^\infty \left(1 - e^{-y^{-p\mu}}\right)\, dy < \infty$ if and only if $\mu > 1/p$. This shows the assertion. ■

The last result shows that the range of eligible $\mu > 0$ is also strongly determined by the *upper* tail behaviour of the distribution of θ, a fact one might not expect when dealing with *lower* tails. The same is true in the case $p = \infty$.

Lemma 2.4. *Let* $\mathbb{P}(|\theta| \leq 1) > 0$. *Then we have*

$$-\int_1^\infty \log \mathbb{P}(|\theta| \leq z^\mu)\, dz < \infty \tag{2.19}$$

if and only if $\mathbb{E}|\theta|^{1/\mu} < \infty$.

Proof: We have, for all $z \geq 1$, $\mathbb{P}(|\theta| \leq z^\mu) \geq \mathbb{P}(|\theta| \leq 1) > 0$. Therefore, by Lemma 1.2, (2.19) holds if and only if $\int_1^\infty \mathbb{P}(|\theta| > z^\mu)\, dz < \infty$. To finish the proof we only have to observe that

$$\int_1^\infty \mathbb{P}(|\theta| > z^\mu)\, dz = \mathbb{E}|\theta|^{1/\mu} - \int_0^1 \mathbb{P}(|\theta| > z^\mu)\, dz,$$

by Fubini's Theorem. ■

2.5.2 Main result

Now we are in the position to prove the main result of this thesis. It complements the necessary conditions in Section 1.4.

Theorem 2.9. *Let $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$, with $\mu > 0$ and $\nu \in \mathbb{R}$, and let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables satisfying condition (O) with $r > \mu$. If $\|(\sigma_n\theta_n)\|_p < \infty$ almost surely, the small deviation probability satisfies:*

$$\lim_{\varepsilon \to 0+} \varepsilon^{\frac{1}{\mu - 1/p}} (-\log \varepsilon)^{\frac{\nu}{\mu - 1/p}} \log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \leq \varepsilon\right) = -C_p,$$

with the constant $C_p \in]0, \infty]$ given by

$$C_p := \begin{cases} \mu^{\nu/\mu}\left(-\int_0^\infty \log \mathbb{P}(|\theta| \leq z^\mu)\, dz\right) & p = \infty, \\ \left[\frac{(\mu-1/p)^{\mu-1/p+\nu}}{\mu^\mu p^{1/p}}\left(-\int_0^\infty \log \mathbb{E}e^{-|z^{-\mu}\theta|^p}\, dz\right)^\mu\right]^{\frac{1}{\mu-1/p}} & p < \infty. \end{cases} \tag{2.20}$$

The constant C_p is finite if and only if $\mathbb{E}|\theta|^{1/\mu} < \infty$ and $\mu > 1/p$.

Proof: Theorem 2.7 and Theorem 2.8, respectively, almost give the assertion. We are only left with conditions (2.9) and (2.11).

The same considerations as in the above Lemma 2.4 can be applied to condition (2.9). It turns out that it is equivalent to the condition $\mathbb{E}S^{-1}(c/|\theta|) < \infty$, which is exactly the conditon for a.s. boundedness by Theorem 1.1.

Similarly, the same considerations as in Lemma 2.2 can be applied to condition (2.11). It turns out that it is equivalent to the finiteness of the expression in (1.8), which is true if and only if $\|(\sigma_n\theta_n)\|_p < \infty$ almost surely. Thus, no new condition appears.

We combine Lemma 2.2, Lemma 2.3, and Lemma 2.4 with Theorem 2.7 and Theorem 2.8, respectively, and obtain the assertion. ■

2.5.3 A discussion on the remaining cases

Theorem 2.9 clarifies the small ball rate for most of the cases where the sequence $(\sigma_n\theta_n)$ is in l_p a.s. However, some cases occur when the small deviation constant in Theorem 2.9 is infinite, even though $(\sigma_n\theta_n) \in l_p$ a.s.

Let us compare Theorem 2.9 to the necessary conditions in Theorem 1.1 and Theorem 1.2. For this purpose, let α be defined as follows.

- If $\mathbb{E}|\theta|^p < \infty$, for all $p > 0$, we set $\alpha := \infty$.
- Otherwise, we assume that there is a number $\alpha > 0$ and a constant $K(\alpha) \in]0, \infty[$ such that $\mathbb{P}(|\theta| > t) \sim K(\alpha)t^{-\alpha}$, as $t \to \infty$.

As in Theorem 1.2, we could add an additional slowly varying function and prove the respective results with the methods developed in this thesis; however, we do not want to make the situation notationally more complicated. We do likewise for the behaviour of the distribution of θ at the origin. Concretely, we assume that, for some $\delta > 0$ and $C > 0$,

$$\mathbb{P}(|\theta| \le t) \ge Ct^\delta, \qquad \text{for all } 0 < t \le 1. \tag{2.21}$$

Notice that this is satisfied for all usually considered distributions, such as Gaussian, stable, Gamma, and even discrete ones, as long as they have mass at the origin. Also it is much more general than the usual assumption that $\mathbb{P}(|\theta| \le t)$ is regularly varying at the origin, as in [12], [36], or [15]. Let us mention that if (2.21) is true then the distribution of θ satisfies condition (O) for all $r > 0$.

Given this situation, let us compare Theorem 2.9 to the necessary conditions. If $\alpha = \infty$ and $p = \infty$, the sequence $(\sigma_n\theta_n)$ is bounded for all at least polynomically decreasing sequences (σ_n). For the other cases, Theorem 1.1 and Theorem 1.2 tell us that $(\sigma_n\theta_n) \in l_p$ a.s. if and only if

$$\begin{cases} \sum_n \sigma_n^p < \infty & p < \alpha, \\ \sum_n \sigma_n^\alpha \log \sigma_n^{-1} < \infty & p = \alpha, \\ \sum_n \sigma_n^\alpha < \infty & p > \alpha. \end{cases}$$

In our example, $\sigma_n \sim n^{-\mu}(1 + \log n)^{-\nu}$, Theorem 2.9 solves most of these cases. However, the following cases remain open, because $C_p = \infty$ in Theorem 2.9, even though $(\sigma_n\theta_n) \in l_p$ almost surely:

- $\mu = \max(1/p, 1/\alpha)$ and $\nu > \max(1/p, 1/\alpha)$ if $p \ne \alpha$,
- $\mu = 1/\alpha$ and $\nu > 2/\alpha$ if $p = \alpha$,

In these cases, the order given by Theorem 2.9 is not the correct one, apparently, as the second part of the same theorem shows. The following theorem determines the rate of (P) in these cases. However, unlike in Theorem 2.9, we only determine the rate in the sense of $\approx$. For convenience, let us define the so-called small deviation function (or small ball function):

$$\phi(\varepsilon) := -\log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right).$$

Theorem 2.10. *With the assumptions stated above, we have:*

1. *If* $\mu > \max(1/p, 1/\alpha)$,

$$\phi(\varepsilon) \sim C_p \varepsilon^{-\frac{1}{\mu - 1/p}} (-\log \varepsilon)^{\frac{\nu}{\mu - 1/p}},$$

where the finite small deviation constant $C_p > 0$ *is given by (2.20).*

2. *If* $p > \alpha$, $\mu = 1/\alpha$, *and* $\nu > 1/\alpha$,

$$\phi(\varepsilon) \approx \varepsilon^{-\frac{1}{\mu - 1/p}} (-\log \varepsilon)^{\frac{\mu - \nu}{\mu - 1/p}}.$$

3. *If* $p < \alpha$, $\mu = 1/p$, *and* $\nu > 1/p$,

$$\log \phi(\varepsilon) \approx \varepsilon^{-\frac{p}{\nu p - 1}}.$$

4. *If* $p = \alpha$, $\mu = 1/\alpha = 1/p$, *and* $\nu > 2/p$,

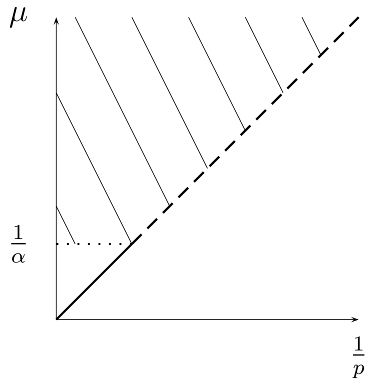
$$\log \phi(\varepsilon) \approx \varepsilon^{-\frac{p}{\nu p - 2}}.$$

For all other values of the parameters $\mu > 0$ *and* $\nu \in \mathbb{R}$, *we have*

$$\mathbb{P}\left(\|(\sigma_n\theta_n)\|_p < \infty\right) = 0,$$

which means that the question for small deviation is solved trivially.

Before we come to the proof of the theorem, let us visualise the results in a diagram and interprete their meaning. The following plot represents the relation of the parameters μ, p, and α (defined above).



Picture 1: Relation of the parameters.

The hatched area in the above diagram ($\mu > \max(1/p, 1/\alpha)$) represents the main case treated by Theorem 2.9, which determines the small deviation rate and the finite small deviation constant. The dashed and dotted lines represent the borderline cases, in which the necessary condition for boundedness is satisfied but the small deviation constant in Theorem 2.9 is infinite.

In the dashed part ($\mu = 1/p$ and $p < \alpha$) and at the intersection of the dashed and dotted line ($\mu = 1/p = 1/\alpha$), the small deviation probability shows super-exponential behaviour (as stated in the third and fourth case of the Theorem 2.10).

However, on the dotted part ($\mu = 1/\alpha$ and $\alpha < p$), an additional logarithmic term appears in the small deviation rate (second case of the Theorem 2.10). Let us stress that this is due to the influence of the *large deviation* behaviour of θ. Note that the latter does not occur for example in the Gaussian case, since we then have $\alpha = \infty$. Even more, the whole phenomenon remains unnoticed in the results based on [28]. This can be seen very well from the formulation of Proposition 2.1 above, where $p = 1$. The proposition assumes $\mathbb{E}|\theta|^2 < \infty$, which requires $\alpha \geq 2$. And since another assumption is $\mu > 1$, we only find ourselves in the area of the main case.

The remaining (white) area of the diagram represents exactly those cases where the problem is solved trivially by the necessary condition and we have

$$\mathbb{P}\left(\|(\sigma_n\theta_n)\|_p < \infty\right) = 0.$$

Thus, the question is solved (on the logarithmic or double logarithmic level) for all those cases when it makes sense.

Proof of Theorem 2.10: Let, as above,

$$S^{-1}(x) := \mu^{\nu/\mu} x^{-1/\mu}(1 - \log x)^{-\nu/\mu}. \tag{2.22}$$

Then

$$S(x) \sim x^{-\mu}(1 + \log x)^{-\nu}.$$

The case $p = \infty$. Here, only the case of finite $\alpha < p = \infty$ (i.e. the second assertion) is interesting. Let $\mu = 1/\alpha$ and $\nu > 1/\alpha$. Theorem 2.1 yields, for all $\varepsilon > 0$,

$$\log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{\sigma_1}\right) + \int_1^\infty \log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{S(x)}\right) dx \\ \leq \log \mathbb{P}\left(\sup_n |\sigma_n \theta_n| \leq \varepsilon\right) \leq \int_1^\infty \log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{S(x)}\right) dx.$$

Note that the remaining probability term on the left-hand side does not pose any problem, using (2.21). Thus, the important term is

$$\int_1^\infty \log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{S(x)}\right) dx \\ = \int_1^{S^{-1}(\varepsilon)} \log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{S(x)}\right) dx + \int_{S^{-1}(\varepsilon)}^\infty \log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{S(x)}\right) dx. \quad (2.23)$$

The first integral can be treated as follows. Since on the domain of integration, $\frac{\varepsilon}{\sigma_1} \leq \frac{\varepsilon}{S(x)}$ holds, we have, by (2.21),

$$0 \geq \int_1^{S^{-1}(\varepsilon)} \log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{S(x)}\right) dx \geq \int_1^{S^{-1}(\varepsilon)} \log \left(\frac{\varepsilon}{\sigma_1}\right)^\delta dx \\ \approx -\delta(-\log \varepsilon) S^{-1}(\varepsilon) \approx -\varepsilon^{-1/\mu}(-\log \varepsilon)^{1-\nu/\mu}.$$

Now let us consider the second integral in (2.23), which actually determines the rate:

$$\int_{S^{-1}(\varepsilon)}^\infty \log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{S(x)}\right) dx.$$

Here, $\frac{\varepsilon}{S(x)} \geq 1$ holds, and we can hence use Lemma 1.2, since (2.21) implies $\mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{S(x)}\right) \geq \mathbb{P}(|\theta| \leq 1) \geq C > 0$. This shows that the last term is bounded from above and below using positive constants by

$$\int_{S^{-1}(\varepsilon)}^\infty \left(\mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{S(x)}\right) - 1\right) dx = -\int_{S^{-1}(\varepsilon)}^\infty \mathbb{P}\left(|\theta| > \frac{\varepsilon}{S(x)}\right) dx.$$

Using tail behaviour defining $\alpha < \infty$, we can conclude that this is again bounded from above and below by

$$-\int_{S^{-1}(\varepsilon)}^\infty \left(\frac{\varepsilon}{S(x)}\right)^{-\alpha} dx = \varepsilon^{-\alpha} \int_0^\varepsilon y^\alpha (S^{-1})'(y)\, dy.$$

Substituting the concrete case (2.22) and recalling (2.13), this equals – up to a constant –

$$\varepsilon^{-\alpha}\int_0^{\varepsilon} y^{-1}(1-\log y)^{-\nu/\mu}\left(1+\frac{\nu}{\log y/e}\right)dy.$$

The term $\left(1+\frac{\nu}{\log y/e}\right)$ is bounded away from zero and infinity for all $y<\varepsilon$, as long as $\varepsilon<\varepsilon_0$. Thus, the rate is finally given by

$$\varepsilon^{-\alpha}\int_0^{\varepsilon} y^{-1}(1-\log y)^{-\frac{\nu}{\mu}}\,dy = -C\varepsilon^{-\alpha}(1-\log\varepsilon)^{1-\frac{\nu}{\mu}} \approx -\varepsilon^{-1/\mu}(-\log\varepsilon)^{\frac{\mu-\nu}{\mu}}.$$

Notice that an additional log term appears, since the upper bound (Theorem 2.3) suggests the rate $\varepsilon^{-1/\mu}(1-\log\varepsilon)^{-\nu/\mu}$.

Let us remark as well that the above considerations are too weak to provide the existence of the small deviation constant.

The case $p<\infty$. Let us consider the cases

$$\begin{cases}\mu=\max(1/p,1/\alpha), \nu>\max(1/p,1/\alpha) & p\neq\alpha,\\ \mu=1/\alpha, \nu>2/\alpha & p=\alpha.\end{cases}$$

Theorem 2.2 yields, for all $\lambda>0$,

$$\log\mathbb{E}e^{-\lambda\sigma_1^p|\theta|^p}+\int_1^\infty \log\mathbb{E}e^{-\lambda|S(x)\theta|^p}\,dx \le \log\mathbb{E}e^{-\lambda\sum_n|\sigma_n\theta_n|^p} \le \int_1^\infty \log\mathbb{E}e^{-\lambda|S(x)\theta|^p}\,dx.$$

Note that the remaining probability term on the left-hand side does not pose any problem, by the exponential Chebyshev Inequality (as in (2.16)) and (2.21). Thus, the important term is

$$\int_1^\infty \log\mathbb{E}e^{-\lambda|S(x)\theta|^p}\,dx = \int_1^{S^{-1}(\lambda^{-1/p})}\log\mathbb{E}e^{-\lambda|S(x)\theta|^p}\,dx+\int_{S^{-1}(\lambda^{-1/p})}^\infty \log\mathbb{E}e^{-\lambda|S(x)\theta|^p}\,dx. \quad (2.24)$$

Let us start with the first term. From above it can be estimated trivially by zero. From below it is bounded by

$$\int_1^{S^{-1}(\lambda^{-1/p})}\log\left(e^{-1}\mathbb{P}\left(|\theta|\le\frac{\lambda^{-1/p}}{S(x)}\right)\right)dx \ge \int_1^{S^{-1}(\lambda^{-1/p})}\log\left(e^{-1}\mathbb{P}\left(|\theta|\le\frac{\lambda^{-1/p}}{\sigma_1}\right)\right)dx,$$

where we used the exponential Chebyshev Inequality (as in (2.16)) in the first step. Using (2.21), we can see that the last term is bounded with positive constants from below and above by

$$-S^{-1}(\lambda^{-1/p})\log\lambda \approx -\lambda^{1/(p\mu)}(\log\lambda)^{-\nu/\mu+1}.$$

Let us now consider the second term in (2.24). On the domain of integration $\lambda S(x)^p \leq 1$ and hence we have $\mathbb{E}e^{-\lambda|S(x)\theta|^p} \geq \mathbb{E}e^{-|\theta|^p} > 0$. Thus, by Lemma 1.2,

$$\int_{S^{-1}(\lambda^{-1/p})}^{\infty} \log \mathbb{E}e^{-\lambda|S(x)\theta|^p}\,dx \approx \int_{S^{-1}(\lambda^{-1/p})}^{\infty} \mathbb{E}(e^{-\lambda|S(x)\theta|^p}-1)\,dx \qquad (2.25)$$

Since on the domain of integration $\lambda S(x)^p \leq 1$, we can use Lemma 1.4.

If $p > \alpha$. Then $\alpha < \infty$ and, by assumption, $\mathbb{P}(|\theta|^p > t) \sim K(\alpha)t^{-\alpha/p}$, as $t \to \infty$. Thus, the term in (2.25) behaves as

$$\int_{S^{-1}(\lambda^{-1/p})}^{\infty} -\lambda^{\alpha/p}S(x)^{\alpha}\,dx = \lambda^{\alpha/p}\int_0^{\lambda^{-1/p}} y^{\alpha}(S^{-1})'(y)\,dy,$$

which can be handled as in the case $p = \infty$ (substituting $\varepsilon = \lambda^{-1/p}$ and using that $\mu = \max(1/p, 1/\alpha) = 1/\alpha$) and turns out to behave (in $\approx$) as

$$-\lambda^{\alpha/p}(\log\lambda)^{-\nu/\mu+1} = -\lambda^{1/(\mu p)}(\log\lambda)^{1-\nu/\mu}.$$

Applying Lemma 1.5 finishes the proof.

If, on the other hand, $p < \alpha \leq \infty$ the term in (2.25) behaves (Lemma 1.4) as

$$\int_{S^{-1}(\lambda^{-1/p})}^{\infty} -\lambda^{1}S(x)^{p}\,dx = \lambda\int_0^{\lambda^{-1/p}} y^{p}(S^{-1})'(y)\,dy,$$

which again can be treated as above (this time $\mu = \max(1/p, 1/\alpha) = 1/p$, however). It turns out to behave as

$$-\lambda\int_0^{\lambda^{-1/p}} y^{p-\frac{1}{\mu}-1}(1-\log y)^{-\nu/\mu}\,dy \approx -\lambda(\log\lambda)^{1-\nu/\mu} = -\lambda(\log\lambda)^{1-\nu p}.$$

Using this estimate for $\log\mathbb{E}e^{-\lambda\sum_n|\sigma_n\theta_n|^p}$ and applying Lemma 1.6 provides the rate of the small deviation probability of $\sum_n|\sigma_n\theta_n|^p$ and thus assertion in this case.

Let us finally consider the case $p = \alpha < \infty$. Here, Lemma 1.4 yields that the term in (2.25) behaves as

$$\int_{S^{-1}(\lambda^{-1/p})}^{\infty} -\lambda S(x)^p \log\left(\frac{1}{\lambda S(x)^p}\right)dx =$$

$$-\int_0^{\lambda^{-\frac{1}{p}}} \lambda y^p\log(\lambda y^p)(S^{-1})'(y)\,dy \approx \lambda\int_0^{\lambda^{-\frac{1}{p}}} y^{-1}(1-\log y)^{-\frac{\nu}{\mu}}\log(\lambda y^p)\,dy.$$

Elementary estimates show that, for $\nu/\mu = \nu p > 2$, this term behaves as

$$-\lambda(\log \lambda)^{2-\nu/\mu} = -\lambda(\log \lambda)^{2-\nu p}.$$

As above, applying Lemma 1.6 gives the assertion. ■

2.6 Applications

2.6.1 Gaussian random variables

Let us consider some examples for concrete random variables. We start with Gaussian random variables. This enables us to compare the results to those from [12]. Of course, on the logarithmic level the result is the same.

Corollary 2.1. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. Gaussian random variables and let $\sigma_n \sim n^{-\mu}(1 + \log n)^{-\nu}$ with $\mu > 0$ and $\nu \in \mathbb{R}$. If $\mu > 1/p$*

$$\lim_{\varepsilon \to 0+} \varepsilon^{\frac{1}{\mu - 1/p}} \left(-\log \varepsilon\right)^{\frac{\nu}{\mu - 1/p}} \log \mathbb{P}\left(\left\|(\sigma_n \theta_n)\right\|_p \le \varepsilon\right) = -C_p,$$

where C_p is the finite, positive constant in (2.20). For $\mu \le 1/p$, the above limit tends to $C_p = \infty$. If $\mu = 1/p$ and $\nu > 1/p$ then

$$\log\left(-\log \mathbb{P}\left(\left\|(\sigma_n \theta_n)\right\|_p \le \varepsilon\right)\right) \approx \varepsilon^{-\frac{p}{\nu p - 1}}.$$

For all other parameter values, $\mathbb{P}\left(\left\|(\sigma_n \theta_n)\right\|_p < \infty\right) = 0$.

Proof: It is well-known that $\mathbb{E}|\theta|^p < \infty$, for all $p > 0$, and that θ satisfies (2.21). Theorem 2.10 shows the assertion. ■

Note that the superexponential estimates do *not* hold for $\mu = 0$, cf. the remark on 'empty balls' after the proof of Theorem 1.1.

For $p = 2$ and any $\mu > 1/2$, the constant in (2.20) can be calculated explicitly. Namely, we have

$$\int_0^\infty \log \mathbb{E} e^{-|z^{-\mu}\theta|^2} \, dz = \int_0^\infty \log\left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-(1+2z^{-2\mu})t^2/2} \, dt\right) dz$$
$$= -\frac{1}{2} \int_0^\infty \log(1 + 2z^{-2\mu}) \, dz = -\frac{2^{1/(2\mu)}\pi}{2\sin\left(\frac{\pi}{2\mu}\right)}$$

Thus, the result is

$$C_2 = \frac{2\mu-1}{2}\left(\frac{2\mu-1}{2}\right)^{2\nu/(2\mu-1)}\left(\frac{\pi}{2\mu\sin\frac{\pi}{2\mu}}\right)^{2\mu/(2\mu-1)}.$$

Note that – on the logarithmic level – Corollary 4.3 in [12] (which is based on Proposition 2.1 above) is a special case of the last statement if one sets $\nu = 0$ and $p = 2$ and substitutes $A = 2\mu$ and $a = \frac{2\mu}{2\mu-1}$. (NB: For $p = 2$, in [12] the probability that the sum is less than $r = \varepsilon$ is calculated. Here, we calculate the probability that the sum is less than $\varepsilon^p = \varepsilon^2$.)

This shows again that the focus of the present treatment is different to the one of [12]. We show less precise results that hold under milder conditions. In particular, we also consider the borderline case and we shall come to the results for non-Gaussian stable distributions in the next section.

2.6.2 Non-Gaussian stable sequences

Application of the main results. Let us consider the example of i.i.d. non-Gaussian, α-stable random variables $\theta, \theta_1, \theta_2, \dots$ with parameters $0 < \alpha < 2$, $\sigma > 0$, $\beta \in [-1, 1]$, and centering constant zero, i.e. random variables with characteristic function

$$\mathbb{E}e^{it\theta} = \begin{cases} \exp\left(-\sigma^\alpha |t|^\alpha (1 - i\beta(\operatorname{sgn} t)\tan\frac{\pi\alpha}{2})\right) & \text{for } \alpha \neq 1, \\ \exp\left(-\sigma^\alpha |t|^\alpha (1 + i\beta\,\frac{2}{\pi}\,(\operatorname{sgn} t)\log|t|)\right) & \text{for } \alpha = 1. \end{cases}$$

We refer to Chapter 1 of [40] for a description of the parameters and their properties. If $|\beta| = 1$ and $0 < \alpha < 1$, we say that θ is totally skewed.

The general results of this thesis lead to the following corollary.

Corollary 2.2. *Let $\theta, \theta_1, \theta_2, \dots$ be a sequence of i.i.d. non-trivial α-stable non-Gaussian random variables that are not totally skewed and let $\sigma_n \sim n^{-\mu}(1 + \log n)^{-\nu}$ with $\mu > 0$ and $\nu \in \mathbb{R}$. If $\mu > \max(1/\alpha, 1/p)$ then*

$$\lim_{\varepsilon \to 0+} \varepsilon^{\frac{1}{\mu - 1/p}} (-\log\varepsilon)^{\frac{\nu}{\mu - 1/p}} \log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \leq \varepsilon\right) = -C_p,$$

where C_p is the finite, positive constant given in (2.20). On the other hand, for $\mu \leq \max(1/p, 1/\alpha)$, the above limit tends to $C_p = \infty$. If $p > \alpha$, $\mu = 1/\alpha$, and $\nu > 1/\alpha$, the correct rate is

$$\log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \leq \varepsilon\right) \approx -\varepsilon^{-\frac{1}{\mu - 1/p}} (-\log\varepsilon)^{\frac{\mu - \nu}{\mu - 1/p}}.$$

If $p < \alpha$, $\mu = 1/p$, and $\nu > 1/p$,

$$\log\left(-\log\mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right)\right) \approx \varepsilon^{-\frac{p}{\nu p-1}}.$$

If $p = \alpha$, $\mu = 1/\alpha = 1/p$, and $\nu > 2/p$,

$$\log(-\log\mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right)) \approx \varepsilon^{-\frac{p}{\nu p-2}}.$$

For the other parameter values, $\mathbb{P}\left(\|(\sigma_n\theta_n)\|_p < \infty\right) = 0$.

Proof: It is well-known (cf. Chapter 1.2 in [40]) that θ has a continuous non-vanishing density on the whole $\mathbb{R}$ if θ is not totally skewed. Therefore, the distribution satisfies (2.21). Furthermore, it is well-known (cf. Chapter 1.2 in [40]) that $\mathbb{P}(|\theta| > t) \sim C_\alpha t^{-\alpha}$, as $t \to \infty$, and thus $\mathbb{E}|\theta|^{1/\mu} < \infty$ if and only if $1/\mu < \alpha$ (i.e. $\mu > 1/\alpha$). Theorem 2.10 gives the assertion. ■

The totally skewed case. In order to clarify the problem for *all* stable distributions, we are only left with the case of totally skewed random variables. The results stated so far do not apply, since those random variables do not satisfy condition (O).

This part is included for completeness w.r.t. stable distributions but as well to see that the situation is very different if condition (O) fails to hold.

Let $0 < \alpha < 1$. Let θ be a totally skewed ($\beta = 1$) α-stable random variable with scale parameter $\sigma = 1$. By Proposition 1.2.12 in [40], this is an a.s. non-negative random variable with Laplace transform

$$\mathbb{E}e^{-\lambda\theta} = \exp\left(-\frac{\lambda^\alpha}{\cos\frac{\pi\alpha}{2}}\right).$$

For i.i.d. copies $\theta_1, \theta_2, \ldots$ of θ and a decreasing sequence (σ_n) we consider problem (P). The result is as follows.

Theorem 2.11. *Let $1 \le p \le \infty$. For any sequence (σ_n) with $\sum_n \sigma_n^\alpha < \infty$, we have*

$$\log\mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right) \approx -\varepsilon^{-\frac{\alpha}{1-\alpha}}, \qquad \text{as } \varepsilon \to 0+,$$

where the finite constants encoded by $\approx$ depend on the sequence (σ_n). For $\sum_n \sigma_n^\alpha = \infty$, $\mathbb{P}\left(\|(\sigma_n\theta_n)\|_p < \infty\right) = 0$.

For $p = 1$, we get even more precisely

$$\log\mathbb{P}\left(\sum_n \sigma_n\theta_n \le \varepsilon\right) \sim -\left(\frac{\sum_n \sigma_n^\alpha}{\cos\frac{\pi\alpha}{2}}\right)^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\varepsilon^{-\frac{\alpha}{1-\alpha}}.$$

Proof for the case $p = 1$: As above, we consider the log-Laplace transform of the random variable in question. It can be calculated – using $\theta_n \geq 0$ and the above formula for the Laplace transform – as follows:

$$\log \mathbb{E} e^{-\lambda \sum_n \sigma_n \theta_n} = \sum_n \log \mathbb{E} e^{-\lambda \sigma_n \theta_n} = \sum_n -\frac{\lambda^\alpha \sigma_n^\alpha}{\cos \frac{\pi\alpha}{2}} = -\frac{\lambda^\alpha}{\cos \frac{\pi\alpha}{2}} \sum_n \sigma_n^\alpha.$$

By Lemma 1.3, this shows that

$$\log \mathbb{P}\left(\sum_n \sigma_n \theta_n \leq \varepsilon\right) \sim -\left(\frac{\sum_n \sigma_n^\alpha}{\cos \frac{\pi\alpha}{2}}\right)^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha) \varepsilon^{-\frac{\alpha}{1-\alpha}}.$$

The case $p = \infty$: Let $c' = 1/\cos \frac{\pi\alpha}{2}$ and let

$$D_0 := \left(\frac{\alpha}{\cos \frac{\pi\alpha}{2}}\right)^{\frac{1}{1-\alpha}}.$$

Then

$$D_0 - c' D_0^\alpha = \left(\frac{\alpha}{\cos \frac{\pi\alpha}{2}}\right)^{\frac{1}{1-\alpha}} \left(1 - \frac{1}{\alpha}\right) = -\left(\frac{1}{\cos \frac{\pi\alpha}{2}}\right)^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha).$$

Furthermore, we put $\lambda_n := \sigma_n^{\frac{\alpha}{1-\alpha}} D_0 \varepsilon^{-\frac{1}{1-\alpha}}$. Then

$$e^{-c' \sigma_n^\alpha \lambda_n^\alpha} = \mathbb{E} e^{-\sigma_n \lambda_n \theta_n} \geq \mathbb{E} e^{-\sigma_n \lambda_n \theta_n} \mathbb{I}_{\{\sigma_n \theta_n \leq \varepsilon\}} \geq e^{-\varepsilon \lambda_n} \mathbb{P}\left(\theta_n \leq \varepsilon / \sigma_n\right).$$

Thus,

$$\begin{aligned} \log \mathbb{P}\left(\sup_n \sigma_n \theta_n \leq \varepsilon\right) &= \sum_n \log \mathbb{P}\left(\theta_n \leq \frac{\varepsilon}{\sigma_n}\right) \\ &\leq \sum_n (\varepsilon \lambda_n - c' \sigma_n^\alpha \lambda_n^\alpha) = -\left(\frac{1}{\cos \frac{\pi\alpha}{2}}\right)^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha) \varepsilon^{-\frac{\alpha}{1-\alpha}} \sum_n \sigma_n^{\frac{\alpha}{1-\alpha}}, \end{aligned}$$

by the definition of λ_n and D_0.

End of the proof: Since, for $1 \leq p \leq \infty$,

$$\begin{aligned} &-\left(\frac{1}{\cos \frac{\pi\alpha}{2}}\right)^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha)\, \varepsilon^{-\frac{\alpha}{1-\alpha}} \sum_n \sigma_n^{\frac{\alpha}{1-\alpha}} \\ &\qquad \geq \log \mathbb{P}\left(\|(\sigma_n \theta_n)\|_\infty \leq \varepsilon\right) \geq \log \mathbb{P}\left(\|(\sigma_n \theta_n)\|_p \leq \varepsilon\right) \\ &\qquad \geq \log \mathbb{P}\left(\|(\sigma_n \theta_n)\|_1 \leq \varepsilon\right) \sim -\left(\frac{\sum_n \sigma_n^\alpha}{\cos \frac{\pi\alpha}{2}}\right)^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha) \varepsilon^{-\frac{\alpha}{1-\alpha}}, \end{aligned}$$

we have that, for all $1 \le p \le \infty$,

$$\log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right) \approx -\varepsilon^{-\frac{\alpha}{1-\alpha}}.$$

Theorem 1.1 and Theorem 1.2 show the necessary condition, since (cf. Chapter 1.2 in [40]) $\mathbb{P}(|\theta| > t) \sim C_\alpha t^{-\alpha}$, as $t \to \infty$. ■

2.6.3 Gamma distributions

Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. Gamma-distributed random variables, i.e.

$$\mathbb{P}(\theta \le t) = \int_0^t \frac{x^{b-1}e^{-x/a}}{\Gamma(b)a^b}\, dx,$$

for some fixed parameters $a, b > 0$. This includes in particular exponential distributions, so-called Erlang distributions, and χ^2-distributions.

Corollary 2.3. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. Gamma-distributed random variables and let $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$ with $\mu > 0$ and $\nu \in \mathbb{R}$. If $\mu > 1/p$ then*

$$\lim_{\varepsilon\to 0+} \varepsilon^{\frac{1}{\mu-1/p}} (-\log \varepsilon)^{\frac{\nu}{\mu-1/p}} \log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right) = -C_p,$$

where C_p is the finite, positive constant in (2.20). For $\mu \le 1/p$, the above limit tends to $C_p = \infty$. If $\mu = 1/p$ and $\nu > 1/p$ then

$$\log\left(-\log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right)\right) \approx \varepsilon^{-\frac{p}{\nu p-1}}.$$

For the other parameter values, $\mathbb{P}\left(\|(\sigma_n\theta_n)\|_p < \infty\right) = 0$.

Proof: It is elementary to check that $\mathbb{E}|\theta|^p < \infty$, for all $p > 0$, and that θ satisfies (2.21) with $\delta = b$. Theorem 2.10 shows the assertion. ■

If $p = 1$ it is possible to calculate the constant explicitly. The calculation is similar to the one in the Gaussian case for $p = 2$. The result is

$$C_1 = (\mu-1)(\mu-1)^{\frac{\nu}{\mu-1}}\, a^{\frac{1}{\mu-1}} b^{\frac{\mu}{\mu-1}} \left(\frac{\pi/\mu}{\sin(\pi/\mu)}\right)^{\frac{\mu}{\mu-1}}.$$

Note that also this recovers the logarithmic level of Corollary 4.3 in [12] if one sets $\nu = 0$, and $\mu = A$, $a = A/(A-1)$, since squared Gaussian random variables are Gamma-distributed with parameters $a = 2$ and $b = 1/2$.

2.6.4 Random Fourier series

In this section, we describe an application of our results to random Fourier series. Random Fourier series are a major tool in harmonic analysis. They have been used to prove results on deterministic Fourier series. For a detailed study we refer to [33], [17], and Chapter 13 in [23]. In particular, conditions for a.s. boundedness and continuity of random Fourier series have been investigated.

One can consider general complex random Fourier series. Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers and $(\xi_n)_{n\in\mathbb{Z}}$ a sequence of (complex valued) i.i.d. random variables. Then one considers the following (complex valued) stochastic process:

$$X_t := \sum_{n\in\mathbb{Z}} a_n \xi_n e^{in2\pi t}, \qquad t \in [0,1]. \tag{2.26}$$

The most classical examples are

- the Rademacher series, where $\mathbb{P}(\xi_n = \pm 1) = 1/2$;
- the Steinhaus series, where $\xi_n = e^{2\pi\omega_n i}$ and (ω_n) are uniformly distributed in $[0,1]$;
- the Gaussian series, where the (ξ_n) are Gaussian.

However, also stable random Fourier series were studied in the context of strongly stationary processes, cf. [34]. In the case of strongly stationary α-stable processes, when $1 \le \alpha < 2$, it is possible to give necessary and sufficient conditions for the a.s. boundedness and continuity of such processes in terms of metric entropy (or majorising measure) conditions. This is surprising, since – unlike in the Gaussian case – for general α-stable processes, there is a huge gap between necessary and sufficient conditions in terms of metric entropy for a.s. boundedness, cf. Section 12.3 in [40].

A natural space for the Fourier series in (2.26) is $L_2([0,1],\mathbb{C})$. Clearly, the norm of X in $L_2([0,1],\mathbb{C})$ equals

$$\|X\|_{L_2([0,1],\mathbb{C})} = \left(\sum_{n\in\mathbb{Z}} |a_n\xi_n|^2 \right)^{1/2}.$$

In some cases, the results of this thesis can solve the small deviation problem for the latter expression, namely, if the $(|a_n|)_{n\in\mathbb{Z}}$ can be rearranged to be a polynomically decreasing sequence. Nevertheless, to avoid having to distinguish different cases, we concentrate on real-valued processes. This is the more important case; and the situation is notationally easier.

Instead of (2.26), we consider the following random Fourier series:

$$X_t := \sum_{n=1}^{\infty} \sigma_n \theta_n \sqrt{2}\, \cos(\pi n t), \qquad t \in [0,1], \tag{2.27}$$

$$X_t := \sum_{n=1}^{\infty} \sigma_n \theta_n \sqrt{2}\, \sin(\pi n t), \qquad t \in [0,1], \tag{2.28}$$

where (σ_n) is a decreasing sequence of positive numbers and the (θ_n) are (real) i.i.d. random variables. Then

$$\|X\|_{L_2[0,1]} = \left(\sum_{n=1}^{\infty} |\sigma_n \theta_n|^2 \right)^{1/2};$$

and Theorem 2.9 yields the following fact.

Corollary 2.4. *Let X be a random Fourier series as in (2.27) or (2.28). If*

$$\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}, \qquad \textit{as } n \to \infty,$$

with $\mu > 1/2$ and $\mathbb{E}|\theta|^{1/\mu} < \infty$ we have

$$\lim_{\varepsilon \to 0+} \varepsilon^{\frac{1}{\mu-1/2}} (-\log \varepsilon)^{\frac{\nu}{\mu-1/2}} \log \mathbb{P}\left(\|X\|_{L_2[0,1]} \le \varepsilon \right) = -C_2,$$

where C_2 is the finite, positive constant in (2.20).

Example: Let $\theta, \theta_1, \theta_2, \ldots$ be an i.i.d. sequence of random variables. Let us consider the process

$$B_t := \theta t + \sqrt{2} \sum_{n=1}^{\infty} \theta_n \frac{\sin \pi n t}{\pi n}, \qquad t \in [0,1]. \tag{2.29}$$

First, we examine the most important case of Gaussian random variables. The following result is well-known, cf. p. 235 in [17] or [21].

Lemma 2.5. *If $\theta, \theta_1, \theta_2, \ldots$ are i.i.d. centred Gaussian random variables, then B defined in (2.29) is a Brownian motion.*

Surely, it makes sense to consider (2.29) also for other types of random variables (θ_n). It is easy to see that

$$\|B\|_{L_2[0,1]} = \left(\frac{|\theta|^2}{3} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} |n^{-1}\theta_n|^2 \right)^{1/2}.$$

By Theorem 1.2, this expression is a.s. finite if and only if $\mathbb{E}|\theta| < \infty$.

In particular, one can study B in $L_2[0,1]$ if (θ_n) are i.i.d. symmetric α-stable with $1 < \alpha \leq 2$. Note that (as can be seen from the next result), unlike in the Gaussian case ($\alpha = 2$), the symmetric α-stable process B is *not* the symmetric α-stable Lévy process.

Corollary 2.4 clarifies the small deviation behaviour of B. We get the following result.

Corollary 2.5. *If $\theta, \theta_1, \theta_2, \ldots$ are i.i.d. random variables with $\mathbb{E}|\theta| < \infty$ then B defined in (2.29) is a.s. in $L_2[0,1]$ and we have*

$$\lim_{\varepsilon \to 0+} \varepsilon^2 \log \mathbb{P}\left(\|B\|_{L_2[0,1]} \leq \varepsilon \right) = -\frac{C_2}{\pi^2},$$

where C_2 is the finite, positive constant defined in (2.20).

Let us mention that, in the Gaussian case, C_2 was calculated in Section 2.6.1. The result was $C_2 = \pi^2/8$. Thus, one obtains the well-known small deviation constant for the Brownian motion in the $L_2[0,1]$ norm: $C_2/\pi^2 = 1/8$ (as noted in the introduction; cf. Proposition 7.2 in [30]).

Chapter 3

Small deviation and metric entropy for stable and Gaussian processes

3.1 The connection between small deviation and metric entropy

In this section, we introduce and discuss the relation between small deviation results and metric entropy for Gaussian and symmetric α-stable stochastic processes.

Let us start with the definition of the (dyadic) entropy numbers. Let us consider two quasi-Banach spaces E and F and a linear bounded operator $u : E \to F$. Then

$$e_n(u) := \inf \left\{ \varepsilon > 0 \mid \exists\, y_1 \dots y_{2^{n-1}} \in F \;:\; u(B_E(0,1)) \subseteq \bigcup_{i=1}^{2^{n-1}} B_F(y_i, \varepsilon) \right\},$$

for every $n \geq 1$, where $B_E(x, \varepsilon)$ denotes the open ball in E with centre x and radius ε and with the same notation for $B_F(y_i, \varepsilon)$. Sometimes, to avoid any confusion, we also write $e_n(u) = e_n(u : E \to F)$.

Entropy numbers are used in many applications, in particular in functional analysis. We refer to [7] for an introduction and the (rather elementary) proofs of the following properties:

- $\|u\| = e_1(u) \geq e_2(u) \geq \ldots \geq 0$, where $\|u\|$ is the operator norm;
- $e_{n+m-1}(u + v) \leq e_n(u) + e_m(v)$, for all operators $u, v : E \to F$ and integers n, m;

- $e_{n+m-1}(u \circ v) \le e_n(u) e_m(v)$, for all compositions $u \circ v$, where $v : D \to E$ and $u : E \to F$ are operators and n, m integers;
- u is compact if and only if $e_n(u) \to 0$, as $n \to \infty$.

Because of the last-mentioned property, entropy numbers are considered to be a measure for the compactness an operator.

Metric entropy tools also have great applications in probability theory. However, it is outside the scope of this thesis to give even an overview; instead we refer to [23].

At this point, we can consider Gaussian processes and the operators related to them. Let $(E, \|.\|)$ be a normed space. Then X is called a centred Gaussian process with values a.s. in the dual space E' if $\langle x, X \rangle$ is a Gaussian random variable for all $x \in E$. We say that X is generated by the operator $u : E \to L_2[0,1]$ if

$$\mathbb{E} e^{i\langle x, X\rangle} = e^{-\|u(x)\|^2_{L_2[0,1]}}, \qquad \text{for all } x \in E.$$

The *analytic* operator u related to X contains all the information on the *stochastic* process X. In particular, the small deviation behaviour of X is encoded by u, as we can see from the following result, which can be found as Theorem 1.2 in [25] – combined with the duality result in [1].

Proposition 3.1. *Let the Gaussian process X be generated by u. Let $\gamma > 1/2$ and $\delta \in \mathbb{R}$ be given. Then the following implications hold.*

(a) We have

$$e_n(u) \succcurlyeq n^{-\gamma}(1 + \log n)^{-\delta}$$

if and only if

$$\log \mathbb{P}\left(\|X\|_{E'} \le \varepsilon\right) \preccurlyeq -\varepsilon^{-\frac{1}{\gamma - 1/2}}(-\log \varepsilon)^{-\frac{\delta}{\gamma - 1/2}},$$

where, for the "if" part, the additional assumption $\log \mathbb{P}\left(\|X\|_{E'} \le \varepsilon\right) \approx \log \mathbb{P}\left(\|X\|_{E'} \le 2\varepsilon\right)$ is required.

(b) We have

$$e_n(u) \preccurlyeq n^{-\gamma}(1 + \log n)^{-\delta}$$

if and only if

$$\log \mathbb{P}\left(\|X\|_{E'} \le \varepsilon\right) \succcurlyeq -\varepsilon^{-\frac{1}{\gamma - 1/2}}(-\log \varepsilon)^{-\frac{\delta}{\gamma - 1/2}}.$$

This result shows that the question of determining the small deviation rate of a Gaussian process is in this polynomial case fully equivalent to the investigation of the compactness properties of the operator generating it.

It seems natural to ask whether similar relations also hold for other classes of processes. Prime candidates for such a generalisation are α-stable processes. In fact, a result of Li and Linde (quoted below as Proposition 3.2) shows that the "only if" part of (a) from Proposition 3.1 is true analogously in the more general setup of symmetric α-stable processes.

The definition of the related operator for stable processes is analogous. We say that X is a symmetric α-stable process with values a.s. in E' if $\langle x, X\rangle$ is a symmetric α-stable random variable for all $x \in E$. We say that X is generated by the operator $u : E \to L_\alpha[0,1]$ if

$$\mathbb{E}e^{i\langle x,X\rangle} = e^{-\|u(x)\|^\alpha_{L_\alpha[0,1]}}, \qquad \text{for all } x \in E.$$

Further details can be found in [26]. The following result can be found as Theorem 4.5 in [26]. Furthermore, the recent work [1] shows that the assumption that the space satisfies property D from [26] can be dropped.

Proposition 3.2. *Let the symmetric α-stable process X be generated by u. Let $\gamma > \max(1 - 1/\alpha, 0)$ and $\delta \in \mathbb{R}$ be given. If*

$$e_n(u) \succcurlyeq n^{-\gamma}(1 + \log n)^{-\delta}$$

then

$$\log \mathbb{P}\left(\|X\|_{E'} \le \varepsilon\right) \preccurlyeq -\varepsilon^{-\frac{1}{\gamma+1/\alpha-1}}(-\log \varepsilon)^{-\frac{\delta}{\gamma+1/\alpha-1}}.$$

Note that Proposition 3.2 becomes the "only if" part of (a) from Proposition 3.1 for the Gaussian case ($\alpha = 2$).

3.2 Independent sequences and diagonal operators

Let us consider the case of our simple stochastic process $X = (\sigma_1\theta_1, \sigma_2\theta_2, \ldots)$, where $\sigma_n \sim n^{-\mu}(1 + \log n)^{-\nu}$, with $\mu > 0$ and $\nu \in \mathbb{R}$, and $\theta_1, \theta_2, \ldots$ are i.i.d. symmetric α-stable (resp. Gaussian) random variables. Since we deal with a particularly simple process and thus operator, it seems promising to investigate the question whether Proposition 3.1 can also hold in all its implications for the non-Gaussian stable case.

In fact, the operator related to the stochastic process $(\sigma_n\theta_n)$ is a diagonal operator generated by the sequence (σ_n).

Lemma 3.1. *Let $1 \le p \le \infty$ and $0 < \alpha \le 2$. The symmetric α-stable process $(\sigma_1\theta_1, \sigma_2\theta_2, \ldots)$ is generated by the operator*

$$D : l_{p'} \to l_\alpha \qquad \text{with} \qquad Dx := (\sigma_1 x_1, \sigma_2 x_2, \ldots).$$

Proof: Let us consider the sequence $X = (\sigma_1\theta_1, \sigma_2\theta_2, \ldots)$ as an element of l_p. Then, for all $x \in l_{p'}$, we have

$$\mathbb{E}e^{i\langle X,x\rangle} = \mathbb{E}e^{i\sum_{n=1}^{\infty}\sigma_n\theta_n x_n} = \prod_{n=1}^{\infty}\mathbb{E}e^{i\sigma_n x_n\theta_n}$$
$$= \prod_{n=1}^{\infty} e^{-|\sigma_n x_n|^\alpha} = e^{-\sum_{n=1}^{\infty}|\sigma_n x_n|^\alpha} = e^{-\|Dx\|_\alpha^\alpha}.$$

This qualifies D as the operator that generates X. ■

Note that the above operator maps to l_α, whereas the definition of an operator generating a stable (resp. Gaussian) process requires an operator mapping to $L_\alpha[0,1]$. In terms of entropy, this is not a difference, since l_α can be isometrically embedded into $L_\alpha[0,1]$.

The entropy numbers of diagonal operators were studied, among others, by Carl and Kühn, cf. [19] and [20] and references therein for the most up-to-date formulation.

In order to be able to make the usual calculations with entropy numbers (in particular to consider dual spaces), we concentrate on the Banach space case, i.e. $1 \le \alpha \le 2$ and $1 \le p \le \infty$.

Let us quote a version of Theorem 1 of [19] (NB: [19] uses $\sim$ for what we call $\approx$ in this thesis).

Proposition 3.3. *Let $1 \le p, q \le \infty$, $\mu > \max(1/q - 1/p, 0)$, and $\nu \in \mathbb{R}$. Then $\sigma_n \approx n^{-\mu}(1+\log n)^{-\nu}$ implies*

$$e_n(D : l_p \to l_q) \approx n^{1/q-1/p-\mu}(1+\log n)^{-\nu}.$$

We can put together the latter result with the connections established by Li and Linde and quoted above as Proposition 3.2.

The Gaussian case: In the Gaussian case, we obtain a different version of Corollary 2.1. It is proven using only the relation between entropy and small deviations and the facts on entropy numbers of diagonal operators.

Corollary 3.1. *Let $\theta_1, \theta_2, \dots$ be i.i.d. standard normal random variables, let $1 \le p \le \infty$, $\mu > 1/p$, and $\nu \in \mathbb{R}$. Then $\sigma_n \approx n^{-\mu}(1+\log n)^{-\nu}$ implies*

$$\log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right) \approx -\varepsilon^{-\frac{1}{\mu-1/p}}(-\log\varepsilon)^{-\frac{\nu}{\mu-1/p}}. \tag{3.1}$$

Proof: We know that $\sigma_n \approx n^{-\mu}(1+\log n)^{-\nu}$ implies, by Proposition 3.3 (since $\mu > 1/p$ implies $\mu > \max(1/p - 1/2, 0) = \max(1/2 - 1/p', 0)$),

$$e_n(D : l_{p'} \to L_2) \approx e_n(D : l_{p'} \to l_2) \approx n^{1/2-1/p'-\mu}(1+\log n)^{-\nu}.$$

This, however, implies (3.1), by Proposition 3.1, since $\mu > 1/p$ implies $\mu + 1/p' - 1/2 > 1/2$. ■

The preceeding fact is not a real surprise. We know from Corollary 2.1 that $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$ with $\mu > 1/p$ implies even

$$\log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right) \sim -C_p\varepsilon^{-\frac{1}{\mu-1/p}}(-\log\varepsilon)^{-\frac{\nu}{\mu-1/p}},$$

as $\varepsilon \to 0+$, for some finite constant $C_p > 0$. So, the entropy methods return the correct rate – it could not have been different.

However, note that Corollary 3.1 is slightly different to Corollary 2.1: While Corollary 2.1 assumes a statement in *strong* asymptotics ($\sim$) and asserts a statement in strong asymptotics, Corollary 3.1 (only) requires *weak* asymptotics ($\approx$) and (only) asserts weak asymptotics.

The non-Gaussian stable case: In the non-Gaussian stable case, the situation is more complicated.

Corollary 3.2. *Let $\theta_1, \theta_2, \dots$ be i.i.d. symmetric α-stable random variables with $1 \le \alpha < 2$. Let $1 \le p \le \infty$, $\nu \in \mathbb{R}$, and assume that $\mu > 1/p$. Then $\sigma_n \approx n^{-\mu}(1+\log n)^{-\nu}$ implies*

$$\log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right) \preccurlyeq -\varepsilon^{-\frac{1}{\mu-1/p}}(-\log\varepsilon)^{-\frac{\nu}{\mu-1/p}}. \tag{3.2}$$

Proof: If $\sigma_n \approx n^{-\mu}(1+\log n)^{-\nu}$ then Proposition 3.3 gives us

$$e_n(D : l_{p'} \to L_\alpha) \approx e_n(D : l_{p'} \to l_\alpha) \approx n^{1/\alpha-1/p'-\mu}(1+\log n)^{-\nu},$$

as long as $\mu > \max(1/\alpha - 1/p', 0) = \max(1/p - 1/\alpha', 0)$, which is satisfied for $\mu > 1/p$. By Proposition 3.2, this implies (3.2), since $\mu - 1/\alpha + 1/p' > \max(1 - 1/\alpha, 0) = 1 - 1/\alpha = 1/\alpha'$. ■

Let us compare this result to Corollary 2.2, which tells us that $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$ implies

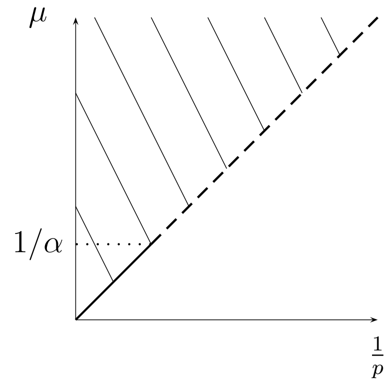
$$\log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right) \sim -C_p \varepsilon^{-\frac{1}{\mu-1/p}}(-\log\varepsilon)^{-\frac{\nu}{\mu-1/p}},$$

as $\varepsilon \to 0+$, for some finite constant C_p, provided $\mu > \max(1/\alpha, 1/p)$. On the other hand, if $\mu \le \max(1/p, 1/\alpha)$, we have $C_p = \infty$. In some of the borderline cases ($\mu = \max(1/p, 1/\alpha)$), Corollary 2.2 gives the order or estimates it.

First of all, we have to notice that – as in the Gaussian case – the entropy technique leads to results assuming weak asymptotics ($\approx$) and asserting weak asymptotics ($\preccurlyeq$), whereas the main result of this thesis assume and assert statements in strong asymptotics ($\sim$).

However, unlike in the Gaussian case, the results that can be obtained using entropy techniques are different to those obtained by the methods developed in this thesis.

Generally, we can say that the (one-sided) estimate in (3.2) – obtained by the entropy technique – is not sharp in all the cases. Corollary 2.2 clarifies the situation for all possible parameter values: For $\mu > \max(1/\alpha, 1/p)$ we get the same rate as in (3.2), for $\mu = \max(1/\alpha, 1/p)$ the rate is different (decreases faster to $-\infty$), and for all other values the small deviation probability is zero. The diagram below shows this situation.



Picture 2: Comparison of Corollary 2.2 with Corollary 3.2.

Note that if $\mu > \max(1/p, 1/\alpha)$, everything is as in the Gaussian case. In the other cases, the rate suggested by the upper bound in (3.2) is not sharp.

Let us demonstrate this with the simple example when $p = \infty$. If $\mu > 1/\alpha$, estimate (3.2) is attained. If we take $\mu = 1/\alpha$ and $\nu > 1/\alpha$, we get (cf. Section 2.5.3) that the rate is

$$\log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right) \approx -\varepsilon^{-\frac{1}{\mu-1/p}}(-\log\varepsilon)^{\frac{\mu-\nu}{\mu-1/p}}.$$

Thus, the rate turns out to be *different* to the one suggested by the entropy result (3.2), i.e. the small deviation function decreases faster than the upper entropy bound. Even more, if $0 < \mu < 1/\alpha$, we have $\mathbb{P}\left(\sup_n |\sigma_n \theta_n| < \infty\right) = 0$, which is not reflected at all by (3.2).

In general, the entropy technique only returns the correct rate in the cases treated by the main theorem, which is $\mu > \max(1/p, 1/\alpha)$ in the α-stable case. The rates are different in the stable case when $1/p < \mu \leq \max(1/p, 1/\alpha)$ due to the tail behaviour of α-stable distributions, cf. discussion after Theorem 2.10. This effect is not represented by the entropy techniques.

Finally, let us discuss the implications for a possible generalisation of Proposition 3.1. It is known (Proposition 3.2) that the "only if" part of (a) is valid analogously. It is imaginable that one can prove the "if" part of (b) analogously in the stable case. However, it is not possible to prove neither the "if" in (a) nor the "only if" in (b) in the general stable case, since this would contradict Corollary 2.2. Cf. Problem 6 on p. 68 and the remark on p. 69.

3.3 Small deviation of stable convolutions via metric entropy

3.3.1 Stable Riemann-Liouville processes

In this final section, let us consider a class of processes where it *is* possible to prove the correct small deviation rate with the help of the metric entropy result quoted in Proposition 3.2. This shows that – even though a general tool as Proposition 3.1 fails to hold in the non-Gaussian stable case – metric entropy is a good way to determine the small deviation rate for non-Gaussian stable processes in *some* cases.

Let us consider processes of the following type:

$$X_t := \sum_{n=0}^{\infty} a_n R_t^{H+n}, \qquad t \in [0,1], \tag{3.3}$$

where $(a_0 := 1, a_1, a_2, \ldots)$ is a sequence of real numbers and

$$R_t^H := \int_0^t (t-s)^{H-1/\alpha}\, dZ_s, \qquad t \in [0,1],$$

is the so-called Riemann-Liouville process with Hurst index $H > 0$. Here, Z is a symmetric α-stable Lévy process. Note that $(R^{H+n})_{n=0}^{\infty}$ are *not* independent; they are defined as fractional integrals of the same α-stable Lévy process Z.

Stable Riemann-Liouville processes have been considered in literature for a long time. They are tightly related to the linear α-stable fractional motion (a possible generalisation of the fractional Brownian motion). For a recent survey we refer to [31]. In the latter work, the small deviation rate of Riemann-Liouville processes are obtained for a wide class of norms.

It is known that R^H is a.s. continuous for $H > 1/\alpha$, a.s. bounded for $H = 1/\alpha$, and a.s. unbounded for $H < 1/\alpha$. This explains the distinction in the following proposition, which can be found as Corollary 8 in [31].

Proposition 3.4. *Let R^H be as defined above with $H \geq 1/\alpha$ and let $\|.\|_p$ be the usual norm in $L_p[0,1]$. Then there exists a constant $0 < K < \infty$ such that*

$$\lim_{\varepsilon \to 0+} \varepsilon^{1/H} \log \mathbb{P}\left(\left\| R^H \right\|_p \leq \varepsilon \right) = -K.$$

In this section, we want to consider the small deviation probabilities under $L_p[0,1]$ norm for the generalisation of Riemann-Liouville processes defined in (3.3). We give examples for this class of processes in Section 3.3.4. A more detailed study has been carried out in [5], where the results are extended to Hölder norms. In the present treatment, we concentrate on the simplest case of the small deviation behaviour of X w.r.t. the $L_p[0,1]$ norm, for $1 \leq p \leq \infty$.

Immediately, the question arises when (3.3) is well-defined, i.e. when the sum converges. It turns out (cf. Lemma 3.3) that the sum converges almost surely in $L_p[0,1]$ if $H \geq 1/\alpha$ and the condition

$$\sum_{n=0}^{\infty} |a_n| n^{H-1/\alpha+\delta} < \infty, \tag{3.4}$$

for holds for $\delta = 0$. However, to prove the upper bound in our small deviation problem, we require (3.4) to hold for some $\delta > 0$.

Let us state the result.

Theorem 3.1. *Let X be as defined in (3.3) and $H \geq 1/\alpha$. Assume that (3.4) holds for some $\delta > 0$. Then, for all $1 \leq p \leq \infty$,*

$$\log \mathbb{P}\left(\left\| X \right\|_p \leq \varepsilon \right) \approx -\varepsilon^{1/H}, \qquad \text{as } \varepsilon \to 0+.$$

The proof is given for the lower and upper bound in the following two subsections, respectively. Finally, we consider some examples in Section 3.3.4.

3.3.2 Lower bound

We start with an easy lemma that states roughly that Riemann-Liouville processes form a semi-group w.r.t. time integration. Let us recall the notation for Euler's Beta function and its relation to the Gamma function:

$$B(p,q) := \int_0^1 (1-r)^{p-1} r^{q-1}\, dr = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad p,q>0.$$

Lemma 3.2. *For every $H \geq 1/\alpha$ and $\beta > 0$, we have a.s.*

$$B(\beta, H-1/\alpha+1) R_t^{H+\beta} = \int_0^t (t-s)^{\beta-1} R_s^H \, ds.$$

Proof: Notice first that the integration on the right-hand side makes sense since almost surely $R^H \in L_\infty[0,1]$ and $s \mapsto (t-s)^{\beta-1}\, \mathbb{I}_{\{s\leq t\}}$ is in $L_1[0,1]$. We have a.s.

$$\begin{aligned}\int_0^t (t-s)^{\beta-1} R_s^H \, ds &= \int_0^1 \mathbb{I}_{\{s\leq t\}}(t-s)^{\beta-1} \left[\int_0^1 \mathbb{I}_{\{u\leq s\}}(s-u)^{H-1/\alpha}\, dZ_u\right] ds \\ &= \int_0^t \left(\int_u^t (t-s)^{\beta-1}(s-u)^{H-1/\alpha}\, ds\right) dZ_u,\end{aligned}$$

where we applied Fubini's Theorem in the second line. Making the substitution $r = (s-u)/(t-u)$, we obtain

$$\begin{aligned}&\int_0^t \left(\int_u^t (t-s)^{\beta-1}(s-u)^{H-1/\alpha}\, ds\right) dZ_u \\ &\quad = \int_0^t \left(\int_0^1 (1-r)^{\beta-1} r^{H-1/\alpha}\, dr\right)(t-u)^{\beta+H-1/\alpha}\, dZ_u \\ &\quad\quad = \mathrm{B}(\beta, H-1/\alpha+1)\int_0^t (t-u)^{\beta+H-1/\alpha}\, dZ_u,\end{aligned}$$

which completes the proof. ■

The last lemma combined with Hölder's Inequality and Stirling's Formula easily leads to the following fact.

Lemma 3.3. *Let $1 \leq p \leq \infty$ and $H \geq 1/\alpha$. There exists a constant $C>0$ independent of n such that a.s., for all $n \geq 1$,*

$$\left\| R^{H+n} \right\|_p \leq C\, n^{H-1/\alpha} \left\| R^H \right\|_p .$$

Now the lower bound for processes (3.3) w.r.t. the $L_p[0,1]$ norm follows.

Proof of the lower bound given in Theorem 3.1: By Lemma 3.3 and the triangle inequality, we have

$$\|X\|_p \le \sum_{n=0}^{\infty} |a_n| \left\|R^{H+n}\right\|_p \le \left(1 + C\sum_{n=1}^{\infty} |a_n| n^{H-1/\alpha}\right) \left\|R^H\right\|_p = C' \left\|R^H\right\|_p ,$$

where condition (3.4) ensures that $C' < \infty$. Hence,

$$\log \mathbb{P}\left(\|X\|_p \le \varepsilon\right) \ge \log \mathbb{P}\left(\left\|R^H\right\|_p \le \varepsilon/C'\right) \ge -C''\varepsilon^{-1/H},$$

where the last inequality comes from Proposition 3.4. ■

3.3.3 Upper bound via metric entropy

In this section we prove the upper bound given in Theorem 3.1. We use metric entropy tools, concretely Proposition 3.2. In order to do so, we have to find lower estimates for the entropy numbers of the operator related to the process (3.3). For the purpose of determining this operator, let us recall the notation of the Riemann-Liouville operator R_β, $\beta > 0$, defined on $L_\infty[0,1]$ as follows:

$$(R_\beta x)(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s)\, ds, \qquad t \in [0,1].$$

The Riemann-Liouville operator – also called fractional integration operator – has been the topic of investigation over the years in many areas of analysis and also probability theory, cf. [38], [37], [30]. Also, the entropy numbers of R_β have been studied e.g. in [25]. However, the following result seemed to be folklore; it is proved rigorously only in [5].

Lemma 3.4. *For all $\beta > 1/2$ and $0 < \alpha \le 2$, we have*

$$e_n(R_\beta : L_\infty[0,1] \to L_\alpha[0,1]) \approx n^{-\beta}.$$

Now let us determine the operator that generates (3.3).

Lemma 3.5. *Let us assume that (3.4) holds for $\delta = 0$. Then the process X is generated by the operator $\bar{R}_X : L_\infty \to L_\alpha$, defined by $\bar{R}_X x := R_X \bar{x}$, where $\bar{x}(t) = x(1-t)$ and the operator $R_X : L_\infty \to L_\alpha$ is given by*

$$R_X := \sum_{n=0}^{\infty} a_n \Gamma(H - 1/\alpha + n + 1)\, R_{H-1/\alpha+n+1}.$$

Furthermore, R_X has the same entropy behaviour as $\bar{R}_X$.

Proof: Let us show first that the operator R_X is well-defined, i.e. that the sum converges in the operator norm. This, however, follows from the elementary fact

$$\|R_\beta : L_\infty \to L_\alpha\| \le \|R_\beta : L_\alpha \to L_\alpha\| = \frac{1}{\Gamma(\beta+1)}, \tag{3.5}$$

and assumption (3.4).

On the other hand, with the help of Theorem 11.4.1 in [40] (interchange of the stochastic integration with the sum and the deterministic integration, respectively) and condition (3.4) it is easy to show that

$$\mathbb{E}e^{i\langle x,X\rangle} = \exp\left(-\left\|\int_0^{.} \sum_{n=0}^{\infty} a_n(.-s)^{H-1/\alpha+n} x(1-s)\,ds\right\|_\alpha^\alpha\right) = e^{-\left\|\bar{R}_X x\right\|_\alpha^\alpha}.$$

It is a standard argument to see that $e_n(\bar{R}_X) \approx e_n(R_X)$. ■

Equipped with these tools we can finally prove the upper bound.

Proof of the upper bound given in Theorem 3.1: By the last lemma and Proposition 3.2, it is sufficient to find a lower estimate for the entropy numbers of R_X defined above. We define

$$u := R_X - \Gamma\left(H - \frac{1}{\alpha} + 1\right) R_{H-\frac{1}{\alpha}+1} = \sum_{n=1}^{\infty} a_n \Gamma\left(H - \frac{1}{\alpha} + n + 1\right) R_{H-\frac{1}{\alpha}+n+1}.$$

By Lemma 3.4 and the additivity property of entropy numbers,

$$\begin{aligned} & e_k\left(R_X : L_\infty \to L_\alpha\right) + e_k\left(u : L_\infty \to L_\alpha\right) \\ & \qquad \ge \Gamma(H-1/\alpha+1) e_{2k+1}\left(R_{H-1/\alpha+1} : L_\infty \to L_\alpha\right) \approx k^{-(H-1/\alpha+1)}, \end{aligned} \tag{3.6}$$

as $k \to \infty$. Fix $0 < \delta < 1$ such that (3.4) holds. By the triangle inequality and the semi-group property of the Riemann-Liouville operator ($R_\beta \circ R_\gamma = R_{\beta+\gamma}$, for all $\beta, \gamma > 0$), we see that for every $x \in L_\infty$

$$\begin{aligned} \|ux\|_2 &\le \sum_{n=1}^{\infty} |a_n| \Gamma(H-1/\alpha+n+1) \|R_{n-\delta} : L_2 \to L_2\| \left\|R_{H-1/\alpha+1+\delta}\, x\right\|_2 \\ &= \left\|R_{H-1/\alpha+1+\delta}\, x\right\|_2 \sum_{n=1}^{\infty} |a_n| \frac{\Gamma(H-1/\alpha+n+1)}{\Gamma(n-\delta+1)} \\ &\le C \left\|R_{H-1/\alpha+1+\delta}\, x\right\|_2 \sum_{n=1}^{\infty} |a_n| n^{H-1/\alpha+\delta} \end{aligned}$$

where the second line comes from (3.5) and the last from Stirling's Formula. By assumption (3.4), we get

$$\|ux\|_2 \le C' \left\|R_{H-1/\alpha+1+\delta} x\right\|_2$$

for all $x \in L_\infty$ and a constant $C' < \infty$; and it follows from Lemma 4.2 in [30] that

$$e_k \left(u : L_\infty \to L_2\right) \le C'' e_k \left(R_{H-1/\alpha+1+\delta} : L_\infty \to L_2\right) \approx k^{-(H-1/\alpha+1+\delta)},$$

the last fact coming from Lemma 3.4. Recalling the obvious inequality

$$e_k \left(u : L_\infty \to L_\alpha\right) \le e_k \left(u : L_\infty \to L_2\right),$$

for every $0 < \alpha \le 2$, and putting this together with (3.6) yields

$$e_k \left(R_X : L_\infty \to L_\alpha\right) \ge D\, k^{-(H-1/\alpha+1)} - e_k \left(u : L_\infty \to L_\alpha\right) \ge D'\, k^{-(H-1/\alpha+1)}$$

as $k \to \infty$. Thus, by Proposition 3.2 and Lemma 3.5,

$$\log \mathbb{P} \left(\|X\|_p \le \varepsilon\right) \le \log \mathbb{P} \left(\|X\|_1 \le \varepsilon\right) = \log \mathbb{P} \left(\|X\|_{L'_\infty} \le \varepsilon\right) \preccurlyeq -\varepsilon^{1/H}.$$

■

3.3.4 Examples

The class of processes X defined in (3.3) contains so-called stable convolutions. By that we mean processes of the following type:

$$X_t = \int_0^t f(t-s)\, dZ_s, \qquad t \in [0,1], \tag{3.7}$$

where $f : [0,1] \to \mathbb{R}$ is a smooth function and Z is as above. In particular, we assume that f can be expanded into a Taylor series about zero in the following way

$$f(s) = \sum_{n=0}^{\infty} a_n s^{H-1/\alpha+n}, \qquad \text{for } s \in [0,1],$$

with $a_0 = 1$ and $H \ge 1/\alpha$. If the radius of convergence of the series is greater than 1, we additionally know that (3.4) holds for some $\delta > 0$.

The class of processes defined in (3.7) is called stable convolutions because it can be interpreted as a convolution of the smooth function f with the symmetric α-stable Lévy process Z. Continuity and boundedness aspects of general stochastic convolutions have been studied recently in [22].

Note that Theorem 11.4.1 in [40] tells us that (3.7) can be represented as in (3.3), by interchanging the sum and the (stochastic) integration.

In particular, stable convolutions include the well-known stable Ornstein-Uhlenbeck process

$$X_t = \int_0^t e^{-\lambda(t-s)}\, dZ_s = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}\, R_t^{1/\alpha+n},$$

for $\lambda > 0$, and so-called stable fractional Ornstein-Uhlenbeck type processes

$$X_t = \int_0^t x^c e^{-\lambda(t-s)}\, dZ_s = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}\, R_t^{c+1/\alpha+n},$$

with $\lambda, c > 0$. The latter processes have been introduced recently (cf. [32], [8], and [41]) and have applications in financial mathematics and network traffic. For references to the applications and a further overview we refer to Section 4 of [5].

Chapter 4

Open problems

Of course, the research presented in this thesis leaves many questions to be solved. Let us list a few of them.

1. The most desirable improvement would be to adapt the main argument in Chapter 2 to the case of regularly varying functions. This was already carried out sucessfully for the upper bound in Section 2.3. The lower bound, however, was only proved for the example $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$. From the results one may guess that similar relations hold if $(1 + \log n)^{-\nu}$ is replaced by a slowly varying function satisfying the additional conditions described in Section 2.3. Apart from the greater generality, the main advantage of this approach would be to avoid the technical calculations in Section 2.4.

2. Certainly, it would be interesting to obtain better estimates for the superexponential cases (third and second case of Theorem 2.10, $\mu = 1/p \geq 1/\alpha$). Here, Theorem 2.10 offers only very rough bounds. This case is of interest as well in the Gaussian setup, i.e. let us take (θ_n) i.i.d. Gaussian random variables and consider

$$\mathbb{P}\left(\sum_{n=1}^{\infty} \frac{\theta_n^2}{n(1+\log n)^b} \leq \varepsilon^2\right), \qquad \text{as } \varepsilon \to 0+; \text{ for } b > 1.$$

3. Similarly to the last point, it is also interesting to investigate the existence of the small deviation constant for the other borderline case (the second case in Theorem 2.10) that only occurs for sufficiently heavy-tailed distributions.

4. The main lemma (Theorem 2.1 and Theorem 2.2, respectively) does not lead to sharp bounds for the case of sequences that are not regularly

varying, e.g. exponentially decreasing sequences (as $\sigma_n \sim e^{-n}$). Here, a different approach is needed to determine the small deviation rate. Also theses cases were handled in literature (e.g. Proposition 4 in [12]) under strong assumptions such as finite variance. However, the necessary conditions in Section 1.4 show that it is not necessary to assume that the involved random variables have finite variance – even less if the sequence decreases sufficiently fast.

5. It is furthermore desirable to clarify the existence of the small deviation constant for the case of totally skewed α-stable random variables, with $0 < \alpha < 1$ and for all $1 \leq p \leq \infty$. One might guess from Theorem 2.11 that this is an easy exercise. However, it has unfortunately not been possible so far. Even though this seems uninteresting from a theoretical point of view, one might recall that totally skewed random variables are important in applications.

6. It is of completely different quality and much higher importance to study the question whether the "if" part of (b) of Proposition 3.1 also holds in the stable case. The results found so far do not contradict such a statement. Of course, it is not clear that this connection can be proved with the help of the methods developed in [26]. Another answer to this question would be a counterexample.

7. One might ask for the subexponential rate in Theorem 2.9, i.e. the rate of

$$\exp\left(C_p \varepsilon^{-\frac{1}{\mu-1/p}} (-\log \varepsilon)^{-\frac{\nu}{\mu-1/p}}\right) \mathbb{P}\left(\left\|(\sigma_n \theta_n)\right\|_p \leq \varepsilon\right), \qquad \text{as } \varepsilon \to 0+.$$

   This was investigated in [12] under the assumption of finite variance (quoted above as Proposition 2.1). As we have shown in this thesis, it is not necessary to assume that the involved distribution have finite variance in order to study the small deviation problem. However, the subexponential rate does depend on the concrete behaviour of θ at the origin, as Proposition 2.1 or the resuls in [15] clearly show. Thus, it is not sufficient to assume only condition (O). It is a very interesting question to which extent it also depends on the concrete upper tail behaviour.

8. There are several open problems related to stable convolutions considered in Section 3.3. They are explained in detail in [5].

Remark

The main ideas of Chapter 2 and Chapter 3.3 will be published, respectively, in *Journal of Theoretical Probability* and *ESAIM: Probability & Statistics*, cf. [3] and [5].

After the submission of this thesis, it has been possible to solve some of the open questions partially. In particular, Problem 6 could be answered affirmatively, cf. [2]. Also, Problem 1 could be solved for regularly varying functions, though only for weak asymptotics, cf. [4].

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Hilfsmittel und Literatur angefertigt habe.

Jena, den 29.05.2006